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ON THE TOPOLOGY OF CLIFFORD ISOPARAMETRIC HYPERSURFACES

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A hypersurface in the unit sphere S^{n+1} is called *isoparametric* [5] if it has constant principal curvatures. The simplest, i.e., those for which the number g of distinct principal curvatures is less than or equal to 2, are the parallels of the equators and the product of spheres. Isoparametric hypersurfaces with $g = 3$ were classified by E. Cartan; they exist in dimensions $n = 3, 6, 12$, and 24. The above examples, being homogeneous, are well understood topologically. All the other isoparametric hypersurfaces have 4 or 6 distinct principal curvatures. Those with $g = 6$ exist only if $n = 6$ or 12.

Isoparametric hypersurfaces with $g = 4$ are the most interesting and have not been completely classified yet. With the exception of two cases in dimensions $n = 8$ and 18, all the known examples belong to the Clifford series discovered by Ferus, Karcher, and Münzner. For every orthogonal representation of the Clifford algebra C_{m-1} on $\mathbb{R}^l$, there corresponds [8] an isoparametric function on S^{2l-1} whose regular level sets are isoparametric hypersurfaces with $g = 4$. If $m \not\equiv 0 \pmod 4$, this function is determined by m and l up to a rigid motion of S^{2l-1}. If, however, $m \equiv 0 \pmod 4$, there are inequivalent representations of C_{m-1} on $\mathbb{R}^l$ parametrized by an integer q, the index of the representation. The unique (up to congruence) zero mean curvature (i.e., minimal) level set of an isoparametric function constructed from an index q representation of C_{m-1} on $\mathbb{R}^l$ is denoted by $M(m, l, q)$.

The aim of the present work is to study the topology of these hypersurfaces. We give a fairly complete classification of the $M(m, l, q)$ as well as their focal varieties in terms of homotopy, homeomorphism, and diffeomorphism types. For "small" l, the $M(m, l, q)$ are of distinct homotopy types, although their cohomological rings are the same. However, it turns out that the diffeomorphic types of $M(m, l, q)$ are periodic in q with a period d_m, the denominator of $B_{m/4}/m$, $B_{m/4}$ being the $(m/4)$th Bernoulli number.

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Finally, as a corollary to our periodicity theorem, we solve negatively a problem of S. S. Chern [7], [6] on the uniqueness of minimal hypersurfaces with given constant scalar curvature in the spheres. This work was done during the author's visit to the Max-Planck-Institut für Mathematik in 1985. The author is grateful to Professor F. Hirzebruch for his hospitality.

1. The Clifford isoparametric hypersurfaces

Let $P_0, P_1, \cdots, P_m$ be elements in $O(2l)$ such that for $i, j = 0, 1, 2, \cdots, m$.

$$P_iP_j + P_jP_i = 2\delta_{ij}I. \tag{1.1}$$

In other words, the P_i's are generators of an orthogonal representation of $C_{0,m+1}$, the Clifford algebra of $\mathbb{R}^{m+1}$ endowed with a positive definite metric on $\mathbb{R}^{2l}$.

Following [8], we denote by $E_+(P_0)$ the $+1$-eigenspace of P_0; then $E_+(P_0) \simeq \mathbb{R}^l$ and is clearly invariant under $E_1 = P_1P_2, \cdots, E_{m-1} = P_1P_m$. These E_i's are elements in $O(l)$ and satisfy

$$E_iE_j + E_jE_i = -2\delta_{ij}I; \tag{1.2}$$

hence they define an orthogonal C_{m-1} module structure on $E_+(P_0)$, where C_{m-1} is, as usual, the Clifford algebra of $\mathbb{R}^m$ endowed with a negative definite metric. Conversely, given $E_1, E_2, \cdots, E_{m-1}$ satisfying (1.2), one can construct $P_0, \cdots, P_m$ such that (1.1) is satisfied (cf. [8, p.482]). There is therefore a 1-1 correspondence between equivalent classes of orthogonal representations of $C_{0,m+1}$ and C_{m-1}.

