J. DIFFERENTIAL GEOMETRY
28 (1988) 309-339

THE HODGE COHOMOLOGY OF A CONFORMALLY COMPACT METRIC

RAFE MAZZEO

Abstract

The Hodge Laplacian acting on differential k-forms is examined for a class of complete Riemannian manifolds with negative sectional curvature near infinity. These manifolds have C^∞ compactifications on which the metric is conformal to one smooth up to the boundary with conformal factor ρ^{-2}, ρ a defining function for the boundary. An example is the Poincaré ball, which serves as a model throughout. The Schwartz kernel of a parametrix for the Laplacian is described for all degrees k except those near half the dimension of the manifold. Its asymptotics are determined in sufficient detail so that we may identify the L^2 harmonic spaces with the relative and absolute cohomology of the compactification for $k < (n-1)/2$ and $k > (n+1)/2$, respectively. In addition, we locate the essential spectrum of the Laplacian in each degree. The construction relies on a calculus of pseudodifferential operators well adapted to the type of degeneracy exhibited by the Laplacian at the boundary of the compactified manifold.

1. Introduction and geometric preliminaries

A. A complete simply-connected Riemannian manifold of negative curvature is diffeomorphic to Euclidean space. If in addition its metric is of bounded geometry then we might expect its Hodge Laplacian, which acts on differential forms, to behave much as the corresponding operator on the constant curvature model space $\mathbf{H}^n$. As we shall see, only sometimes is this the case.

The salient properties of the hyperbolic Laplacian are well known. Those which concern us here involve its spectrum on $L^2(dg_H)$, dg_H =hyperbolic measure, and the existence of bounded harmonic functions or forms with given "asymptotic" boundary values—including representation theorems in terms of these boundary values. Thus, if Δ_k denotes the Laplacian on k-forms, then Δ_k has no point spectrum unless $k = n/2$, in which case it has only an infinite dimensional eigenspace at zero (Dodziuk [5]). From Donnelly [6] its continuous spectrum is known to be unitarily equivalent to a finite sum of multiplication operators $a^2 + x^2$ acting on $L^2(\mathbf{R}^+, H)$, H an auxiliary Hilbert

Received January 28, 1987 and, in revised form, May 4, 1987.

space. In particular, the range of Δ_k is closed and the continuous spectrum is bounded away from zero except when $k = (n \pm 1)/2$. This behavior for values of k near the middle degree results from the conformal equivariance of the coboundary operator δ, and the conformal invariance of the L^2 norm, in the middle degree. As for bounded harmonic functions, these exist in abundance. There are classical Poisson representation formulas for them in terms of their boundary values and, due to the representation theorists, one may even allow hyperfunction boundary value (although the corresponding harmonic functions are no longer bounded then). Similar theorems hold in the setting of differential forms.

Let us now quote some of the relevant variable curvature results. To fix notation, let M be a simply-connected complete Riemannian manifold of dimension n, the sectional curvatures of which satisfy bounds $-b^2 \leq K_M \leq -a^2 < 0$. There is a geometrically natural compactification $\overline{M} = M \cup S_\infty$ homeomorphic to the ball B^n, first defined in Eberlein and O'Neill [8]. Points of S_∞ may be thought of as classes of mutually asymptotic geodesics. A priori $\overline{M}$ exists only as a topological manifold.

In this generality rather more is known about Δ_0 than other degrees. For example, it is quite trivial that there are no L^2 harmonic functions since such a function must be constant and the volume of M is infinite. More strongly, McKean [13] proved that $\operatorname{spec}\Delta_0$ lies in $[(n-1)^2a^2/4, \infty)$. The multiplicity of the continuous spectrum presumably varies in a complicated manner in $[(n-1)^2a^2/4, (n-1)^2b^2/4]$ and there are no other L^2 eigenfunctions. However there are many bounded harmonic functions. By the work of Anderson [1] and Sullivan [18], for each continuous function f on S_∞ there is a unique harmonic function u on M which assumes the boundary values f in the topology of $\overline{M}$; Yau's asymptotic maximum principle shows that u is bounded. Later Anderson and Schoen [3] gave a representation formula for u in terms of f, in the process showing that S_∞ has a natural Hölder structure with exponent $\alpha = a/b$.

The situation for other values of k is rather more subtle and less is known. A strong vanishing theorem, which Donnelly and Xavier [7] were able to prove by assuming that the curvature is tightly pinched, $|a/b-1| < \varepsilon(n,k)$, concerns the L^2 harmonic space $\mathscr{H}^k = \ker \Delta_k$:

$$\dim \mathscr{H}^k = \begin{cases} 0, & k \neq n/2, (n \pm 1)/2, \\ \infty, & k = n/2. \end{cases} \tag{1.1}$$

They also establish nonzero lower bounds on the spectrum. Dodziuk [5] proved a vanishing theorem of the form (1.1) for complete manifolds with rotational symmetry. (He actually used hypotheses on the growth of the metric, and assumed nothing about the curvature.) Rather surprisingly then, Anderson [2] produced examples of manifolds with the correct geometry but for which $\mathscr{H}^k$ is infinite dimensional for any one fixed value of $k \neq 0$, n, in the process showing that the pinching constants used by Donnelly and Xavier to obtain (1.1) are sharp. Finally, very little is known about the solvability of the asymptotic Dirichlet problem for forms. It is closely related to the L^2 theory above, in the same relation for example as between the homogeneous and inhomogeneous Dirichlet problem for the Laplacian on a finite smoothly bounded domain.

B. In this paper we describe and study a class of manifolds for which a finiteness theorem for the Hodge cohomology spaces $\mathscr{H}^k$ analogous to (1.1) may be proved. The manifolds in this class are modelled closely after $\mathbf{H}^n$ in their metric asymptotics. In particular they possess a smooth, albeit extrinsically defined, ideal boundary.

Thus let M be a compact manifold with boundary and ρ a positive defining function for ∂M:

$$\rho \geq 0, \quad \rho^{-1}(0) = \partial M, \quad d\rho|_{\partial M} \neq 0.$$

For any smooth metric h on M define

$$g = \rho^{-2} h. \tag{1.2}$$

This metric g, which we term *conformally compact*, and its Laplace operator are the focus of this paper. It is a complete metric since the singular factor ρ^{-2} has the effect of pushing ∂M out to infinity. The relevant geometry is summed up in the:

Proposition. *(M, g) is a complete manifold. If γ is a nontrapped geodesic then $\gamma(t)$ tends to a definite point $\gamma_\infty \in \partial M$ as $t \to \infty$ and $\gamma'(t)$ tends in direction to the h-unit normal to the boundary ν at γ_∞. Furthermore, the curvature tensor at $\gamma(t)$ becomes increasingly isotropic for t large, with sectional curvatures all tending to $-(\partial\rho/\partial\nu)^2(\gamma_\infty)$.*

This is verified by straightforward computation. So although the topology of M and the geometry of g are arbitrary in the interior, their behavior at infinity is distinctly reminiscent of the hyperbolic model.

The Hodge Laplacian Δ_k for the metric g, for which we derive the explicit expression in the next section, is also quite similar to its hyperbolic analogue. Indeed, in a sense to be made precise later, the hyperbolic Laplacian is an asymptotic limit of the variable curvature operator. Hence it may be regarded as an elliptic operator on the compactification $\overline{M} = M \cup \partial M$ which degenerates rather thoroughly at ∂M. Our main result is

(1.3) Theorem. *For the conformally compact metric g of* (1.2) *there are natural isomorphisms between the (finite dimensional) spaces*

$$\mathscr{H}^k = \begin{cases} H^k(M, \partial M), & k < (n-1)/2, \\ H^k(M), & k > (n+1)/2. \end{cases}$$

If $-a_0^2 = \sup -(\partial p/\partial \nu)^2$ *is the maximum limiting curvature at infinity, then the essential spectrum of* Δ_k *is* $[a_0^2(n-2k-1)^2/4, \infty)$, $\{0\} \cup [a_0^2/4, \infty)$, $[a_0^2(n-2k+1)^2/4, \infty)$ *for* $k < n/2$, $k = n/2$, $k > n/2$, *respectively. In particular,* $\lambda = 0$ *is an isolated eigenvalue of infinite multiplicity for* $\Delta_{n/2}$.

This result provides a topological interpretation of the harmonic spaces on certain open manifolds. Notice that, in place of the relative or absolute boundary conditions for the Laplacian necessary for the standard Hodge theorem for compact manifolds with boundary, the L^2 condition alone provides the boundary condition here. (1.3) may be regarded as asserting Fredholm properties for a certain asymptotic elliptic boundary problem.

The proof is microlocal. §2 contains a description of a space of pseudodifferential operators large enough to contain parametrices for the degenerate operators Δ_k. These have genuine symbol ellipticity once the definition of symbol has been suitably reinterpreted. But, as with classical elliptic boundary problems, an additional boundary ellipticity condition is required before Fredholm properties will hold. Whereas the classical Lopatinski-Schapiro condition is algebraic, the condition here is differential and requires the invertibility of certain model operators. This analysis is the subject of §3. Finally in §4 we gather the machinery developed and complete the proof.

This approach to certain degenerate elliptic equations originates with Richard Melrose, to whom the author is indebted for thoughtful supervision during the writing of the thesis upon which this paper is based. More thorough treatment of many of the results here is to be found in that thesis [11]. These techniques have also been used to prove the existence of a meromorphic continuation of the resolvent for Δ_0—as well as of the Eisenstein series for certain quotients of $\mathbf{H}^n$— for those conformally compact metrics with asymptotically constant curvature at infinity [12]. Finally, the asymptotic Dirichlet problem for differential forms may be studied by these methods.

C. We now briefly derive an expression for the Laplacian Δ_k corresponding to the metric (1.2). For reasons to become clear later, it is natural to consider instead the conjugate

$$P = \rho^k \Delta_k \rho^{-k}.$$

From $\Delta_k = d\delta + \delta d$ it follows that

$$P = (\rho^k d\rho^{-k-1})(\rho^{k+1}\delta\rho^{-k}) + (\rho^k \delta \rho^{-k+1})(\rho^{k-1} d\rho^{-k}). \tag{1.4}$$

Thus we first compute $\rho^{j+1}d\rho^{-j}$ and $\rho^{j-1}\delta\rho^{-j}$ for each j. The first of these is trivial:

$$\rho^{j+1}d\rho^{-j}\omega = \rho d\omega - jd\rho\wedge\omega, \qquad \omega\in\Omega^j.$$

As for the second, a duality argument shows that

$$\rho^{j-1}\delta\rho^{-j}\omega = \rho^{n-j+1}\delta_h\rho^{j-n}\omega = \rho\delta_h\omega + (n-j)\iota(\nabla\rho)\omega, \qquad \omega\in\Omega^j, \tag{1.5}$$

since $\delta(f\omega) = f\delta\omega + \iota(\nabla f)\omega$. In (1.5) $\iota(\;)$ is contraction, $\nabla\rho$ is the gradient with respect to h, and likewise δ_h is the coboundary for the metric h.

Insert these identities into the factorization for P. A bit of bookkeeping then leads to the desired expression

$$\begin{aligned} P\omega = {} & \rho^2\Delta_h\omega + (2-k)\rho d\rho\wedge\delta_h\omega - k\rho\delta_h(d\rho\wedge\omega) \\ & + (n-k)\rho L_{\nabla_\rho}\omega - 2\rho\iota(\nabla\rho)\,d\omega + (n-2k)d\rho\wedge\iota(\nabla\rho)\omega \\ & - k(n-k-1)\iota(\nabla\rho)(d\rho)\omega, \qquad \omega\in\Omega^k. \end{aligned} \tag{1.6}$$

L_{∇_ρ} is the Lie derivative and Δ_h is the Laplacian for the metric h.

