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THE ORIENTATION OF YANG-MILLS MODULI SPACES AND 4-MANIFOLD TOPOLOGY

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1. Introduction

This paper has two separate purposes. The first is to modify the proofs of [3] and [6] (which considered simply connected manifolds) to obtain results on the intersection forms of 4-manifolds in the presence of fundamental groups. As an extension of the theorem of [3] we shall prove:

Theorem 1. *If X is a closed, oriented smooth 4-manifold whose intersection form*

$$Q : H^2(X;\mathbb{Z})/\text{Torsion} \to \mathbb{Z}$$

is negative definite, then the form is equivalent over the integers to the standard form $(-1) \oplus (-1) \oplus \cdots \oplus (-1)$.

In short, the result of [3] (Theorem A in [6]) extends without change to manifolds with arbitrary fundamental groups. For indefinite forms we shall prove:

Theorem 2. *Let X be a closed, oriented smooth 4-manifold with the following three properties*:

(i) $H_1(X;\mathbb{Z})$ *has no 2-torsion.*

(ii) *The intersection form Q on* $H^2(X)/\text{Torsion}$ *has a positive part of rank* 1 *or* 2.

(iii) *The intersection form is even.*

Then Q is equivalent over the integers to one of the forms

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In short, Theorems B and C of [6] extend to manifolds with no 2-torsion in their first homology group.

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This second result seems less satisfactory and it is possible that more is true. Recall that the intersection form on a 4-manifold X is even provided $w_2(X) \in H^2(X; \mathbb{Z}/2)$ maps to zero in the universal coefficient sequence:

$$(1.1)\quad \operatorname{Ext}(H_1(X;\mathbb{Z});\mathbb{Z}/2) \to H^2(X;\mathbb{Z}/2) \to \operatorname{Hom}(H_2(X;\mathbb{Z},\mathbb{Z}/2)).$$

The manifold admits a spin structure if and only if w_2 is zero. Since the group $\operatorname{Ext}(H_1(X;\mathbb{Z}),\mathbb{Z}/2)$ is zero if H_1 has no 2-torsion, hypotheses (i) and (iii) of Theorem 2 together imply that X is spin, but are presumably strictly stronger.

There is an example by Habegger [12] showing that Theorem 2 would be false without hypothesis (i). Habegger's manifold is a quotient of a $K3$ surface: it has fundamental group $\mathbb{Z}/2$ and the nonstandard intersection form $(-E_8) \oplus \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, with a positive part of rank 1. At the same time this example shows that the hypothesis of Rohlin's Theorem is sharp: the signature of Habegger's manifold is 8 while Rohlin's theorem asserts that of a spin 4-manifold is divisible by 16. In this example the manifold is not spin although the intersection form is even; w_2 corresponds to the nonzero element in $\operatorname{Ext}(H_1,\mathbb{Z}/2) \cong \mathbb{Z}/2$. Thus an interesting open problem, suggested by this example of Habegger, is to find whether hypotheses (i) and (iii) of Theorem 2 can be replaced by the condition that the manifold be spin.

The proofs of Theorems 1 and 2 follow the pattern explained in §III of [6]. We use the solutions of the anti-self-dual (ASD) Yang-Mills equations over the 4-manifolds to obtain compact manifolds-with-boundary parametrizing families of connections, and exploit the zero pairing between the boundary of these and suitable cohomology classes. The first new feature that arises is the greater complexity of the ends of the Yang-Mills moduli spaces themselves. In general the moduli spaces M_k of ASD connections on a bundle with $c_2 = k$ have compactifications $\overline{M}_k$ involving contributions from the lower spaces M_j $(j < k)$. If the 4-manifold has fundamental group π_1, then the space M_0, parametrizing representations $\pi_1 \to \mathrm{SU}(2)$, may itself be complicated. However from the point of view of the differential equations these flat solutions are degenerate. They can be perturbed away and the same perturbation is then used to modify the ends of the higher moduli spaces. Given this basic idea the detailed constructions of perturbations in §2 below are not very enlightening.

The second new feature, which is needed here only for the proof of Theorem 1, is an account of the orientation of Yang-Mills moduli spaces. The development of this is the other main purpose of the paper. We show the spaces are orientable, define canonical orientations, compare these at different points in the moduli spaces, and compute the action of the diffeomorphisms on the orientation. These results are needed for certain other applications of gauge theory to topology [5], [7], [8], and are really a part of index theory. We calculate by using excision arguments but in order to make contact with the explicit models of [6] these are done with differential rather than pseudo-differential operators. For manifolds without 2-torsion in H_1, Fintushel and Stern have given a simpler argument to show that many nonstandard intersection forms do not occur. In §4 we remove the assumption on H_1 from their argument using these results on orientations. Meanwhile, M. Furuta has given a proof of Theorem 1 for manifolds having $H_1 = 0$ [11]. His proof is similar to the one we give in §§2, 3 but introduces some interesting new constructions.

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2. Description and deformation of moduli

(a) Flat connections over negative definite manifolds. If a smooth, oriented 4-manifold X has a negative definite intersection form, then the index Theorem predicts the "virtual dimension" of $M_k(X)$—the moduli space parametrizing ASD connections on an SU(2) bundle with $c_2 = k$—to be

$$\dim M_k(X) = 8k - 3 + 3b_1(X). \tag{2.1}$$

To prove Theorem 1 it suffices to consider manifolds X with first Betti number $b_1(X)$ equal to 0. We can use the argument of Fintushel and Stern [9]: If surgeries are performed on loops γ_i representing an integral basis for the free part of $H_1(X; \mathbb{Z})$, we get a new manifold with the same form on $H_2/\text{Torsion}$, the same torsion in H_1 and with $b_1 = 0$. (Another approach is to fix the manifold X but "cut down" the moduli spaces M_k to the subsets M_k' representing connections whose monodromy around the loops γ_i is 1. This imposes $3b_1(X)$ constraints on the connections, $\dim M_k' = 8k - 3$, and all the arguments below may be carried out using the cut down moduli spaces.)

According to Freed and Uhlenbeck [10] the moduli space M_1 is, for generic Riemannian metrics on X, a smooth manifold of the dimension given by (2.1) except for singularities associated to Abelian reductions of the bundle. When $b_1 = 0$ (so $\dim M_1 = 5$) there is one such singularity for each reduction and so for each pair:

$$\pm e, e \in H^2(X; \mathbb{Z}), \qquad e^2 = -1. \tag{2.2}$$

Let A be the finite abelian group $H_1(X; \mathbb{Z})$ and $\hat{A} = \text{Hom}(A, S^1) \cong \text{Ext}(A, \mathbb{Z})$ the dual group. The reductions in M_1 corresponding to a given element in $H^2/\text{Torsion}$ form a principle $\hat{A}$ set since $\hat{A}$ is the torsion subgroup of H^2.

The moduli space $M_0(X)$ parametrizes flat SU(2) connections and hence the conjugacy classes of representations $\rho : \pi_1(X) \to \mathrm{SU}(2)$. We divide these representations into four kinds:

(i) The trivial representation $\pi_1 \to \{1\}$ corresponding to the product connection θ. This has isotropy group $\Gamma_\theta \cong \mathrm{SU}(2)$ in the gauge group of bundle automorphisms.

(ii) Nontrivial representations $\pi_1 \to \{\pm 1\}$ mapping to the center of SU(2). These are in (1-1) correspondence with the elements of order 2 in $\hat{A}$ and also give connections with isotropy SU(2).

(iii) Reducible representations, not of type (i) or (ii), which map to a copy of S^1 in SU(2). Up to conjugacy in SU(2) these correspond to pairs $\pm\alpha$ where $\alpha \in \hat{A}$, $2\alpha \neq 0$. The corresponding connections have isotropy S^1.

(iv) Irreducible representations associated to connections with isotropy ± 1.

If the only representation is the trivial type (i) the arguments in [3], [10] or [6, §III] go through unchanged. The moduli space M_1 has a natural compactification $\overline{M}_1 = M_1 \cup X$ and, since $H_1(X)$ is necessarily zero, the count of internal singularities is the same. In general there is a compactification $M_1 \cup (M_0 \times X)$ [6, §III] but rather than analyzing this we will deform the equations defining M_0 and hence the ends of M_1. The key point is that the virtual dimension of M_0 is negative.

(b) Deforming the equations. Let X be a Riemannian 4-manifold with $b_1 = 0$ and negative definite intersection form. If $\gamma : S^1 \to X$ is a loop based at a point x in X and A a connection on an SU(2) bundle P over X, let $h_\gamma(A) \in (\operatorname{Aut} P)_x$ be the holonomy of the connection around γ. We will use these to define gauge invariant perturbations of the ASD equations $F_+(A) = 0$.

Choose a map

$$\psi : \mathrm{SU}(2) \to \mathfrak{su}(2),$$

equivariant under the adjoint actions, which inverts the exponential map when restricted to the complement of a small ball around $-1 \in \mathrm{SU}(2)$. The equivariance of ψ gives corresponding maps

$$\psi_x : (\operatorname{Aut} P)_x \to (\mathfrak{g}_P)_x$$

to the bundle of Lie algebras $\mathfrak{g}_P$ associated to P. If $\nu \in \Omega^2_+(X)$ is a self-dual 2-form supported in a small neighborhood of x, define a section

$$\tau = \tau(\nu, \gamma, A) \in \Omega^2_+(\mathfrak{g}_P) \tag{2.3}$$

by first spreading $\psi_x(h_\gamma(A)) \in (\mathfrak{g}_P)_x$ to a section of $\mathfrak{g}_P$ defined over a neighborhood of x (using parallel transport along radial geodesics) then taking the tensor product with ν. For fixed ν, γ this gives a gauge invariant map from the connections on P to $\Omega^2_+(\mathfrak{g}_P)$. Let Σ be the set of maps defined by finite linear combinations of these:

$$\sigma(A) = \sum_{i=1}^{N} \varepsilon_i \tau(\nu_i, \gamma_i, A),$$

and for each $\sigma \in \Sigma$ let M_0^σ be the space of equivalence classes of solutions to the equations

$$F_+(A) + \sigma(A) = 0. \tag{2.4}$$

When $\sigma = 0$ this is the usual moduli space of ASD, hence flat, connections described in (a). The global analytical properties of the perturbed equations fit into the framework of the infinite dimensional Fredholm equations described in [6, §IV], to which we refer for notation: The maps $A \to \sigma(A)$ from, say, L_1^p connections (with $p > 2$) to L^p 2-forms are smooth and their derivatives are compact operators factoring through the inclusion of L_1^p in L^p. So the spaces M_0^σ have virtual dimension -3.

Begin with the case when $H_1(X; \mathbb{Z}) = 0$. Then $M_0(X)$ is the union of a compact set V parametrizing irreducible representations of type (iv) and a single point $[\theta]$ of type (i), which is isolated from V, since $H_1(X; \mathbb{R}) = 0$.

Lemma (2.5). *If A is any flat irreducible connection, then there are finite sets $\{\gamma_i\}_{i=1}^n$ of loops in X and 2-forms $\{\nu_i\}_{i=1}^n$ supported in small balls around the base points of the γ_i such that:*

(i) *The sections $\tau(\nu_i, \gamma_i, A)$ generate the vector space $H_A^2 = \Omega_+^2(\mathfrak{g}_P)/\operatorname{Im} d_A^+$.*

(ii) $(\gamma_i \cup \operatorname{supp}\nu_i) \cap (\gamma_j \cup \operatorname{supp}\nu_j)$ *is empty for $i \neq j$.*

(iii) *Any 2-dimensional homology class in X may be represented by a surface disjoint from the γ_i,* $\operatorname{supp}\nu_i$.

Proof. There is a finite set of points $x_1, \cdots, x_m$ in X such that the harmonic lift $H_A^2 \subset \Omega_+^2(\mathfrak{g}_P)$ of H_A^2 restricts monomorphically to

$$\bigoplus_{\alpha=1}^{m} \Lambda_+^2(\mathfrak{g}_P)_{x_\alpha}.$$

We take $N = 9m$ and for each α choose a small ball round x_α over which the sections in H_A^2 have small variation and 9 distinct points inside it.

Now for each point x_α the set of possible holonomies $h_\gamma(A)$ for loops γ based at x_α is dense in $(\operatorname{Aut} P)_{x_\alpha}$ since the connection is irreducible. So there are three loops $\gamma_{1,\alpha}, \gamma_{2,\alpha}, \gamma_{3,\alpha}$ such that the $\psi h_{\gamma_{i,\alpha}} = e_{i,\alpha}$ form a basis of $(\mathfrak{g}_P)_{x_\alpha}$. Choose a basis $\omega_{1,\alpha}, \omega_{2,\alpha}, \omega_{3,\alpha}$ for $(\Lambda_+^2)_{x_\alpha}$ and label the nine points near x_α by $x_{i,j,\alpha} = x(e_{i,\alpha}, \omega_{j,\alpha})$. Then we can choose loops $\gamma_{i,j,\alpha}$ based at $x_{i,j,\alpha}$ whose holonomy is close to that of $\gamma_{i,\alpha}$, and 2-forms $\nu_{i,j,\alpha}$ approximating "δ-functions" at the $x_{i,j,\alpha}$, close to multiplies of $\omega_{j,\alpha}$ in a local trivialization of Λ_+^2.

