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VOLUME, DIAMETER AND THE FIRST EIGENVALUE OF LOCALLY SYMMETRIC SPACES OF RANK ONE

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1. Introduction

Let X be a Riemannian symmetric space of noncompact type and $\mathbb{R}$-rank 1. Then X is a hyperbolic space $H^n_{\mathbb{K}}$, where $\mathbb{K}$ is either $\mathbb{R}$, $\mathbb{C}$, the quaternions $\mathbb{H}$ or the Cayley numbers $\mathbb{O}$ (in the last case $n = 2$). These spaces carry canonical Riemannian metrics with sectional curvature $K \equiv -1$ for $H^n_{\mathbb{R}}$ and $-1 \leqslant K \leqslant -1/4$ in the other cases.

We consider compact quotients $V = \Gamma \backslash X$ of X by a discrete, freely operating group of isometries. We derive relations between the volume $\operatorname{vol}(V)$, the diameter $\operatorname{diam}(V)$, and the first eigenvalue λ_1 of the Laplacian on V.

Theorem 1. *Let X be a symmetric space of rank 1 with compact quotient V.*

(1) *If $X = H^n_{\mathbb{R}}$ for $n \geqslant 4$ or $X = H^n_{\mathbb{C}}$, then*

$$\operatorname{diam}(V) \leqslant a_n \operatorname{vol}(V).$$

(2) *If $X = H^n_{\mathbb{H}}$ or $H^2_{\mathbb{O}}$, then*

$$\operatorname{diam}(V) \leqslant b_n + \frac{13}{2} \ln \operatorname{vol}(V),$$

where the constants a_n, b_n depend only on n.

Remarks. (i) Gromov [4] proved (1) under the weaker hypothesis that X satisfies the curvature condition $-1 \leqslant K < 0$. In this case, he had to assume $\dim X \geqslant 8$. For $4 \leqslant \dim X \leqslant 7$ he proved the inequality $\operatorname{diam} V \leqslant a_n^*(\operatorname{vol} V)^3$. A modification of Gromov's proof shows that the linear estimate remains true in dimensions 4–7 for locally symmetric spaces.

(ii) Inequality (1) does not hold for compact quotients of $H^2_{\mathbb{R}}$ and $H^3_{\mathbb{R}}$. It is easy to construct surfaces V_i with $K \equiv -1$ such that $\operatorname{vol} V_i \leqslant$ constant and $\operatorname{diam} V_i \to \infty$. Jørgenson and Thurston constructed 3-dimensional examples with the same property [14].

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(iii) The linear estimate (1) is optimal for $H^n_{\mathbb{R}}$, $n \geqslant 4$. For every $n \in \mathbb{N}$ there exists a sequence V_i of compact quotients of $H^n_{\mathbb{R}}$ such that the ratio $\operatorname{diam} V_i/\operatorname{vol} V_i$ is uniformly bounded from below: Take a compact quotient V of $H_{\mathbb{R}}$ with infinite homology group $H_1(V)$ (see [9]), and a sequence of finite cyclic coverings V_i of V (for details, also compare the proof of the Corollary below).

(iv) The logarithmic estimate (2) and the argument of (iii) implies that the group $H_1(V)$ is finite for compact quotients of $H^n_{\mathbb{H}}$ and $H^2_{\mathbb{O}}$ (one can even say more, see the Corollary). This is known and a consequence of the fact that the isometry groups $\mathrm{Sp}(n,1)$ of $H^n_{\mathbb{H}}$ and $\mathbb{F}_{4(-20)}$ of $H^2_{\mathbb{O}}$ and every lattice subgroup satisfy the Kazhdan property (T) ([7], [3]). We also use this property in our argument. In fact we use the explicit determination of the irreducible unitary class one representations of $\mathrm{Sp}(n,1)$ and $\mathbb{F}_{4(-20)}$. From this description it follows that the identity representation of these groups is isolated in the set of class one representations, hence in the unitary dual of these groups. (Compare the Lemma in §2.)

By remark (iii) there is no logarithmic estimate for the diameter of compact quotients of $H^n_{\mathbb{R}}$. If this estimate fails (and $n \geqslant 4$), then the first eigenvalue of the Laplacian has to tend to zero. This is made precise by the following result.

Theorem 2. *Let $X = H^n_{\mathbb{K}}$ be a symmetric space of rank 1 other than $H^2_{\mathbb{R}}$, $H^3_{\mathbb{R}}$ with compact quotient V. Then the first eigenvalue λ_1 of the Laplacian on V is estimated by*

$$\lambda_1 \leqslant \frac{\alpha_n + \beta_n \ln \operatorname{vol}(V)}{\operatorname{diam}(V)}$$

with constants α_n, β_n depending only on n.

Remark. (v) This inequality is not true for the real hyperbolic 2- and 3-spaces. By a result of Schoen, Wolpert and Yau [12] the first eigenvalue λ_1 of a surface F of genus $g \geqslant 2$ with $K \equiv -1$ is estimated by $a_g l(F) \leqslant \lambda_1(F)$, where a_g is a positive constant depending only on g and $l(F)$ is the length of a smallest chain of closed geodesics separating F into two pieces. It is easy to construct a family F_i of Riemann surfaces of genus 2 such that $\operatorname{diam} F_i \to \infty$, but $l(F_i) \geqslant$ constant: take metrics with only one short closed geodesic not separating the surface.