2. Analysis

A. Any differential operator $L\in \mathrm{Diff}(M)$ is locally the sum of products of vector fields. Correspondingly, natural subrings of $\mathrm{Diff}(M)$ may be defined as generated in this sense by certain geometrically natural subalgebras of the Lie algebra $\Gamma(TM)$ of all smooth vector fields. Such a subalgebra $\mathscr{V}$ which differs (in its sheaf of germs) from $\Gamma(TM)$ only at the boundary is called a "boundary structure." The associated space of operators is denoted $\mathrm{Diff}_{\mathscr{V}}(M)$.

Elements of a subring of $\mathrm{Diff}(M)$ defined in this manner all degenerate in some uniform fashion at the boundary. However, there is still a good notion of ellipticity in this context, and a class of pseudodifferential operators exists which contains parametrices with compact remainder for elliptic elements in $\mathrm{Diff}_{\mathscr{V}}(M)$. Concrete examples of this theory (work in progress of Melrose-Mendoza) have been worked out fully in only two cases. The first, by Melrose [14]—whence the general program originates—and Melrose and Mendoza [15], involves the algebra $\mathscr{V}_b$ of totally characteristic vector fields—those tangent to the boundary and otherwise unconstrained in the interior. Totally characteristic operators, the space of which is abbreviated $\mathrm{Diff}_b(M)$, arise naturally in geometric problems on conic spaces. The other example involves the algebra $\mathscr{V}_0$ of vector fields which vanish at ∂M (and are unconstrained in the interior). The associated space of operators, now abbreviated $\mathrm{Diff}_0(M)$, also arise naturally in geometric problems as we shall see. The theory of $\mathscr{V}_0$ pseudodifferential operators is developed fully in Mazzeo [11], and less completely in Mazzeo and Melrose [12]. Here we outline the main ideas and results, and refer to these papers for complete details and proofs.

Near a point $q \in \partial M$ choose coordinates $z^1, \cdots, z^n$ such that z^n vanishes on ∂M. For convenience we shall usually write $y = (z^1, \cdots, z^{n-1})$, $x = z^n$. Now $\mathscr{V}_0$ as defined above is obviously closed under Lie bracket, and is generated over $C^\infty(M)$ by

$$\left\{ x\frac{\partial}{\partial y^1}, \cdots, x\frac{\partial}{\partial y^{n-1}}, x\frac{\partial}{\partial x} \right\}. \tag{2.1}$$

Thus, in these coordinates, a typical $L \in \mathrm{Diff}_0^m(M)$ has the expression

$$L = \sum_{r+|\alpha|\leq m} a_{\alpha,r}(y,x)(x\partial_y)^\alpha (x\partial_x)^r. \tag{2.2}$$

Notice that P of (1.6) is of $\mathscr{V}_0$-type; $\mathscr{V}_0$ has many close connections with hyperbolic geometry.

A global way to regard $\mathscr{V}_0$ is as the *full* set of sections of a certain bundle ${}^\circ TM$—the $\mathscr{V}_0$ tangent bundle. It is canonically isomorphic to TM over the interior of M, and there is a natural map ${}^\circ TM \to TM$ induced by the inclusion $\mathscr{V}_0 \to \Gamma(TM)$. (2.1) gives a spanning set of sections for ${}^\circ TM$; conversely they may be used to define it, although invariant definitions are easy to come by. Dual to ${}^\circ TM$ is the $\mathscr{V}_0$ cotangent bundle ${}^\circ T^*M$. A spanning basis of sections for it, dual to those in (2.1), is

$$\left\{ \frac{dy^1}{x}, \cdots, \frac{dy^{n-1}}{x}, \frac{dx}{x} \right\}. \tag{2.3}$$

Any $L \in \mathrm{Diff}_0^m(M)$ has a symbol which is a smooth function on ${}^\circ T^*M$ and polynomial (in fact homogeneous) on the fibers. If L is as in (2.2) then

$${}^\circ\sigma_m(L)(y,x,\eta,\xi) = \sum_{|\alpha|+r=m} a_{\alpha,x}(y,x)\eta^\alpha \xi^x. \tag{2.4}$$

Here (y,x,η,ξ) refers to the point $(y, x, \sum \eta_i \, dy^i/x + \xi \, dx/x)$ of ${}^\circ T^*M$. Now, L is defined to be elliptic (in the sense of $\mathscr{V}_0$) if

$${}^\circ\sigma_m(L)(y,x,\eta,\xi) = 0 \Rightarrow (\eta,\xi) = 0.$$

Finally, each $L \in \mathrm{Diff}_0^m(M)$ has an important model at every $q \in \partial M$ which plays a seminal role in the theory. Choose a chart ϕ around $q \in \partial M$ mapping into M_q with $\phi(q) = 0$, $\phi_*|_0 = I$. Also let R_r denote the dilation by r on M_q. Then define the normal operator to L at q by

$$N_q(L)u = \lim_{r \to 0} R_r^* \phi^* L (\phi^{-1})^* R_{1/r}^* u. \tag{2.5}$$

It is easy to check that for L as in (2.2)

$$N_q(L) = \sum_{|\alpha|+r\leq m} a_{\alpha,r}(0,0)(x\partial_y)^\alpha(x\partial_x)^r, \qquad q=(0,0). \tag{2.6}$$

Note that this operator is invariant not only under the dilation group, but also under translations by vectors in ∂M_q. In fact, it is left invariant for the group G_q, the semi-direct product of these two actions.

B. We now proceed to define a microlocalized version of Diff_0^* which contains parametrices for the elliptic $\mathscr{V}_0$ differential operators. Elements of this new space of pseudodifferential operators are best approached through their Schwartz kernels. Although these are distributions on $M\times M$, they live more naturally on a slightly larger manifold, $M\times_0 M$, obtained by "blowing up" $M\times M$ along the diagonal in its boundary. Let us first recall the geometry of the manifold with corner $M\times M$. It has two natural codimension one boundary components: $\partial_1^l = \partial M\times M$, $\partial_1^r = M\times\partial M$ and the corner $\partial_2 = \partial M\times\partial M$. The only other natural submanifolds are the diagonal $\Delta\iota$ and its boundary $\partial\Delta\iota\subset\partial_2$. If (y,x) are coordinates of the usual type on the first M factor, and $(\tilde{y},\tilde{x})$ identical ones on the second, then these submanifolds have the defining equations

$$\partial_l = \{x=0\}, \qquad \partial_r = \{\tilde{x}=0\},$$
$$\Delta\iota = \{x=\tilde{x}, y=\tilde{y}\}, \qquad \partial\Delta\iota = \{x=\tilde{x}=0, y=\tilde{y}\}.$$

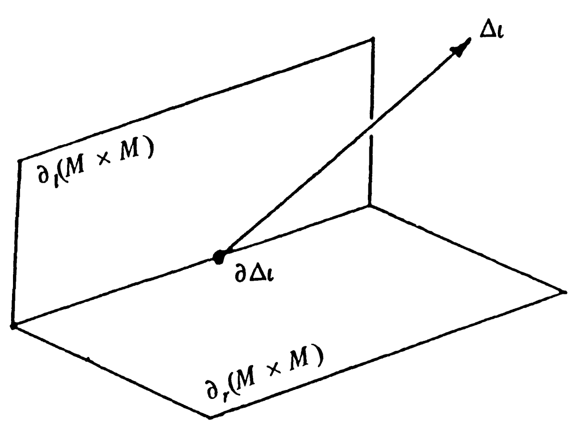

FIGURE 1

Next we define $M\times_0 M$ by a C^∞ process of blowing up analogous to the usual algebraic one. Set theoretically $M\times_0 M$ is $M\times M\backslash\partial\Delta\iota$ union $S_{++}N\partial\Delta\iota$, the inward pointing quarter of the spherical normal bundle to $\partial\Delta\iota$

in $M \times M$. This has a natural C^∞ structure [16] such that polar coordinates on $M \times M$ around $\partial\Delta\iota$ lift to a nonsingular chart via the canonical "blow-down" map

$$b\colon M \times_0 M \to M \times M, \tag{2.7}$$

which collapses the new face $S_{++}N\partial\Delta\iota$ to $\partial\Delta\iota$ and is the identity elsewhere.

We proceed with a local coordinate description of its geometry. Thus, define the new (polar) variables

$$Y = y - \tilde{y}, \quad R = [x^2 + |Y|^2 + \tilde{x}^2]^{1/2}, \quad \omega = R^{-1}(x, Y, \tilde{x}), \tag{2.8}$$

so that $R \geq 0$ and $\omega = (\omega_0, \omega', \omega_n) \in S^n_{++}$, i.e., ω_0, $\omega_n \geq 0$. We shall use $(R, \omega, \tilde{y})$ as a full set of coordinates. Note that

$$x = R\omega_0, \quad y = \tilde{y} + R\omega', \quad \tilde{x} = R\omega_n.$$

This $\mathscr{V}_0$ stretched product $M \times_0 M$ has *three* codimension one boundary components:

$$F = \{R = 0\}, \quad T = \{\omega_0 = 0\} = b^{-1}(\partial_l)/F, \quad B = \{\omega_n = 0\} = b^{-1}(\partial_r)/F,$$

the front, top and bottom faces, respectively. In addition, the interior of $\Delta\iota$ lifts to a submanifold whose closure

$$\Delta\iota_0 = \{\omega = (1/\sqrt{2}, 0, 1/\sqrt{2})\}$$

has boundary $\partial\Delta\iota_0 = \{\omega = (1/\sqrt{2}, 0, 1/\sqrt{2}), R = 0\}$ contained in the interior of F—away from all corners

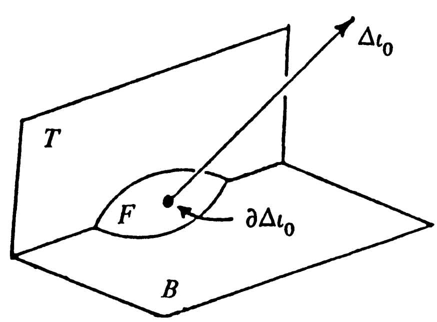

FIGURE 2

The front face F has the structure of a quarter-sphere bundle over $\partial\Delta\iota$. Each fiber F_q over (q, q), $q \in \partial M$, may be identified with three distinct objects. On the one hand, it is equivalent (away from its bottom edge $F_q \cap B$) with the half tangent space M_q. M_q, as the first summand of $T_{(q,q)}(M \times M)$, projects stereographically to the quarter-sphere $S_{++}N_q\partial\Delta\iota = F_q$. On the other hand, F_q as a quarter-sphere is diffeomorphic to the manifold-with-corner obtained by blowing up the ball B^n around a point p in its boundary. If the blow-down map is called β, $\beta\colon F_q \to B^n$, then we make the identification so that $\beta^{-1}(p) = F_q \cap B$. Finally, F_q may also be thought of as the compactification of the group G_q mentioned above.

$M \times_0 M$ has been defined because it is well adapted to the algebra $\mathscr{V}_0$ in a variety of ways. For example, we now calculate how $\mathscr{V}_0$ itself, initially lifted to the first factor of $M \times M$, then lifts nonsingularly to $M \times_0 M$. It suffices to examine the lifts of the generators $x\partial_x$, $x\partial_{y_i}$. In the coordinates $(y, x, \tilde{y}, \tilde{x})$ on $M \times M$ these become vector fields which are written the same way. Next, in the coordinates (2.8) we may check that

$$b_* X = x\partial_x, \qquad b_* Y_i = x\partial_{yi},$$

where

$$X = \omega_0^2 R\partial_R + \omega_0 \left\{ \partial_{\omega_0} - \omega_0 \sum_{j=0}^{n} \omega_j \partial_{\omega_0} \right\}, \tag{2.9}$$
$$Y_i = \omega_0\omega_i R\partial_R + \omega_0 \left\{ \partial_{\omega_i} - \omega_i \sum_{j=0}^{n} \omega_j \partial_{\omega_0} \right\}.$$

These vector fields X, Y_i, the unique smooth lifts of the generators, actually lie tangent to all faces of $M \times_0 M$ including F. In other words, vector fields need to be at least as degenerate as those in $\mathscr{V}_0$ to lift nonsingularly from M to $M \times_0 M$.