By general position we can arrange that these sets of loops and forms satisfy conditions (ii) and (iii) of the lemma. Property (i) follows from the fact that, when the approximations in the construction are made sufficiently fine, no nonzero element of H_A^2 can be orthogonal to all of the $\tau(\nu_{i,j,\alpha}, \gamma_{i,j,\alpha}, A)$.

Since the set V of flat irreducible connections is compact we can suppose the γ_i, ν_i $(i = 1, \cdots, N)$ chosen so that the three conditions of Lemma (2.5) hold for all the points $[A]$ in V simultaneously. We fix such a choice and consider the N-parameter family of deformed equations:

$$F_+(A) + \sum_{i=1}^{N} \varepsilon_i \tau(\nu_i, \gamma_i, A) = 0.$$

Proposition (2.6). *Suppose $H_1(X; \mathbb{Z}) = 0$ and choose ν_i, γ_i as above. Then for any $r > 0$ we can choose $(\varepsilon_i) \in \mathbb{R}^N$ with $|(\varepsilon_i)| < r$ such that for each index j and any t in $[0,1] \subset \mathbb{R}$ the only gauge equivalence class of solutions to the equation*

$$E_{j,t}: F_+(A) + \sum_{\substack{i=1 \\ i \neq j}}^{N} \varepsilon_i \tau(\nu_i, \gamma_i, A) + t\tau(\nu_j, \gamma_j, A) = 0$$

is that of the trivial flat connection θ.

Proof. (Compare [13].) Consider the universal equation

$$F_+(A) + \sum_{i=1}^{N} \varepsilon_i(\nu_i, \gamma_i, A) = 0$$

over the product $\mathscr{B} \times \mathbb{R}^N$ of the space of equivalence classes of connections $\mathscr{B} = \mathscr{A}/g$ with the parameter space $\mathbb{R}^N$. By property (i) of (2.5) this equation has maximal rank over $V \times \{0\}$ so when restricted to the product of an L_1^p (manifold) neighborhood U of V in $\mathscr{B}$ and a ball $|\varepsilon| < r_0$ the universal zero set Z is a manifold of dimension

$$\begin{aligned} \dim \ker(d_A^+ \oplus \tau_i) &: \ker d_A^* \oplus \mathbb{R}^N \to \Omega_+^2(\mathfrak{g}_P) \\ &= \operatorname{index}(d_A^* \oplus d_A^+) + N = N - 3. \end{aligned}$$

Now for each index j in $\{1, \cdots, N\}$ consider the obvious projection

$$Z \to \mathbb{R}^N \to \mathbb{R}_j^{N-1},$$

forgetting the jth coordinate. By Sard's Theorem the image in $\mathbb{R}_j^{N-1}$ has empty interior. It follows that for a second category subset of vectors ε in $B(r) \subset \mathbb{R}^N$ there are no solutions of $E_{j,t}$ (for any t) in U. Considering the N conditions simultaneously we arrange the same thing for all j.

But when ε is small enough the only solution of the equation $E_{j,t}$ (for t in $[0,1]$) outside U is $[\theta]$. For, since $H^2_+(X) = 0$, this flat connection is a regular solution of the original equation. So under small deformations it persists as an isolated solution of the new equation. If there were other solutions A_ε, then, letting ε tend to 0, we obtain a sequence of equivalence classes of connections with $\|F\|_{L^2}$, $\|F_+\|_{L^p} \to 0$ but with no subsequence converging to L^p_1 to a flat connection. This would contradict Uhlenbeck's compactness Theorem.

We consider next the solutions of the perturbed equations when $H_1(X;\mathbb{Z})$ is nonzero and there are more reducible connections in M_0. We need only consider separately these of type (iii)—the reductions of type (ii) define the trivial flat connection on the adjoint bundle $\mathfrak{g}_P$ and their local deformation behavior is the same as $[\theta]$.

If A is a reduction of type (iii), corresponding to a splitting $\mathfrak{g}_P = \mathbb{R} \oplus L^{\otimes 2}$, where the flat complex line bundle $L^{\otimes 2}$ is nontrivial, then a neighborhood of $[A]$ in M_0 is modelled on the zeros of an equivariant map

$$\mathbb{C}^p \cong H^1(X; L^{\otimes 2}) \underset{\phi}{\to} \mathbb{C}^{p+1} \cong H^2_+(X; L^{\otimes 2}) \tag{2.7}$$

divided by the action of $\Gamma_A \cong S^1$. Here we have used the index theorem to relate the dimensions of H^1, H^2_+ and these spaces can be identified with those obtained from the cohomology of X in the twisted coefficient system $L^{\otimes 2}$. In just the same way the solutions of the universal equation

$$F_+(A) + \Sigma \varepsilon_i \tau_i = 0$$

are modelled by a quotient of the zeros of a map:

$$\chi : \mathbb{C}^p \times \mathbb{R}^N \to \mathbb{C}^{p+1}, \qquad \chi|_{\mathbb{C}^p \times \{0\}} = \phi. \tag{2.8}$$

Let D be the component of the second derivative of χ which maps $\mathbb{R}^N$ to $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^p, \mathbb{C}^{p+1})$. The first derivative of φ at 0 vanishes, so if $D(\varepsilon)$ is an injection for some small ε in $\mathbb{R}^N$ then the corresponding deformed equation has an isolated solution associated to the bundle reduction.

Lemma (2.9). *There are finite sets* $\{\gamma_i\}$, $\{\nu_i\}$ *satisfying conditions* (ii) *and* (iii) *of Lemma* (2.5) *such that for each reduction of type* (iii) *and an open dense subset of the vectors* $\varepsilon = (\varepsilon_i)_{\varepsilon_j = 0}$ *in* $\mathbb{R}^N$, *the map* $D(\varepsilon)$ *above is injective.*

Proof. Write $D(\varepsilon) = \Sigma_{i=1}^N \varepsilon_i D_i$. It suffices to show that $D(\varepsilon)$ is injective for some ε in $\mathbb{R}^N$. We begin with a simple algebraic fact: suppose on the contrary that $D(\varepsilon)$ fails to be injective for all ε; then either $\bigcap_i \mathrm{Ker}\, D_i$ is a proper subspace of $\mathbb{C}^p$ or $\Sigma_i \mathrm{Im}\, D_i$ is a proper subspace of $\mathbb{C}^{p+1}$. The proof is left to the reader. We need, then, to choose the γ_i, ν_i so that neither of these alternatives occur.

Now D_i is obtained from the derivative $(\delta\tau/\delta A)(\nu_i, \gamma_i, A)$ by projecting to the quotient H^2_A and restricting to the "transversal" $A + H^1_A$. Suppose that γ is a loop in X such that $h_\gamma(A) = \pm 1$. Then:

$$\frac{\delta}{\delta A}\tau(\nu, \gamma, A) = \left(\frac{\delta}{\delta A}(h_\gamma(A))\right)_{\text{P.T.}} \otimes \nu,$$

where P.T. denotes the local parallel transport near the base point x of γ, using the flat connection A. In turn the derivative

$$\frac{\delta h_\gamma}{\delta A}(A) : \left(H^1(X; L^{\otimes 2}) \subset \Omega^1(\mathfrak{g}_P)\right) \to [L^{\otimes 2}]_x$$

is given by

$$\left(\frac{\delta h_\gamma}{\delta A}\right)(a) = \int_{S^1} P_\gamma(\dot{\gamma} a) \in L^{\otimes 2} \tag{2.10}$$

for $a \in \Omega^1(L^{\otimes 2})$. Here P_γ denotes the A-horizontal pull-back of sections of $\gamma^*(\mathfrak{g}_P)$ to the fiber over the base point. So for any element $\zeta \in (L^{-2})_x$ we have a number $(\zeta, (\delta h_\gamma/\delta A)a)$ obtained by integrating around γ.

Let $\Pi : \tilde{X} \to X$ be the finite covering given by a fixed leaf of the horizontal foliation of (L^{-2}). The twisted cohomology $H^1(X; L^{\otimes 2})$ is isomorphic to a subspace of the ordinary cohomology $H^1(\tilde{X}; \mathbb{C})$—an eigenspace of the generator of the covering group. Dually the loop γ lifts to a loop $\tilde{\gamma}$ in $\tilde{X}$ and if ζ is the lift of x the number $(\zeta, (\delta h_\gamma/\delta A)a)$ is the usual pairing between $H^1(\tilde{X})$ and $H_1(\tilde{X})$.

But any class in $H_1(\tilde{X}; \mathbb{C})$ is represented by an $\mathbb{R}$-linear combination of horizontal lifts of loops γ. It follows from the fact that the pairing between $H^1(\tilde{X})$ and $H_1(\tilde{X})$ is perfect that there are finitely many loops γ_i for which

$$\bigoplus_i \frac{\delta h_{\gamma_i}}{\delta A} : H^1(X; L^{\otimes 2}) \to \bigoplus_i L^{\otimes 2}_{x_i}$$

is a monomorphism. Then, for suitable ν_i, no nonzero vector in H^1 is annihilated by all the D_i and, as in (2.5) we can arrange that no harmonic form is perpendicular to all of $\mathrm{Im}(\delta\tau_i/\delta_A)$.

The general position arguments for (2.6), (2.9) combine to give:

Corollary (2.11). *If X has a negative definite form and $H_1(X; \mathbb{R}) = 0$, there are finite sets $\{\gamma_i\}$, $\{\nu_i\}$ satisfying conditions* (ii) *and* (iii) *of* (2.5) *and an $\varepsilon = (\varepsilon_i)$ in $\mathbb{R}^N$ such that the equation*

$$F_+(A) + \sum_{i=1}^{N} \varepsilon_i \tau(\nu_i, \gamma_i, A) = F_+ + \sigma = 0$$

in $\mathscr{B}$ has only isolated solutions corresponding to the abelian reductions. The cokernels $H_A^{2,\sigma}$ of the differential of $F_+ + \sigma$ have complex dimension 1 *at the solutions of type* (iii) *and* 0 *at other solutions. Moreover we can suppose that the only solutions of the equations $E_{j,t}$ of Proposition* (2.6) *(j in $\{1,\cdots,N\}$, t in* $[0,1]$) *are either flat of types* (i), (ii) *or in small L_1^p neighborhoods of the flat reductions of type* (iii).

(c) Indefinite forms. As $b = b_2^+$ grows, the virtual dimension of M_0, $-3-3b$, decreases (assuming always that $H^1(X;\mathbb{R}) = 0$). Irreducible flat connections can be perturbed away just as in (2.6). Moreover we are able to avoid solutions in larger families of equations. For a vector ε in $\mathbb{R}^N$, any b indices $j_1,\cdots,j_b$ in $\{1,\cdots,N\}$, and numbers $t_1,\cdots,t_b$ in $[0,1]$ we consider the equation

$$E_{j_1,\cdots,j_b,t_1,\cdots,t_b}: F_+ + \sum_{i\notin\{j_\alpha\}} \varepsilon_i\tau_i + \sum_{\alpha=1}^{b} t_\alpha \varepsilon_{j_\alpha}\tau_{j_\alpha} = 0,$$

obtained by contracting any b coordinates. In contrast to the negative definite case, the abelian reductions of type (iii) also disappear after small deformations.

Proposition (2.12). *If $b_2^+(X) > 0$ and $H^1(X;\mathbb{R}) = 0$, then there are loops and forms, as in* (2.11), *and a perturbation ε in $\mathbb{R}^N$ such that the only solutions of the perturbed equation*

$$F_+ + \sum \varepsilon_i \tau(\nu_i,\gamma_i,A) \equiv F_+ + \sigma = 0$$

correspond to the flat reductions of type (i), (ii). *Moreover we can suppose that the only solutions of the b dimensional family of equations $E_{j_1,\cdots,j_b,t_1,\cdots,t_b}$ are either flat of type* (i), (ii) *or in small L_1^p neighborhoods of flat reductions of type* (iii).

