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REPRESENTATIONS OF SURFACE GROUPS IN COMPLEX HYPERBOLIC SPACE

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Introduction

Let S be a closed Riemann surface of genus $g > 1$ and let $\rho\colon \pi_1(S) \to \mathrm{PSU}(1,n)$ be a representation of its fundamental group in the group of motions of complex hyperbolic n-space. There is a natural characteristic number associated with ρ, which can be defined as follows. Let D denote the hyperbolic plane (i.e., the universal cover of S), and let B^n denote the unit ball in $\mathbf{C}^n$ with its Bergmann metric (complex hyperbolic n-space) and Kähler form ω. The representation ρ determines a flat bundle over S with fiber B^n. Since B^n is contractible this bundle has a section, which is equivalent to an equivariant mapping

$$f\colon D \to B^n, \qquad f(\gamma x) = \rho(\gamma) f(x) \quad \text{for all } \gamma \in \pi_1(S).$$

The form $f^*\omega$ is invariant under the action of $\pi_1(S)$ on D, hence descends to a form on S that will still be denoted by $f^*\omega$. The characteristic number in question is

$$c(\rho) = \int_S f^*\omega.$$

Since B^n is contractible, any two equivariant maps are equivariantly homotopic, so the value of $c(\rho)$ is independent of the choice of f. Using the techniques of bounded cohomology, it is proved in [2] that this characteristic number satisfies the inequality $|c(\rho)| \le 4\pi(g-1)$.

The purpose of this paper is to prove the following theorem, which completely characterizes the case in which equality holds.

Theorem. *If $|c(\rho)| = 4\pi(g-1)$, then the image of ρ leaves a complex line in B^n invariant.*

Observe that a complex line in B^n is the same as a complex totally geodesic subspace (with respect to the Bergmann metric) of complex dimension one.

If we think of $\mathrm{SU}(1,n)$ as the subgroup of $\mathrm{SL}(n+1,\mathbf{C})$ that leaves invariant the Hermitian form $|z_0|^2 - |z_1|^2 - \cdots - |z_n|^2$, and $\mathrm{PSU}(1,n)$ as $\mathrm{SU}(1,n)$ modulo

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its center (a cyclic group of order $n+1$), any subgroup that leaves a complex line invariant is conjugate to the subgroup of all matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \qquad A \in U(1,1),\ B \in U(n-1),\ \det(A)\det(B) = 1.$$

From ρ we thus obtain a representation ρ_1 into $\mathrm{PSU}(1,1)$ namely by restricting to the invariant line (the matrices A in the above decomposition), and ρ_1 also has characteristic number of absolute value $4\pi(g-1)$ since the Kähler class of B^n restricts to that of the invariant line. Now the representations of $\pi_1(S)$ in $\mathrm{PSU}(1,1) = \mathrm{PSL}(2,\mathbf{R})$ with maximum characteristic number have been classified by W. Goldman in his thesis [3]: these are precisely the Fuchsian representations, i.e., the faithful representations with discrete and co-compact image. Goldman's theorem and some simple considerations give the following corollary:

Corollary. *If* $|c(\rho)| = 4\pi(g-1)$, *then there exist representations* $\rho_1 : \pi_1(S) \to \mathrm{PSU}(1,1)$ *and* $\rho_2 : \pi_1(S) \to \mathrm{U}(n-1)$, *with* ρ_1 *Fuchsian, such that* ρ *is conjugate to*

$$\begin{pmatrix} (\det \rho_2)^{-1/2} & 0 \\ 0 & \rho_2 \end{pmatrix}.$$

From this corollary one concludes easily that the set of isomorphism classes of flat $\mathrm{PSU}(1,n)$-bundles over S with $c(\rho) = 4\pi(g-1)$ forms a component of the space of all flat $\mathrm{PSU}(1,n)$-bundles over S, and that this component is homeomorphic to the Cartesian product of the Teichmüller space of S with the space of isomorphism classes of flat $\mathrm{U}(n-1)$-bundles over S (cf. [4, Theorem 6]).

The proof of the Theorem is motivated by the proof presented by Thurston of the "strict version of Gromov's theorem" [8, Theorem 6.4]. We show that the equivariant map f as above has a measurable extension to the boundary of D. Then we show that the image of the boundary of D is the boundary of a complex line in B^n, and that this line is invariant under ρ. The referee points out that the existence of the boundary map follows from a quite general theorem of Zimmer [10, Theorem 4.3.9], but, since some of the details of the construction given here are needed for the proof, the arguments cannot really be shortened by appealing to this general principle.