For later use we introduce projective coordinates which are typically simpler to use than the polar ones in computations. Set

$$s = x/\tilde{x}, \qquad u = (y - \tilde{y})/\tilde{x}, \tag{2.10}$$

and use $(s, u, \tilde{x}, \tilde{y})$ as a full chart. These are nonsingular except along B, where $\tilde{x} = 0$. Then we see that $X = s\partial_s$ and $Y_i = s\partial_{u_i}$.

The last bit of structure on the stretched product which we require are a family of density bundles. First generally, suppose X is any manifold with codimension one boundary components $\partial X_1, \cdots, \partial X_N$ which have defining functions $\rho_1, \cdots, \rho_N$, respectively. Now define

$$\Gamma_0(X) = \{\text{smooth densities } \nu \text{ on the interior of } X \text{ such that } \rho_1^n \cdots \rho_N^n \cdot \nu \text{ extends smoothly to all of } X\}.$$

Let $\mathscr{L}_0(X)$ be the line bundle for which $\Gamma_0(X)$ is the full set of sections. Then, specializing to $X = M \times M$ or $M \times_0 M$, we have the

Lemma. $b^*\mathscr{L}_0(M \times M) \cong \mathscr{L}_0(M \times_0 M)$.

Proof. Away from F and $\partial\Delta\iota$ there is nothing to prove since b is a diffeomorphism then. At F use the polar coordinate chart. Then

$$|dx\, dy\, d\tilde{x}\, d\tilde{y}/(x^n\tilde{x}^n)| = |dR\, d\omega\, d\tilde{y}/(R^n\omega_0^n\omega_n^n)|.$$

The initial term generates $\Gamma_0(M \times M)$ while the final one generates $\Gamma_0(M \times_0 M)$, which gives the result.

C. Having carefully set up the geometry, it is now straightforward to introduce the space of $\mathscr{V}_0$ Schwartz kernels. Recall first that a standard pseudodifferential operator on M is one whose Schwartz kernel is a distribution on $M \times M$ conormal along the diagonal and C^∞ elsewhere (see [9]). (A distribution is said to be conormal to a submanifold S if all its derivatives by vector fields tangent to S are of fixed Sobolev regularity. Equivalently, its Fourier transform in directions normal to S must be a symbol.) However, since $\Delta\iota$ intersects the corner ∂_2, such a kernel could potentially behave in various ways at this submanifold of intersection. For example, a standard elliptic parametrix will have a kernel which is a restriction of a kernel conormal along the diagonal of $\tilde{M} \times \tilde{M}$, where $\tilde{M}$ is an open extension of M. We shall use another type of extension behavior. Let $[M \times_0 M]^2$ be the $\mathscr{V}_0$ stretched product doubled across the front face. It has the structure of C^∞ manifold with corners up to codimension two. Furthermore, the doubled diagonal $[\Delta\iota_0]^2$ lies in its interior.

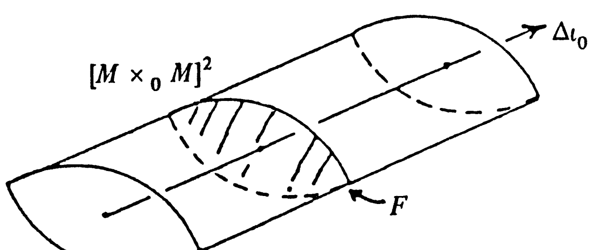

FIGURE 3

Definition. $K_0^m(M; \Gamma_0^{1/2})$ is the space of distributional sections of $\Gamma_0^{1/2}(M \times_0 M)$ which extend to the double as distributional sections of $\Gamma_0^{1/2}([M \times_0 M]^2)$ conormal along the doubled diagonal of order m, and vanishing to infinite order on the doubled boundary faces T and B.

$\Psi_0^m(M; \Gamma_0^{1/2})$ is the space of operators with Schwartz kernels on $(M \times M)$ obtained as pushforwards of elements of $K_0^m(M; \Gamma_0^{1/2})$.

Here $\Gamma_0^{1/2}$ is the space of half densities associated with Γ_0. In an abuse of notation, the kernel $\kappa(A)$ of $A \in \Psi_0^m$ will itself be said to belong to K_0^m, disregarding the intermediary pushforward. Also, more significantly, $b_*\kappa$, $\kappa \in K_0^m$, is always well defined by wave front considerations. For the intersection of F, where b is only a submersion, and $\Delta\iota_0$, where κ is singular, is transversal, and away from F b is an isomorphism.

The action of $A \in \Psi_0^m(M;\Gamma_0^{1/2})$ near the boundary is easily represented. Set as standard half-densities

$$\mu = \left|\frac{dx\,dy}{x^n}\right|^{1/2}, \quad \gamma = \left|\frac{dx\,dy\,d\tilde{x}\,d\tilde{y}}{x^n\tilde{x}^n}\right|^{1/2} = \left|\frac{dx\,dy\,ds\,du}{sx^n}\right|^{1/2}. \tag{2.11}$$

Then $\kappa(A) = \kappa(s,u,\tilde{x},\tilde{y})\cdot\gamma$ where κ is conormal of order m along $\{s=1, u=0\}$ and decreases rapidly as $s \to 0,\infty$ or $|u|\to\infty$. On the half density $f(x,y)\cdot\mu$,

$$(Af)(x,y) = \left\{\int \kappa\left(s,u,\frac{x}{s},y-\frac{x}{s}u\right)f\left(\frac{x}{s},y-\frac{x}{s}u\right)\frac{ds}{s}du\right\}\cdot\mu,$$

since $\tilde{x} = x/s$, $\tilde{y} = y - xu/s$. In particular the kernel of the identity is

$$\kappa(I) = \delta(s-1)\delta(u)\cdot\gamma \in K_0^0.$$

As for any space of conormal distributions there is a symbol map

$$\sigma_m\colon K_0^m(M;\Gamma_0^{1/2})/K_0^{m-1} \to S^m(N^*\Delta\iota_0, \Gamma_0(M)\otimes\Gamma(\text{fiber}))/S^{m-1}.$$

The symbol spaces on the right take values in the density bundle on the fibers of the conormal bundle by use of the invariant Fourier transform. The extra $\Gamma_0(M)$ factor results from restricting $\Gamma_0^{1/2}(M\times_0 M)$ to $\Delta\iota_0$. It is now possible to "divide" by a canonical density and also to identify $N^*\Delta\iota_0$ with ${}^\circ T^*M$ (see [11] for details) to arrive at the $\mathscr{V}_0$ symbol isomorphism

$$^\circ\sigma_m\colon \Psi_0^m(M;\Gamma_0^{1/2})/\Psi_0^{m-1} \xrightarrow{\sim} S^m({}^\circ T^*M)/S^{m-1}. \tag{2.12}$$

This leads to a satisfactory symbol calculus since the product formula

$$^\circ\sigma_{m_1+m_2}(A_1\cdot A_2) = {}^\circ\sigma_{m_1}(A_1)\,{}^\circ\sigma_{m_2}(A_2) \tag{2.13}$$

is also valid. The principal difficulty in establishing this last formula is verifying that $A_1\cdot A_2$ still lies in the same space of operators. However, if A_1 is differential this fact is quite trivial. We should also note that the symbol map as defined in (2.4) agrees with that in (2.12).

The space Ψ_0^* is slightly too small to contain parametrices for the elliptic elements in Diff_0^*, due to the infinite order vanishing of its kernels on the top and bottom faces. To remedy this, suppose $\mathscr{V}_b$ is the Lie algebra of vector fields on $[M \times_0 M]^2$ tangent to the (doubled) top and bottom faces T and B, and let ρ_T and ρ_B be defining functions for these faces. Now set

$$\mathscr{A}^{a,b} = \{u \in \mathscr{D}'([M \times_0 M]^2): V_1 \cdots V_j u \in \rho_T^a \rho_B^b (\log \rho_T \log \rho_B)^N L^\infty \text{ for } V_i \in \mathscr{V}_b,\ i = 1, \cdots, j, \text{ for all } j \text{ and for some } N \text{ independent of } j\}.$$

This has a natural subspace of polyhomogeneous elements

$$\mathscr{A}^{a,b}_{\mathrm{phg}} = \left\{ u \in \mathscr{A}^{a,b}: u \sim \sum_{i=0}^{\infty} \sum_{j=0}^{N} u_{ij} \rho_T^{a+i} (\log \rho_T)^j \text{ near } T, \text{ with an analogous expansion near } B \right\}. \tag{2.14}$$

Here the u_{ij} are C^∞ and depend only on the tangential variables

Definition.

$$K_0^{-\infty,a,b}(M; \Gamma_0^{1/2}) = \mathscr{A}^{a,b}_{\mathrm{phg}}|_{M \times_0 M} \otimes \Gamma_0^{1/2}(M \times_0 M),$$
$$K_0^{m,a,b}(M; \Gamma_0^{1/2}) = K_0^m(M; \Gamma_0^{1/2}) \oplus K_0^{-\infty,a,b}(M; \Gamma_0^{1/2}),$$
$$\Psi_0^{m,a,b}(M; \Gamma_0^{1/2}) = \{A: \kappa(A) \in K_0^{m,a,b}\}.$$

In other words, the kernels in $K_0^{m,a,b}$ are sections of $\Gamma_0^{1/2}$ which are C^∞ in the interior and up to F and have classical expansions at T and B.

The symbol map (2.13) extends readily to $\Psi_0^{m,a,b}$ by setting

$${}^\circ\sigma_m(A) = {}^\circ\sigma_m(A_1) \quad \text{if } A = A_1 + A_2, \qquad A_1 \in \Psi_0^m \text{ and } A_2 \in \Psi_0^{-\infty,a,b}.$$

As a final embellishment on these spaces of operators we shall need to consider the form bundles and $\mathscr{V}_0$ operators acting between them. Over M itself the basic ingredients are the $\mathscr{V}_0$ cotangent bundle ${}^\circ T^*M$ and its exterior powers ${}^\circ\Lambda^k(M) = \Lambda^k({}^\circ T^*M)$. These have spanning sets of sections:

$$\left\{ \frac{dy^I}{x^k}, \frac{dy^J \wedge dx}{x^k}, \quad \text{where } |I| = k,\ |J| = k-1 \right\}.$$

The operator P of (1.7) is the one induced by Δ_k on sections of ${}^\circ\Lambda^k$ written in this basis, i.e.,

$$\Delta_k\left(\frac{\omega}{x^k}\right) = \frac{P\omega}{x^k}.$$

We may pull ${}^\circ\Lambda^k$ back to either the left (first) or right (second) factor of $M \times M$, these pullbacks being denoted ${}^\circ\Lambda_l^k$, and ${}^\circ\Lambda_r^k$, respectively. These in turn may be lifted to $M \times_0 M$ and will be denoted by the same symbols there.

In order that Ψ_0^* act between these bundles, it is simply necessary to tensor with $\mathrm{Hom}({}^\circ\Lambda_l^k, {}^\circ\Lambda_r^k)$. Unfortunately though we shall require operators with slightly byzantine boundary behavior. By specifying a conformal structure the k-form bundle splits at the boundary into tangential and normal components:

$$\Lambda^k(M)|_{\partial M} = \Lambda_t^k \oplus \Lambda_n^k,$$

and similarly for ${}^\circ\Lambda^k$. In fact, we may even assume that ${}^\circ\Lambda^k$ splits in a neighborhood of ∂M; this is less natural, but any two splittings compatible with the conformal structure agree to first order at the boundary. It is then well defined to require that the tangential and normal components of a form $\omega = \omega_t + \omega_n$ vanish at different rates at the boundary, or in fact to require that

$$\omega_t \in \mathscr{A}_{\mathrm{phg}}^a, \qquad \omega_n = \mathscr{A}_{\mathrm{phg}}^b,$$

($\mathscr{A}_{\mathrm{phg}}^*$ being defined as in (2.14) with reference to a single boundary component) but only when $|a-b| \leq 1$. This restriction arises because ω_t and ω_n are only defined to first order at ∂M.