Proof. At an abelian flat connection A of type (ii) there is now a component of $H_A^2 = H_+^2(X) \oplus H_+^2(X; L^2)$ fixed by the isotropy group Γ_A. The local universal model has the shape

$$\chi: \mathbb{C}^p \times \mathbb{R}^N \to \mathbb{C}^{p+1+b_2^+} \times \mathbb{R}^{b_2^+}.$$

Let E be the component of the derivative of χ mapping $\mathbb{R}^N$ to $\mathbb{R}^{b_2^+} \cong H_+^2(X)$. Solutions near $[A]$ are removed by the small deformation ε in $\mathbb{R}^N$ if $E(\varepsilon)$ is nonzero. But

$$E(\varepsilon) = \Pi\left(\sum_{i=1}^{N} \varepsilon_i h_{\gamma_i}(A)\cdot \nu_i\right),$$

where $\Pi: \Omega_+^2(X) \to H_+^2(X)$ is projection. (Note that the $h_{\gamma_i}(A)$ lie in the trivial component of $\mathfrak{g}_P = \mathbb{R} \oplus L^{\otimes 2}$.) This can be made nonzero by choosing the loops so that $h_{\gamma_i}(A) \neq 0$, using the fact that $L^{\otimes 2}$ is not trivial.

(d) Deforming the ends. Let α_1, α_2 be two classes in $H_2(X, \mathbb{Z})$ where X satisfies the hypotheses of Theorem 1 and has zero first Betti number. We will compute the intersection pairing α_1, α_2 using the moduli space $M_1^{\sigma'}$ of solutions to a suitable perturbation $F_+(A) + \sigma'(A) = 0$ of the ASD equations for connections with $c_2 = 1$.

Fix a perturbation $\sigma = \sum \varepsilon_i \tau(\nu_i, \gamma_i, -)$ on the connections with $c_2 = 0$, as in Corollary (2.11), with ε_i small. The loops γ_i and supports of the forms ν_i are disjoint from surfaces Σ_1, Σ_2 representing α_1, α_2. Let δ be small compared with the separation between $(\gamma_i \cup \operatorname{supp} \nu_i)$, $(\gamma_j \cup \operatorname{supp} \nu_j)$ $(i \neq j)$ and between the $(\gamma_i \cup \operatorname{supp} \nu_i)$ and Σ_k. For any connection A we define a "scale" or "inverse concentration" $\lambda(A) > 0$ as in [3]. Choose the perturbation σ' so that:

(i) $\sigma'(A) = 0$ if the scale $\lambda(A) > \delta$.

(ii) If $\lambda(A) < \delta/2$, then

$$\sigma'(A) = \sum_{i=1}^{N} \rho_i(A) \varepsilon_i \tau_i(\nu_i, \gamma_i, A),$$

where ρ_i is a smooth function, $\rho_i(A) \in [0,1]$, $\rho_i = 0$ if the scale of A restricted to the δ-neighborhood of $\gamma_i \cup \operatorname{supp} \nu_i$ is less than $\delta/4$, and $\rho_i = 1$ if this scale is bigger than $\delta/2$. The definitions in the different regions are smoothly patched together using bump functions.

Now choose representatives $V_{\Sigma_1}, V_{\Sigma_2}$ for the cohomology classes over spaces of connections associated to Σ_1, Σ_2, as in [6, §III]. By general position these can be chosen so that V_{Σ_1} and V_{Σ_2} do not meet any of the discrete set of flat reducible connections over X. The V's are closed so, by Corollary (2.11), when ε is small then do not meet any of the solutions of the equations $E_{j,t}$ on the bundle with $c_2 = 0$.

We analyze the ends of the space

$$N = M_1^{\sigma'} \cap V_{\Sigma_1} \cap V_{\Sigma_2}.$$

If $[A_i]$ is an infinite sequence of gauge equivalence classes in N containing no convergent subsequences, then the same arguments as for the ASD equations themselves show that $\lambda(A_i)$ tends to 0. Moreover if we "blow up" neighborhoods of points where the concentration is large the rescaled connections converge to the standard instanton with total action $8\pi^2$. The total action of any solution to the equation $F_+(A) + \sigma'(A) = 0$ on the bundle with $c_2 = 1$ is $8\pi^2 + \|\sigma'(A)\|^2$.

We can suppose the ε_i were chosen originally to be so small that $\|\sigma'(A)\|^2$ is less than $8\pi^2$ for any connection A, hence there is at most one center of concentration of A_i when $i \gg 0$ and we can assume these converge to a point

p in x. Then p must lie on one of the intersection points $\Sigma_1 \cap \Sigma_2$. For, by the defining property (ii) of σ', when i is large the connections A_i satisfy the ASD equations near their centers of concentration. So Uhlenbeck's Removal of Singularities theorem applies as in the usual case to show that the A_i converge on the complement of p to a limit A_∞, a connection on the trivial bundle. This connection must satisfy one of the equations $E_{j,t}$ since the functions ρ_i take values in $[0, 1]$. Hence it does not lie in either V_Σ. This implies that the point p must lie in $\Sigma_1 \cap \Sigma_2$ (cf. [6, §III]). Hence we also see that A_∞ satisfies the equation

$$(F_+ + \sigma)(A_\infty) = 0.$$

Thus the ends of N are made up of connections with one center of concentration near a point of $\Sigma_1 \cap \Sigma_2$ and close, away from this point, to one of the reducible connections making up M_0^σ. Conversely, the connection of this kind satisfying the equation $F_+(A) + \sigma(A) = 0$ can be analyzed in the same way as the ASD connections themselves using the alternating method of [6, §§IV, V]. This is quite clear since the perturbing term σ is supported away from the center of concentration. (Note however that we do not have a canonical harmonic lift of the cohomology spaces $H_A^{2,\sigma}$ given by the zeros of a formal adjoint.)

We can read off the contributions to the ends of N using the (perturbed analogue of) Theorem (5.5) of [6]. By Proposition (3.19) of that reference we can suppose the $V_{\Sigma_1} \cap V_{\Sigma_2}$ represented locally by connections where the center lies exactly on an intersection point of $\Sigma_1 \cap \Sigma_2$. Initially we ignore signs.

Proposition (2.13). *Let X satisfy the hypotheses of Theorem 1 and have $H_1(X; \mathbb{R}) = 0$. If $N = M_1^{\sigma'} \cap V_{\Sigma_1} \cap V_{\Sigma_2}$ is chosen as above, the end of N associated to a point p of $\Sigma_1 \cap \Sigma_2$ and a reducible connection A in M_0^σ has the form of an open interval if A is of type* (i) *or* (ii). *If A is of type* (iii) *the end is modelled on the quotient by S^1 of the zeros of an equivariant map*

$$\varphi : \mathrm{SO}(3) \times \mathbb{R}^+ \to \mathbb{C}.$$

Here $S^1 \subset \mathrm{SO}(3)$ acts on SO(3) *by multiplication and acts on $\mathbb{C}$ with weight* 1.

Perturb the situation slightly, if necessary, to get transversality and truncate N to a compact 1-manifold-with-boundary $\hat{N}$, as in [6, §III]. Using Lemma (2.27) of [6] we can arrange that the contribution to the boundary of $\hat{N}$ from each reduction of the $c_2 = 1$ bundle—labelled by $\pm e$ where $e^2 = -1$—consists of

$$(\alpha_1 \cdot e)(\alpha_2 \cdot e) \tag{2.14}$$

points. For each intersection point of Σ_1, Σ_2 in x we can use Proposition (2.13) to arrange that the contribution to $\partial\hat{N}$ from a reduction consists of

$$(2.15) \qquad \begin{cases} 1 \text{ point if the reduction is of type (i) or (ii)}, \\ 2 \text{ points if the reduction is of type (iii)} \end{cases}$$

(since the degree of the bundle $SO(3) \times_{S^1} \mathbf{C}$ over S^2 is 2).

In §§3 and 4 below we will define an orientation of the moduli spaces, and hence $\hat{N}$, and calculate the orientation at the different points of $\partial\hat{N}$ to show that the oriented boundary is

$$(2.16) \qquad \partial\hat{N} = \frac{1}{2} \sum_{e^2=-1} (\alpha_1 \cdot e)(\alpha_2 \cdot e) + \sum_{\substack{\text{Reductions} \\ \text{of type (i), (ii)}}} \alpha_1 \cdot \alpha_2 + 2 \sum_{\substack{\text{Reductions of} \\ \text{type (iii)}}} \alpha_1 \cdot \alpha_2 .$$

So:

$$-\frac{1}{2}|A| \cdot \left(\sum_{\substack{e \in H^2/\text{Torsion} \\ e^2=-1}} (\alpha_1 \cdot e)(\alpha_2 \cdot e) \right) = |A| \cdot (\alpha_1 \cdot \alpha_2),$$

and the intersection form is standard, as asserted by Theorem 1.

The proof of Theorem 2 is easier. If X satisfies the hypotheses there, and $H_1(X; \mathbf{R}) = 0$, then there are no reductions of type (ii). Hence we can find deformed equations as in Proposition (2.12) whose only solution is the flat product connection θ. Make a further small deformation so that $\sigma(A) = 0$ if all the $h_{\gamma_i}(A)$ are very close to 1. Then modify the ends of the higher moduli spaces, as above, and use the argument of [6]. The description of the links of the perturbed moduli spaces is unchanged since the equations are the same for connections close to θ over $\bigcup(\gamma_i \cup \operatorname{supp} \nu_i)$, no orientations are involved since the proof uses mod 2 cohomology.

3. Orientations and the determinant line bundle

(a) Determinants. The "determinant line" of a real elliptic operator

$$D : \Gamma(\xi_1) \to \Gamma(\xi_2)$$

defined over a compact manifold is the 1-dimensional vector space

$$(3.1) \qquad \Lambda(D) = \det(\operatorname{Ker} D) \otimes \det(\operatorname{Coker} D)^*.$$

(Here det( ) denotes the highest exterior power of a finite dimensional vector space.) If $s_1, \cdots, s_N$ are sections of ξ_2 generating $\operatorname{coker} D$ and $S: \mathbb{R}^N \to \Gamma(\xi_2)$ is the corresponding map with $S(e_i) = -s_i$, then the exact sequence

$$(3.2) \qquad 0 \to \operatorname{Ker} D \to \operatorname{Ker}(D \oplus S) \to \mathbb{R}^N \to \operatorname{coker} D \to 0$$

defines a natural isomorphism:

$$(3.3) \qquad \Lambda(D) \cong \det(\operatorname{Ker}(D \oplus S)) \otimes (\det \mathbb{R}^N)^*.$$

It follows that if the operator D varies in a continuous family, then the determinant lines of the family form a bundle over their parameter space [2]. If the bundles ξ_1 and ξ_2 have complex structures, commuting with D, then the determinant line has a standard orientation induced by the complex structures on $\operatorname{Ker} D$ and $\operatorname{coker} D$. (Recall that the usual orientation of a complex vector space with basis $e_1, \cdots, e_n$ is $e_1 \wedge Je_1 \wedge \cdots \wedge e_n \wedge Je_n$.)

Let X be any compact oriented Riemannian 4-manifold, $E \to X$ a rank 2 unitary bundle, and $\mathfrak{g}_E$ the associated SO(3) bundle. If A is a connection on E let $\mathscr{D}_A$ be the operator:

$$(3.4) \qquad \mathscr{D}_A = -d_A^* \oplus d_A^+ : \Omega^1(\mathfrak{g}_E) \to (\Omega^0 \oplus \Omega^2_+)(\mathfrak{g}_E).$$

An orientation of $\Lambda(\mathscr{D}_A)$ will define an orientation of an appropriate Yang-Mills moduli space—spelled out in §4 below. We shall make a small abuse of language by talking of canonical isomorphisms between determinant lines $\Lambda(D)$ where more precisely we mean isomorphisms of their orientations $\Lambda(D)/\mathbb{R}^+$.

The action of the gauge group on the connections lifts to the determinant lines. For any connection A the stabilizer Γ_A has a connected image in $\operatorname{Aut}(\mathfrak{g}_E)$, so the determinant lines $\Lambda_A \equiv (\mathscr{D}_A)$ descend to form a bundle Λ_E over the space $\mathscr{B}_E$ of gauge equivalence classes. Topologically such unitary bundles E correspond exactly to pairs $(c_1(E), c_2(E))$ so there are infinite families of determinant line bundles $\Lambda_E = \Lambda(c_1, c_2)$, indexed by $H^2(X; \mathbb{Z}) \times \mathbb{Z}$.