The above Theorem was the motivation for the work of Goldman and Millson on the local rigidity of "Fuchsian" representations of $\mathrm{SU}(1,n)$ in $\mathrm{SU}(1,n+k)$ [5], and for their conjecture on the global rigidity of such representations. Their rigidity conjecture has been proved recently, for $n > 1$, by K. Corlette [1]. He establishes the existence of a harmonic section of the corresponding flat bundle and then applies Siu's rigidity technique [7]. Thus the present paper, combined with [1], shows that the rigidity result desired by Goldman and Millson holds for all n and k. We refer to [5] for the precise meaning of rigidity.

We remark that our Theorem can also be proved by using Corlette's existence theorem of a harmonic section and then applying word for word the arguments used to prove the main theorem of [9], which is the special case of the present theorem in which the image of ρ is a co-compact discrete subgroup. In fact our motivation in writing [9] was to prove the present theorem, but the existence of a harmonic section was not available at that time. The relation between these two approaches is, heuristically speaking, that here we construct the boundary values (at infinity) of the harmonic section without constructing the section itself.

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1. Bounded cohomology

For the proof of the Theorem we assume some familiarity with the techniques of bounded cohomology as presented in [6, Chapter 2]. We will compute the cohomology of S by the complex of "straight cochains". This is the complex of Borel measurable functions on $D \times \cdots \times D \to \mathbf{R}$ which are invariant under the action of $\Gamma = \pi_1(S)$: $c(\gamma x_0, \cdots, \gamma x_p) = c(x_0, \cdots, x_p)$ for all $\gamma \in \Gamma$ and all $x_0, \cdots, x_p \in D$. Straight k-cochains are therefore functions on $\Gamma \backslash D^{k+1}$, and the points of $\Gamma \backslash D^{k+1}$ are in one-to-one correspondence with the geodesic singular k-simplices on S, hence the terminology straight cochain. The coboundary operator in this complex is defined by the usual formula $\delta c(x_0, \cdots, x_{p+1}) = \sum((-1)^i c(x_0, \cdots, \hat{x}_i, \cdots, x_{p+1})$.

Finite linear combinations of points of $\Gamma \backslash D^{k+1}$ are called straight chains, and finite Borel (signed) measures on $\Gamma \backslash D^{k+1}$ are called (straight) measure chains. Either of these two complexes of chains computes the homology of S. The differential on the complex of measure chains is defined by duality. In other words, the boundary of a measure cycle μ is uniquely characterized by the requirement that $\partial \mu(c) = \mu(\delta c)$ for all continuous straight cochains c. This definition agrees with the usual definition of boundary on the subcomplex of straight chains, i.e., the measures that are finite linear combinations of Dirac masses.

The characteristic number in question is represented by the straight 2-cochain, denoted $f^*\omega$, defined by the formula

$$f^*\omega\langle x_0, x_1, x_2\rangle = \int_{\langle fx_0, fx_1, fx_2\rangle} \omega,$$

where $\langle fx_0, fx_1, fx_2\rangle$ denotes a geodesic triangle in B^n with vertices fx_0, fx_1, fx_2, and $f\colon D \to B^n$ is the equivariant map as in the introduction. Observe that since B^n does not have constant curvature, the vertices fx_i do not determine a unique geodesic triangle. For the purpose of the above formula, by a geodesic triangle we mean any two-simplex whose edges are the geodesic segments joining the vertices fx_i. Since ω is an exact form on B^n, its integral over a triangle depends only on the boundary, hence the value of the integral is independent of the choice of triangle filling in the edges. If a particular choice of a spanning triangle is desired, the center of gravity construction in [2, p. 462] is a natural choice.

In [2] it is proved that for all $y_0, y_1, y_2 \in B^n$, $|\int_{\langle y_0,y_1,y_2\rangle} \omega| < \pi$, hence $f^*\omega$ represents a bounded cohomology class on S with sup norm at most π. The proof was based on the following formula, which is a special case of the proposition in §2 of [2]. We assume that $y_0 = 0$. Then

$$\int_{\langle 0,y_1,y_2\rangle} \omega = \arg((1 - y_1 \cdot \bar{y}_2)/(1 - \bar{y}_1 \cdot y_2)), \tag{1.1}$$

where the dot denotes the sum of products of the components of the corresponding vectors in $\mathbf{C}^n$.