Now, any $G \in \mathrm{Hom}({}^\circ\Lambda_l^k, {}^\circ\Lambda_r^k)$ corresponds to a matrix

$$G = \begin{pmatrix} G_{tt} & G_{nt} \\ G_{tn} & G_{nn} \end{pmatrix},$$

where

$$G_{ij} \colon ({}^\circ\Lambda_l^k)_i \to ({}^\circ\Lambda_r^k)_j, \qquad i,j = t,n.$$

The G_{ij} are called the components of G.

Finally, suppose σ and τ are two-by-two real matrices

$$\sigma = (\sigma_{ij}), \quad \tau = (\tau_{ij}), \qquad i,j = t,n.$$

Definition.

$$K_0^{-\infty,\sigma,\tau}(M; {}^\circ\Lambda^k \otimes \Gamma_0^{1/2}) = \{G = (G_{ij}),\ i,j = t,n,$$
$$G_{ij} \in K_0^{-\infty,\sigma_{ij},\tau_{ij}} \otimes \mathrm{Hom}(({}^\circ\Lambda_l^k)_i, ({}^\circ\Lambda_r^k)_j)\}$$

with $K_0^{m,\sigma,\tau}$ and $\Psi_0^{m,\sigma,\tau}$ defined in the obvious way.

As before, these spaces make sense only when

$$|\min(\sigma_{tt}, \sigma_{nt}) - \min(\sigma_{tn}, \sigma_{nn})| \leq 1,$$

and similarly for τ.

D. The continuity properties of operators in $\Psi_0^{m,a,b}$ are easily stated. Attention here is restricted to the scalar case for simplicity: all results generalize easily to the vector case, for example by referring to local trivializations. Introduce first the Sobolev-type spaces

$$H_b^m(M; \Gamma_0^{1/2}) = \{u \colon (x\partial_x)^i (x\partial_y)^\alpha u \in L^2(M; \Gamma_0^{1/2}), i + |\alpha| \leq m\}$$

for any nonnegative integer m.

(2.15) Proposition. *$A \in \Psi_0^{-m,a,b}$ is bounded between*

$$x^r H_b^s(M;\Gamma_0^{1/2}) \to x^{r'} H_b^{s+m}(M;\Gamma_0^{1/2})$$

provided $r' \leq r$, $a - r > (n-1)/2$, $b + r' > (n-1)/2$, $a + b > n - 1$, and for $s, m = 0, 1, 2, \cdots$.

The proof appears fully in [11]. As usual, the main point is to establish boundedness of residual operators, $m = -\infty$, on L^2. The rest follows from the symbol calculus and by commuting vector fields through A.

In this context though, the residual operators are somewhat nontrivial to analyze. For it is one of the central issues of this theory that these residual operators, while smoothing in the interior, are not compact on any reasonable space. To get compactness one requires slightly more regular kernels.

Definition. $R^i K_0^{-\infty,a,b}$ denotes the space of kernels in $K_0^{-\infty,a,b}$ vanishing to order i, $i = 1, 2, \cdots, \infty$, at the front face F. $R^i \Psi_0^{-\infty,a,b}$ is the associated space of operators.

(2.16) Proposition. *If $A \in R^\infty \Psi_0^{-\infty,a,b}$ then for a, b, r, r' as in* (2.15)

$$A: x^r L^2(M;\Gamma_0^{1/2}) \to x^{r'} L^2(M;\Gamma_0^{1/2})$$

is compact.

This is proved by using the vanishing hypothesis (which is far stronger than it need be) to provide the uniform smallness near ∂M in applying the L^2 Arzela-Ascoli theorem.

We mention one other result of a slightly different nature, which shall be strengthened considerably later in a special case. Let $\dot{C}^\infty$ denote the space of functions vanishing to infinite order at ∂M, and define $\mathscr{A}_{\text{phg}}^a$ as in (2.14) but with respect to the single boundary component ∂M. Then from the formula for Af we easily get

(2.17) Proposition. *For any $a, b, m \in \mathbf{R}$ and $A \in \Psi_0^{m,a,b}$, $A: \dot{C}^\infty \to \mathscr{A}_{\text{phg}}^a$ is bounded.*

E. To conclude this discussion of the general theory, we now examine how $\mathscr{V}_0$ pseudodifferential operators are modelled at ∂M. The normal operator of $L \in \text{Diff}_0^m(M)$ defined in (2.5) is already an example of such a model. In fact a normal operator exists for any $A \in \Psi_0^{m,a,b}$. Set, for $q \in \partial M$,

$$N_q(A) = \kappa(A)|_{F_q}. \tag{2.18}$$

For $m = -\infty$, which is the only case we need, this is an element of

$$\mathscr{A}_{\text{phg}}^{a,b}(F_q) \otimes \Lambda_0^{1/2}(M \times_0 M)|_{F_q}.$$

Although as it stands there is nothing "operator-like" about $N_q(A)$, recall that the interior of F_q may be thought of as the group G_q, so that $N_q(A)$ may be interpreted as a convolution operator. Recalling also that F_q may be identified with the half tangent space M_q, the operator $N_q(L)$ of (2.5) when regarded in this sense, agrees with its alternate definition (2.18). There is also a product formula of great importance

$$N_q(L \cdot A) = N_q(L)N_q(A). \tag{2.19}$$

If $L \in \mathrm{Diff}_0^m(M)$ is elliptic, a parametrix may be constructed for it in two or three stages. The first is quite familiar from the usual theory; the symbol calculus is used to find a first approximation to the parametrix, $E_1 \in \Psi_0^{-m}$, such that

$$LE_1 = I - Q_1, \qquad Q_1 \in \Psi_0^{-\infty}.$$

Since the remainder Q_1 is not compact, this step alone will not guarantee finite dimensionality of the null-space of L in L^2. What is needed, as noted in the last section, is a remainder the kernel of which vanishes to infinite—or even just high—order at F. Thus, write

$$\kappa(Q_1) \sim \sum \kappa_j(\tilde{y}, \omega)R^j$$

valid asymptotically near F. We seek an operator E_2, also residual, such that $L(E_1 + E_2) = I - Q_2$, i.e. $LE_2 = Q_1 - Q_2$ where $\kappa(Q_2)$ vanishes to infinite order on F. That is, LE_2 must cancel the terms in the expansion for $\kappa(Q_1)$. This is possible under certain additional ellipticity assumptions, but at the expense of adding boundary terms: $E_2 \in \Psi_0^{-\infty,a,b}$ for some a, b.

The additional ellipticity hypothesis arises from trying to solve $LE_2 = Q_1$ asymptotically near F. E_2 must satisfy the normal equation

$$N_q(L)N_q(E_2) = N_q(Q_1).$$

Obviously it is necessary to require that $N_q(L)$ be invertible on some reasonable space in order that this equation (as well as others corresponding to higher terms in the series) have solutions.

The construction is almost complete, save for a relatively minor third step involving another model, the indicial operator. For $A \in R^\infty\Psi_0^{-\infty,a,b}$, define $I(A)$ by the projection

$$R^\infty\Psi_0^{-\infty,a,b} \xrightarrow{I} \mathscr{A}^{a,b} \otimes \Lambda_0^{1/2} / \mathscr{A}^{a+1,b} \otimes \Lambda_0^{1/2}.$$

Roughly, $I(A)$ captures the leading term of the expansion for $\kappa(A)$ at the top face T. Any $L \in \mathrm{Diff}_0^m$ has a somewhat differently defined indicial operator $I(L)$ which satisfies

$$I(L \cdot A) = I(L)I(A). \tag{2.20}$$

Strictly speaking there are many choices for $I(L)$ such that (2.20) is satisfied. But a natural choice is

$$I(L) = \sum_{r \leq m} a_{0,r}(0,y)(x\partial_x)^r \tag{2.21}$$

for L as in (2.2).

Finally, provided $I(L)$ is invertible, the parametrix construction is completed by finding an $E_3 \in \Psi_0^{-\infty,a,b}$, with $\kappa(E_3)$ vanishing to infinite order on F, and such that

$$L(E_1 + E_2 + E_3) = I - Q_3,$$

i.e., $LE_3 = Q_2 - Q_3$ with $Q_3 \in R^\infty \Psi_0^{-\infty,\infty,b}$. This requires solving

$$I(L)I(E_3) = I(Q_2).$$

Now, with $E = E_1 + E_2 + E_3$, elliptic finiteness theorems will follow immediately.

3. Model operators

A. Before we may apply the general theory sketched in the last section to the specific operator P of (1.6) we must examine the invertibility of its normal and indicial operators. Indeed, this is the only part of the analysis that depends on more than general symbol ellipticity notions. First we must compute these operators explicitly in the coordinates $(y^1, \cdots, y^{n-1}, x)$.

From the limit definition it is readily verified that

$$N_q(x^j \partial_x) = \begin{cases} 0, & j > 1, \\ x\partial_x, & j = 1, \end{cases} \qquad N_q(x^j \partial_x^2) = \begin{cases} 0, & j > 2, \\ x^2\partial_x^2, & j = 2 \end{cases}$$

(no terms with $j < 1$ or $j < 2$, respectively, occur in a $\mathscr{V}_0$ operator), and similarly for terms with y derivatives. Hence computation of the normal operator becomes purely formal: one discards all but the first term in the series expansion for each coefficient so that only dilation invariant expressions remain. Applying this to P itself, and using the notation $\rho(y,x) = a(y)x + b$,

$b = O(x^2)$, we finally arrive at the formula at $q = (0,0)$:

$$
\begin{aligned}
&a(0)^{-2}N_q(P)(\omega_I\, dy^I + \omega_J dy^J \wedge dx) \\
&\qquad = \left[-x^2\Delta_E\omega_I + (n-2)x\frac{\partial\omega_I}{\partial x} - k(n-k-1)\omega_I\right] dy^I \\
&\qquad + \left[-x^2\Delta_E\omega_J + (n-2)x\frac{\partial\omega_J}{\partial x} - (k-1)(n-k)\omega_J\right] dy^J \wedge dx \\
&\qquad + 2(-1)^k \sum_s x\frac{\partial\omega_I}{\partial y^{i_s}}(-1)^{s-1}\, dy^{I/i_s} \wedge dx \qquad (3.1)\\
&\qquad + 2(-1)^{k+1}\sum_j x\frac{\partial\omega_J}{\partial y^j}\, dy^j \wedge dy^J.
\end{aligned}
$$

Here $\Delta_E = \partial_x^2 + \sum \partial_{y_i}^2$. The product formula (2.19) shows that

$$N_q(\Delta_k)\left(\frac{\omega}{x^k}\right) = \frac{N_q(P)}{x^k},$$

where Δ_k is the Laplacian on k-forms for the conformally compact metric g. It is interesting (and quite useful) to realize the

Proposition. *$N_q(\Delta_k)$ is the Hodge Laplacian on k-forms for the metric* $(dx^2 + dy^2)/a(0)^2x^2$, $q = (0,0)$.

This identification is immediate if one specializes g to the hyperbolic metric above and notes that its Laplacian is already dilation invariant.