(b) Excision. Suppose that a compact Riemannian manifold Z is written as a union of open sets $Z = U \cup V$ and that $D: \Gamma(\xi) \to \Gamma(\eta)$ is a first order real differential operator over Z. Suppose that over U there is a bundle isomorphism $\Theta: \Gamma(\xi) \to \Gamma(\eta)$ relative to which D is skew adjoint: $\langle Df, \Theta f \rangle = 0$ for $f \in C_c^\infty(\xi|_U)$. Choose cut-off functions β, γ with

$$0 \leqslant \beta, \gamma \leqslant 1, \quad \beta = 1 \text{ on } \operatorname{supp}(\nabla \gamma), \quad \operatorname{supp}(\beta) \subset U,$$
$$\operatorname{supp}(1 - \beta) \subset V, \quad \operatorname{supp}(\gamma) \subset V, \quad \operatorname{supp}(1 - \gamma) \subset U.$$

Let $\hat{Z} = \hat{U} \cup \hat{V}$ be another manifold with a corresponding set-up: $\hat{D}$, $\hat{\Theta}$, $\hat{\xi}$, $\hat{\eta}$, $\hat{\beta}$, $\hat{\gamma}$. Suppose there is a diffeomorphism σ from V to $\hat{V}$, lifting to bundle isomorphisms, and relative to this diffeomorphism the two sets of data correspond over the V's. We will construct an excision isomorphism between the determinant lines Λ_D, $\Lambda_{\hat{D}}$. This could be done easily using pseudo-differential operators, as in [1], but those would be harder to combine with the isomorphisms needed in §3(d), (e) below.

For $u > 0$ define

$$(3.5) \qquad D_u = D + u\beta\Theta : \Gamma(\xi) \to \Gamma(\eta)$$

over Z and similarly $\hat{D}$ over $\hat{Z}$. Suppose $\lambda > 0$ and f is a section of ξ with $\|D_u f\|^2 \leqslant \lambda \|f\|^2$ (all norms are L^2); then

$$\begin{aligned}\lambda\|f\|^2 &\geqslant \langle D_u f, \Theta(\beta f)\rangle \geqslant \langle Df, \Theta(\beta f)\rangle + u\|\beta f\|^2 \\ &\geqslant \tfrac{1}{2}\{\langle Df, \Theta(\beta f)\rangle + \langle D(\beta f), \Theta(f)\rangle\} - c\|f\|^2 + u\|\beta f\|^2\end{aligned}$$

for a fixed constant c, independent of u. The first term on the right vanishes so

$$(3.6) \qquad \|\beta f\|^2 \leqslant ((\lambda + c)/u)\|f\|^2.$$

We can suppose that the same constant c gives similar inequalities for D_u^*, $\hat{D}_u$, and $\hat{D}_u^*$.

Define four maps, all of which we denote by σ_γ, from $\Gamma(\xi)$ to $\Gamma(\hat{\xi})$, from $\Gamma(\eta)$ to $\Gamma(\hat{\eta})$, from $\Gamma(\hat{\xi})$ to $\Gamma(\xi)$, and from $\Gamma(\hat{\eta})$ to $\Gamma(\eta)$, by cutting-off using the functions γ, $\hat{\gamma}$ and then applying the identification σ over the V's. So for f in $\Gamma(\xi) : \|\hat{D}_u \sigma_\gamma f - \sigma_\gamma D_u f\|^2 \leqslant c'\|\beta f\|^2$, and we can suppose that the same constant c' (independent of u) gives similar inequalities for the other "commutators."

For each fixed u choose a map $S : \mathbb{R}^N \to \Gamma(\eta)$ so that

$$(3.7) \qquad \|(D_u \oplus S)^* \phi\| > \|\phi\| \quad \text{for all nonzero } \phi \text{ in } \Gamma(\eta_1),$$

$$(3.8) \qquad \|D_u^* S v\|^2 < 2\|v\|^2 \quad \text{for all nonzero } v \text{ in } \mathbb{R}^N.$$

This can be done by mapping the basis elements of $\mathbb{R}^N$ to suitable eigenvectors of $D_u D_u^*$. Let $\hat{S} = \sigma_\gamma S : \mathbb{R}^N \to \Gamma(\hat{\eta})$ and π, $\hat{\pi}$ be L^2-projections onto the kernels of $D_u \oplus S$, $\hat{D}_0 \oplus \hat{S}$.

Lemma (3.9). *There is a constant $u_0 = u(c, c')$ such that when $u > u_0$, $\hat{D}_u \oplus \hat{S}$ is surjective, and*

$$\hat{\pi} \circ \sigma_\gamma : \operatorname{Ker}(D_u \oplus S) \to \operatorname{Ker} \hat{D}_u \oplus \hat{S}$$

is an isomorphism.

Proof. This is quite routine, using repeatedly the fact expressed by (3.6) that the mass of the relevant sections is concentrated over the V's.

First we show that if u is fixed large enough then

$$\left\|(\hat{D}_u \oplus \hat{S})^* \hat{\phi}\right\| \geqslant \tfrac{1}{2}\|\hat{\phi}\|,$$

say, for all $\hat{\phi}$ in $\Gamma(\hat{\eta})$. For if

$$\left\|(\hat{D}_u \oplus \hat{S})^* \hat{\phi}\right\| < \|\hat{\phi}\|,$$

then $\|\hat{\beta}\hat{\phi}\| < \sqrt{(c+1)/u}\,\|\hat{\phi}\|$ (by (3.6) for $\hat{D}^*$) so:

$$\left\|\sigma_\gamma(\hat{\phi})\right\| \geqslant \|\hat{\phi}\| - \|\beta\hat{\phi}\| \geqslant \left(1 - \sqrt{(c+1)/u}\right)\|\hat{\phi}\|.$$

Whereas

$$\begin{aligned}\left\|D_u^*(\sigma_\gamma\hat{\phi})\right\| &\leqslant \left\|\left(D_u^*\sigma_\gamma - \sigma_\gamma\hat{D}_u^*\right)\hat{\phi}\right\| + \left\|\sigma_\gamma\hat{D}_u^*\hat{\phi}\right\| \\ &\leqslant \sqrt{c}' \cdot \|\hat{\beta}\hat{\phi}\| + \|\hat{\phi}\|,\end{aligned}$$

using (3.6) for D_u^* and the hypothesis on $\hat{\phi}$. Also $\|S^*\sigma_\gamma(\hat{\phi})\| = \|\hat{S}^*\hat{\phi}\|$ so

$$\begin{aligned}\left(1 - \sqrt{\frac{c+1}{u}}\right)\|\hat{\phi}\| &\leqslant \left\|\sigma_\gamma\hat{\phi}\right\| \leqslant \left\|(D_u \oplus S)^*\sigma_\gamma\hat{\phi}\right\| \\ &\leqslant \left\|(\hat{D}_u \oplus \hat{S})^*\hat{\phi}\right\| + \sqrt{\frac{(c+1)(c'+1)}{u}}\,\|\hat{\phi}\|\end{aligned}$$

and the assertion follows when $u > 4(c+1)(1 + \sqrt{c'+1}\,)^2$.

For such u, $\hat{D}_u \oplus \hat{S}$ is surjective and to prove that $\hat{\Pi}(\sigma_\gamma \oplus 1)$ is injective it suffices to show that for any nonzero (f, v) in $\mathrm{Ker}(D_u \oplus S)$ we have

$$\left\|(\hat{D}_u \oplus \hat{S})(\sigma_\gamma f, v)\right\| < \tfrac{1}{2}\left\|(\sigma_\gamma f, v)\right\|.$$

But

$$\left\|D_u^* D_u f\right\| = \left\|D_u^* S v\right\| \leqslant \sqrt{2}\,\|v\|,$$

by (3.8), so $\|D_u f\|^2 \leqslant 2\|v\|\,\|f\|$ and

$$\begin{aligned}\|\beta f\|^2 &\leqslant \frac{2\|v\|\|f\| + c\|f\|^2}{u} \\ &\leqslant \left[\mathrm{Max}\left(\frac{1}{10}, \frac{1}{10c'}\right)\right]^2 \|(f, v)\|^2,\end{aligned}$$

say, if u is large enough. Then

$$\left\|(\hat{D}_u \oplus \hat{S})(\sigma_\gamma f, v)\right\| \leqslant \sqrt{c'}\,\|\beta f\| \leqslant \tfrac{1}{10}\|(f, v)\|$$

and $\|\sigma_\gamma f\| \geqslant \|f\| - \|\beta f\|$, so

$$\left\|(\sigma_\gamma f, v)\right\| \geqslant \tfrac{9}{10}\|(f, v)\|$$

and the assertion follows. The same argument shows that $\Pi(\sigma_\gamma \oplus 1)$ gives a monomorphism $\mathrm{Ker}(\hat{D}_u \oplus \hat{S}) \to \mathrm{Ker}(D_u \oplus S)$, completing the proof.

Composing the isomorphism of (3.9) with those of (3.3) gives an isomorphism

$$e_{\Theta,\hat{\Theta},\sigma,S,u}:\Lambda(D_u)\to\Lambda(\hat{D}_u),$$

say. This plainly changes continuously with variations of the stabilizing map S, subject to conditions (3.7), (3.8). In particular, if the map S is extended to $S\oplus S_1:\mathbb{R}^{N+M}\to\Gamma(\eta)$, then the isomorphism is changed only by a positive scalar, since we can consider the family $S\oplus\varepsilon S_1$, $0\leqslant\varepsilon\leqslant 1$. So the isomorphism is independent of S. Similarly, using continuity in u, we get an isomorphism:

$$e_{\Theta,\hat{\Theta},\sigma}:\Lambda(D)\to\Lambda(\hat{D}).$$

We then have the standard fact

Proposition (3.10). *If D_t and $\hat{D}_t$ are families of elliptic operators parametrized by a compact space T and for each $t\in T$ there are maps $\sigma_t,\Theta_t,\hat{\Theta}_t$ as above, varying continuously with t, then there is a continuous family of isomorphisms*

$$e_{\Theta_t,\hat{\Theta}_t,\sigma_t}:\Lambda(D_t)\to\Lambda(\hat{D}_t).$$

Moreover if $\Theta_t,\hat{\Theta}_t$ can be extended over all of X compatibly with σ_t, then the isomorphism agrees with the composite of

$$\Lambda(D)\cong\det\operatorname{Ker}D\otimes\Theta_t(\det\operatorname{Ker}D)\cong\mathbb{R},$$
$$\Lambda(\hat{D})\cong\det\operatorname{Ker}\hat{D}\otimes\hat{\Theta}_t(\det\operatorname{Ker}\hat{D})\cong\mathbb{R}.$$

This proposition follows immediately from the lemma, the fact that the conditions (3.7), (3.8) are open, and that the constant u_0 depends only on c, c' and hence on the symbol of D. The point of this section is that we have obtained the isomorphism using only local operators.

(c) Orientations and instantons: linear algebra preliminaries. In §3(d) below we relate the different determinant lines $\Lambda(c_1,-)$ by an excision argument. Here we fix some conventions needed for explicit calculations.

Our guide for fixing orientations is the case when X is a complex Kähler surface. Then the ASD connections may be identified with certain holomorphic vector bundles [3] and their moduli space has a complex structure. Similarly, at the linear level, the operators $\mathscr{D}_A$ are compatible with a complex structure and their determinant lines have a standard trivialization.

The complexified de Rham complex of a Kähler surface X decomposes into

(3.11) $$d=\partial\oplus\bar{\partial}:\Omega_X^{p,q}\to\Omega_X^{p+1,q}\to\Omega_X^{p,q+1}.$$

Contraction with the metric form ω gives an operator $\Lambda:\Omega_X^{p+1,q+1}\to\Omega_X^{p,q}$ which obeys the Kähler identities

(3.12) $$\partial^*=i[\Lambda,\bar{\partial}],\qquad\bar{\partial}^*=-i[\Lambda,\partial]$$

[17, p. 193]. Identify the real 1-forms Ω^1_X with $\Omega^{0,1}_X$ by taking the $(0,1)$ component and similarly the real self-dual forms $\Omega^2_{+,X}$ with $\Omega^0_X \cdot \omega \oplus \Omega^{0,2}_X$. Then the operator $\mathscr{D} = -d^* \oplus d^+$ is identified with

$$(-\bar{\partial}^* \oplus \bar{\partial}) : \Omega^{0,1}_X \to \left(\Omega^0_X\right)^{\mathbb{C}} \oplus \Omega^{0,2}_X .$$

Here we make $\Omega^0 \oplus \Omega^0_X \cdot \omega$ into a complex space $(\Omega^0_X)^{\mathbb{C}}$ by:

$$(3.13) \qquad I \cdot \omega/\sqrt{2} = -1, \qquad I \cdot 1 = \omega/\sqrt{2}.$$

In the same way the twisted operators $\mathscr{D}_A$ are identified with operators $-\bar{\partial}^*_A \oplus \bar{\partial}_A$ commuting with complex structures.