Let $\delta(m)$ be the dimension of irreducible C_{m-1}-modules (e.g., $\delta(4) = 4$, $\delta(8) = 8$); then $l = k\delta(m)$, where k is a positive integer. It is well known that when $m \equiv 0 \pmod 4$, there are two irreducible C_{m-1}-modules Δ_m^+ and Δ_m^-, distinguished by $E_1 E_2 \cdots E_{m-1} = \mathrm{Id}$ or $-\mathrm{Id}$. If we write

$$E_+(P_0) = a\Delta_m^+ \oplus b\Delta_m^- \tag{1.3}$$

as C_{m-1}-modules, then

$$\mathrm{tr}(P_0P_1 \cdots P_m) = q2\delta(m), \tag{1.4}$$

where $q = a - b$. q is the only algebraic invariant of equivalent classes of representations of C_{m-1} on $\mathbb{R}^l$ and will be called the index of the representation. Note that

$$q \equiv k \pmod 2. \tag{1.5}$$

According to E. Cartan [5], a smooth function f defined on a space-form is called *isoparametric* if $\|df\|^2$ and Δf are functions of f. The latter conditions are equivalent to the condition that the regular level sets $f^{-1}(c)$ have constant principal curvatures. It is still an open problem to classify all such functions on the standard sphere. However, with few exceptions, most isoparametric functions on the standard sphere discovered after E. Cartan belong to the "Clifford family" due to Ferus, Karcher, and Münzner and are given by

$$f(x) = \langle x, x\rangle^2 - 2\sum_{i=1}^{m} \langle P_i x, x\rangle, \tag{1.6}$$

where $x \in S^{2l-1}$ and $P_0, P_1, \cdots, P_m$ satisfy (1.1). f maps S^{2l-1} onto $[-1,1]$. Two systems defined by (1.1) with the same index q give rise to equivalent functions on S^{2l-1}. If $m \not\equiv 0 \pmod 4$, then q is always zero. If $m \equiv 0 \pmod 4$, there are $k+1$ inequivalent functions on S^{2l-1} corresponding to the $k+1$ distinct values of q. We have the following algebraic varieties in S^{2l-1}:

$$\begin{aligned} M(m,l,q) &= f^{-1}\left(\frac{l-2m-1}{l-1}\right), \\ M_+(m,l,q) &= f^{-1}(1), \\ M_{-1}(m,l,q) &= f^{-1}(-1). \end{aligned} \tag{1.7}$$

These varieties are determined by the three numbers m, $l = k\delta(m)$ and q up to a rigid motion of S^{2l-1}. Note that a change of sign of q does not change the corresponding varieties. We also write M, $M(q)$ etc. for $M(m,l,q)$ etc. when the missing numbers are understood. The following were shown in [8]:

(i) $M(m,l,q)$ is a compact connected minimal hypersurface with constant scalar curvature $4l^2 - 16l + 12$ in S^{2l-1}, $M \cong M_+ \times S^m$;

(ii) $M_-(m,l,q)$ is diffeomorphic to a S^{l-1}-bundle over S^m;

(iii) if $m \equiv 0 \pmod 4$, then $M(m,l,q)$ and $M(m,l,q')$ are not congruent to each other unless $q = \pm q'$.

It was also shown that for any $m \geqslant 1$, $M_+(m,l,q)$ is a compact connected submanifold in S^{2l-1} of codimension $m+1$. In fact, it is a complete intersection of $m+2$ quadrics in $\mathbb{R}^{2l}$ hence has trivial normal bundle in S^{2l-1} (cf. (3.6)).

In view of (i) and (iii) above, one would immediately get counterexamples to Chern's problem (cf. §4) if all of the minimal hypersurfaces $M(m,l,q)$ were diffeomorphic. At first glance, this seems plausible since they all have isomorphic cohomology rings as was shown by Münzner [11]. However, a closer look at the first examples proves that this is wrong. In fact the hypersurfaces may have distinct homotopy types (cf. Theorem 2(b) in §3).

Example. It can be shown that $M_+(4,8,0) = S^3 \times S^7$ while $M_+(4,8,2) = \mathrm{Sp}(2)$. Since $\pi_6(\mathrm{Sp}(2)) = 0 \neq \pi_6(S^3 \times S^7)$ and $M \cong M_+ \times S^4$, it follows that $M(4,8,0)$ and $M(4,8,2)$ have distinct homotopy types. In fact, this is the case for any $m \equiv 0 \pmod 4$ provided l is not big. But, when l is big, some of the hypersurfaces become diffeomorphic, hence provide us with the desired counterexamples.

2. Geometry and topology of M_-

It is necessary to review some known facts on Clifford modules and K-theory which will be needed in this paper. The reader is referred to [2] for definitions and proofs.