The indicial operator is obtained from (2.20):

$$
\begin{aligned}
&a(0)^{-2}I_q(P)(\omega_I\, dy^I + \omega_J\, dy^J \wedge dx) \\
&\qquad = \left[-x^2\frac{d^2\omega_I}{dx^2} + (n-2)x\frac{d\omega_I}{dx} - k(n-k-1)\omega_I\right] dy^I \qquad (3.2)\\
&\qquad + \left[-x^2\frac{d^2\omega_J}{dx^2} + (n-2)x\frac{d\omega_J}{dx} - (k-1)(n-k)\omega_J\right] dy^J \wedge dx.
\end{aligned}
$$

Hence the solutions to $I_q(P)\omega = 0$ are of the form

$$\omega = (c_1x^k + c_2x^{n-k-1})\, dy^I + (d_1x^{k-1} + d_2x^{n-k})\, dy^J \wedge dx. \qquad (3.3)$$

A most fortuitous property of the operator P, resulting from its invariance under a group, is the constancy of the exponents in (3.3). This need not be the case for the so-called indicial roots of a general $\mathscr{V}_0$ operator and the final analysis would then be more complicated.

It is also possible to reduce $N_q(P)$ to an ordinary differential operator by conjugating with the partial Fourier transform in the y variables. If η is dual to y, then having performed this conjugation we may choose the basis elements, confusingly also called dy^1, etc., so that $\eta = |\eta|\, dy^1$. Then the normal operator simplifies substantially , and almost uncouples. Neglecting the harmless $a(0)^2$ factor, the operator—which we relabel N_P—has three separate actions:

(i) If $\omega = \omega_I\, dy^I$ and the first index $i_1 > 1$, then

$$N_P\hat{\omega} \equiv (P_1\hat{\omega}_1)\, dy^I = \left[-x^2\frac{d^2\hat{\omega}_I}{dx^2} + (n-2)x\frac{d\hat{\omega}_I}{dx} + (x^2|\eta|^2 - k(n-k-1))\omega_I\right] dy^I.$$

(ii) If $\hat{\omega} = \hat{\omega}_J\, dy^J \wedge dx$ and $j_1 = 1$, then

$$N_P\hat{\omega} \equiv (P_2\hat{\omega}_J)\, dy^J \wedge dx = \left[-x^2\frac{d^2\hat{\omega}_J}{dx^2} + (n-2)x\frac{d\hat{\omega}_J}{dx} + (x^2|\eta|^2 - (k-1)(n-k))\hat{\omega}_J\right] dy^J \wedge dx. \tag{3.4}$$

(iii) If $\hat{\omega} = \hat{\omega}_I\, dy^I + \hat{\omega}_J\, dy^J \wedge dx$ and $I = (i_1, \cdots, i_k) = (1, j_1, \cdots, j_k)$, then

$$N_P\hat{\omega} = [P_1\hat{\omega}_I + 2(-1)^{k+1}ix|\eta|\hat{\omega}_J]\, dy^I + [P_2\hat{\omega}_J + 2(-1)^k ix|\eta|\hat{\omega}_I]\, dy^J \wedge dx,$$

where $\hat{\omega}_I$, $\hat{\omega}_J$ are the Fourier transforms in y of ω_I, $\hat{\omega}_J$.

B. The primary task facing us is to fully understand the kernel which inverts $N_q(P)$ on $L^2(x^{-n}\, dx\, dy)$. The choice of this space is both natural—since the measure is the invariant hyperbolic one—and tantamount to imposing a vanishing Dirichlet condition on the problem $N_q(P)u = f$. At any rate, it suffices to understand the kernels inverting each of the three Bessel-type operators in (3.4) on the space $L^2(\mathbf{R}^+, x^{-n}\, dx; L^2(\mathbf{R}^{n-1}, d\eta))$. We do this first for fixed η and then examine the dependence on η.

Being uncoupled, the operators P_1 and P_2 are much simpler to analyze than the one in (iii). The first step is to understand all solutions to $P_i u = 0$, $i = 1, 2$. These may be found explicitly:

$$P_i u_i = 0 \Rightarrow u_i(x) = c_1 x^{(n-1)/2} I_{\nu_i}(x|\eta|) + c_2 x^{(n-1)/2} K_{\nu_i}(x|\eta|), \quad i = 1, 2, \quad \nu_1 = \frac{n-2k-1}{2}, \quad \nu_2 = \frac{n-2k-1}{2}. \tag{3.5}$$

I_ν and K_ν are the modified Bessel function of the first kind and Macdonald's function, respectively, of order ν.

We consider only the values $k < (n-1)/2$. Values $k > (n+1)/2$ are treated, for example, by duality and the analysis breaks down for $k = (n-1)/2$, $n/2$, $(n+1)/2$. Hence we now assume that both $\nu_1, \nu_2 > 0$. The asymptotics of the solutions in (3.5) are well known [10]. Thus near $x = 0$.

$$\text{(3.6)}\quad \begin{array}{ll} x^{(n-1)/2}I_{\nu_1}(x|\eta|) \sim x^{n-k-1}|\eta|^{\nu_1}, & x^{(n-1)/2}K_{\nu_1}(x|\eta|) \sim x^{k}|\eta|^{-\nu_1}, \\ x^{(n-1)/2}I_{\nu_2}(x|\eta|) \sim x^{n-k}|\eta|^{\nu_2}, & x^{(n-1)/2}K_{\nu_2}(x|\eta|) \sim x^{k-1}|\eta|^{-\nu_2}, \end{array}$$

whereas as $x \to \infty$

$$\text{(3.7)}\quad \begin{aligned} x^{(n-1)/2}I_{\nu_i}(x|\eta|) &\sim x^{n/2-1}e^{x|\eta|}\Big/\sqrt{2\pi|\eta|}, \\ x^{(n-1)/2}K_{\nu_i}(x|\eta|) &\sim x^{n/2-1}e^{-x|\eta|}\sqrt{\pi/2|\eta|} \end{aligned}$$

for $i = 1, 2$. Notice that the rates of vanishing of the solutions in (3.6) match those in (3.3); that is, the operators I_P and N_P have the same indicial roots (in the sense of Fuchsian operators). Furthermore, in order that a solution u_i of P_i lie in $L^2(x^{-n}\,dx)$ near $x = 0$ it is necessary that $c_2 = 0$ in (3.5). On the other hand, a solution u_i increases exponentially unless $c_1 = 0$. Hence $P_i u = 0$ has no solutions which lie in $L_2(\mathbf{R}^+, x^{-n}\,dx)$.

Inverting kernels are now easily obtained. These will be functions $G_i(x, \tilde{x}, \eta)$ satisfying $G_i \in L^2(x^{-n}\,dx)$:

$$P_i G_i(x, \tilde{x}, \eta) = x^n \delta(x - \tilde{x}), \qquad i = 1, 2,$$

which is the kernel of the identity on $L^2(x^{-n}\,dx)$. The usual methods [4] for self-adjoint operators (both P_1 and P_2 are formally self-adjoint with respect to $x^{-n}\,dx$) yield

$$\text{(3.8)}\quad \begin{aligned} G_i(x, \tilde{x}, \eta) = c_{k,n} x^{(n-1)/2}\tilde{x}^{(n-1)/2}[&I_{\nu_i}(x|\eta|)K_{\nu_i}(\tilde{x}|\eta|)H(\tilde{x} - x) \\ &+ I_{\nu_i}(\tilde{x}|\eta|)K_{\nu_i}(x|\eta|)H(x - \tilde{x})]; \end{aligned}$$

H is of course the Heaviside function. It is important to note the constant $c_{k,n}$ in front does not depend on η. One may then employ Schur's criterion for the boundedness of integral kernels along with the asymptotics in (3.6), (3.7) to prove the

Proposition. *For $i = 1, 2$, $f(x) \mapsto \int G_i(x, \tilde{x}, \eta)\tilde{x}^{-n}\,d\tilde{x}$ is a bounded transformation on $L^2(\mathbf{R}^+, x^{-n}\,dx)$.*

It is not possible to find quite as explicit an inverse for the coupled operator L of (3.4)(iii). The first issue is to determine the nature of the solutions to $L\hat{\omega} = 0$. There are no solutions in $L^2(x^{-n}\,dx)$ on all of $\mathbf{R}^+$, reasonably enough, since the existence of such a solution is equivalent to the existence of a form ω in $L^2(x^{2k-n}\,dx\,dy)$ satisfying the hyperbolic Laplace equation $\Delta_H \omega = 0$. But for $k < (n-1)/2$ such a form does not exist [5]. (This is a good moment to point out that one reason for using the conjugate $P = x^k \Delta x^{-k}$ is to eliminate the dependence of the measure on k—a technical convenience in proofs.)

General perturbation considerations indicate that there should be a two-dimensional family of solutions to $L\hat{\omega} = 0$ which increase exponentially but lie in $L^2(x^{-n}\,dx)$ near 0, and another two-dimensional family of exponentially decreasing solutions which do not lie in $L^2(x^{-n}\,dx)$ near 0. This is actually the case, and we may even find one function in each family. For, corresponding to the decomposition $\Delta = d\delta + \delta d$ corresponds another, by virtue of (2.19):

$$N_P = N(x^{-k}\,dx^{k-1})N(x^{-k+1}\delta x^k) + N(x^{-k}\delta x^{k+1})N(x^{-k-1}\,dx^k).$$

Thus we look for joint solutions to

$$N(x^{-k-1}\,dx^k)\hat{\omega} = 0, \qquad N(x^{-k+1}\delta x^k)\hat{\omega} = 0,$$

which then automatically satisfy $N_P\hat{\omega} = 0$. These two equations when written out may, by a simple change of basis, actually be uncoupled. Thus with straightforward computation one finds the solutions

$$\hat{\omega}_1 = x^{(n+1)/2}(I_{\nu_2}(x|\eta|)\,dy^I + cI_{\nu_1}(x|\eta|)\,dy^J \wedge dx),$$
$$\hat{\omega}_2 = x^{(n+1)/2}(K_{\nu_2}(x|\eta|)\,dy^I + c'K_{\nu_1}(x|\eta|)\,dy^J \wedge dx).$$

The constants c, c' depend only on k, n and (linearly on) $|\eta|$. ν_1 and ν_2 are the same numbers as in (3.5). Thus $\hat{\omega}_1$ belongs to the first, exponentially increasing, family and $\hat{\omega}_2$ belongs to the second.

We now seek two additional functions, one in each family, so as to have a full basis of solutions to $L\hat{\omega} = 0$. Inasmuch as L is an operator of regular singular type, all solutions are given by Frobenius series. The indicial roots of L are the same as those of the uncoupled system (P_1, P_2), namely $k, n-k-1$ and $k-1, n-k$. In fact, Frobenius theory implies that the two additional functions we seek have expansions (setting $|\eta| = 1$ for the time being)

$$\hat{\omega}_3 = \begin{pmatrix} x^{n-k-1} + \cdots \\ cx^{n-k} + \cdots \end{pmatrix} + A(\log x)\hat{\omega}_1,$$
$$\hat{\omega}_4 = \begin{pmatrix} x^k + \cdots \\ c'x^{k-1} + \cdots \end{pmatrix} + B(\log x)\hat{\omega}_2.$$

These are exponentially increasing and decreasing, respectively, and column vector notation replaces the use of dy^I and $dy^J \wedge dx$.