By a theorem of Schoen [11], λ_1 of a compact hyperbolic 3-manifold V with $\operatorname{vol}(V) \leqslant v$ is estimated by $\lambda_1(V) \geqslant c_v$, where c_v depends only on v. Thus the examples of Jørgenson and Thurston show that the inequality does not hold in dimension 3.

The logarithmic estimate of Theorem 1(2) is not homogeneous with respect to finite cyclic coverings of a given manifold V, since the diameter and the volume both grow like a power of the degree of the covering. This can be used to obtain an explicit bound on the order of the first homology group $H_1(V) = \Gamma/[\Gamma, \Gamma]$:

Corollary. *Let V be a compact quotient of $H_{\mathbb{H}}^n$ or $H_{\mathbb{O}}^2$ with* $\mathrm{vol}(V) = v$. *Then the order of the homology group $H_1(V)$ is bounded by $|H_1(V)| \leqslant (v+1)^{(v+1)^{c_n}}$.*

Outline of the proof. There are two ingredients of the argument. The first one is a bound of the distance of two subsets A, B in V by λ_1, $\mathrm{vol}(A)$, $\mathrm{vol}(B)$, and $\mathrm{vol}(V)$ (§2). This gives Theorem 1(2) and Theorem 2 in the case that the injectivity radius is everywhere big. We have to investigate separately the pieces of the manifold, where the injectivity radius is small (§3).

We note that Gromov and Milman [5] obtained an estimate of the distance between two sets A and B in a general compact Riemannian manifold M in terms of λ_1, $\mathrm{vol}(A)$, $\mathrm{vol}(B)$, and $\mathrm{vol}(M)$, but in our special case we need a better estimate.

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2. An isoperimetric inequality

Let $V = \Gamma \backslash X$ be a compact locally symmetric space of rank one, i.e. $X = G/K$ is a symmetric space of noncompact type endowed with the Riemannian metric coming from the Killing form of the Lie algebra of G. For A and B measurable subsets of V, we bound their distance $d(A, B)$ in terms of the first eigenvalue of the Laplacian of V and the volumes of A, B, and V. We remark that this inequality also gives a lower bound for the growth of the volume of an ε-neighborhood of a set $A \subset V$.

Proposition 1. *Let A and B be measurable subsets of V, λ_1 the first nonzero eigenvalue of the Laplacian of V, $d(A, B)$ the distance between A and B, a, b, v the volume of A, B, V, and ρ the half sum of the positive roots. Then*

(1) $G = \mathrm{SO}_0(n,1)$ *or* $\mathrm{SU}(n,1)$:

$$\min\left(\rho^2, \lambda_1\right) d(A, B) \leqslant c(n) + \frac{3\rho^2}{3\rho - 1} \ln\left[\left(\frac{v}{a} - 1\right)\left(\frac{v}{b} - 1\right)\right],$$

where $c(n) > 0$ depends only on n.

(2) $G = \mathrm{Sp}(n,1)$:

$$\cosh d(A, B) \leqslant \left[\left(\frac{v}{a} - 1\right)\left(\frac{v}{b} - 1\right)\right]^{1/4}.$$

(3) $G = \mathbb{F}_{4(-20)}$:

$$\cosh d(A, B) \leqslant \left[1764\left(\frac{v}{a} - 1\right)\left(\frac{v}{b} - 1\right)\right]^{1/12}.$$

We first remark that the distance $d(A, B)$ can be written as:

$$d(A, B) = \inf\left\{t > 0 \middle| \int_{T_1 V} \chi_A \circ T_t(x) \chi_B(x)\, d\mu(x) > 0\right\},$$

where $T_1 V$ is the unit tangent bundle, T_t the geodesic flow operating on $T_1 V$, and χ_A, χ_B the characteristic functions of A, B viewed as functions on $T_1 V$. Let $R, S \in L^2(V)$: we estimate the difference

$$\left|\int_{T_1 V} R \circ T_t(x) \bar{S}(x)\, d\mu(x) - \frac{\int_v R \int_v \bar{S}}{v}\right|$$

for t big in terms of spherical functions of X. The properties at infinity of these functions will give the result.

A. We describe the action of the geodesic flow T_t on the unit tangent bundle $T_1 V$: Let $x_0 \in X$, K be the stabilizer of x_0 in G, $\mathfrak{k}$ and $\mathfrak{g}$ the Lie algebras of K and G, and let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ for the Killing form. Thus $\mathfrak{p}$ can be identified with the tangent space of X at x_0. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Then $\mathfrak{a} = \mathbb{R} H$, where $H \in \mathfrak{p}$ is a unit vector. If M is the centralizer of A in K, then $M = \{k \in K \mid \mathrm{Ad}(k)(H) = H\}$. One verifies that the map

$$\begin{aligned} \phi : G/M &\to T_1 X \\ \dot{g} &\to \left(g x_0, T_{x_0} L_g(H)\right) \end{aligned}$$

is a C^∞-isomorphism where $T_{x_0} L_g$ is the derivative at x_0 of the isometry L_g of X defined by g. Furthermore, $\phi(\dot{g} \exp(tH)) = T_t \phi(\dot{g})$, where exp is the exponential map of G.