Recall that $\mathrm{Spin}(m) \subset C_m^0 = C_{m-1}$, hence a C_{m-1}-module is a $\mathrm{Spin}(m)$ module in a canonical way. The unit sphere S^m in $\mathbb{R}^{m+1}$ is canonically diffeomorphic to $\mathrm{Spin}(m+1)/\mathrm{Spin}(m)$. For every C_{m-1} module V, we can construct the associated vector bundle $\alpha(V) = \mathrm{Spin}(m+1) \times_{\mathrm{Spin}(m)} V$ via the induced representation of $\mathrm{Spin}(m)$ on V. The characteristic map $S^{m-1} \to \mathrm{SO}(V)$ of the bundle $\alpha(V)$ is [12] given by regarding S^{m-1} as the unit sphere in the vector subspace of C_{m-1} spanned by $1, E_1, \cdots, E_{m-1}$, and then using the Clifford multiplication. Let $N(C_{m-1})$ be the free abelian group generated by isomorphic classes of irreducible C_{m-1} modules. Then $N(C_{m-1}) = N(C_m^0)$ is isomorphic to $M(C_m)$, the free abelian group generated by isomorphic classes of irreducible *graded* C_m modules. The construction $V \to \alpha(V)$ extends to a ring homomorphism α: $M(C_m) \to \widetilde{KO}(S^m)$. Since α annihilates the image of i^*: $M(C_{m+1}) \to M(C_m)$, where i: $C_m \to C_{m+1}$ is the inclusion, it induces a homomorphism α: $A_m \to \widetilde{KO}(S^m)$, where A_m is the cokernal of i^*. The following is a special case of Theorems 14.3 and 11.5 in [2].

(2.1) Theorem (*Atiyah-Bott-Shapiro*). *α is an isomorphism.*

In particular, when $m \equiv 0 \pmod 4$, the vector bundles $\xi_m^+ = \alpha(\Delta_m^+)$ and $\xi_m^- = \alpha(\Delta_m^-)$ both are generators of $\widetilde{KO}(S^m) \cong \mathbb{Z}$ and $\xi_m^+ + \xi_m^- = 0$ in $\widetilde{KO}(S^m)$. Specializing to the case of $V = E_+(P_0)$, where $P_0, \cdots, P_m$ are given as in (1.1), we get a vector bundle ξ of rank l over S^m. It is related to M_- via

Proposition 1. *M_- is diffeomorphic to $S(\xi)$, the unit sphere bundle of ξ.*

Proof. $P_0, \cdots, P_m$ are orthonormal in $\mathbb{R}(2l)$ endowed with the inner product

$$\langle A, B\rangle = \frac{1}{2l}\mathrm{tr}(A'B). \tag{2.2}$$

$P_0, \cdots, P_m$ span a vector subspace U in $\mathbb{R}(2l)$ of dimension $m+1$. The unit sphere in U is denoted by $\Sigma(P_0, \cdots, P_m)$ and is called the *Clifford Sphere*. It is known that M_- is the disjoint union of the great spheres $S^{2l-1} \cap E_+(P) = SE_+(P)$, where $P \in \Sigma(P_0, \cdots, P_m)$. In other words, M_- is the unit sphere bundle $S(\eta)$ of the vector bundle η over $\Sigma(P_0, \cdots, P_m)$, the fiber over $P \in \Sigma(P_0, \cdots, P_m)$ being $E_+(P)$.

Consider the map j of U into the associative algebra $\mathbb{C}(2l)$ defined by

$$j(A) = \sqrt{-1}A. \tag{2.3}$$

Clearly $j(A)^2 = -\langle A, A\rangle I$, hence j extends to an embedding of the Clifford algebra C_{m+1} on U into $\mathbb{C}(2l)$ by the universality property of the algebra C_{m+1}. If we identify C_{m+1} with its image $j(C_{m+1})$ in $\mathbb{C}(2l)$, it is obvious that $\mathrm{Spin}(m+1) \subset \mathbb{R}(2l)$. Now define a C^∞ map

$$\varphi\colon \mathrm{Spin}(m+1) \times E_+(P_0) \to E(\eta) \tag{2.4}$$

by $\varphi(A, Y) = Ay$. Since $Ay \in E_+(AP_0A^{-1})$ and $AP_0A^{-1} \in \Sigma(P_0, \cdots, P_m)$, φ maps $\mathrm{Spin}(m+1) \times E_+(P_0)$ into $E(\eta)$, the total space of η. It is surjective because $\mathrm{Spin}(m+1)$ acts on $\Sigma(P_0, \cdots, P_m)$ transitively. If $\varphi(A_1, Y_1) = \varphi(A_2, Y_2)$, then $A_2 = A_1B$ and $By_2 = y_1$, where $B \in \mathrm{Spin}(m)$, the isotropy subgroup fixing P_0. Hence φ induces a C^∞ bundle isomorphism

$$\bar{\varphi}\colon E(\xi) \to E(\eta), \tag{2.5}$$

and therefore a diffeomorphism of their associated sphere bundles $S(\xi)$ and M_-.

Corollary 1. *M_- is diffeomorphic to the product $S^m \times S^{l-1}$ in each of the following cases*:

(i) $m \equiv 1$ *or* $2 \pmod 8$ *and k is even*;

(ii) $m \equiv 3, 5, 6$ *or* $7 \pmod 8$;

(iii) $m \equiv 0 \pmod 4$ *and* $q = 0$.