Although we know the behavior of these solutions at $x = 0$, it is also important to know their precise asymptotics as $x \to \infty$. Of course, such asymptotics for $\hat{\omega}_1$ and $\hat{\omega}_2$ follow from (3.7). Up to constant factors,

$$\hat{\omega}_1 \sim e^x \begin{pmatrix} x^{n/2} + \cdots \\ c_1 x^{n/2} + \cdots \end{pmatrix}, \qquad \hat{\omega}_2 \sim e^{-x} \begin{pmatrix} x^{n/2} + \cdots \\ c_2 x^{n/2} + \cdots \end{pmatrix};$$

the dots indicate a formal series in descending power of x. The operator L has an irregular singularity at $x = \infty$, but a satisfactory theory exists for determining asymptotics of its solutions (see [4, Chapter 5]). Thus, if one can find formal solutions of the form above, then real solutions exist with these asymptotics. Such formal solutions are readily obtained for L, and two of these match precisely the asymptotics of $\hat{\omega}_1$, $\hat{\omega}_2$. The other two must therefore correspond to $\hat{\omega}_3$ and $\hat{\omega}_4$, as follows from the absence of solutions to $L\hat{\omega} = 0$ in $L^2(\mathbf{R}^+, x^{-n}\, dx)$. This absence is proved in turn by the previously noted fact that such a solution, were it to exist, could be used to construct a nontrivial harmonic L^2 k-form on hyperbolic space, which is impossible. Alternately, the obvious integration by parts as above show that an L^2 solution to $L\hat{\omega} = 0$ must also satisfy $N(x^{-k-1}\, dx^k)\hat{\omega} = 0$, $N(x^{-k+1}\delta x^k)\hat{\omega} = 0$, and hence be a combination of $\hat{\omega}_1$ and $\hat{\omega}_2$, neither of which lie in L^2. This integration by parts is justified by estimates analogous to those by which one concludes that an L^2 harmonic k-form on a complete manifold is both closed and coclosed, and in any case is completely straightforward. Hence we get that

$$\hat{\omega}_3 \sim e^x \begin{pmatrix} x^{n/2-2} + \cdots \\ c_3 x^{n/2-2} + \cdots \end{pmatrix}, \qquad \hat{\omega}_4 \sim e^{-x} \begin{pmatrix} x^{n/2-2} + \cdots \\ c_4 x^{n/2-2} + \cdots \end{pmatrix},$$

as $x \to \infty$, again all assuming $|\eta| = 1$. We should note that none of the constants $c_1, c_2, c_3, c_4, c, c', A, B$ may vanish.

Finally we may construct the inverting kernel $G_c(x, \tilde{x}, \hat{\eta})$, $|\hat{\eta}| = 1$. (We shall analyze η-dependence later.) It has a form similar to that of the scalar kernels:

$$G_c(x, \tilde{x}, \hat{\eta}) = U(x)V(\tilde{x})^* H(\tilde{x} - x) + V(x)U(\tilde{x})^* H(x - \tilde{x}), \tag{3.9}$$

where U and V are two-by-two matrices the columns of which are linear combinations of $\hat{\omega}_1, \hat{\omega}_3$ and $\hat{\omega}_2, \hat{\omega}_4$, respectively. The condition $LG_c = x^n\delta(x - \tilde{x})\,\mathrm{Id}$ entails, among other relations, that

$$U'(x)V(x)^* - V'(x)U(x)^* = -\tfrac{1}{2}x^{n-2}\,\mathrm{Id}\,.$$

This implies that

$$U(x) = \begin{pmatrix} a\hat{\omega}_3 & b\hat{\omega}_1 \end{pmatrix}, \qquad V(x) = \begin{pmatrix} c\hat{\omega}_2 & d\hat{\omega}_4 \end{pmatrix} \tag{3.10}$$

for some nonvanishing constants a, b, c, d. By estimates similar to those used for G_1, G_2 we have the

Proposition. *G_c induces a bounded integral transformation on $L^2(x^{-n}\, dx)$.*

Finally then we may reinsert η-dependence and prove the

Proposition. *There exists a unique kernel $G(x, \tilde{x}, \eta)$ which is bounded on $L^2(\mathbf{R}^+, x^{-n}\, dx; L^2(\mathbf{R}^{n-1}, d\eta))$ and such that*

$$N_P(xd_x, ix\eta)G(x, \tilde{x}, \eta) = x^n\delta(x - \tilde{x})\,\mathrm{Id}, \qquad k < (n-1)/2.$$

Proof. Uniqueness follows from the nonexistence of L^2 harmonic forms for the stated range of values of k. Let $G_0(x, \tilde{x}, \hat{\eta})$ be the kernel obtained by piecing together the kernels G_1, G_2, G_0 for each fixed $\hat{\eta}$ of unit modulus. Replace $x, \tilde{x}$ by $x|\eta|$, $\tilde{x}|\eta|$ and use the homogeneity of δ to deduce the relationship

$$N_P(xd_x, ix|\eta|\hat{\eta})G_0(x|\eta|, \tilde{x}|\eta|, \hat{\eta}) = |\eta|^{n-1}x^n\delta(x-\tilde{x})\,\mathrm{Id}\,.$$

Hence

$$G(x,\tilde{x},\eta) = |\eta|^{1-n}G_0(x|\eta|, \tilde{x}|\eta|, \eta/|\eta|)$$

for any $\eta \neq 0$.

The L^2 boundedness of G is inherited from that of G_0 as follows. First observe that the dilation

$$f(x) \mapsto f_0(x) = a^{(1-n)/2}f(ax), \qquad a \in \mathbf{R}^+,$$

is an isometry on $L^2(x^{-n}\,dx)$. Replace f by f_a in

$$\int\left|\int G_0(x,\tilde{x},\hat{\eta})f(\tilde{x})\tilde{x}^{-n}\,d\tilde{x}\right|^2 x^{-n}\,dx \leq c\int |f|^2x^{-n}\,dx,$$

where c is independent of $\hat{\eta} \in S^{n-2}$. The new right-hand side is independent of a, whereas the inner integral of the left-hand side, after a change of variables, becomes

$$\int a^{(n-1)/2}G_0(x, a^{-1}\tilde{x}, \hat{\eta})f(\tilde{x})\tilde{x}^{-n}\,d\tilde{x}.$$

Dilating this function of x by a^{-1}, then setting $a = |\eta|^{-1}$, we arrive at the inequality

$$\int\left|\int G_0(x,\tilde{x},\eta)f(\tilde{x})\tilde{x}^{-n}\,d\tilde{x}\right|^2 x^{-n}\,dx \leq c\int |f|^2x^{-n}\,dx$$

for each η. Finally, if f is allowed to depend on η also, then one need merely integrate both sides of this inequality with respect to η and the proof is complete.

The inverse for the partial differential operator N_P is given by

$$G(x,y,\tilde{x},\tilde{y}) = c_n\int e^{i(y-\tilde{y})\cdot\eta}G(x,\tilde{x},\eta)\,d\eta.$$

The reason for the painstaking explicit construction of this kernel, particularly inasmuch as its existence is well known, is so we may fit it into the framework of the last section.

(3.11) Proposition. *$G(z,\tilde{z})\cdot\gamma \in K_0^{-2,\sigma,\tau}(\mathbf{R}^n_+, {}^\circ\Lambda^k \otimes \Gamma_0^{1/2})$ where*

$$\sigma = \begin{pmatrix} n-k-1 & n-k+1 \\ n-k & n-k \end{pmatrix}, \quad \tau = \sigma^t \quad \text{when } k < (n-1)/2,$$

$$\sigma = \begin{pmatrix} k & k \\ k-1 & k+1 \end{pmatrix}, \quad \tau = \sigma^t \quad \text{when } k > (n+1)/2,$$

$z, \tilde{z} \in \mathbf{R}^n_+$ and γ is the half-density of (2.11).

The proof is fairly long, but the main point is to use formulas (3.8), (3.9), (3.10) and the known asymptotics of the special functions involved to derive symbol estimates for $G(x, \tilde{x}, \eta)$, uniform down to $x = 0$ or $\tilde{x} = 0$. These transform to the necessary estimates for $G(z, \tilde{z})$ in the Fourier integral. As always, full details appear in [11].

Recall the discussion in §2.B, wherein the interior of the half space M_q ($\simeq \mathbf{R}^n_+$) is naturally identified with the interior of F_q ($\simeq S^n_{++}$) for each $q \in \partial M$. F_q should also be thought of as the ball B^n blown up around a point p on its boundary. Through the combination of these identifications G may be transferred to a kernel G_B on the ball. It is the inverse of the operator induced by Δ_H on ${}^\circ\Lambda^k(B^n)$. If G_Δ is the inverse of the Laplacian on Λ^k, then it follows from (3.11) that

(3.12) Corollary. $G_\Delta(w, \tilde{w}) \cdot \gamma \in K_0^{-2,\sigma-k,\tau-k}(B^n, \Lambda^k \otimes \Gamma_0^{1/2})$.

Here σ, τ are as in (3.11), $W, \tilde{W}$ are coordinates in B^n, and for any real number c, $\sigma + c$ is the matrix with entries $\sigma_{ij} + c$.

C. The final parametrix construction requires specialized mapping properties of G_Δ which depend on the fact that it is the inverse of a differential operator. To state these efficiently, let us first define analogues of the spaces in (2.14).

(3.13) Definition. For $\alpha = (\alpha_t, \alpha_n)$ a pair of real numbers, set

$$\mathscr{A}^{\alpha}_{\text{phg}}(X, \Lambda^k) = \{\omega = \omega_t + \omega_n \text{ near } \partial X \text{ such that } \omega_i \in \mathscr{A}^{\alpha_i}_{\text{phg}}(X) \otimes \Gamma(\Lambda^k)\} \quad \text{for } X = M \text{ or } B^n.$$

Similarly, recall the blow-down map $\beta \colon S^n_{++} \to (B^n, p)$ which collapses one edge of S^n_{++} to $p \in \partial B^n$. For $\delta = (\delta_t, \delta_n)$ another pair of real numbers, set

$$\mathscr{A}^{\alpha,\delta}_{\text{phg}}(B^n, \Lambda^k) = \beta^* \mathscr{A}^{\alpha,\delta}(S^n_{++}, \Lambda^k).$$

Thus elements in this space have expansions, with leading terms regulated by α, δ, near the two codimension one boundaries of S^n_{++}, respectively, or equivalently, near $B^n \backslash \{p\}$ and $\{p\}$.

As always we require $|\alpha_t - \alpha_n| \le 1$, $|\delta_t - \delta_n| \le 1$.

(3.14) Proposition. *Let* $\mu \in \mathscr{A}^{\alpha+q}_{\text{phg}}(B^n, \Lambda^k)$, $\alpha = (n - 2k - 1, n - 2k)$ *if* $k < (n-1)/2$ *and* $\alpha = (0, -1)$ *if* $k > (n+1)/2$, $q \in \mathbf{Z}^+$ *and where* $\alpha + q = (\alpha_t + q, \alpha_n + q)$ *in either case. Then there exists a unique* $\omega \in \mathscr{A}^{\alpha}_{\text{phg}}$ *such that* $\Delta\omega = \mu$.

Proof. We reduce to the case of Proposition (2.16) by solving inductively for the terms in the expansion for μ at the boundary. Thus, if ρ is a defining function for ∂B^n

$$\mu_i \sim \sum_{j=0}^{\infty} \mu_{ij}, \qquad i = t, n,$$

$$\mu_{ij} \sim \sum_{l=0}^{N} c_{ijl}(\theta)\rho^{\alpha_i+q+j}(\log\rho)^l.$$

Here θ is the coordinate on the boundary. Since $\Delta - I_\Delta : \mathscr{A}^\beta \to \mathscr{A}^{\beta+1}$ for any β, we first choose ω_{i0}, $i = t, n$, so that

$$I_\Delta \omega_{i0} = \mu_{i0}, \qquad \omega_{i0} = \sum_{l=0}^{N} c'_{i0l}(\theta)\rho^{\alpha_i+q}(\log\rho)^l.$$

This is always possible since I_Δ is an ordinary differential operator of Euler type. Successively choose ω_{ij} so that $I_\Delta\omega_{ij} = \mu'_{ij}$ where the μ'_{ij} have the correct homogeneity and depend only on μ_{ij} and $(\Delta - I_\Delta)\omega_{im}$, $m < j$. Each of these equations has a unique solution since each exponent $\alpha_i + q + j$ is greater than any indicial root of I_Δ.