For a discrete subgroup Γ of G operating without fixed points on X, we have an isomorphism $\phi_\Gamma : \Gamma \backslash G/M \to T_1 V$.

B. We consider on G, K, M Haar-measures such that the corresponding measure on G/K coincides with the Riemannian measure on X. There exists a unique measure μ_1 on $\Gamma \backslash G/M$ such that

$$\int_{\Gamma \backslash G/M} d\mu_1(\dot{g}) \sum_{\gamma \in \Gamma} \int_M R(\gamma g m)\, dm = \int_G R(g)\, dg,$$

where R is a continuous integrable function on G. This measure induces via ϕ_Γ a measure μ on T_1V. Let $R, S \in L^2(V)$, $R_1 = R \circ \phi_\Gamma$, $S_1 = S \circ \phi_\Gamma$, $a_t := \exp(tH)$. Then

$$\int_{T_1V} R \circ T_t(x) \bar{S}(x)\, d\mu(x) = \int_{\Gamma\backslash G/M} R_1(\dot{g}a_t)\overline{S_1(\dot{g})}\, d\mu_1(\dot{g})$$

$$= \frac{1}{\operatorname{vol}(M)} \int_{\Gamma\backslash G} R_1(\dot{g}a_t)\overline{S_1(\dot{g})}\, d\dot{g}.$$

Let $\{\phi_n\}_{n\geqslant 0}$ be a complete orthonormal system of eigenfunctions of the Laplacian of V and $c_n = \langle R, \phi_n\rangle$ and $d_n = \langle S, \phi_n\rangle$ the Fourier coefficients of R and S. Then one has

$$\int_{\Gamma\backslash G} R_1(\dot{g}a_t)\overline{S_1(\dot{g})}\, dg = \sum_{n,m\geqslant 0} c_n \bar{d}_m \int_{\Gamma\backslash G} \phi_n(\dot{g}a_t)\overline{\phi_m(\dot{g})}\, d\dot{g},$$

where ϕ_n are viewed as right K-invariant functions on $\Gamma\setminus G$. We remark that $\Psi_{n,m}(h) = \int_{\Gamma\backslash G} \phi_n(\dot{g}h)\overline{\phi_m(\dot{g})}\,d\dot{g}$ is a bi-K-invariant function on G, thus we can view $\Psi_{n,m}$ as functions on X only depending on the distance to x_0. If Δ is the Laplacian of the symmetric space, then

$$\Delta\Psi_{n,m} = \lambda_n \Psi_{n,m},$$

$$\Psi_{n,m}(e) = \operatorname{vol}(K)\int_V \phi_n(x)\overline{\phi_m(x)}\, dx = \operatorname{vol}(K)\delta_{n,m}.$$

However one knows that for all $\lambda \in \mathbb{C}$ there exists a unique C^∞-function F_λ on X only depending on the distance to x_0 such that:

(1) $F_\lambda(x_0) = 1$, (2) $\Delta F_\lambda = -\lambda F_\lambda$ [8, §3.2]. It follows that $\Psi_{n,m} = \delta_{n,m}\operatorname{vol}(K) F_{-\lambda_n}$. The equality becomes

$$\frac{\operatorname{vol}(M)}{\operatorname{vol}(K)} \int_{T_1V} R \circ T_t(x)\overline{S(x)}\, d\mu(x) = \sum_{n=0}^{\infty} c_n \bar{d}_n F_{-\lambda_n}(a_t).$$

From the integral representation of F_λ as integral over K [8, §4.1],

$$F_\lambda(g) = \int_K e^{(s-\rho)H(g^{-1}k)}\, dk,$$

where $-\lambda + \rho^2 = s^2$ and $g = k\exp(H(g)H)n$ in $G = KAN$ is the Iwasawa decomposition of g, we derive the following properties:

(1) $|F_\lambda(x)| \leqslant F_{\rho^2}(x)$ if $\lambda \geqslant \rho^2$.

(2) $0 \leqslant F_{\lambda_2}(x) \leqslant F_{\lambda_1}(x)$ if $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \rho^2$.

Using the above equality and the properties of F_λ we see that

$$\left|\frac{\operatorname{vol}(M)}{\operatorname{vol}(K)}\int_{T_1V} RT_t(x)\overline{S(x)}\,d\mu(x) - \frac{1}{v}\left[\int_V R\int_V S\right]\right|$$
$$\leqslant \max\left(F_{\rho^2}(a_t), F_{\lambda_1}(a_t)\right)\left\|R - \frac{1}{v}\int_V R\right\|_{L^2(V)}\left\|S - \frac{1}{v}\int_V S\right\|_{L^2(V)}.$$

The rest of the proof consists of estimating $F_{-\lambda}$.