In fact, when $m \equiv 0 \pmod 4$, $\xi = q\xi_m^+ \oplus$ *trivial bundle.*

Proof. Since $\operatorname{codim}(M_-) = l - m - 1 \geqslant 1$, $l \geqslant m + 2$, the vector bundle ξ is trivial iff it is stably trivial. If $m \not\equiv 0 \pmod 4$, then ξ is k times a generator of $\widetilde{KO}(S^m)$ by Theorem 2.1. The corollary follows immediately from the table of $\widetilde{KO}(S^m) = \pi_{m-1}O$ as given by Bott periodicity. If $m \equiv 0 \pmod 4$, $\xi_m^+ + \xi_m^-$ is trivial.

Corollary 2. *If* $m \equiv 0 \pmod 4$, *then the $(m/4)$th Pontrjagin class of ξ is*

$$p_{m/4}(\xi) = cq\gamma_m \tag{2.6}$$

where $\gamma_m \in H^m(S^m, \mathbb{Z})$ is a suitable generator and c is an integer depending only on m.

Proof. The complexification homomorphism $C\colon \widetilde{KO}(S^m) \to \tilde{K}(S^m)$ maps Δ_m^+ to $2g_m$ if $m \equiv 4 \pmod 8$ and to g_m if $m \equiv 0 \pmod 8$, where g_m is a generator in $\widetilde{KO}(S^m)$. The Chern character ch: $\tilde{K}(S^m) \to H^m(S^m, \mathbb{Z})$ is isomorphic onto [4]. The corollary follows from $p_{m/4}(\xi) = c_{m/2}(\xi \otimes \mathbb{C})$.

We are now in a position to give a complete classification of the $M_-(q)$'s according to homotopy, diffeomorphic, and homeomorphic types:

Theorem 1. *Assume that* $m \equiv 0 \pmod 4$. *Then*

(a) $M_-(m,l,q)$ *and* $M_-(m,l,q')$ *have the same homotopy type iff* $q = \pm q' \pmod{d_m}$, *where* d_m *is the denominator of* $B_{m/4}/m$, $B_{m/4}$ *being the* $(m/4)$*th Bernoulli number.*

(b) $M_-(m,l,q)$ *and* $M_-(m,l,q')$ *are homeomorphic* (*resp. diffeomorphic*) *iff* $q = \pm q'$.

Proof. Write $\xi = \xi_1 \oplus \theta^1$, where θ^1 is the trivial line bundle over S^m. Being a sphere bundle with cross-section, the homotopy type of M_- is, according to James-Whitehead [10], completely determined by the subset $\{J\chi, -J\chi\}$ of $\pi_{m+l-2}(S^{l-1})$, where χ, $\chi' \in \pi_{m-1}O(l-1)$ are the characteristic maps of ξ_1 and ξ_1' respectively, J is the Hopf-Whitehead J-homomorphism $J\colon \pi_{m-1}O(l-1) \to \pi_{m+l-2}(S^{l-1})$. Since $l \geqslant m+2$, $\pi_{m-1}O(l-1) = \pi_{m-1}(O)$, $\pi_{l+m-2}(S^{l-1}) = \pi_{m-1}^S$, the $(m-1)$th stem of the stable homotopy group of the sphere. Using the isomorphism $\widetilde{KO}(S^m) \cong \pi_{m-1}(O)$, we find $\chi = q \cdot g_m$ and $\lambda' = q' g_m$ for a suitable generator $g_m \in \pi_{m-1}(O) = Z$. It is well known following the solution of the Adams conjecture that J is isomorphic onto a cyclic subgroup of π_{m-1}^s of order d_m. This proves (a).

Let $\tau(M_-)$ be the tangent bundle of M_-. Then

$$\tau(M_-) \oplus \theta^1 = \pi^*\tau(S^m) \oplus \pi^*\xi, \tag{2.7}$$

where $\pi\colon M_- \to S^m$ is the bundle map. Hence $p_{m/4}(M_-) = \pi^* p_{m/4}(\xi)$. $\pi^*\colon H^m(S^m, Z) \to H^m(M_-, Z)$ is an isomorphism by Gysin's sequence, the $(m/4)$th-Pontrjagin class of $M_-(m,l,q)$ is $cq\gamma_m$, γ_m being a generator of $H^m(M_-, Z) = Z$. Part (b) follows at once from the topological invariance of the rational Pontrjagin class [12]. This completes the proof of Theorem 1.