The space of complex structures on $\mathbb{R}^4$ compatible with a given metric and orientation is connected (a copy of S^2). Choosing one such structure, with complex coordinates $z_1 = x_0 + ix_1$, $z_2 = x_2 + ix_3$, we orient the 3-space $\Lambda^2_+(\mathbb{R}^4)^*$ of self-dual forms by

$$(3.14) \qquad \omega \wedge \alpha \wedge I\alpha,$$

where $\omega = dx_0 dx_1 + dx_2 dx_3$ is the metric form and α is the $(0,2)$ form $d\bar{z}_1 \wedge d\bar{z}_2$. Then this orientation is independent of the choice of complex coordinate system. Together with the metric it makes $\Lambda^2_+(\mathbb{R}^4)^*$ into a Lie algebra with the rule $e_1 = [e_2, e_3]$, where e_1, e_2, e_3 is an oriented orthonormal basis. Of course this Lie algebra is one of the factors of $\mathbb{R}^+ \times \mathrm{SO}(4)$—the conformal linear transformation of $\mathbb{R}^4$. For (ϕ, w) in $\mathbb{R} \oplus \Lambda^2_+(\mathbb{R}^4)^*$ the corresponding vector field $\delta(\phi, w)$ on $\mathbb{R}^4$ has component $\phi r\, \partial/\partial r$ radially and induces the action $-\mathrm{ad}(w)$ on $\Lambda^2_+(\mathbb{R}^4)^*$.

In standard quaternionic coordinates $Z = x_0 + x_1 i + x_2 j + x_3 k$ we may identify the Lie algebras $\Lambda^2_+(\mathbb{R}^4)^*$ and $\mathrm{Im}\,\mathbb{H}$ by mapping i, j, k to their coefficients in the quaternionic differential form: $-\mathrm{Im}(dZ d\bar{Z})$. Then $\mathbb{R} \oplus \Lambda^2_+(\mathbb{R}^4)$ is identified with $\mathbb{H}$ and the map δ is given by left quaternionic multiplication. So for any compatible $\mathbb{C}^2 \cong \mathbb{R}^4$, δ intertwines the complex structures on the vector fields over $\mathbb{C}^2$ and on $\mathbb{R} \oplus \Lambda^2_+(\mathbb{R}^4)^* \cong \mathbb{C} \oplus \Lambda^{0,2}(\mathbb{C}^2)^*$. Note also that with this convention the curvature form of the basic instanton over S^4 is *minus* the identity, as in [6, §V(i)].

Next, let v be a vector field on a Riemannian 4-manifold X and $\mathscr{A}$ be the affine space of connections on a bundle E over X. For each point A in $\mathscr{A}$ put

$$(3.15) \qquad a(v, A) = -v \lrcorner F_A \in \Omega^1(\mathfrak{g}_E) = T\mathscr{A}.$$

This defines a vector field $a(v, *)$ on $\mathscr{A}$ which is related to v in the following way: If f_t is the flow on X generated by v and Φ_t is the flow on $\mathscr{A}$ generated by $a(v, *)$, then

$$\Phi_t(A) \cong f^*_{-t}(A).$$

We can write $a(v, A) = a^+ + a^- = (-v \lrcorner F_A^+) + (-v \lrcorner F_A^-)$, so $a = a^-$ if A is ASD. A short calculation shows that, in general,

$$(3.16) \qquad \begin{aligned} &\text{(i)} \quad -d_A^* a^- = (d^- v^*) \cdot F_A + v^* \cdot d_A^* F_A^-, \\ &\text{(ii)} \quad -d_A^+ a^8 - = [v \lrcorner (d_A F_A^-)]^+ + \Pi(\mathscr{L}_v g) \cdot F_A^-. \end{aligned}$$

Here v^* is the 1-form dual to v, and $\Pi(\mathscr{L}_v g)$ is the trace-free component of the Lie derivative of the metric which pairs with F_A by the isomorphism between trace free symmetric 2-tensors and $\mathrm{Hom}(\Lambda^2_-, \Lambda^2_+)$. Over Euclidean 4-space the 1-forms dual to vector fields $\delta(\phi, w)$ are annihilated by d^-, so for any ASD connection I over $\mathbb{R}$ we have a map $i : \mathbb{R} \oplus \Lambda^2_+(\mathbb{R}^4)^* \to \mathrm{Ker}\,\mathscr{D}_I$; $i(\phi, w) = a^-(\delta(\phi, w), I)$ (cf. [15, Lemma 8.2]).

Finally note that if X is Kähler and A is any connection the map $v \to a^-(v, A)$ from vector fields to $\Omega^1(\mathfrak{g}_E) \cong \Omega^{0,1}(\mathfrak{g}_E)$ is complex linear, since F_A^- is of type $(1, 1)$.

(d) Addition of instantons. Let x be a point in X, λ be a small positive number, and

$$\rho : (\mathfrak{g}_E)_x \to (\Lambda^2_{+,X})_x$$

be an isomorphism of SO(3) spaces. For any U(2) connection A on a bundle E over X, we denote by $\tilde{A} = A' \#_\rho J_\lambda$ a connection formed as in [6, §III(ii)]. This is done by flattening A over the annulus:

$$\Omega = \{ y \in X \mid MN^{-1}\sqrt{\lambda} < d(x, y) < MN\sqrt{\lambda} \}$$

and attaching a "flattened instanton" J_λ of scale λ. Here $N > 0$ is a fixed number chosen as in [6, §IV] and M will be fixed below.

$\tilde{A}$ is carried by a bundle $\tilde{E}$ with $c_1(\tilde{E}) = c_1(E)$, $c_2(\tilde{E}) = c_2(E) + 1$. We will compare the determinant line bundles Λ_E, $\Lambda_{\tilde{E}}$ by explicitly comparing the kernels and cokernels of $\mathscr{D}_A$ and $\mathscr{D}_{\tilde{A}}$, after stabilization. Define

$$(3.17) \qquad V_x = \mathbb{R} \oplus \Lambda^2_+(T^*X)_x \oplus TX_x,$$

an 8-dimensional vector space whose orientation is fixed by the conventions of §3(c) above. When X is Kähler, V_x has a complex structure. An element (ϕ, u, ζ) of V_x defines a conformal Killing-vector field $\delta(\phi, u) + \zeta$ on TX_x, as in §3(c). Define a map

$$\delta_x : V_x \to (\text{Vector fields on } X)$$

in a similar way, using a normal coordinate system and cutting-off with a function β, $\mathrm{supp}\nabla\beta \subset \Omega$. Then for any connection B over X and v in V_x let $i_B(v) = a^-(B, \delta_x(v))$.

If $v = (\phi, u, \lambda\zeta) \in V_x$ we have

$$(3.18) \qquad \| i_{\tilde{A}}(v) \|_{L^2(X)} \geqslant \mathrm{const}\, \lambda(|\phi| + |u| + |\xi|).$$

This follows from the approximate homogeneity of the construction with respect to the scale λ. On the other hand the form of $\mathscr{D}_{\tilde{A}}(i_{\tilde{A}}(v))$ can be estimated using (3.16): $d^-(\delta_x(v)^*)$ is small and $\delta_x(v)$ is approximately a conformal Killing field. Each of the four right-hand terms in (3.16) gives one contribution supported in the $MN\sqrt{\lambda}$ ball due to the curvature of X and another supported in the annulus Ω due to the cut-off. Calculations, very similar to those in [3, Theorem 19] and [10, Proposition 9.29] give

$$\|\mathscr{D}_{\tilde{A}}(i_{\tilde{A}}(v))\|_{L^2(X)} \leqslant \text{const}\left(\frac{\lambda}{M^2} + \varepsilon(M,\lambda)\right),$$

where, for fixed M, $\varepsilon(M,\lambda) \leqslant \text{const}\,\lambda^{3/2}$.

Let τ be an isomorphism of the bundles $\mathfrak{g}_E$, $\mathfrak{g}_{\tilde{E}}$ away from x which intertwines the connections A, $\tilde{A}$. Taubes' argument in [14, Proposition (8.8)] gives a uniform bound on the eigenfunctions belonging to the low-lying spectrum of $\mathscr{D}_{\tilde{A}}\mathscr{D}_{\tilde{A}}^*$. It follows that if we choose a stabilizer map:

$$S : \mathbb{R}^N \to (\Omega^0 \oplus \Omega^2_+)(\mathfrak{g}_E)$$

with

(3.19)

$$\left\|(\mathscr{D}_A \oplus S)^*\alpha\right\|^2_{L^2} \geqslant \|\alpha\|^2, \quad \left\|\mathscr{D}_A^* S\Theta\right\|^2_{L^2} \leqslant 2\|\Theta\|^2, \quad \|S\Theta\|_{L^\infty} \leqslant \text{const}\|\Theta\|.$$

Then the map $\tilde{S}_\tau : \mathbb{R}^N \to (\Omega^0 \oplus \Omega^2_+)(\mathfrak{g}_{\tilde{E}})$, $\tilde{S}_\tau(\Theta) = \tau(1-\beta)S(\Theta)$ stabilizes $\mathscr{D}_{\tilde{A}}$. Inequalities like (3.19) hold for $\mathscr{D}_{\tilde{A}}$, $\tilde{S}_\tau$ with a change in the multiplying constants to $1 - O(\sqrt{\lambda})$, $2 + O(\sqrt{\lambda})$ since $\|\nabla\beta\|_{L^2}$ is $O(\sqrt{\lambda})$. Let π be L^2-projection and define

$$p_\tau = \pi \circ (i_{\tilde{A}} \oplus (1-\beta)\tau \oplus 1) : V_x \oplus \operatorname{Ker}(\mathscr{D}_A \oplus S) \to \operatorname{Ker}(\mathscr{D}_{\tilde{A}} \oplus \tilde{S}_\tau).$$

If M is made sufficiently large and then λ made small, (3.18) and (3.19) give

$$\|\mathscr{D}_{\tilde{A}}(i_{\tilde{A}}v)\|^2_{L^2} \leqslant \tfrac{1}{2}\|i_{\tilde{A}}(v)\|^2_{L^2}.$$

Arguing as in Lemma (3.7) (compare also [16]) shows that p_ρ is a monomorphism. Then the Atiyah-Singer index theorem gives: $\operatorname{index}(\mathscr{D}_{\tilde{A}}) - \operatorname{index}(\mathscr{D}_A) = 8 = \dim V_x$ so p_ρ is an isomorphism. Note that the conditions on M, λ which must be fulfilled are independent of the connection A.

Using the orientation fixed on V_x we get an isomorphism $j_x : \Lambda(A) \to \Lambda(\tilde{A})$, induced by p_τ. This does not depend on the choice of τ since two choices differ by the image of Γ_A in Aut $\mathfrak{g}_E$ and this is a connected group. We have

Proposition (3.20). *The isomorphism $j_x : \Lambda(A) \to \Lambda(\tilde{A})$ extends continuously to any family of gauge equivalence classes of connections $[A]$ and points x in X. If X is Kähler it is compatible with the complex orientations on $\Lambda(A)$ and $\Lambda(\tilde{A})$.*

The last property holds because if X is Kähler we can choose holomorphic normal coordinates so that δ_x and hence $i_{\tilde{A}}$ are complex linear. Then the whole construction is compatible with the complex structures.

Remark (3.21). Suppose, more generally, $D: \Gamma(\xi) \to \Gamma(\eta)$ and $\tilde{D}: \Gamma(\xi) \to \Gamma(\tilde{\eta})$ are elliptic differential operators over X which, away from x, are intertwined by a bundle isomorphism and near x have the form

$$D = \mathscr{D}_A \oplus D', \qquad \tilde{D} = \mathscr{D}_{\tilde{A}} \oplus D'.$$

Then the argument above gives an isomorphism $j: \Lambda(D) \to \Lambda(\tilde{D})$ which agrees with $j_x \cdot l$ in the case when the direct sum decomposition extends over all of X.

Corollary (3.22). *For any 4-manifold X, $l > 0$, and $U(l)$ bundle $E \to X$ the line bundle Λ_E over the space $\mathscr{B}_E$ of connections on E is trivial.*

Proof. We use the same stabilization as in [3, Lemma 10]. If $\phi: S^1 \to \mathscr{B}_E$ is a loop and $\phi': S^1 \to \mathscr{B}_{E'}$ is the corresponding loop representing connections on $E' = E \oplus L$, where L is a complex line bundle, then $\langle w_1(\Lambda_E), \phi \rangle = \langle w_1(\Lambda_{E'}), \phi' \rangle$, since $\mathfrak{g}_{E'} = \mathfrak{g}_E \oplus \mathbb{R} \oplus E \otimes L^*$. This means that, considering $E \oplus (\det E)^* \oplus \mathbb{C}^p$, we may reduce to the case of SU(l) bundles with $l \gg 0$. Then (as in [3]) the pairing of the loop with $w_1(\Lambda_E)$ depends only on the class it defines in $[X, SU] \cong K^{-1}(X)/H^1(X; \mathbb{Z}) \cong H^3(X; \mathbb{Z})$.