We observe that this formula has the following geometric interpretation. Let $[y_0, y_1]$ denote the intersection with B^n of the complex line in $\mathbf{C}^n$ through y_0 and y_1, and let $\pi\colon B^n \to [y_0, y_1]$ be the orthogonal projection along geodesics (which, when $y_0 = 0$, is easily checked to agree with the orthogonal projection with respect to the Hermitian inner product in $\mathbf{C}^n$). Then

$$\int_{\langle y_0,y_1,y_2\rangle} \omega = \int_{\langle y_0,y_1,\pi y_2\rangle} \omega. \tag{1.2}$$

This formula can be easily proved, independently of (1.1), as follows. Since ω is an exact form on B^n, its integral over the boundary of the tetrahedron $\langle y_0, y_1, y_2, \pi y_2\rangle$ vanishes. Therefore, if we can show that the integral of ω over each of the faces $\langle y_0, y_2, \pi y_2\rangle$ and $\langle y_1, y_2, \pi y_2\rangle$ vanishes, then the vanishing of the integral over the boundary would be equivalent to (1.2).

To this end, consider the face $\langle y_0, y_2, \pi y_2\rangle$. It is formed by geodesics from the vertex y_2 and orthogonal not just to the edge $\langle y_0, \pi y_2\rangle$, but also to the whole complex line $[y_0, y_1]$ which is the complexification of this edge. From this it is easy to see that this face is a totally real submanifold of B^n, hence the restriction of ω to it is identically zero. The same reasoning applies to the face $\langle y_1, y_2, \pi y_2\rangle$, thereby completing the proof of formula (1.2).

Observe that formula (1.2) holds also when any of the y_i lie in the boundary of B^n, so we may let the y_i be arbitrary points in the closed ball $\bar{B}^n$.

Lemma (1.3). *For all $y_0, y_1, y_2 \in \bar{B}^n$, $|\int_{\langle y_0,y_1,y_2\rangle} \omega| \le \pi$, and equality holds if and only if there is a complex line in $\mathbf{C}^n$ so that y_0, y_1, y_2 lie in the intersection of this line with the boundary of B^n.*

Proof. The inequality is clear from (1.2), since $\langle y_0, y_1, \pi y_2\rangle$ is a triangle in the hyperbolic plane $[y_0, y_1]$. If equality holds, $y_0, y_1, \pi y_2$ must all belong to the boundary of $[y_0, y_1]$. But, by the strict convexity of the boundary of B^n, if πy_2 belongs to the boundary of $[y_0, y_1]$, so must y_2, and the proof is complete.

With this lemma we get a geometric proof of the bound for $c(\rho)$, independent of the computations with the potential for the Bergmann metric in [2]:

Proposition (1.4). $|c(\rho)| \le 4\pi(g-1)$.

Proof. The number $c(\rho)$ is the evaluation on $[S]$, the fundamental cycle of S, of the straight cochain $f^*\omega$. By Lemma (1.3), the sup norm of this cochain is at most π: $\|f^*\omega\|_\infty \le \pi$. But the value of the L^1 norm of $[S]$ is well known: $\|[S]\|_1 = 4(g-1)$, and a proof of this fact can be found in [8, Theorem 6.2], or, by a more elementary argument, in the introduction to [6]. By the duality between the L^1 and L^∞-norms,

$$|c(\rho)| = |f^*\omega([S]) \le \|f^*\omega\|_\infty \|[S]\|_1 = 4\pi(g-1),$$

which completes the proof of the proposition.

The case of equality in Lemma (1.3) strongly suggests that if equality holds in this proposition then ρ must leave a complex line invariant. In order to make this precise we will need the following estimate for the sides of a geodesic triangle in B^n so that the integral of ω is very nearly maximal:

Lemma (1.5). *If $\langle y_0, y_1, y_2\rangle$ is a geodesic triangle in B^n such that $|\int_{\langle y_0,y_1,y_2\rangle} \omega| = \pi - e^{-x}$, $x \gg 0$, then $d(y_i, y_j) > x$.*

Proof. We may assume that $y_0 = 0$ and that $\int_{\langle 0,y_1,y_2\rangle} \omega > 0$. By (1.2),

$$\int_{\langle 0,y_1,y_2\rangle} \omega = \int_{\langle 0,y_1,\pi y_2\rangle} \omega = \int_{\langle 0,r,z\rangle} \omega',$$

where $r = |y_1|$, $\pi y_2 = z y_1$, $|z| < 1$, $\mathrm{Re}(z) > 0$, and ω' is the Kähler form of the hyperbolic plane $|z| < 1$. Applying (1.1) to this hyperbolic plane we get