3. Geometry and topology of M_+ and M

Since $M = M_+ \times S^m$, the topology of M_+ has a more direct bearing on M than that of M_-. It turns out that the topology of M_+ depends essentially on the homotopy of the so-called Clifford cross-sections of Stiefel manifolds. The reader is referred to [9] for definitions and proofs which are not given here.

Definition 1. Let $E_1, \cdots, E_{m-1}$ be an orthogonal representation of C_{m-1} on $\mathbb{R}^l$. The map $\sigma\colon S^{l-1} \to V_{l,m}$, $x \mapsto (x, E_1 x, \cdots, E_{m-1}x)$ is a cross-section of $V_{l,m}$ over S^{l-1}. σ is called the *Clifford cross-section* of the representation $E_1, \cdots, E_{m-1}$. σ can be identified as an element in $\pi_{l-1}(V_{l,m})$, referred to as Clifford elements in [3]. If $m \not\equiv 0$ (mod 4), all Clifford cross-sections are homotopic. The interesting case is $m \equiv 0 \pmod 4$. In this case, there are at most $k+1$ homotopy classes of Clifford cross-sections given by the $k+1$ inequivalent representations. (Recall that $l = k\delta(m)$.) The Clifford cross-section of the Clifford module Δ_m^+ (resp. Δ_m^-) is denoted by σ_m^+ (resp. σ_m^-).

An important notion for cross-sections is that of the *intrinsic join* $\sigma * \tau$ for $\sigma \in \pi_i(V_{r,m})$ and $\tau \in \pi_j(V_{s,m})$, defined by James. It is a bilinear map

$$\pi_i(V_{r,m}) \times \pi_j(V_{s,m}) \to \pi_{i+j-1}(V_{r+s,m}),$$

and is associative. It is also commutative when acted on joins of a finite number of σ_m^+'s and σ_m^-'s. The Clifford cross-section of a direct sum of C_{m-1} modules is the intrinsic join of the cross-sections of the summands. Hence the Clifford cross-section of the module $a\Delta_m^+ + b\Delta_m^-$ is precisely

$$\sigma_{a,b} = \underbrace{\sigma_m^+ * \cdots * \sigma_m^+}_{a} * \underbrace{\sigma_m^- * \cdots * \sigma_m^-}_{b}. \tag{3.1}$$

The σ_m^+'s and σ_m^-'s on the right-hand side of (3.1) can be rearranged in any order.

As was mentioned in §2, an orthogonal representation of C_{m-1} on $\mathbb{R}^l$ also induces, besides $\sigma\colon S^{l-1} \to V_{l,m}$, a map $\sigma'\colon S^{m-1} \to O(l)$ which is just the characteristic map of the bundle $\alpha(\mathbb{R}^l)$ over S^{l-1}. The mapping $\sigma \mapsto \sigma'$ corresponds to the isomorphism

$$A_m \cong \widetilde{KO}(S^m) = \pi_{m-1}O. \tag{3.2}$$

Let $\Delta\colon \pi_{l-1}(V_{l,m}) \to \pi_{l-2}(S^{l-m-1})$ be the boundary homomorphism in the homotopy exact sequence of the fibration $S^{l-m-1} \to V_{l,m+1} \to V_{l,m}$ and S be the suspension. James [9] showed that

$$S^{m+1} \circ \Delta\sigma = J\sigma', \tag{3.3}$$

where $J\colon \pi_{m-1}O(l) \to \pi_{l+m-1}(S^l)$ is the Hopf-Whitehead J-homomorphism. Since we are in the stable range, $\pi_{l+m-1}(S^l) = \pi_{m-1}^s$ and $\pi_{m-1}O(l) = \pi_{m-1}O$. Combining (3.2) and (3.3) gives

$$\Delta\sigma_{a,b} = qg_m, \tag{3.4}$$

where g_m is a generator of $J\pi_{m-1}O \subset \pi_{m-1}^s$ and $q = a - b$. $J\pi_{m-1}O$ is cyclic of order d_m. In view of (1.5) it is clear that the cardinality of the Δ-image in π_{m-1}^s of the set of Clifford cross-sections of $V_{l,m}$ is $\min\{k+1, \frac{1}{2}d_m\}$.

Lemma 1. *For all $m \equiv 0 \pmod 4$, $\sigma_{1,1}$ and $\sigma_{0,2}$ are not homotopic.*

Proof. If $\sigma_{1,1}$ and $\sigma_{0,1}$ were the same in $\pi_{2\delta(m)-1}V_{2\delta(m),m}$ it would follow by killing σ_m^+ one by one that every $\sigma_{a,b}$ is homotopic to either $\sigma_{k,0}$ or $\sigma_{0,k}$. On the other hand, the number of homotopy classes of Clifford cross-sections of $V_{l,m}$ is at least $\min\{k+1, \frac{1}{2}d_m\}$ as was shown above. $\min\{k+1, \frac{1}{2}d_m\} = \frac{1}{2}d_m$ for big k, since $d_m \geqslant 24$ (in fact, $24 \mid d_m$, cf. [1]). Hence there are at least 12 homotopy classes of Clifford cross-sections when k is big, a contradiction.