Let γ be a loop in X and $E \oplus \mathbb{C}^q$ an SU(l) bundle, where E has rank 2. Choose a connection A^* on an SU(2) bundle E^* with $c_2(E^*) = c_2(E) - 1$. Then define a family of connections

$$\phi_\gamma(t) = \left(A^* \#_{\rho(t)} J_\lambda\right) \oplus \theta$$

on $E \oplus \mathbb{C}^q$ using a left ρ of γ to $\operatorname{Hom}(\mathfrak{g}_E, \Lambda^2_{+,x})$. By Proposition (3.20) the determinant line bundle is trivial over ϕ_γ. On the other hand, arguing as in [6, Lemma (3.8), Proposition (3.19)], we see that the class defined by ϕ_γ in $H^3(X; \mathbb{Z})$ is the Poincaré dual of $[\gamma]$. Since Poincaré Duality gives an isomorphism $H_1(X; \mathbb{Z}) \to H^3(X; \mathbb{Z})$, $w_1(\Lambda_{E \oplus \mathbb{C}^q})$ pairs trivially with all loops and the determinant bundles are trivial.

(e) Reductions and complex structures. The vector space $H^2_+(X)$ of self-dual harmonic 2-forms on a Riemannian 4-manifold X depends upon the choice of metric. But the determinants, $\det H^2_+$, may all be identified since the set of maximal positive subspaces for the intersection form is contractible. We will call an orientation α_x of the line

$$\det H^1(X) \otimes \det\left(H^2(X) \oplus H^2_+(X)\right)$$

a *homology orientation* of the 4-manifold. A choice of homology orientation clearly trivializes the line Λ_θ corresponding to the trivial SU(2) connection, since the bundle-valued harmonic forms are then copies of the ordinary ones.

However there is a choice in the conventions one might adopt and we must make explicit the one that we use.

If A is any reducible connection on a U(2) bundle, compatible with a decomposition

$$E = (\mathbb{C} \oplus L) \otimes L', \qquad \mathfrak{g}_E \cong \mathbb{R} \oplus L,$$

and α_x is a homology orientation of X we can define an orientation $o(L, L', \alpha_x)$ of Λ_A. First, we fix the decomposition of $\mathfrak{g}_E$ by decreeing that the generator "1" of the trivial factor acts with positive weight on L in the (left) adjoint representation. Then we write

$$\Lambda_A \cong \Lambda_{A|\mathbb{R}} \cdot \Lambda_{A|L}$$

and use α_x to orient the first term and the complex structure on L to orient the second. Explicitly, if

$$\alpha_x = (\theta_1 \cdots \theta_p) \otimes (\phi_1 \cdots \phi_q),$$

and $(\rho_1, \cdots, \rho_r)$; and $(\sigma_1, \cdots, \sigma_s)$ are complex bases for $\operatorname{Ker} D_{A|L}$, and $\operatorname{Ker} D^*_{A|L}$, then

$$\begin{aligned} &o(L, L', \alpha_x) \\ (3.23) \qquad &= (\theta_1 \cdot 1)(\theta_2 \cdot 1) \cdots (\theta_p \cdot 1)(\rho_1 \cdot I\rho_1)(\rho_2 \cdot I\rho_2) \cdots (\rho_r \cdot I\rho_r) \\ &\quad \otimes (\phi_1 \cdot 1)(\phi_2 \cdot 1) \cdots (\phi_q \cdot 1)(\sigma_1 \cdot I\sigma_1) \cdots (\sigma_s \cdot I\sigma_s). \end{aligned}$$

If L is the trivial bundle, so

$$\mathfrak{g}_E \cong \mathfrak{su}(2) = \langle e_1, e_2, e_3 \rangle \quad \text{with } e_1 = 1, \text{ say, and } Ie_2 = e_3,$$

then this agrees with the orientation

$$\begin{aligned} &(\Theta_1 e_1)(\Theta_1 e_2)(\Theta_1 e_2) \cdots (\Theta_p e_3) \\ &\quad \otimes (\phi_1 e_1)(\phi_1 e_2)(\phi_1 e_3)(\phi_2 e_1)(\phi_2 e_2) \cdots (\phi_q e_3) \end{aligned}$$

and compares with

$$\begin{aligned} &(\Theta_1 e_1)(\Theta_2 e_1) \cdots (\Theta_p e_1)(\Theta_1 e_2) \cdots (\Theta_p e_3) \\ &\quad \otimes (\phi_1 e_1)(\phi_2 e_1) \cdots (\phi_q e_1)(\phi_1 e_2) \cdots (\phi_q e_3) \end{aligned}$$

by the sign $(-1)^{[p(p-1)/2 + q(q-1)/2]}$.

Definition (3.24). *The "standard orientation" of the determinant line Λ_E when E is an* SU(2) *bundle over a homology oriented 4-manifold (X, α_x) is that obtained from $o(\mathbb{C}, \mathbb{C}, \alpha_x)$ on the trivial bundle and repeated application of the isomorphism of Proposition* (3.20).

When X is Kähler these standard orientations agree with the complex orientation defined in §3(c) if we fix the correct homology orientation. Use the Hodge decomposition to write:

$$H^1(X;\mathbb{R}) \simeq H^{1,0}; \qquad H^0 \oplus H^2_+ \cong \mathbb{R} \oplus \mathbb{R}\omega \oplus H^{2,0}$$

and let the element α_x be defined by the complex structures on these spaces, where we set $I \cdot 1 = -\omega/\sqrt{2}$, *opposite* to (3.13). It is easy to check that the two orientations agree for the trivial bundle; then the general case follows from the last sentence of Proposition (3.20).

Any two U(2) bundles with the same first Chern class differ topologically by a number of "instanton additions." So we may compare the orientations defined at different reductions.

Proposition (3.25). *Let E_0, E_1 be* U(2) *bundle over X with $c_1(E_0) = c_1(E_1)$ which have reductions*

$$E_0 \cong (\mathbb{C} \oplus L_0) \otimes L_0', \qquad E_1 \cong (C \oplus L_1) \otimes L_1'.$$

Then the orientations $o(L_1, L_1', \alpha_x)$ and $o(L_0, L_0', \alpha_x')$ compare, via repeated applications of the isomorphisms of (3.20), *with the sign $(-1)^{[c_1(L_0') - c_1(L_1')]^2}$.*

Proof. If X is Kähler we can use the index theorem to compare the orientations at the reductions with the complex orientation. The operator

$$\mathscr{D}_A = \Omega^1(\mathbb{R} \oplus L) \to (\Omega^1 \oplus \Omega^2_+)(\mathbb{R} \oplus L)$$

decomposes into three parts:

$$(-\bar\partial^* \oplus \bar\partial) \ :\Omega^{0,1} \to (\Omega^0)^{\mathbb{C}} \oplus \Omega^{0,2},$$

$$(-\bar\partial^* \oplus \bar\partial)_A : \Omega^{0,1} \otimes_{\mathbb{C}} L \to \{(\Omega^0) \cup \mathbb{C} \oplus \Omega^{0,2}\} \otimes_{\mathbb{C}} L,$$

$$(-\bar\partial^* \oplus \bar\partial)_A : \Omega^{0,1} \otimes_{\mathbb{C}} \bar{L} \to \{(\Omega^0) \oplus \Omega^{0,2}\} \otimes_{\mathbb{C}} \bar{L}.$$

The complex structures defined by L and by the base space agree on the second term and are opposite on the third. Similarly, for the first term, our homology orientation on the base space uses the opposite complex structure to that defined by $(-\bar\partial^* \oplus \bar\partial)$. The complex orientation of a vector space W and its conjugate $\overline{W}$ differ by $(-1)^{\dim_{\mathbb{C}} W}$. So an orientation $o(L, L', \alpha_x)$ compares with the complex orientation with the sign

$$(-1)^{[\operatorname{ind}(-\bar\partial^* \oplus \bar\partial)_{\bar{L}} - \operatorname{ind}(-\bar\partial^* \oplus \bar\partial)]}.$$

By the index theorem this is equal to

$$(-1)^{[c_1(L)^2 + K_x \cdot c_1(L)]/2}.$$

Since the isomorphism of (3.20) is compatible with a Kähler structure we see that $o(L_0, L_0', \alpha_x)$ and $o(L_1, L_1', \alpha_x)$ compare according to the parity of

$$\tfrac{1}{2}\left(c_1(L_0)^2 - c_1(L_1)^2\right) + K_X \cdot \left(c_1(L_0) - c_1(L_1)\right).$$

This is the same as the parity of $K_X \cdot (c_1(L'_1) - c_1(L'_0))$, since $c_1(L_0) + 2c_1(L'_0) = c_1(L_1) + 2c_1(L'_1)$. Finally $K_X \cdot D = D^2 \bmod 2$, so Proposition (3.25) is true when X is Kähler.

The same proof works if the base manifold X has an almost complex structure. Then the spaces $\Omega^1(\mathfrak{g}_E)$ and $(\Omega^0 + \Omega^2_+)(\mathfrak{g}_E)$ have complex structures and the symbols of the $\mathscr{D}_A$ operators are complex linear. There is a linear deformation through elliptic operators

$$\mathscr{D}^t_A = (1-t)\mathscr{D}_A - tI\mathscr{D}_A I$$

from $\mathscr{D}_A$ to the complex linear operator $\frac{1}{2}(\mathscr{D}_A - I\mathscr{D}_A I)$. We can suppose that the almost complex structure is Kähler in a neighborhood of a point x in X. So, by Remark (3.21), the isomorphism of (3.20) extends to compare the determinant lines of $\mathscr{D}^t_A$ and $\mathscr{D}^t_{\hat{A}}$ when the instantons are added near x. When $t = \frac{1}{2}$ the whole discussion for the Kähler case applies and this can then be transferred back to the $\mathscr{D}_A$ operators by continuity in t.

The proof of Proposition (3.25) is completed by using an excision argument. An oriented 4-manifolds admits an almost complex structure if there is an integral class c lifting w_2 and such that $c^2 = 3\tau + 2e$. An integral lift of w_2 always exists and its square necessarily equals $3\tau + 2e \bmod 4$. It follows easily that for any oriented 4-manifolds X there is a connected sum $X\#l(S^2 \times S^2)$ which admits an almost complex structure. Hence Proposition (3.25) follows from the lemma below.

Lemma (3.26). *Let X, W be closed, oriented 4-manifolds and $\hat{X} = X\#W$. Suppose α_X, $\alpha_{\hat{X}}$ are homology orientations and L_0, L'_0 and L_1, L'_1 are complex line bundles over X with*

$$c_1\big((\mathbb{C} \oplus L_1) \otimes L'_0\big) = c_1\big((\mathbb{C} \oplus L_1) \otimes L'_1\big).$$

Let $\hat{L}$, $\hat{L}_0$ and $\hat{L}_1$, $\hat{L}'_1$ be the corresponding bundles over $\hat{X}$. Then the sign

$$o\big(L_1, L'_1, \alpha_X\big)/o\big(L_0, L'_0, \alpha_{\hat{X}}\big)$$

by which the reductions compare is equal to

$$o\big(\hat{L}, \hat{L}'_1, \alpha_{\hat{X}}\big)/o\big(\hat{L}_0, \hat{L}'_0, \alpha_{\hat{X}}\big).$$

Proof. Suppose, without loss, that

$$c_2\big((\mathbb{C} \oplus L_1) \otimes L'_1\big) - c_2\big((\mathbb{C} \oplus L_0) \otimes L'_0\big) = l \geqslant 0.$$

There is a 1-parameter family B_t of U(2) connections over X with $B_0 = A_0\#J\#J\# \cdots \#J$ and $B_1 = A_1$, where the A_i are reducible connections, compatible with bundle splittings $(\mathbb{C} \oplus L_i) \otimes L'_i$, and the instantons are added at points $X_1, \cdots, X_l$ outside the region $\Omega \subset X$ where the connected sum is formed.

Define $D_t = -\mathscr{D}_{B_t} \oplus \mathscr{D}^*_{A_0}$. We may suppose that all the connections B_t are flat over Ω so over this region there is an excision isomorphism Θ for the D_t, as in §3(b). If $\hat{B}_t$ denotes the corresponding family over $\hat{X}$, trivialized over $\hat{\Omega} \subset W$, Proposition (3.10) gives a continuous family of isomorphisms

$$e_t : \Lambda(D_t) \to \Lambda(\hat{D}_t).$$

It suffices to show that these are compatible with the isomorphisms

$$j : \Lambda_{A_0} \to \Lambda_{B_0}, \qquad \hat{j} : \Lambda_{\hat{A}_0} \to \Lambda_{\hat{B}_0}$$

and with the orientations at the reductions.