$$\int_{\langle 0,r,z\rangle} \omega' = \arg((1 - rz)/(1 - r\bar{z})).$$

Fix a positive angle $\theta < \pi$, and let L be the ray in the upper half-plane of the complex variable z that starts at the point $z = 1$ and makes an angle $\theta/2$ with the interval $[0, 1]$. Then $\arg((1 - rz)/(1 - r\bar{z})) = \theta$ precisely when rz lies in the intersection of L with the unit disk. The minimum value of $r|z|$ occurs when the segment from 0 to rz makes a right angle with L, and in this case $r|z| = \cos(\pi/2 - \theta/2)$. Therefore for all r, z such that $\arg((1-rz)/(1-r\bar{z})) = \theta$ we have the inequality

$$r|z| \geq \cos(\pi/2 - \theta/2).$$

Now let $\theta = \pi - e^{-x}$. Then for all r, z under consideration we have the inequality

$$r \geq r|z| \geq \cos(e^{-x}/2) \geq 1 - e^{-2x}/8.$$

Substituting this inequality in the formula for the distance $d(y_0, y_1)$:

$$d_{B^n}(y_0, y_1) = d_{\{|z|<1\}}(0, r) = \log((1 + r)/(1 - r)),$$

we get the desired inequality for $d(y_0, y_1)$:

$$d(y_0, y_1) \geq \log((2 - e^{-2x}/8)/(e^{-2x}/8)) \geq 2x - C > x$$

for $x \gg 0$ and for some positive constant C. Since we can apply the same argument replacing y_0 and y_1 by any two vertices of the triangle, we get $d(y_i, y_j) > x$, as desired.

2. Proof of the Theorem

To prove the theorem we make use of measure cycles, as in [6], [8], to represent the fundamental cycle of S. We start by reviewing this technique.

We write G for the group of isometries of D; it has two components, the identity component being $\mathrm{PSL}(2, \mathbf{R})$. Let X denote the quotient space $\Gamma\backslash G$ and let μ denote Haar measure on G, normalized so that $\mu(X) = \mathrm{Area}(S) = 4\pi(g - 1)$. We let σ_i be a fixed equilateral triangle in D with sides of length i, and $A(\sigma_i)$ denotes its area. Finally we let Z_i denote the measure cycle consisting of all G-translates of σ_i each weighted with coefficient $1/A(\sigma_i)$. By this we mean that Z_i is the linear functional on straight two-cochains on S whose value on the cochain c is given by the formula

$$c(Z_i) = \int_X A(\sigma_i)^{-1} c(g\sigma_i)\, d\mu(g).$$

This generalized chain is actually a generalized cycle because each edge in its boundary belongs to precisely two translates of σ_i with opposite orientations. To compute the homology class it represents we evaluate it on the area form dA of S:

$$dA(Z_i) = \mu(X) = 4\pi(g - 1) = dA([S]),$$

therefore Z_i represents the fundamental cycle $[S]$.

From this it follows that, if ρ is a representation with maximum characteristic number, then

$$(f^*\omega)(Z_i) = \int_X A(\sigma_i)^{-1}(f^*\omega)(g\sigma_i)\,d\mu = 4\pi(g-1),$$

or equivalently,

$$\int_X \int_{st(fg\sigma_i)} \omega\, d\mu = A(\sigma_i)\mu(X), \tag{2.1}$$

where $st(fg\sigma_i)$ denotes the geodesic triangle in B^n with vertices f(vertices of $g\sigma_i$). This formula, which makes sense by itself without any reference to measure cycles and bounded cohomology, is actually the only place where we make any essential use of these techniques. We will now follow Thurston [8] in showing that (2.1) implies that f extends to a measurable map $f\colon \bar{D} \to \bar{B}^n$, and that the extended f maps the boundary of D to the boundary of B^n. Our Theorem will follow easily from the existence of this extension. First we need some lemmas.

Lemma (2.2). *$\pi - A(\sigma_i) \sim 6e^{-i/2}$ as $i \to \infty$.*

Proof. From the hyperbolic law of cosines,

$$\cosh C = \cosh A \cosh B - \cos\gamma \sinh A \sinh B,$$

applied to our equilateral triangle σ_i, it follows that $1 - \cos\gamma \sim 1/\cosh i$ as $i \to \infty$, therefore $\gamma \sim 2e^{-i/2}$. Since $A(\sigma_i) = \pi - 3\gamma$, we get the lemma.