Lemma 2. *For all* $m \equiv 0 \pmod 4$ *and nonnegative integers* a, b, a', b', s, *and* t, $\sigma_{a,b} \simeq \sigma_{a',b'}$ *implies* $\sigma_{a-s,b-t} \simeq \sigma_{a'-s,b'-t}$, *provided that* $a + b - (s + t) \geqslant 2$.

Proof. The generalized suspension theorem of James says that when $\theta \in \pi_{m-1}V_{m,k}$ is the class of a cross-section, then

$$\pi_j(V_{n,k}) \to \pi_{j+m}(V_{m+n,k})$$

defined by $\theta_{\#}(\alpha) = \theta * \alpha$ is injective for $j < 2(n - k) - 1$. Lemma 2 follows by observing that when $a + b - (s + t) \geqslant 2$, the condition of the above theorem of James is satisfied. Note that $\delta(m) \geqslant m$ for $m \equiv 0 \pmod 4$.

The key to the topology of M_+ is the following elementary observation which, however, escaped the attention of [8].

Lemma 3. *For all* m, M_+ *is diffeomorphic to the unit sphere bundle* $S(\xi)$ *of the rank* $l - m$ *vector bundle* ξ *over* $S^{l-1} = SE_+(P_0)$, *the fiber of* ξ *over* $x \in S^{l-1}$ *being the orthogonal complement in* $E_+(P_0)$ *of the m-plane spanned by* $\{x, E_1x, \cdots, E_{m-1}x\}$.

Proof. Define a map π: $M_+ \to S^{l-1} = SE_+(P_0)$ by

$$\pi(x) = \frac{1}{\sqrt{2}}(x + P_0x). \tag{3.5}$$

Since M_+ is defined by the equations

$$M_+ = \{x \in S^{2l-1}: \langle P_0x, x\rangle = 0, \cdots, \langle P_mx, x\rangle = 0\}, \tag{3.6}$$

clearly $\pi(x) \in SE_+(P_0)$. Straightforward computation shows that

$$\pi^{-1}(y) = \left\{\frac{1}{\sqrt{2}}(y + z) \mid z \in SE_-(P_0),\right.$$
$$\left.\langle P_1z, y\rangle = \langle P_1z, E_1y\rangle = \cdots = \langle P_1z, E_{m-1}y\rangle = 0\right\},$$

hence the lemma follows.

In view of Lemma 2, we would like to know the number of homotopy classes of Clifford cross-sections of $V_{l,m}$. This number will have an upper bound h depending only on m if there exists a positive integer h, such that

$$\sigma_{h,0} \simeq \sigma_{0,h}. \tag{3.7}$$

Any such number h has to be a multiple of $\frac{1}{2}d_m$ as can be seen from (3.4).

To show the existence of h, observe that $\sigma_{0,h} = \lambda\sigma_{h,0}$, where λ is the involution on $V_{l,m}$ which changes the sign of the last vector in the m-frame and leaves the other vectors unchanged. It was shown by James [9, 13.2] that

$$1 - \lambda_* = \mathscr{U}_* S\Delta, \tag{3.8}$$

where $\mathscr{U}_*$: $\pi_{l-1}S^{l-m} \to \pi_{l-1}V_{l,m}$ comes from the homotopy exact sequence of the fibration $S^{l-m} \to V_{l,m} \to V_{l,m-1}$. It is clear from (3.8) that $h = d_m$ satisfies (3.7).

Definition 2. For $m \equiv 0 \pmod 4$, let

$$h_m = \inf\{h \mid \sigma_{h,0} \simeq \sigma_{0,h}\}.$$

Proposition 2. *Suppose $m \equiv 0 \pmod 4$. Then*

(i) $h_m = \frac{1}{2}d_m$ *or* d_m;

(ii) $h_4 = d_4$ $(= 24)$;

(iii) *for any nonnegative integers a, b, a', and b', $\sigma_{a,b} \simeq \sigma_{a',b'}$ iff $q \equiv q' \pmod{2h_m}$, where $q = a - b$, $q' = a' - b'$.*

Proof. (i) is obvious in view of the remarks preceding Definition 2.