We may choose the bundle isomorphisms Θ, $\hat{\Theta}$ to be compatible with the splitting into real and complex parts at $t = 0, 1$. So when $t = 1$

$$e_1 : \Lambda_{B_1} \cdot \Lambda_{A_0} \to \Lambda_{\hat{B}_1} \cdot \Lambda_{\hat{A}_1}$$

splits into

$$e_1^{\mathbb{R}} : \Lambda_{A_1|\mathbb{R}} \cdot \Lambda_{A_0|\mathbb{R}} \to \Lambda_{\hat{A}_1|\mathbb{R}} \cdot \Lambda_{\hat{A}_0|\mathbb{R}},$$
$$e_1^{\mathbb{C}} : \Lambda_{A_1|L_1} \cdot \Lambda_{A_0|L_0} \to \Lambda_{\hat{A}_1|\hat{L}_1} \cdot \Lambda_{\hat{A}_1|\hat{L}_0}.$$

$e_1^{\mathbb{R}}$ maps $\alpha_X \cdot \alpha_X$ to $\alpha_{\hat{X}} \cdot \alpha_{\hat{X}}$, by the last clause of Proposition (3.10). Also $e_1^{\mathbb{C}}$ is induced by a complex linear map. Hence e_1 maps

$$o(L_1, L_1', \alpha_X) \cdot o(L_0, L_0', \alpha_X)$$

to the corresponding element in $\Lambda_{\hat{B}_1} \cdot \Lambda_{\hat{A}_1}$.

When $t = 0$ there is a diagram:

$$\begin{array}{ccc} \Lambda_{B_0} \cdot \Lambda_{A_0} & \xrightarrow{e_0} & \Lambda_{\hat{B}_0} \cdot \Lambda_{\hat{A}_0} \\ \downarrow j & & \downarrow \hat{j} \\ \Pi \det V_{x_i} \cong \mathbb{R} & \longrightarrow & \Pi \det V_{\hat{x}_i} \cong \mathbb{R} \end{array}$$

Lemma (3.26) is equivalent to the commutativity of this diagram. In turn this is equivalent to the commutativity of the similar diagram for the operators $D_0 + u\beta\Theta$ and $\hat{D}_0 + u\hat{\beta}\hat{\Theta}$ defined using Remark (3.21). But this last fact is visibly true when $u \gg 0$ since the injections i_{x_i} and $i_{\hat{x}_i}$ correspond under σ and the maps j are made from the composite of these with small L^2-projections.

Recall that the group $H^1(X; \mathbb{Z}/2)$ acts on the space $\mathscr{B}_E$ of U(2) connections and the quotient by the action is the space $\mathscr{B}_{\mathfrak{g}_E}$ of connections on the SO(3) bundle $\mathfrak{g}_E$. The operators $\mathscr{D}_A$ act on $\mathfrak{g}_E$-valued forms so the determinant lines descend to give a line bundle $\Lambda_{\mathfrak{g}_E}$ over $\mathscr{B}_{\mathfrak{g}_E}$.

Corollary (3.27). *The line bundles $\Lambda_{\mathfrak{g}_E}$ are trivial.*

Proof. Suppose first that E admits a reduction $E \cong (\mathbb{C} \oplus L) \otimes L'$. An element ρ of $H^1(X; \mathbb{Z}/2)$ maps a connection compatible with this reduction to one compatible with the reduction $E \cong (\mathbb{C} \oplus L) \otimes L' \otimes L_\rho$ where $C_1(L_\rho)$ is the image of ρ by the Bockstein map:

$$\beta : H^1(X; \mathbb{Z}/2) \to H^2(X; \mathbb{Z}).$$

ρ sends the element $o(L, L', \alpha_X)$ of the determinant line bundle to $o(L, L' \otimes L_\rho, \alpha_X)$. But $\int c_1(L_\rho)^2 = 0$ so these are equal by Proposition (3.25). Hence $\Lambda_{\mathfrak{g}_E}$ is trivial in this case. The general case can be reduced to this by using Proposition (3.20) to compare the actions for different values of c_2.

Let $f : X \to X$ be an orientation preserving diffeomorphism. Associate to f the sign $\alpha_f = \pm 1$, by which $f^* : H^*(X; \mathbb{R}) \to H^*(X; \mathbb{R})$ acts on the homology orientations. Suppose w is in the image of the reduction map $H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z}/2)$ and is fixed by f^*. If c is any lift of w, $f^*(c) - c$ vanishes mod 2 and

$$\beta(w, F) = (-1)^{((f^*(c)-c)/2)^2}$$

is independent of the lift. Now if ζ is an SO(3) bundle with $w_2(\zeta) = w$ there is a natural way in which f acts on the orientations of the line bundle Λ_ξ. We form an SO(3) bundle Ξ over the mapping torus X_f which restricts to ξ on the fibers of $X_f \to S^1$. This gives a family of $\mathscr{D}_A$ operators parametrized by the circle. The diffeomorphism acts on the determinant line according to the (reduced) index of this family.

Corollary (3.28). *The diffeomorphism f acts on the orientations of Λ_ξ by the sign $\alpha_f \cdot \beta(w_2(\mathfrak{g}_E), f)$.*

The proof is a simple application of Propositions (3.20) and (3.25).

4. Applications to moduli spaces

(a) Interior local models. If the connection induced by A on $\mathfrak{g}_E$ in ASD there is a deformation complex

$$\Omega^0(\mathfrak{g}_E) \xrightarrow{-dA} \Omega^1(\mathfrak{g}_E) \xrightarrow{d_A^+} \Omega^2_+(\mathfrak{g}_E). \tag{4.1}$$

The kernels of $\mathscr{D}_A$ and $\mathscr{D}_A^*$ are isomorphic to the parts H_A^1 and $H_A^0 \oplus H_A^2$, respectively, of the cohomology of this complex. When H_A^0 and H_A^2 are 0, H_A^1 is the tangent space at $[A]$ to a moduli space M of ASD connections on $\mathfrak{g}_E$. So an orientation of the determinant line bundle orients the moduli space and, in particular, a homology orientation of X defines standard orientations of the SU(2) moduli spaces, as in §3.

In general a neighborhood of $[A]$ in M has a finite dimensional model $\phi^{-1}(0)/\Gamma_A \subset H_A^1/\Gamma_A$. Here $\Gamma_A \subset \mathcal{G}$ is the isotropy group of A, with Lie algebra H_A^0. It acts on the left on H_A^1 and H_A^2 via the adjoint representation. For each p in H_A^1 let $r_p : H_A^0 \to H_A^1$ be the derivative of the action; $r_p(u) = [u, p]$. The map ψ is the H_A^2 component of the curvature of a connection close to $A + p$.

Linearizing this Kuranishi description at a point p in $\phi^{-1}(0) \subset H_A^1$ gives a complex:

$$H_A^0 \underset{r_p}{\to} H_A^1 \underset{\delta\phi_p}{\to} H_A^2. \tag{4.2}$$

If p represents a smooth point of the moduli space, the cohomology of this complex is the tangent space there. Moreover the orientation of the moduli space near $[A]$ which is derived from a trivialization of Λ_A and the local triviality of the determinant line bundle agrees with that obtained from the complex (4.2). This is just a matter of writing out the definitions and using the fact that ρ represents a part of the left action of $\mathcal{G}$ on $\mathcal{A}$, whose derivative is $-d_*$, while $\delta\phi$ represents a part of the derivative d_*^+ of the curvature F_+ on $\mathcal{A}$.

Explicit calculations with these determinant lines can be very confusing. The same point set is given the structure of a continuous line bundle in different ways depending on the conventions used in (3.2) and (3.3). Similarly the identity map is not continuous between bundles $\Lambda(D_t)$ and $\Lambda(-D_t)$. We fix conventions by saying that if $(\alpha_1, \cdots, \alpha_{q+p})$ is a basis for $\operatorname{Ker} D$ and $\beta_1, \cdots, \beta_q$ for $\operatorname{Ker} D^*$, and if $D'(\alpha_i) = \beta_i$, $i = 1, \cdots, q$, and $D'(\alpha_j) = 0$, $j > q$, for a nearby operator D', then $(\alpha_1 \wedge \cdots \wedge \alpha_{q+p}) \otimes (\beta_1 \wedge \cdots \wedge \beta_q)$ and $(\alpha_{q+1} \wedge \cdots \wedge \alpha_{q+p})$ represent nearby elements in $\Lambda(D)$ and $\Lambda(D')$.

Suppose that $b_1(X)$ and $b_2^+(X)$ are 0 so that the generator 1 in $H^0(X)$ defines a homology orientation. Let E be an SU(2) bundle with $c_2(E) = 1$ which admits a reduction $E = L \oplus L^{-1}$. Assume, for simplicity, that $H_A^2 = 0$ for the ASD connection A corresponding to the reduction. Then we can compute the standard orientation of the moduli space M_1 near $[A]$ in terms of its explicit description as a cone on $\mathbb{CP}^2$. For we know, by (3.23), that the standard orientation is $-o(L^2, L^{-1}, \alpha_X)$. But at a point p in $H_A^1 \cong \mathbb{C}^2$ the orientation $o(L^2, L^{-1}, \alpha_X)$ is

$$(1) \otimes \left(n \wedge \rho_p(1) \wedge v_1 \wedge v_2 \wedge v_3 \wedge v_4\right),$$

where n denotes the normal vector pointing away from the reduction, $\rho_p(1)$ is $i \cdot n$, and the v_i are lifts of a standard oriented frame in $T\mathbb{CP}^2$. To obtain the volume element in the moduli space corresponding to $o(L^2, L^{-1}, \alpha_X)$ we "cancel" (1) with $(\rho_p(1))$ introducing one minus sign because of their separation by n. Thus we have:

Example (4.3). *The standard orientation of* M_1 *near a link* $P_e \cong \mathbb{CP}^2$ *is* $n \wedge$ (*standard orientation of* P_e) *where* n *is the normal pointing away from the reduction.*

Of course the same is true for the perturbed moduli spaces of §1 and we see that there is no cancellation between the homology contributions from the reductions.

(b) Local models of the ends. Let A be an ASD SU(2) connection on a bundle E with $c_2(E) = k$ and x a point in X. Theorem (5.5) of [6] gives a description of a neighborhood of the "ideal" ASD connection (A, x) in the compactification of the moduli space M_{k+1}. We let N be the product of $\mathbb{R}^+ \times \mathrm{Hom}((\mathfrak{g}_E)_x, \Lambda^2_{+,x}) \times \{\text{nbd. of } x \text{ in } X\}$ with a neighborhood of 0 in H^1_A. There is a map $\phi: N \to H^2_A$ representing, as in (a), a projection of the curvature. ϕ is equivariant under the left action of Γ_A on N and H^2_A and a part of the end of M_{k+1} is modelled on $\phi^{-1}(0)/\Gamma_A$. So at a point n in N representing a smooth point in the moduli space there is, again, a finite dimensional complex

$$H^0_A \xrightarrow{r_n} \left(TN \cong V_x \oplus H^1_A\right) \xrightarrow{\delta\phi_n} H^2_A, \tag{4.4}$$

with cohomology TM_{k+1}. Here we have identified a factor V_x in the tangent space of N using the obvious left action of the conformal affine group of $(TX)_x$. Then, since N has a fixed orientation, the exact sequence (4.4) gives an isomorphism between the determinant of TM_{k+1} and Λ_A.

Proposition (4.5). *The isomorphism of determinants given by* (4.4) *is the same as that defined using the isomorphism of Proposition* (3.20).

Proof. This proposition asserts the rather obvious fact that the parametrizations of solutions in [16] and [6] agree, up to a small error. The main point is to get the right signs.