C. We recall some properties of spherical functions: There exist representations ρ_s of G in a Hilbert space $\mathscr{H}_s$ such that

$$F_\lambda(g) = \langle \rho_s(g)v_s, v_s\rangle,$$

where $-\lambda + \rho^2 = s^2$ and v_s is a $\rho_s(K)$-invariant vector in $\mathscr{H}_s$ of length 1. One shows that ρ_s is unitarizable, i.e. F_λ is positive definite, exactly in the following cases (see [13, Corollaire, p. 566]):

(1) $G = \mathrm{SO}_0(n,1)$ and $\mathrm{SU}(n,1)$: $\mathbb{R}e(s) = 0$ or $-\rho \leqslant s \leqslant \rho$.

(2) $G = \mathrm{Sp}(n,1)$: $\mathbb{R}e(s) = 0$ or $2-\rho \leqslant s \leqslant \rho - 2$, $s = \pm\rho$.

(3) $G = \mathbb{F}_{4(-20)}$: $\mathbb{R}e(s) = 0$ or $-5 \leqslant s \leqslant 5$, $s = \pm 11$.

The spherical functions can be interpreted as hypergeometrical functions: in the case of $\mathrm{SO}_0(n,1)$, $\mathrm{SU}(n, \text{therefore}1)$, $\mathrm{Sp}(n,1)$

$$F_\lambda(a_t) = (\operatorname{ch} t)^{s-\rho}{}_2F_1\left(\frac{dn-\rho-s}{2}, \frac{\rho-s}{2}, \frac{dn}{2}, \operatorname{th}^2(t)\right), \quad \text{where } d = 1,2,4.$$

In the exceptional case

$$F_\lambda(a_t) = (\operatorname{ch} t)^{-(11+s)}{}_2F_1\left(\frac{11+s}{2}, \frac{5+s}{2}, 8; \operatorname{th}^2(t)\right)$$

[13, p. 565]. (We use ch, th instead of cosh, tanh.)

Lemma. (1) Let $G = \mathrm{SO}_0(n,1)$, $\mathrm{SU}(n,1)$. *Then*

$$F_\lambda(a_t) \leqslant c(n)(1+t)^{1-s/\rho}(\operatorname{ch} t)^{s-\rho},$$

where $0 \leqslant \lambda \leqslant \rho^2$, $-\lambda + \rho^2 = s^2$, $0 \leqslant s \leqslant \rho$.

(2) $G = \mathrm{Sp}(n,1)$: $F_{8n+8}(a_t) = (\operatorname{ch} t)^{-2}$.

(3) $F_{96}(a_t) \leqslant 42(\operatorname{ch} t)^{-6}$.

Proof. (1) We treat only the case $\rho \geqslant 2$, the other cases require the same techniques with minor modifications. We consider the representation of ${}_2F_1$ as Euler integral:

$$F_{\rho^2}(a_t) = (\operatorname{ch} t)^{-\rho}c(n)\int_0^1 u^{\rho/2-1}(1-u)^{(dn-\rho)/2-1}(1-u\operatorname{th}^2 t)^{(\rho-dn)/2}\,du.$$

We make the change of variables $u\operatorname{th}^2 t = v$. Since we have $\rho \geqslant 2$ and in particular $(dn-\rho)/2 - 1 \geqslant 0$, it follows that

$$\left(1 - \frac{v}{\operatorname{th}^2 t}\right)^{(dn-\rho)/2-1} \leqslant (1-v)^{(dn-\rho)/2-1}$$

for $0 \leqslant v \leqslant \mathrm{th}^2 t$. Therefore

$$(\mathrm{ch}\, t)^{\rho} F_{\rho^2}(a_t) \leqslant c(n)(\mathrm{th}\, t)^{-\rho} \int_0^{\mathrm{th}^2 t} u^{\rho/2-1}(1-u)^{-1}\, du$$

$$\leqslant c(n) \frac{2 \ln(\mathrm{ch}\, t)}{\mathrm{th}^2 t} < c(n)(1 + \ln(\mathrm{ch}\, t)) < c(n)(1+t).$$

We used also $\rho/2 - 1 \geqslant 0$ in the second inequality. The function $s \to F_\lambda(a_t)$ is holomorphic in $s \in \mathbb{C}$ and one verifies:

(a) $\mathbb{R}e(s) = 0 : |F_\lambda(a_t)| \leqslant F_{\rho^2}(a_t) < c(n)(1+t)(\mathrm{ch}\, t)^{-\rho}$.

(b) $\mathbb{R}e(s) \leqslant \rho : |F_\lambda(a_t)| \leqslant F_0(a_t) = 1$.

(c) $\mathbb{R}e(s) = \rho : |F_\lambda(a_t)| \leqslant 1$.

The theorem of Phragmen-Lindelöf completes the proof.

(2) is clear from the integral representation for ${}_2F_1$.