(ii) Consider the homotopy exact sequence of the fibration

$$S^{l-4} \to V_{l,r} \to V_{l,3}\colon \pi_{l-2}(S^{l-4}) \to \pi_l(V_{l,4}) \to \pi_l(V_{l,3}) \\ \to \pi_{l-1}(S^{l-4}) \to \pi_{l-1}(V_{l,4}) \to \cdots.$$

We have $\pi_l(S^{l-4}) = \pi_4^s = 0$. Moreover [13], $\pi_l V_{l,4} = (Z_2)^3$, $\pi_l V_{l,3} = (Z_2)^3$, hence u_* is injective. The suspension S is an isomorphism. Hence $\sigma_{h,0} \simeq \sigma_{0,h}$ iff $\Delta\sigma_{h,0} = 0$ or $h \equiv 0 \pmod{d_4}$. This proves (ii).

(iii) Assuming $\sigma_{a,b} \simeq \sigma_{a',b'}$, without loss of generality, we may assume that $a = \min\{a, b, a', b'\}$. If $b \leqslant 1$, then either $a = a'$ and $b = b'$ or $\sigma_{0,1} \simeq \sigma_{1,0}$, i.e. $\sigma_m^+ \simeq \sigma_m^-$, hence $\sigma_{1,1} \simeq \sigma_{0,2}$, contradicting Lemma 1. Hence (iii) holds when $b \leqslant 1$. If $b \geqslant 2$, Lemma 2 gives

$$\sigma_{0,b} \simeq \sigma_{a'',b'}, \tag{3.9}$$

where $a'' = a' - a$. We claim that $b - b' \geqslant 2$, otherwise one would get $\sigma_{0,2} \simeq \sigma_{1,1}$ by Lemma 2. Applying Lemma 2 to (3.9) yields

$$\sigma_{0,b''} \simeq \sigma_{a'',0} \tag{3.10}$$

with $b'' = b - b'$, $a'' = a' - a$. Since $h_m \leqslant a'' = b''$, write $a'' = ph_m + r$ with $p > 0$ and $0 \leqslant r < h_m$. If $r > 0$, then $r > 2$ as h_m and a'' are both multiples of $\frac{1}{2}d_m$. Lemma 2 applied to (3.10) yield $\sigma_{r,0} \simeq \sigma_{0,r}$, contradicting the definition of h_m. Hence r has to vanish. This proves (iii).

Remark 1. We are, at present, unable to determine if $h_m = d_m$ holds for $m = 8, 12$, etc.

Let Δ': $\pi_{l-1}(V_{l,m}) \to \pi_{l-2}O(l-m)$ be the boundary operator in the homotopy exact sequence of the fibration $O(l-m) \to O(l) \to V_{l,m}$ and Δ'': $\pi_{l-1}S^{l-1} \to \pi_{l-2}S^{l-m-1}$ the boundary operator for that of $S^{l-m-1} \to M_+ \to S^{l-1}$.

Lemma 4. *Suppose $m \equiv 0 \pmod 4$. Let σ be a Clifford section of $V_{l,m}$ and ξ the vector bundle over S^{l-1} defined by σ as in Lemma 3. Then*

(i) *the characteristic map for the $O(l-m)$-bundle ξ is $\Delta'\sigma$,*

(ii) *$\Delta''\gamma_m = (P_* \circ \Delta')\sigma = \Delta\sigma$, where $P_*\colon \pi_{l-2}O(l-m) \to \pi_{l-2}(S^{l-m-1})$ is induced by the bundle map $P\colon O(l-m) \to S^{l-m-1}$ and γ_m is a generator of $\pi_{l-1}(S^{l-1})$.*

Proof. This follows easily by examining the definitions of Δ, Δ', Δ'', and P. Details are left to the reader.

Theorem 2. (a) *$M(m, l, q)$ and $M(m, l, q')$ are isotopic in S^{2l-1} if $q \equiv \pm q' \pmod{2d_m}$. The same is true for M_+.*

(b) *$M(m, l, q)$ and $M(m, l, q')$ are of distinct homotopy types if $q \not\equiv \pm q' \pmod{d_m}$. The same is true for M_+.*

Proof. (a) Consider the following general construction of sphere bundles over spheres as submanifolds in odd-dimensional spheres. Take 2 copies of $\mathbb{R}^l$ to form $\mathbb{R}^{2l} = \mathbb{R}^l \oplus \mathbb{R}^l$. For any smooth map $h\colon S^{l-1} \to G_p(\mathbb{R}^l)$, the set

$$V_h = \left\{ \frac{1}{\sqrt{2}}(x, v) \,|\, \|x\| = \|v\| = 1,\ v \in h(x) \right\}$$

is clearly a smooth submanifold in S^{2l-1} of dimension $l + p - 2$. It is obvious that if $h \simeq h'\colon S^{l-1} \to G_p(\mathbb{R}^l)$, then V_h and $V_{h'}$ are isotopic in S^{2l-1} (rotate the fibers S^{p-1}).