Asymptotically sum the ω_{ij} to an ω_0 with $(\omega_0)_i \sim \sum \omega_{ij}$, $i = t, n$. Then $\Delta\omega_0 - \mu = f \in C^\infty$. Apply (2.16) now, that is, set $\omega_1 = G_B f \in \mathscr{A}^\alpha_{\text{phg}}$. Hence $\omega = \omega_0 - \omega_1 \in \mathscr{A}^\alpha_{\text{phg}}$ is the desired solution.

(3.15) Proposition. *For any $\mu \in \mathscr{A}^{\alpha+q,\beta}_{\text{phg}}(B^n, \Lambda^k)$, $q \in \mathbf{Z}^+$, α the two-vector of (3.14) and β any two-vector with integer entries (and $|\beta_t - \beta_n| \le 1$) there is always a solution ω to $\Delta\omega = f$, $\omega \in \mathscr{A}^{\alpha,\beta}_{\text{phg}}(B^n, \Lambda^k)$.*

The style of proof here is rather more involved than that for (3.14), but the idea is first to subtract off the additional singularity, thereby reducing to (3.14). This is a local question, so we may replace B^n by $\mathbf{R}^n_+$, with p corresponding to 0, and also assume that μ is supported in $|z| \le 1$, say.

The principle tool we use is the Mellin transform in the (polar) variable r,

$$u(r,\theta) \to u_M(\varsigma,\theta) = \int r^{-\varsigma-1}u(r,\theta)\,dr,$$
$$u(r,\theta) = (2\pi)^{-1}\int r^{\varsigma}u_M(\varsigma,\theta)\,d\varsigma. \tag{3.16}$$

The inversion formula, and all of the facts we need about this transform follow from the observation that it is just the Fourier transform in logarithmic coordinates; in any case they also appear in [15].

If u vanishes for $r > 1$, as we shall always assume, then u_M is defined and analytic in a half-space $\operatorname{Re}\varsigma < a$. The integral for the inverse transform is taken over a contour $\operatorname{Re}\varsigma = c$. The number a is closely related to the decay rate of u as $r \to 0$. If there is a definite power of vanishing here then u_M is rapidly decreasing as $|\operatorname{Im}\varsigma| \to \infty$, $\operatorname{Re}\varsigma$ fixed. But most importantly, if u has a classical expansion as $r \to 0$, then u_M continues meromorphically to the whole plane. The poles occur only at $\varsigma \in \beta_0 + \mathbf{Z}^+$ if β_0 is the most singular exponent in the expansion. The order of the pole at $\beta_0 + i$ is $N_i + 1$ if there are N_i logarithmic factors in the ith term of the expansion. Conversely, if $v(\varsigma, \theta)$ is meromorphic in ς with only real poles and decreases rapidly on each line $\operatorname{Re}\varsigma$ =constant, then (3.16) defines a function with an expansion as $r \to 0$ in powers of r, these powers corresponding to the locations of the poles of v—and the order of the poles related to the number of logarithmic factors. In addition, only those powers occur which correspond to poles to the right of the contour of integration.

Now, define the operator L_ς by $(\Delta u)_M = L_\varsigma u_M(\varsigma, \theta)$ so that

$$L_\varsigma = |z|^{-\varsigma} \Delta |z|^\varsigma.$$

Since Δ is invariant under the homothety $z \to az$, L_ς is also. Hence it is algebraic in ς, and differential in θ. In fact, as an operator on $S_+^{n-1} = \{\theta : \theta_n \geq 0\}$ it lies in Diff_0^2. Its indicial operator is nothing but that for Δ,

$$I(L_\varsigma) = I_\Delta.$$

The strategy now is to invert L_ς. Its inverse G_ς does not exist for every value of ς, but rather is meromorphic on the ς plane. It is defined formally as follows. Let $\rho(r) \in C_0^\infty(\mathbf{R})$ equal 1 near $r = 0$. Then for $f \in \dot{C}^\infty(S_+^{n-1}) \otimes \Lambda^k$ set

$$G_\varsigma f(\theta) = \lim_{t \to 0} M_t^*(r^{-\varsigma} G_\Delta(r^\varsigma \rho(r) f(\theta))),$$

where G_Δ is the inverse of the hyperbolic Laplacian and M_t is the dilation map. By substituting the integral defining G_Δ we get

$$G_\varsigma f(\theta) = \int_{S_+^{n-1}} H_\varsigma(\theta, \tilde{\theta}) f(\tilde{\theta}) (\tilde{\theta}_n)^{-(n-1)} \, d\tilde{\theta}, \tag{3.17}$$

where

$$H_\varsigma(\theta, \tilde{\theta}) = \int_0^\infty G_\Delta(\theta, \tau\tilde{\theta}) \tau^\varsigma \frac{d\tau}{\tau \tilde{\theta}_n}. \tag{3.18}$$

If ς assumes a value for which (3.18) converges, then G_ς is a well-defined inverse for L_ς. To determine these values of ς, fix $\theta \neq \tilde{\theta}$ and note that by homothety invariance $G_\Delta(\theta, \tau\tilde{\theta}) = G_\Delta(\tau^{-1}\theta, \tilde{\theta})$. The components G_{ij} of G_Δ decay at various rates as $\tau \to 0$ or ∞, but if $k < (n-1)/2$, then G_{tt} is the extreme case. From (3.12) one sees that

$$G_{tt}(\theta, \tau\tilde{\theta}) \sim \tau^{n-2k-1}, \qquad \tau \to 0,$$
$$G_{tt}(\tau^{-1}\theta, \tilde{\theta}) \sim \tau^{-(n-2k-1)}, \qquad \tau \to \infty.$$

Thus (3.18) converges when $-(n-2k-1) < \operatorname{Re}\varsigma < n-2k-1$. Bookkeeping with the expansions for the various components of G_Δ shows that

Lemma. *H_ς extends meromorphically to the whole ς-plane with poles at $\varsigma \in \pm(n-2k-1+\mathbf{Z}^+)$ if $k < (n-1)/2$ and $\varsigma \in \mathbf{Z}$ if $k > (n+1)/2$. These poles are at most of order two, and the residues at $\varsigma = N$ of H_ς and $(\varsigma - N)H_\varsigma$ are operators of finite rank.*

Since $L_\varsigma G_\varsigma = (\theta_n)^{n-1}\delta(\theta - \tilde{\theta})$ when (3.18) converges, by analytic continuation this equation must also hold on the whole ς-plane away from the poles. Furthermore, as an operator on S_+^{n-1}, H_ς almost lies in $K_0^{-2,\sigma-k,\tau+k-1}$, σ and τ the matrices of (3.11). In fact all its singularities and expansions near $\Delta\iota_0$, T and B (of $S_+^{n-1} \times_0 S_+^{n-1}$) are of the correct type, as is seen from (3.18). The only difference is that it also has logarithmic growth near F. This does not affect any of its continuity properties, so the proof of (3.14) may be repeated to show that

$$L_\varsigma \omega = \mu \in \mathscr{A}_{\text{phg}}^{\alpha+q}(S_+^{n-1}, \Lambda^k)$$

has a unique solution $\omega \in \mathscr{A}_{\text{phg}}^{\alpha}(S_+^{n-1}, \Lambda^k)$ depending meromorphically on ς. Its poles are precisely those of H_ς.

Proof of (3.15). As above, consider μ as a form on $\mathbf{R}_+^n$ supported in $|z| \leq 1$. Its Mellin transform in r, $\mu_M(\varsigma, \theta)$, is meromorphic with poles at $\min(\beta_t, \beta_n) + \mathbf{Z}^+$, μ_M still lies in $\mathscr{A}_{\text{phg}}^{\beta+q}$ on S_+^{n-1}. Next let $\omega_M(\varsigma, \theta) = G_\varsigma \mu_M(\varsigma, \theta)$. It too is meromorphic in ς and lies in $\mathscr{A}_{\text{phg}}^{\beta}(S_+^{n-1})$. Since the β_i are integers, its poles occur at $\varsigma \in \pm(n-2k-1+\mathbf{Z}^+)$ (or $\mathbf{Z}$) and $\min(\beta_t, \beta_n) + \mathbf{Z}^+$. Define

$$\omega_1(r, \theta) = (2\pi)^{-1} \int_{\operatorname{Re}\varsigma = \min(\beta_t, \beta_n) - 1/2} r^\varsigma \omega_M(\varsigma, \theta)\, d\varsigma.$$

Then $\Delta\omega_1 - \mu = g$ vanishes to infinite order at $r = 0$. Transferred back to B^n, g lies in $\mathscr{A}_{\text{phg}}^{\alpha+q}(B^n, \Lambda^k)$. Now apply (3.14) to solve

$$\Delta\omega_2 = g, \qquad \omega_2 \in \mathscr{A}_{\text{phg}}^{\alpha}.$$

$\omega_1 - \omega_2$ is the desired solution.

4. Hodge cohomology

A. It is now a simple matter to use the machinery thus far developed to construct a parametrix for Δ_g by the method sketched in §2.E. The first of the three steps is valid for all degrees f since it requires only symbol ellipticity. Thus with the aid of the symbol map (2.12) we find

$$E_0 \in \Psi_0^2(M, \Lambda^k)$$

satisfying

$$\Delta_0 E_0 = I - Q_0, \qquad Q_0 \in \Psi_0^{-\infty}.$$

(Notice that we are using the "less natural" bundle Λ^k rather than ${}^\circ\Lambda^k$ to simplify the asymptotics exponents later on.)

In an ordinary problem this would complete the process. But Q_0, while smoothing, is not compact on L^2. Hence we must continue. In the next step we seek an operator

$$E_1 \in \Psi_0^{-\infty,\sigma',\tau'}, \qquad \sigma' = \sigma - k, \tau' = \tau - k,$$

σ, τ the matrices of (3.11), which satisfies

$$\Delta_g E_1 = Q_0 - Q_1, \qquad Q_1 \in R^\infty \Psi_0^{-\infty,\sigma',\tau'}.$$

This step requires the invertibility of the normal operator and thus fails in the middle degrees $k = n/2$, $(n \pm 1)/2$.

We need only determine the power series expansion of E_1 at F, and then choose an operator with this as its Taylor series, in order that this last formula obtain. So, its first term $E_{1,0}$ is chosen so that

$$N_q(\Delta_g) N_q(E_{1,0}) = N_q(Q_0), \qquad q \in \partial M.$$

This is possible since $N_q(Q_0)$ vanishes to all orders on the boundary of F_q, hence also on ∂B^n if the equation is transferred to the ball. The solution is well behaved:

$$N_q(F_{1,0}) \in \mathscr{A}_{\mathrm{phg}}^{\sigma',\tau'}(F_q, \Lambda^k).$$

Now set

$$\Delta_g E_{1,0} = Q_0 - Q'_{1,1}, \qquad Q'_{1,1} \in R\Psi_0^{-\infty,\sigma',\tau'}.$$

Actually $Q'_{1,1}$ lies in the slightly better space $R\Psi_0^{-\infty,\sigma'+1,\tau'}$ since $I(\Delta_0)$ annihilates the leading term of the expansion for $E_{1,0}$ at the top face. Thus we may write

$$Q'_{1,1} = x Q''_{1,1}, \qquad Q''_{1,1} \in \Psi_0^{-\infty,\sigma',\tau'}.$$

To find the other items in the series for E_1 it is convenient to use a Taylor expansion in $\tilde{x}$ rather than in R:

$$E_1 \sim \sum \tilde{x}^j E_{1,j}.$$

The advantage is that Δ_g acts directly on the coefficient operators

$$\Delta_g E_1 \sim \sum \tilde{x}^j \Delta_g E_{1,j}.$$

We also define $Q_{1,j} = (x/\tilde{x}) Q''_{1,j}$ for each j. The inductive step is to solve

$$\Delta_g E_{1,j} = Q_{1,j} - x Q''_{1,j+1}$$

for each j. Here we expect

$$Q_{1,j} \in \Psi^{-\infty,\sigma'+1,\tau'+j} \Leftrightarrow Q''_{1,j} \in \Psi_0^{-\infty,\sigma',\tau'-j+1}.$$

But (3.15) guarantees the existence of a solution $E_{1,j}$, and the remainder $Q''_{1,j+1}$ is thereby determined. It is because $N_q(Q_{1,j})$ becomes increasingly singular near the bottom edge of F_q that we require the full strength of (3.15).