Lemma (2.3). *Let $\epsilon_i = \pi - A(\sigma_i)$, and let $Y = \{g \in X\colon |\int_{st(fg\sigma_i)}\omega| < A(\sigma_i) - i^2\epsilon_i\}$. Then $\mu(Y) < \mu(X)/i^2$.*

Proof. Let $\phi(g) = \int_{st(fg\sigma_i)}\omega$. Since $\phi(g) > A(\sigma_i) - i^2\epsilon_i$ on Y, and $\phi(g) < A(\sigma_i) + \epsilon_i = \pi$ on X, it follows from (2.1) that

$$A(\sigma_i)\mu(X) = \int_X \phi(g)\,d\mu < A(\sigma_i)\mu(X) + \epsilon_i(\mu(X-Y) - i^2\mu(Y)),$$

therefore $\mu(X) - i^2\mu(Y) > \mu(X-Y) - i^2\mu(Y) > 0$, in other words, $\mu(Y) < \mu(X)/i^2$, as desired.

Proposition (2.4). *For every $x \in D$ and for almost every geodesic ray γ with $\gamma(0) = x$, $\lim_{i\to\infty} f(\gamma(i))$ exists and belongs to the boundary of B^n. Moreover, if $y \in \partial D$ is the endpoint of such a ray, then the value of $\lim_{i\to\infty} f(\gamma(i))$ depends only on y.*

Proof. Let F be the interior of a fundamental domain for the action of Γ on D. The set of all geodesic triangles $g\sigma_i$ with first vertex in F has full measure in the set of all triangles forming the cycle Z_i, and from now on it will be sufficient to consider only such triangles. By Lemma (2.3), for any i_0 we have that for all $g\sigma_i$, except those in a set of measure at most $\mu(X)\sum_{i_0}^{\infty} 1/i^2$, the inequality

$$\left|\int_{st(fg\sigma_i)} \omega\right| \geq \pi - i^2\epsilon_i$$

holds for all $i > i_0$. In particular for all triangles except those in the small exceptional set, $|\int_{st(fg\sigma_i)} \omega|$ converges exponentially fast to π as $i \to \infty$. Therefore, by Lemma (1.5), the length of each edge converges linearly to infinity. Letting $i_0 \to \infty$ we obtain that for almost all triangles $g\sigma_i$, with the first vertex in F, the length of the edges of $st(fg\sigma_i)$ converge linearly to infinity as $i \to \infty$.

There exists then a point $x_0 \in F$ so that for almost all triangles $g\sigma_i$, with first vertex x_0, the lengths of the edges of $st(fg\sigma_i)$ converge linearly to infinity. These triangles are in 2-1 correspondence with their first edges. Thus we get, in particular, that for almost all geodesics γ with $\gamma(0) = x_0$, the distance $d(f(\gamma(0)), f(\gamma(i)))$ converges linearly to infinity. Since f is a Lipschitz map, $d(f(\gamma(i), f(\gamma(i+1)))$ is bounded, therefore, since B^n has negative curvature, bounded away from zero, by standard angle comparison theorems the angles at $f(\gamma(0))$ between the geodesic segments $\langle f(\gamma(0)), f(\gamma(i))\rangle$ and $\langle f(\gamma(0)), f(\gamma(i+1))\rangle$ form an exponentially decreasing function of i. Therefore, the sequence of angles formed by the geodesic segments $\langle f(\gamma(0)), f(\gamma(i))\rangle$ with a fixed direction at $f(\gamma(0))$ is a Cauchy sequence, hence the sequence of directions of these segments is convergent. But this means precisely that the sequence $f(\gamma(i))$ converges, in the topology of $\bar{B}^n$, to a point in the boundary of B^n.

Finally, if x is an arbitrary point of D, then every geodesic ray from x is asymptotic to a geodesic ray from x_0. Since f is a Lipschitz map, the limits obtained from rays starting at x must also exist for almost all rays and give the same limit as the corresponding ones from x_0, thereby concluding the proof of the proposition.

We can now conclude the proof of the Theorem. Proposition (2.4) provides us with a measurable extension of f, still denoted by f, and $f\colon \partial D \to \partial B^n$. Taking the limit as $i \to \infty$ in (2.1), it is clear that f maps almost every triple of points in ∂D to the vertices of an ideal triangle in B^n for which $f\omega$ assumes its maximum value π. By Lemma (1.3), the vertices of each such simplex lie in the intersection of ∂B^n with a complex line in $\mathbf{C}^n$. Since a complex line is determined by two points, this complex line must be the same for almost all triple of points in ∂D. Since the set of geodesics joining the ideal vertices is invariant under $\rho(\Gamma)$, this complex line is invariant under $\rho(\Gamma)$, thereby completing the proof of the Theorem.

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