It follows from the proof of Lemma 3 that M_+ can be obtained this way by putting $p = l - m$ and $h =$ the Clifford cross-section composed with the projection $V_{l,m} \to G_m(\mathbb{R}^l)$ and the canonical isometry $G_m(\mathbb{R}^l) = G_{l-m}(\mathbb{R}^l)$. Now (a) follows since M_+ has trivial normal bundle.

To every m-sphere bundle M over S^n with $n \leqslant 2m - 1$, the subset $\{\alpha(M), -\alpha(M)\} \subset \pi_{n-1}(S^m)$ is (cf. [10, pp. 148–149]) an invariant of the homotopy type of M, where $\alpha(M)$ is the image of the generator of $\pi_n(S^n)$ under the boundary homomorphism of the homotopy exact sequence. For the special case of $M = M_+(q)$, in view of (3.3) and Lemma 4 one has $\alpha(M_+(q)) = qg_m$, where g_m is a generator of the image of the J-homomorphism in $\pi_{l-2}S^{l-m-1} = \pi^S_{m-1}$. Hence $M_+(q) \simeq M_+(q')$ iff $q \equiv \pm q' \pmod{d_m}$.

The corresponding assertion for $M(q)$ follows at once from that for $M_+(q)$ and the following observation:

Let X and Y be 1-connected CW-complexes such that $H_i(X) = H_i(Y)$ for all i, and assume that $H_i(X) \neq 0$ implies that $H_{i-m}(X)$ and $H_{i+m}(X)$ both vanish. Then $X \times S^m \simeq Y \times S^m$ iff $X \simeq Y$. (For a proof, look at the composition $X \xrightarrow{i_X} X \times S^m \xrightarrow{\sim} Y \times S^m \xrightarrow{\pi_Y} Y$ and use Whitehead's theorem. Note that M_+ has the same homology as $S^{l-1} \times S^{l-m-1}$ and that m and l are even.)

Corollary 3. *For any* $l = k\delta(m)$, *the number of diffeomorphic types of the* $M(m, l, q)$ *in* S^{2l-1} *is less than or equal to* $\frac{1}{2}d_m + 1$ *and bigger than* $\frac{1}{4}d_m - 1$.

4. A problem of S. S. Chern on minimal hypersurfaces in the sphere

Contrary to compact minimal submanifolds of higher codimensions, examples of minimal hypersurfaces in the standard sphere S^{n+1} are hard to produce. The first known examples are all homogeneous in nature hence have constant scalar curvatures or, equivalently, constant lengths of second fundamental forms. It was therefore natural to try to classify these minimal hypersurfaces. One of the first questions in this respect is that of uniqueness, namely,

(4.1) Problem. Let f, g: $M^n \to S^{n+1}$ be closed embedded minimal hypersurfaces with the same constant scalar curvature. Does there exist an isometry T in S^{n+1} such that $g = T \circ f$?

This problem was first asked by S. S. Chern in 1968 (cf. [6, p. 43]). Later, in his joint work [7] with Do Carmo and Kobayashi, Chern raised the same problem once again and conjectured that the answer seemed likely to be affirmative.

In view of (i) and (ii) in §1, it is natural to check the uniqueness conjecture on $M(m, l, q)$. The key question is whether there are distinct positive integers q and q' such that $M(m, l, q)$ and $M(m, l, q')$ are diffeomorphic. The lowest dimension in which this phenomenon occurs is not known. But it follows from Theorem 2 that this is always the case when the dimension of the sphere is big enough. In fact, 199 suffices:

Corollary 4. *There are two compact embedded minimal hypersurfaces in* S^{199} *which are diffeomorphic but noncongruent in* S^{199}, *both have constant scalar curvature* 38412, *namely*, $M(4, 100, 25)$ *and* $M(4, 100, 23)$.

Remark 2. For each $m = 8, 12, \cdots$ and $l > \frac{1}{2}d_m\delta(m)$, the $M(m, l, q)$ are divided into disjoint classes according to their diffeomorphic types. The cardinality of these classes grows indefinitely when l tends to infinity.

Remark 3. It can be shown by using Weyl's formula for the volumes of tubes that the $M(m, l, q)$ have the same volume (i.e., independent of q).

Remark 4. The $M(m, l, q)$ also provide us with diffeomorphic but nonisometric compact simply-connected Riemannian manifolds such that the curvature tensors at all points of any of the manifolds are all orthogonally equivalent.

Added in proof. The author is grateful to Professor N. H. Kuiper for informing me of an error in the original proof of Theorem 2(b).

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