Finally, since

$$\tilde{x}^j E_{1,j} \in \Psi_0^{-\infty,\sigma',\tau'}$$

for each j, we may asymptotically sum the series of these terms so as to obtain E_1—which lies in the same space.

The third and last step is to find E_2 for which

$$\Delta_g E_2 = Q_1 - Q_2, \qquad Q_2 \in R^\infty \Psi_0^{-\infty,\infty,\tau'},$$

that is, to cancel off the expansion for Q_1 at the top face. This is accomplished by solving a sequence of equations where the operator in question is simply $I(\Delta_g)$, and is straightforward.

The sum $E = E_1 + E_2 + E_3 \in \Psi_0^{-2,\sigma',\tau'}$ is our final parametrix.

(4.1) Theorem. $\Delta_g E = I - Q$, $Q \in R^\infty \Psi_0^{-\infty,\infty,\tau'}$. *The remainder* Q *and its adjoint* $Q^* \in R^\infty \Psi_0^{-\infty,\sigma',\infty}$ *are compact on* $L^2\Omega^k(M, dg)$, *and so* Δ_g *is Fredholm on that space.*

B. An immediate consequence of (4.1) is the finite dimensionality of

$$\mathscr{H}^k = \{\omega \in L^2\Omega^k(M, dg) \colon \Delta_g \omega = 0\}$$

for k away from the middle degrees. In fact, (3.14) also shows that for such ω, since then $\omega = Q^*\omega$, we have

$$(4.2) \quad \omega \in \mathscr{A}^\alpha_{\mathrm{phg}}(M, \Lambda^k), \qquad \alpha = \begin{cases} (n-2k-1, n-2k), & k < (n-1)/2, \\ (0,-1), & k > (n+1)/2. \end{cases}$$

It remains for us to compute the dimensions of the $\mathscr{H}^k$. Since any harmonic form in L^2 on a complete manifold is closed, there is a map

$$(4.3) \qquad \mathscr{H}^k \ni \omega \mapsto [\omega] \in H^k(M)$$

into the absolute cohomology. When $k < (n-1)/2$ (4.2) shows that $i^*\omega = 0$, $i \colon \partial M \hookrightarrow M$ the inclusion. Hence we obtain a map into the relative cohomology for these degrees:

$$(4.4) \qquad \mathscr{H}^k \ni \omega \mapsto [\omega] \in H^k(M, \partial M),$$

since these relative groups may be defined using forms pulling back to zero on the boundary [17]. A modicum of care is required since the forms in $\mathscr{H}^k$ are not C^∞ up to the boundary, but their use is easily justified.

Proof of the Main Theorem (1.4). By duality it suffices to prove that (4.4) is injective when defined (i.e. below the middle degrees) and (4.3) is surjective above the middle degrees. Let us prove injectivity first. We need to show that if $\omega \in \mathscr{H}^k$, $k < (n-1)/2$, and $\omega = d\eta$, $i^*\eta = 0$, then $\omega = 0$. Choose coordinates near ∂M so that the components of the metric satisfy $h_{in} = 0$, $i < n$, $h_{nn} = 1$, and so that z^n is globally defined in a collar neighborhood of the boundary. Then since $\delta\omega = 0$

$$\begin{aligned} \|\omega\|^2 &= (d\eta, \omega) - \langle \eta, \delta_g \omega \rangle = \int d(\eta \wedge *_g \omega) \\ &= \lim_{\varepsilon \to 0} \int_{\{z^n = \varepsilon\}} \eta \wedge *_h \omega \cdot \rho^{2k-n}. \end{aligned}$$

For the last equality we used Stokes' Theorem and the equality $*_g = \rho^{2k-n} *_h$ on Ω^k. By the assumption on the coordinates only η_t and $\omega_n = (*_h\omega)_t$ enter into this last integral. From (4.2) we know that

$$\omega_n = O(\rho^{n-2k}(\log\rho)^N)$$

for some N (actually $N = 1$ works). We therefore need to show that η_t vanishes as some power of ρ to ensure that the integral vanishes in the limit so that we may conclude $\omega = 0$.

For this we use the chain homotopy operator R of [17]. If $\omega = \omega_I\, dy^I + \omega_J\, dy^J \wedge dx$ and $\omega_I(y, 0) = 0$, i.e., $i^*\omega = 0$, then define $R\omega = (R\omega)_J\, dy^J$ by

$$(R\omega)_J = (-1)^{k-1} \int_0^1 \omega_J(y, tx)\, dt.$$

It may be shown that this is coordinate independent and $dR\omega + Rd\omega = \omega$ (provided $i^*\omega = 0$). Note also that $i^*R\omega = 0$. Thus if $d\omega = 0$ then $\omega = dR\omega$ near ∂M where R is defined. Hence if ψ is supported in this neighborhood and is identically one near ∂M then $\omega - d(\psi R\omega)$ represents the same relative class as ω and vanishes near the boundary. Thus there exists a *smooth* form β with

$$i^*\beta = 0, \qquad \omega - d(\psi R\omega) = d\beta.$$

Hence with $\eta = \beta + \psi R\omega$ we see that $\eta_t = O(\rho\log\rho)$ and the integral vanishes as $\varepsilon \to 0$.

On the other hand, when $k > (n+1)/2$, let α be a smooth representative for an arbitrary absolute cohomology class. Obviously $\alpha \in L^2\Omega^k(dg)$ so by (4.1) we may decompose α as

$$\alpha = \Delta_g\beta + \omega, \qquad \omega \in \mathscr{H}^k, \beta \in L^2\Omega^k.$$

But $d\alpha = 0$ so that $\alpha = d\delta\beta + \omega$; hence $[\alpha] = [\omega]$ and so (4.3) is surjective.

Next, any $n/2$-form on M smooth up to the boundary and satisfying $d\omega = 0$, $\delta_h\omega = 0$ is in $\mathscr{H}^{n/2}$; hence this space is infinite dimensional. The spectrum is nonessential in $0 < \lambda < a_0^2(n-2k-1)^2/4$, $k \le n/2$ (the case $k \ge n/2$ following by duality), because for these values of λ the general methods of this paper also yield a parametrix for $\Delta_g - \lambda$. In fact, since we are not interested in precise asymptotics, we may take this $E(\lambda)$ of the form $E_1 + E_2$ — E_1 is constructed using the symbol calculus and E_2 is chosen to cancel only the constant term in the Taylor expansion of the error Q_1 left by E_1. This in turn requires the invertibility on L^2 of $a^2\Delta_H - \lambda$, $a = \partial\rho/\partial\nu(q)$, $q \in \partial M$. Since by definition $a_0 = \inf a$, this last step is always possible by the results of [6].

Finally, to show that each $\lambda \geq a_0^2(n-2k-1)^2/4$ (again for $k \leq n/2$) lies in the essential spectrum we must produce for each such λ and each $\varepsilon > 0$ an infinite dimensional family of forms for which $\|(\Delta_g - \lambda)\omega\| \leq \varepsilon\|\omega\|$. These are obtained from the generalized eigenforms $f_{\mu,I}(x) = (x^{(n-2k-1)/2+i\mu})\,dy^I$ on $\mathbf{H}^n$. These satisfy $(a_0^2\Delta_H - \lambda)f_{\mu,I} = 0$ with $\lambda = a_0^2((n-2k-1)^2/4 + \mu^2)$, but of course just miss lying in L^2. Then, a brief calculation shows that one may choose $\phi(x,y)$ with support in $\{|y| \leq x, a \leq x \leq b\}$ and such that $\phi(x,y)f_{\mu,I}$ is an approximate eigenform for $a_0^2\Delta_H$ provided a, b are suitably small. Now, if $q \in \partial M$ is chosen so that $\partial\rho/\partial\nu(q) = a_0$, and if coordinates are chosen so that $q = (0,0)$, then $\phi \cdot f_{\mu,I}$ may be considered as a form on M supported near q. It will satisfy the necessary inequality if a, b are small enough, and the infinite dimensional family is spanned by such forms. Hence λ is in the essential spectrum. The proof is complete.

A finer analysis of this essential spectrum, perhaps to rule out the existence of point spectrum in this range, seems a difficult problem in general. In the special case when $k = 0$ and the limiting curvature is constant one may show (see [12]) that the spectrum in this range is absolutely continuous with no embedded eigenvalues.

We conclude with a simple observation. A space closely related to $\mathscr{H}^k$, but usually larger, is the more widely studied L^2 cohomology space L^2H^k. The two spaces are identical when the range of Δ_k is closed, so we obtain the

Corollary. *$L^2H^k \simeq \mathscr{H}^k$ for all values of $k \neq (n \pm 1)/2$. In particular, if also $k \neq n/2$, L^2H^k is of finite dimension (and is identified with the topological cohomology of M).*

References

[1] M. Anderson, *The Dirichlet problem at infinity for manifolds of negative curvature*, J. Differential Geometry **18** (1983) 701–721.

[2] ——, *L^2 harmonic forms and a conjecture of Dodziuk-Singer*, Bull. Amer. Math. Soc. (N.S.) **13** (1985) 163–165.

[3] M. Anderson & R. Schoen, *Positive harmonic functions on complete manifolds of negative curvature*, Ann. of Math. (2) **121** (1985) 429–461.

[4] E. Coddington & N. Levinson, *Theory of ordinary differential equations*, McGraw Hill, New York, 1955.

[5] J. Dodziuk, *L^2 harmonic forms on rotationally symmetric Riemannian manifolds*, Proc. Amer. Math. Soc. **77** (1979) 395–400.
[6] H. Donnelly, *The differential form spectrum on hyperbolic space*, Manuscripta Math. **33** (1981) 365–385.
[7] H. Donnelly & F. Xavier, *On the differential form spectrum of negatively curved Riemannian manifolds*, Amer. J. Math. **106** (1984) 169–185.
[8] P. Eberlein & B. O'Neill, *Visibility manifolds*, Pacific J. Math. **46** (1973) 45–109.
[9] L. Hörmander, *The analysis of linear partial differential operators*, Vol. III, Springer, Berlin, 1985.
[10] N. N. Lebedev, *Special functions and their applications*, Dover, New York, 1972.
[11] R. Mazzeo, *Hodge cohomology of negatively curved manifolds*, Thesis, MIT, 1986.
[12] R. Mazzeo & R. B. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature*, J. Functional Analysis, to appear.
[13] H. P. McKean, *An upper bound to the spectrum of Δ on a manifold of negative curvature*, J. Differential Geometry **4** (1970) 359–366.
[14] R. B. Melrose, *Transformation of boundary problems*, Acta Math. **147** (1981) 149–236.
[15] R. B. Melrose & G. Mendoza, *Elliptic operators of totally characteristic type*, Math. Sci. Res. Inst. Reprint, also to appear in J. Differential Equations.
[16] R. B. Melrose & N. Ritter, *Interaction of progressing waves for semilinear wave equations. II*, Ark. Mat., to appear.
[17] D. Ray & I. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Advances in Math. **7** (1971) 145–210.
[18] D. Sullivan, *The Dirichlet problem at infinity for a negatively curved manifold*, J. Differential Geometry **18** (1983) 723–732.

STANFORD UNIVERSITY