We can suppose that the Riemannian metric on X is flat near x. Let $\{\tilde{A}(n) \mid n \in N\}$ be the family of connections,

$$\tilde{A}(\lambda, \rho, \eta, p) = A' \#_\rho J_{\lambda,\eta} + (1 - \beta)p,$$

where, as in §3(d), $J_{\lambda,\eta}$ is the flattened instanton with scale λ and center $\exp_x(\eta)$. The construction of [6] gives a nearby family $A^\infty(n)$ of connections such that $F_+(A^\infty(n)) \in (1 - \beta)H^0_A$. Here we have suppressed the map τ of §3(d). The bundles $\tilde{E}(n)$ carrying $\tilde{A}(n)$ and $A^\infty(n)$ are identified with E away from x. Since $\tilde{E}(n)$ varies with n it does not make sense to define a derivative $\partial\tilde{A}/\partial n$ mapping to $\Omega^1(\mathfrak{g}_{\tilde{E}})$ but we can define

$$\frac{\partial}{\partial n}(A^\infty - \tilde{A}): TN_n \to \Omega^1(\mathfrak{g}_{\tilde{E}(n)}).$$

Identifying TN with $V_x \oplus H_A^1$, estimates like (4.24), (4.32), and (4.54) in [6] give

$$\left\| \frac{\partial}{\partial n}(A^\infty - \tilde{A})(\phi, u, \lambda\xi, q) \right\|_{L^2(X)} \leqslant \text{const}\big(\lambda^{3/2}(|\phi| + |u| + |\xi|) + \lambda \cdot |q| + |p|\,|q||\big). \tag{4.6}$$

The ambiguity in comparing the bundles $\tilde{E}(n)$ for different values of n is represented by a gauge transformation supported in the ball B inside the inner boundary of Ω. So $\partial\tilde{A}/\partial n$ maps to

$$\Omega^1(\mathfrak{g}_{\tilde{E}})/d_{\tilde{A}}\,(\text{sections supported in } B)$$

and hence to

$$\Omega^1(\mathfrak{g}_{\tilde{E}})/d_{\tilde{A}}\big([(1-\beta)H^0A]^\perp\big).$$

$\partial A^\infty/\partial n$ can be defined similarly. Taking L^2-horizontal lifts gives

$$\operatorname{Im}[\partial\tilde{A}/\partial n] \in \{a \in \Omega^1(\mathfrak{g}_{\tilde{E}}) \mid d_{\tilde{A}}^* a \in (1-\beta)H_A^0\},$$

$$U_n = \operatorname{Im}[\partial A^\infty/\partial n] \in \{a \in \Omega^1(\mathfrak{g}_{\tilde{E}}) \mid d_{A^\infty}^* a \in (1-\beta)H_A^0\}.$$

By the conditions on $F_+(A^\infty)$ we have

$$U_n = \{a \in \Omega^1(\mathfrak{g}_{\tilde{E}}) \mid \mathscr{D}_{A^\infty} a \in (1-\beta)(H_A^0 \in H_A^2)\}$$

and $\partial A^\infty/\partial n$ gives an isomorphism

$$\alpha : TN_n = V_x \oplus H_A^1 \to U_n.$$

Define $s: U_n \to H_A^0 \oplus H_A^2$ by $s(a) = h$ if $\mathscr{D}_{A^\infty} a = (1-\beta)h$. Then

$$s \circ \alpha : TN \to H_A^0 \oplus H_A^2$$

is equal to $r_n^* \oplus \delta\phi_n$ (cf. (4.4)) where the adjoint r_n^* is formed using the metric on TN_n pulled back by α from the L^2-metric on U_n and the metric

$$\|h\|_{H_A^0}^2 = \int_X (1-\beta)|h|^2\, d\mu$$

on H_A^0.

On the other hand we can follow the approach of §3(d) using stabilizing maps defined by H_A^0 and H_A^2. The construction works equally well for the connections in the linear path from $\tilde{A} = \tilde{A}(n)$ and $A^\infty(n)$. We get an isomorphism $p: V_x \oplus H_A^1 \to U_n$ defined by L^2-projection of $i_{A^\infty} \oplus (1-\beta)$, and we have defined in Proposition (3.20) an orientation of the moduli space using

$$s \circ P : V_x \oplus H_A^1 \to H_A^0 \oplus H_A^2.$$

Thus to show the orientations agree we need to show that $\alpha^{-1} \circ P : TN \to TN$ has a positive determinant.

But in fact, when λ and $|p|$ are small, $\alpha^{-1} \circ P$ is close to the identity. For, since $A^\infty(n)$ is ASD,

$$i_{A^\infty(n)}(v) = a^-(A^\infty(n), \delta_x(v)) = -\delta_x(V) \lrcorner F_{A^\infty(n)}.$$

But we can show, as in [6, (4.30)], that

$$\| F_{A^\infty(n)} - F_{\tilde{A}(n)} \|_{L^2(X)} \leqslant \text{const}(\lambda + |p|\sqrt{\lambda} + |p|^2),$$

and plainly

$$\| \delta_x(\phi, u, \lambda\xi) \|_{L^\infty} \leqslant \text{const}\sqrt{\lambda}(|\phi| + |u| + |\xi|).$$

So

$$(4.7) \quad \| I_{A^\infty}(v) + \delta_x(v) \lrcorner F_{\tilde{A}} \| \leqslant \text{const}\sqrt{\lambda}(|\phi| + |u| + |\xi|)(\lambda + |p|\sqrt{\lambda} + |p|^2).$$

But, as explained in §3(c), $-\delta_x(v) \lrcorner F_{\tilde{A}}$ is the tangent vector at $\tilde{A}$ to the flow $f_{-t}^*(\cdot)$ on the connections generated by the flow f_t of the vector field $\delta_x(v)$ on X. Now, if $v = (\phi, u, o)$ then

$$f_{-t}^*(\tilde{A}(\lambda, \rho, o, p)) \cong \tilde{A}(e^\phi \lambda, e^u \cdot \rho, o, p)$$

since the rotation $e^{-\phi}$ of $TX_x \cup \{\infty\} \cong S^4$ lifts to the basic instanton bundle Λ^2_-, preserving the connection and acting on the fiber of infinity. Λ^2_+, by $\operatorname{ad}(e^\phi)$ (cf. §3(c) for the signs). If the translation vector $\lambda\xi$ is not zero a similar equation holds with a cut-off error of L^2-norm $O(\lambda^{9/4} \cdot |\xi|^{3/4})$. Combining (4.6) and (4.7) and supposing $|p| < \lambda$ we get

$$\left\| i_{A^\infty}(v) - \frac{\partial A^\infty}{\partial n}(v, o) \right\|_{L^2} \leqslant \text{const}\,\lambda^{3/2}(|\phi| + |u| + |\xi|).$$

But P is defined by projecting to $U_n = \operatorname{Im}[\partial A^\infty / \partial n]$ so, if $|p| < \lambda$,

$$\| (P - [\partial A^\infty / \partial n])(\phi, u, \lambda\xi, q) \|_{L^2} \leqslant \text{const}(\lambda^{3/2}(|\phi| + |u| + |\xi|) + \lambda |q|).$$

Whereas

$$\| P(\phi, u, \lambda\xi, q) \|_{L^2} \geqslant \text{const}[\lambda(|\phi| + |u| + |\xi|) + |q|]$$

(cf. (3.17)). Hence $\alpha^{-1} \circ P$ is close to the identity when λ is small, and Proposition (4.5) is proved. Clearly there are similar statements for the addition of many instantons and for the perturbed equations of §2.

(4.8) Examples. (i) $b^1(X) = b^2_+(X) = 0$, *A a flat, reducible connection of type* (i) *or* (ii). Then $\Gamma_A \cong SU(2)$ and the model (4.4) is

$$\mathfrak{su}(2) \xrightarrow{r_n} V_x \to 0.$$

There is an oriented basis

$$V_x = \langle 1, e_1, e_2, e_3; \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3 i \rangle,$$

where ε_i correspond to translations and e_i to a standard basis of Λ^2_+. The map r_n takes a standard basis (f_1, f_2, f_3) of $\mathfrak{su}(2)$ to $(-1e_1, -e_2, -e_3)$. In the model of the end of the moduli space as a collar on X the vector "1" in TN corresponds to an inward pointing normal. Hence, using the obvious homology orientation of X, the standard orientation on this piece of M_1 is

$$(\text{inward pointing normal}) \wedge (\text{standard orientation of } X).$$

(ii) $b^1(X) = b^2_+(X) = 0$, *A a flat, reducible connection of type* (iii) *with* $H^1_A = 0$. Γ_A is a copy of S^1 and the model is

$$(H^0_A \cong \mathbb{R}) \to V_x \to (H^2_A \cong \mathbb{C}).$$

Deform the map ϕ to its leading term, say (cf. [6, §V]). If α, $i\alpha$ is a real basis for H^2_A then at a point x we can choose a frame (f_1, f_2, f_3) for $\mathfrak{g}_E$ so that $\alpha_x = \theta f_2$, $i\alpha_x = \theta f_3$ where $\theta \in \Lambda^2_{+,x}$. Then

$$R(\lambda, \rho) = \lambda^2([\rho(\theta)\cdot f_2]\alpha + [\rho(\theta)\cdot f_3] i\alpha).$$

If θ is not 0 then the points lying over x where R vanishes correspond to maps ρ taking θ to a multiple $l \cdot f_1$. There are two components, distinguished by the sign of l. Let $\theta = le_1$ where (e_1, e_2, e_3) is a standard frame for a $\Lambda^2_{+,x}$. Then $\rho(e_i) = f_i$ defines a point in $R^{-1}(0)$. The linearized model there is given by

$$\begin{aligned} r(f_1) &= -e_1, \\ \delta R(e_2) &= \lambda^2([e_2, le_1]\cdot e_3)\alpha_3 = -\lambda^2 l\alpha, \\ \delta R(e_3) &= \lambda^2([e_3, le_1]\cdot \alpha_2)\alpha_2 = \lambda^2 l(i\alpha). \end{aligned}$$

Away from the zero set of α the end of the moduli space consists of two sheets, each a collar on the 4-manifold and the standard orientation of the moduli space is

$$(\text{inward pointing normal}) \wedge (\text{standard orientation of } X)$$

on each sheet.

Deforming this picture to the perturbed equations of §2 we see that the homology contributions of all the boundary components are of the same sign, completing the proof of Theorem 1.

(iii) $b_i(X) = 0$, $b_2^+(X) = 1$; *A a flat reducible connection of type* (i) *or* (ii) (cf. [5]). Pick a generator ω for $H^2_+(X)$ and give X the homology orientation $-1 \wedge \omega \in \det(H^0 \oplus H^2_+)$. The complex (4.4) is

$$(H^0_A \cong \mathbb{R}^3) \xrightarrow[r]{} V_x \xrightarrow[\delta\phi]{} H^2_A \cong \mathbb{R}^3 \cdot \omega$$

and $\Gamma_A \cong \mathrm{SU}(2)$. First we can divide N by SU(2) to obtain a reduced model:

$$(o, \varepsilon) \times X \xrightarrow[(\phi\cdot\omega)]{} \Lambda^2_+.$$

The 5-manifold $(o, \varepsilon) \times X$ should be oriented by

$$(\text{inward pointing normal}) \wedge (\text{standard orientation of } X).$$

Suppose x is a point where the harmonic form ω vanishes transversally. Then we can choose local oriented coordinates x_0, x_1, x_2, x_3 and an oriented frame e_1, e_2, e_3 for Λ^2_+ so that $\omega = \Sigma_{i=1}^3 x_i e_i$. Then the zero set of $\phi \cdot \omega$ is approximated by that of ω. The standard orientation of the moduli space is that corresponding to $\partial/\partial x_0 \wedge n$, where n is the normal pointing into the moduli space.

(c) The technique of Fintushel and Stern. R. Fintushel and R. Stern prove Theorem 1 for manifolds whose intersection form represents -2 or -3 and whose first homology has no 2-torsion [9]. Their argument uses mod 2 homology and cohomology. Using the oriented moduli spaces we can extend their argument to remove the hypothesis on H_1.

Suppose L is a line bundle over a negative definite manifold X with $c_1(L)^2 = -2$ or -3, and suppose this is the largest nonzero number represented by the form. We assume $H_1(X; \mathbb{R})$ is zero as in §2. Then Fintushel and Stern show that the moduli space M of ASD connections on the U(2) bundle $E = \mathbb{C} \oplus L$ is a compact space of dimension $-2c_1(L)^2 - 3 = 1$ or 3. The reducible connections present are in 1-1 correspondence with splittings $L_1^{-1}L \oplus L_1$, where $c_1(L_1) = c_1(L)$ mod torsion. (Here we are working with U(2) bundles, so our spaces are finite coverings of Fintushel and Stern's moduli spaces of SO(3) connections.) In the case when $c_1(L)^2 = -2$ the moduli space is, generically, a 1-manifold with $|H_1(X; \mathbb{Z})|$ boundary points. By Corollary (3.22) and Proposition (3.25) the orientations of the boundary points agree so we get a contradiction to the existence of such a 4-manifold X in this case. When $c_1(L)^2 = -3$ Fintushel and Stein show that a (truncated) moduli space would be an oriented 3-manifold with boundary $|H_1(X; \mathbb{Z})|$ copies of S^2. Again we can suppose that the orientations of the boundary are all equal to those defined by the $o(L_1^2, L_1^{-1}L, \alpha_X)$ at the reductions. Define a map

$$\mu : H_2(X; \mathbb{Z}) \to H^2(M \setminus \text{reductions}, \mathbb{Z}),$$

$$\mu([\Sigma]) = -c_1\left(\det(\operatorname{ind} \partial_{\Sigma,\Sigma})^2 \otimes \det(\operatorname{ind} \partial_{\Sigma, E \otimes L})^{-1}\right)$$

(cf. [6, §II]). Then, as in [6, Lemma (2.27)], $\mu(\Sigma)$ restricts to $-2(c_1(L), \Sigma)$ times the generator on each oriented boundary component and, choosing Σ so that $(c_1(L), \Sigma) \neq 0$, we again obtain a contradiction.

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