(3)

$$F_{96}(a_t) = (\mathrm{ch}\, t)^{-6} 105 \int_0^1 u^4 (1-u)^2 (1 - u\, \mathrm{th}^2 t)^{-8}\, du$$

$$= 105 (\mathrm{ch}\, t)^{-6} (\mathrm{th}\, t)^{-10} \int_0^{\mathrm{th}^2 t} u^4 \left(1 - \frac{u}{\mathrm{th}^2 t}\right)^2 (1-u)^{-8}\, du.$$

We have to estimate $x^{-5} \int_0^x t^4 (1 - t/2)^2 (1-t)^{-8}\, dt$ for $x \to 1$: $(1 - t/x)^2 \leqslant (1-t)^2$ for $0 \leqslant t \leqslant x$, hence, the integral is bounded by

$$x^{-5} \int_0^x t^4 (1-t)^{-6}\, dt \leqslant \frac{1}{5x}\left[(1-x)^{-5} - 1\right].$$

On the other hand we have

$$x^{-5} \int_0^x t^4 (1-t)^{-6}\, dt \leqslant \frac{(1-x)^{-6}}{5}$$

so that the integral is estimated by

$$\tfrac{1}{5}(1-x)^{-5} \min\left((1-x)^{-1}, x^{-1}\right) \leqslant \tfrac{2}{5}(1-x)^{-5}.$$

This proves (3).

D. We are now able to prove the proposition:

(1) Let $G = \mathrm{SO}_0(n,1)$, $\mathrm{SU}(n,1)$, and let $R = \chi_A$, $S = \chi_B$. Then we have

$$\frac{\mathrm{vol}(M)}{\mathrm{vol}(K)} \int_{T_1 V} \chi_A \circ T_t(x) \chi_B(x)\, d\mu(x)$$

$$\geqslant \frac{ab}{v} - \max\left(F_{\rho^2}(a_t), F_{\lambda_1}(a_t)\right) \sqrt{ab\left(1 - \tfrac{a}{v}\right)\left(1 - \tfrac{b}{v}\right)},$$

where we can suppose $\lambda_1 \leqslant \rho^2$. We use the estimate of $F_{\lambda_1}(a_t)$ and the remark that $1 + t \leqslant 3e^{t/3}$ for $t \geqslant 0$. Hence $F_{\lambda_1}(a_t) \leqslant 3c(n)e^{t(s-\rho)(1-1/3\rho)}$, where $-\lambda_1 + \rho^2 = s^2$ and $0 \leqslant s \leqslant \rho$. This shows that if

$$\frac{ab}{v} - 3c(n)e^{t(s-\rho)(1-1/3\rho)}\sqrt{ab\left(1-\tfrac{a}{v}\right)\left(1-\tfrac{b}{v}\right)} > 0,$$

then $t \geqslant d(A, B)$, from which (1) follows.

(2), (3) $G = \mathrm{Sp}(n,1)$ and $\mathbb{F}_{4(-20)}$: one observes that the spherical functions F_λ with λ an eigenvalue of the Laplacian of V are clearly of positive type, which shows that $\lambda_1 \geqslant 8n + 8$ in the case $\mathrm{Sp}(n,1)$ and $\lambda_1 \geqslant 96$ in the case $\mathbb{F}_{4(-20)}$. The proof is now the same as in (1).

3. Volume of tubes

Let $X = H^n_{\mathbb{K}}$ with $\dim X = dn$, $d = \dim_{\mathbb{R}} \mathbb{K}$. We study the region of a compact quotient $V = \Gamma \setminus X$ where the injectivity radius is small. Based on results of Margulis and Heintze there is a nice description of the connected components of the set $\{p \in V \mid \mathrm{Inj\,Rad}(p) \leqslant \mu\}$, where $\mu = \mu(dn)$ is a constant depending only on the dimension of V (see e.g., [1, §10]): let U be a connected component of this set and let W be a component of $\pi^{-1}(U)$, where $\pi\colon X \to V$ is the canonical projection. Then W is precisely invariant under the action of Γ, i.e. either $\gamma W \cap W = \varnothing$ or $\gamma W = W$ for all $\gamma \in \Gamma$. The group $\Delta := \{\gamma \in \Gamma \mid \gamma W = W\}$ is an infinite cyclic group of hyperbolic isometries with common axis c. Thus $c\colon \mathbb{R} \to W$ is a geodesic, such that every $\gamma \in \Delta$ translates c, i.e. $\gamma c(t) = c(t + \omega)$. It is now possible to identify U with $\Delta \setminus W$.

For $\varepsilon > 0$, let $T_\varepsilon U$ be the ε-neighborhood of U. We can assume that for $0 < \varepsilon \leqslant \mu/4$ the distance set $T_\varepsilon W \subset X$ is also precisely invariant with $\Delta = \{\gamma \in \Gamma \mid \gamma T_\varepsilon W = T_\varepsilon W\}$, thus $T_\varepsilon U = \Delta \setminus T_\varepsilon W$. (This property of $T_\varepsilon U$ follows with a slight modification of the description of U as in [1] by choosing our Margulis constant μ smaller than the "usual" constant.)

For $\gamma \in \Delta$ we consider the displacement function $d_\gamma\colon x \to d(x, \gamma x)$. Let $d_\Delta(x) := \min_{\gamma \in \Delta - \mathrm{id}} d_\gamma(x)$. We then have for $x \in W$, that $\mathrm{Inj\,Rad}(\pi(x)) = \frac{1}{2} d_\Delta(x)$. Note that $W = \{x \in X \mid d_\Delta(x) \leqslant 2\mu\}$. For $p \in U$ let $r(p) := d(x, \partial U)$ be the distance of p to the boundary of U. Then $r(p) = d(x, \partial W)$, where $x \in W$ is a point with $\pi(x) = p$. Let

$$r(U) := \max_{p \in U} d(p, \partial U).$$

Proposition 2. *Let $X = H^n_{\mathbb{K}}$ be a symmetric space of rank 1 other than $H^2_{\mathbb{R}}$, $H^3_{\mathbb{R}}$ of dimension dn. Let V be a compact quotient of X and let U be a component of $\{p \in V \mid \mathrm{Inj\,Rad}(p) \leqslant \mu(dn)\}$. Then*

$$\mathrm{vol}\big(T_{\mu/8}(U)\big) \geqslant d_n \sinh\left(\tfrac{1}{3} r(U)\right).$$

Proof. We consider $U = \Delta \setminus W$ as above, where W is a tubular neighborhood of a geodesic c in X and Δ is an infinite cyclic group of hyperbolic isometries with axis c. We claim:

(1) The geodesic c is contained in a complete totally geodesic Δ-invariant submanifold $\overline{M}$ with $1 < \dim \overline{M} < dn$.

We first prove (1) in the case that X is the real hyperbolic space $H^n_{\mathbb{R}}$, $n \geqslant 4$. Let α be a generator of the cyclic group Δ and $x \in c(\mathbb{R})$. Let $N_c(x)$ be the $(n-1)$-dimensional space of all tangent vectors at x normal to c. The isometry α together with the parallel translation along c defines an orthogonal map of $N_c(x)$. Since $\dim N_c(x) \geqslant 3$, this map has a nontrivial proper invariant subspace $A \subset N_c(x)$. We consider the subspace $A \times \mathbb{R} \subset N_c(x) \times \mathbb{R} = T_x X$. By construction, the image $\overline{M}$ of $A \times \mathbb{R}$ under the exponential map satisfies the required properties.

If $X = H^n_{\mathbb{K}}$ for $\mathbb{K} = \mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$, then c is contained automatically in a totally geodesic subspace $\overline{M}$ of dimension 2, 4, or 8 coming from the $\mathbb{K}$-structure on X. This subspace is clearly Δ-invariant.

Let $\pi_{\overline{M}}$: $X \to \overline{M}$ be the orthogonal projection. We consider $\overline{M}' := \{x \in X \mid \pi_{\overline{M}}(x) \in c(\mathbb{R})\}$, hence $\overline{M} \cap \overline{M}' = c(\mathbb{R})$. $\overline{M}'$ is Δ-invariant since $\overline{M}$ and c are, and we define the manifolds $M := \pi(W \cap \overline{M}) \subset U$ and $M' := \pi(W \cap \overline{M}') \subset U$. It is not difficult to prove that $r(p) = r(q)$ for points p and q on the closed geodesic $\pi(c)$. (Use the fact that every $\alpha \in \Delta$ commutes with a transvection σ along the geodesic c, thus $d_\alpha(\sigma c) = d_\alpha(x)$ and hence $d_\Delta(\sigma x) = d_\Delta(x)$.) Let $r(c) := r(p)$ for a point $p \in \pi(c)$. We now prove:

(2) $\operatorname{vol}(T_{\mu/8}(U)) \geqslant d_n' \operatorname{sh}(\frac{1}{2} r(c))$.

For that reason we consider sequences $p_0, \cdots, p_k$ of points in ∂U with the following properties $p_0 \in M$, $p_k \in M'$, $\mu/4 \leqslant d(p_i, p_{i+1}) \leqslant \mu/2$ for $0 \leqslant i \leqslant k-2$, and $d(p_{k-1}, p_k) \leqslant \mu/2$.

Since $\dim X \geqslant 3$, ∂U is connected and a sequence with these properties exists. We now fix a sequence with k minimal. It follows that $d(p_i, p_j) \geqslant \mu/4$ for all $i \neq j$ and $0 \leqslant i, j \leqslant k-1$. Thus the balls $B_{\mu/8}(p_i)$, $0 \leqslant i \leqslant k-1$, are disjoint and since $\operatorname{Inj Rad}(p_i) = \mu$ the volume of each ball is bigger than a constant h. Thus $\operatorname{vol}(T_{\mu/8}(U)) \geqslant h_n k$ and it remains to estimate k.

Note that p_i and p_{i+1} can be joined by a unique minimal geodesic, since $\operatorname{Inj Rad}(p_i) = \mu$. Let g: $[0, k] \to T_{\mu/4}(U)$ be the piecewise geodesic with $g(i) = p_i$ for $0 \leqslant i \leqslant k$ and let $\bar{g}$ be a lift of g to $T_{\mu/4}(W)$. Thus $\bar{g}(0) \in \partial W \cap \overline{M}$, $\bar{g}(k) \in \partial W \cap \overline{M}'$. Let $x := \pi_c(\bar{g}(k))$, where π_c is the projection onto c. By the definition of $\overline{M}$ and $\overline{M}'$ we have $\sphericalangle_x(\bar{g}(0), \bar{g}(k)) = \pi/2$. The definition of $r(c)$ implies $d(x, \bar{g}(i)) \geqslant r(c)$ for all i. Let B be the distance ball of radius $r(c)$ at the point x and let π_B: $X \to B$ be the orthogonal projection onto B. We consider the points $x_i := \pi_B(\bar{g}(i))$. Since the projection π_B is distance decreasing, $d(x_i, x_{i+1}) \leqslant \mu/2$. Let d^S be the spherical distance measured on the distance sphere B. Using the results of Heintze and Im Hof [6] we see $d^S(x_i, x_{i+1}) \leqslant 2\operatorname{sh}(\mu/4)$. Since μ is very small and $\mu(n) \to 0$, we can assume $d^S(x_i, x_{i+1}) \leqslant \mu$. Let g^*: $[0, k] \to B$ be the piecewise spherical geodesic joining the points x_i. Since the curvature is bounded by $-\frac{1}{4}$ we obtain with the usual comparison theorems:

$$\operatorname{Length}(g^*) \geqslant \tfrac{\pi}{2}\operatorname{sh}\left(\tfrac{1}{2}r(c)\right), \quad \text{thus } k \geqslant \tfrac{\pi}{\mu}\operatorname{sh}\left(\tfrac{1}{2}r(c)\right).$$

We now prove the proposition in general. If $r(c) \geqslant \frac{2}{3}r(U)$, then the result follows from (2). Thus we assume $r(c) \leqslant \frac{2}{3}r(U)$. Let p be a point with $r(p) = r(U)$. Then $d(p, \pi(c)) \geqslant \frac{1}{3}r(U)$, since $r(p) \leqslant d(p, \pi(c)) + r(c)$. Let x be a lift of p to W and $x_0 := \pi_c(x)$. Let $x' \in \partial W$ be the point where the geodesic ray h: $[0, \infty) \to X$ with $h(0) = x_0$, $h(1) = x$ meets ∂W. Let $\bar{x} = h(t_0)$ be the point on h between x and x' with $d(x, \bar{x}) = \frac{1}{3}r(U)$. Then $d(\bar{x}, c) \geqslant \frac{2}{3}r(U)$ and $d(\bar{x}, \partial W) \geqslant \frac{2}{3}r(U)$. Let $y \in \partial W$ be a point with $d(y, c) = r(c) \leqslant \frac{2}{3}r(U)$. As in the proof of (2) we consider a sequence $p_0, \cdots, p_k$ in ∂U with $p_0 = \pi(x')$, $p_k = \pi(y)$, and $\mu/4 \leqslant d(p_i, p_{i+1}) \leqslant \mu/2$ such that k is minimal. As in (2) we consider the piecewise geodesic g joining the points p_i and the lift $\bar{g}$ to $T_{\mu/4}(W)$ with $\bar{g}(0) = x'$, $\bar{g}(k) = \bar{y}$ with $\pi(\bar{y}) = \pi(y)$. Thus $d(\bar{y}, c) \leqslant \frac{2}{3}r(U)$ and $\bar{y}$ is contained in the $\frac{2}{3}r(U)$ and in particular in the $d(\bar{x}, c)$-distance tube of c. Since this last tube is convex and $\dot{h}(t_0) \in T_{\bar{x}}X$ is normal to this tube, we see that $\sphericalangle_{\bar{x}}(x', \bar{y}) \geqslant \pi/2$. Since all points $\bar{g}(i)$ have distance $\geqslant \frac{2}{3}r(U)$ from $\bar{x}$, we can estimate $k \geqslant d_n^* \operatorname{sh}(\frac{1}{2}\frac{2}{3}r(U))$ as in (2).

4. Proofs

Proof of Theorem 1. (1) (Compare Gromov's proof [4].) Let p and q be points of V such that $d(p, q) = \operatorname{diam}(V) = d$. Choose p_1, q_1 such that the injectivity radius at p_1, q_1 is bigger than μ with $d_1 := d(p, p_1)$ and $d_2 := d(q, q_1)$ minimal; let $d' := d(p_1, q_1)$. Since $\dim V \geqslant 3$ the description of the tubes in §3 implies that $W := \{p \in V \mid \operatorname{Inj Rad}(p) \geqslant \mu\}$ is connected. By covering a path from p_1 to q_1 in W with balls of radius μ we see that $v \geqslant a_1(n)d'$. If $d' \geqslant d/2$ then $v \geqslant a_n d$ with $a_n = a_1(n)/2$. If $d' < d/2$ then $\max(d_1, d_2) \geqslant d/4$ and the result follows from the volume estimate of Proposition 2 with a suitable constant a_n.

(2) Let $p, q, p_1, q_1, d_1, d_2, d$, and d' be as in the proof of (1). If $d' \geqslant d/13$, then we apply Proposition 1 to $A := B(p_1, \mu)$, $B := B(q_1, \mu)$ and obtain

$$d \leqslant c_1(n) + \frac{13}{2}\ln(v).$$

If $d' \leqslant d/13$, then $\max(d_1, d_2) \geqslant 6/13d$. By Proposition 2, $v \geqslant \mathrm{vol}(T_{\mu/8}(U)) \geqslant c_2(n)\mathrm{sh}(2d/13)$; from this we conclude again

$$d \leqslant c_1(n) = \frac{13}{2}\ln(v).$$

A similar computation gives the result for $\mathbb{F}_{4(-20)}$.

Proof of Theorem 2. We make a preliminary remark: from Cheng's inequality [2] $\lambda_1 \leqslant \rho^2 + c_3(n)/\mathrm{diam}(V)^2$ and the fact that $\mathrm{diam}(V) \geqslant c_4(n) > 0$, we conclude that $\lambda_1 \leqslant c_5(n)$. In particular (using Theorem 1(2)) we have to prove Theorem 2 only for $H^n_{\mathbb{R}}$, $n \geqslant 4$, and $H^n_{\mathbb{C}}$.

Suppose first $\lambda_1 \geqslant \rho^2$; then we have by Proposition 1

$$\rho^2 d(A, B) \leqslant c(n) + \frac{3\rho^2}{3\rho - 1}\ln\left[\left(\frac{v}{a} - 1\right)\left(\frac{v}{b} - 1\right)\right],$$

hence

$$\lambda_1 \leqslant c_5(n) \leqslant c_5(n)\frac{c(n) + (3\rho^2/(3\rho - 1))\ln[(\frac{v}{a} - 1)(\frac{v}{b} - 1)]}{\rho^2 d(A, B)}.$$

We therefore always have

$$\lambda_1 \leqslant \frac{c_6(n) + c_7(n)\ln[(\frac{v}{a} - 1)(\frac{v}{b} - 1)]}{d(A, B)}.$$

Now let p, q, p_1, d' be as in the proof of Theorem 1. If $d' \geqslant d/2$, then we apply the above inequality to $A := B(p_1, \mu)$, $B := B(q_1, \mu)$ and obtain the desired result. If $d' \leqslant d/2$, then $\max(d_1, d_2) \geqslant d/8$, and by Proposition 2 we have $v \geqslant c_2(n)\,\mathrm{sh}(d/24)$ so that $(c_8(n) + c_9(n)\,d\ln(v))/d \geqslant 1$. The fact that $\lambda_1 \leqslant c_5(n)$ again implies the inequality.

Proof of the Corollary. Let $\Gamma' = [\Gamma, \Gamma]$, V' be the corresponding covering of V, and $G = \Gamma/\Gamma'$ be the Galois group of $p\colon V' \to V$. Let $x_0 \in V'$ such that the injectivity radius of V at $p(x_0)$ is bigger than the Margulis constant μ, and let $S := \{\gamma \in G \mid d(x_0, \gamma x_0) \leqslant 3d\}$, where $d = \mathrm{diam}(V)$. Certainly S generates G and by a similar argument as in [10] one shows that $d(x_0, \gamma x_0) \geqslant L(\gamma)d - 1$, where $L(\gamma)$ is the length of γ with respect to the generator system S, and this holds for $d(x_0, \gamma x_0) \geqslant 3d$. If $d(x_0, \gamma x_0) < 3d$, then $L(\gamma) = 1$ and by the assumption on x_0, $d(x_0, \gamma x_0) \geqslant \mu$. Hence we have in all cases $d(x_0, \gamma x_0) \geqslant c(n)L(\gamma)$. This proves $\mathrm{diam}(V') \geqslant c(n)\mathrm{diam}(\mathfrak{g})$, where $\mathfrak{g}$ is the graph of G with respect to S. Certainly

$$k := |S| \leqslant \frac{\mathrm{vol}(B(x_0, 4d))}{\mathrm{vol}(V)} \leqslant \frac{c_1(n)e^{8\rho d}}{v}.$$

But $d \leqslant c_2(n) + 7\ln(v)$ (by Theorem 1), thus $|S| \leqslant c_3(n)v^{56\rho-1}$. If S^m denotes the set of elements of length at most m, then $|S^m| \leqslant (2m)^k$, since G is abelian. From this it follows that $\operatorname{diam}(\mathfrak{g}) \geqslant \frac{1}{2}|G|^{1/k}$. Combining this with the inequality $c(n)\operatorname{diam}(\mathfrak{g}) \leqslant \operatorname{diam}(V') \leqslant c_2(n) + 7\ln(|G|v)$ we obtain for $|G| \geqslant 2$:

$$\begin{aligned}\frac{c(n)}{2}|G|^{1/k} &\leqslant c_2(n) + 7\ln(|G|v) \leqslant \ln(|G|)(c_3(n) + \ln(v))\\ &\leqslant 2k|G|^{1/2k}(c_3(n) + \ln(v)).\end{aligned}$$

We now conclude

$$|G| \leqslant \left[\frac{4k(c_3(n) + \ln(v))}{c(n)}\right]^{2k}.$$

Combining this with the estimate for k we derive the result.

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