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HOMOLOGY OF CLOSED GEODESICS IN A NEGATIVELY CURVED MANIFOLD

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0. Introduction

Let M be a compact Riemannian manifold whose geodesic flow on the unit tangent bundle is of *Anosov type*. It is known that each free homotopy class of closed paths in M contains a unique closed geodesic (W. Klingenberg [9]). In this paper, we show, by using a modified Perron-Frobenius theorem, that there exist infinitely many *prime* closed geodesics in each homology class in $H_1(M, \mathbb{Z})$. More precisely, we prove

Theorem 1. *If we denote by $N(x, \alpha)$ the number of prime geodesics $\mathfrak{p}$ in M whose homology class is a given α and length $l(\mathfrak{p}) \leqslant x$, then the exponential growth rate $\lim_{x\to\infty} x^{-1} \log N(x, \alpha)$ is equal to the topological entropy h of the geodesic flow* (> 0).

The above theorem is intuitively anticipated from the fact that the fundamental group of M has exponential growth, while $H_1(M, \mathbb{Z})$ has polynomial growth. In this view, one may ask whether there exist infinitely many closed orbits with a given homology class for a general Anosov flow (X, ϕ_t). It is known that the exponential growth rate of the number of closed orbits with respect to the period is always equal to the topological entropy. But one can easily construct an Anosov flow such that every homology class contains only finitely many closed orbits.

If $H_1(M, \mathbb{Z})$ is of finite order, Theorem 1 can be proven by means of dynamical L-functions, combining the idea of proof of the Chebotarev density theorem with the fact that there is a resemblance between prime closed geodesics and prime ideals in number fields (see [13], [17], [18]). In fact, we have

$$N(x, \alpha) \sim \left(\operatorname{Card} H_1(M, \mathbb{Z})\right)^{-1} \frac{e^{hx}}{hx}$$

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as x goes to $+\infty$ ([2], [13]). If we regard $H_1(M, \mathbb{Z})$ as an "ideal class group," this is a geometric analogue of the Dirichlet theorem for arithmetic progressions.

The heart of the matter is in the case rank $H_1(M, \mathbb{Z}) > 0$. Since the classical argument in number theory cannot apply to the ideal class group of *infinite order*, we need to develop a new idea to handle this case. Indeed, if we would try to go along the same line as a proof of the case of finite order, we have to investigate the singularity around $s = h$ of the function

$$\int_{\hat{H}_1(M,\mathbb{Z})} \chi(-\alpha) \frac{L'(s,\chi)}{L(s,\chi)}\, d\chi,$$

where $d\chi$ denotes the normalized Haar measure on the character group of $H_1(M, \mathbb{Z})$, and $L(s, \chi)$ is the L-function defined by

$$L(s,\chi) = \prod_{\mathfrak{p}} (1 - \chi([\mathfrak{p}]) \exp(-sl(\mathfrak{p})))^{-1}.$$

Even in a special case (for instance, in the case of constant negatively curved surfaces), the singularity of the above function seems very complicated (a difficulty is caused by the presence of small eigenvalues of the Laplacian acting on sections of the flat line bundle associated to a character χ).

The method we will take up in this paper is rather combinatorial, and is motivated by Bowen [4] and Pollicott [14]. To explain the underlying idea, we recall that one of basic tools in the study of Anosov flows is a *Markov family*, which defines an *oriented finite graph* embedded in the base manifold as a 1-dimensional CW-complex. In view of Bowen's symbolic dynamics [4], each closed path in the graph approximates a closed orbit of the flow. The problem of counting closed orbits of Anosov flows is therefore reduced to that of closed paths in oriented graphs. But as we mentioned above, this is not enough to establish Theorem 1. A special feature of a geodesic flow ϕ_t is that it has a reversible property: $-\phi_t(v) = \phi_{-t}(-v)$. This enables us to construct a Markov family with an involution (see [1]). The presence of an involution yields an interesting property of eigenvalues of "twisted" *Perron-Frobenius matrices* associated to the graph, which plays a crucial role in counting closed paths and leads us to the conclusion.

It is known that if the curvature of M is strictly negative, then the geodesic flow is of Anosov type (see [6]). It should be noted that our assertion is not always valid for nonpositively curved Riemannian manifolds. Indeed, a homology class α of a flat torus contains a prime closed geodesic if and only if α is not a nontrivial multiple of another homology class.

We should point out that there are many compact constant negatively curved manifolds with positive first Betti number (cf. [11]). For adequate background on the subject of Anosov flows we refer to [3], [4], and [16].

To avoid confusion, we employ the following terminology throughout: For a general flow (X, ϕ_t) an *orbit cycle* $\mathfrak{p}$ with period $\tau(\mathfrak{p})$ means an oriented cycle in X defined by a map: $\mathbb{R}/\tau(\mathfrak{p})\mathbb{Z} \to X$ $(t \mapsto \phi_t(x))$, where x is a periodic point with $\phi_{\tau(\mathfrak{p})}(x) = x$, $\tau(\mathfrak{p}) > 0$. If $\tau(\mathfrak{p})$ is the minimal period of x, the orbit cycle $\mathfrak{p}$ is a closed orbit in the usual sense (we also say in this case that $\mathfrak{p}$ is *prime*). In case of geodesic flows, orbit cycles correspond to (not necessarily prime) closed geodesics, and $\tau(\mathfrak{p})$ is the length of the corresponding closed geodesic.

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1. Twisted Perron-Frobenius theorem

Throughout, we shall use the terminology given in [2].

Let (V, E) be an oriented irreducible finite graph, where V is the set of *vertices* and $E \subset V \times V$ is the set of *edges*. We assume that (V, E) is not a circuit graph. Given a *path* $c = (v_0, \cdots, v_m)$, $(v_i, v_{i+1}) \in E$, we put $|c| = m$. When c is closed, i.e. $v_0 = v_m$, we denote by $\langle c \rangle$ the *cycle* represented by c.

It is known that the set of vertices of an irreducible graph can be decomposed into disjoint subsets $V_1, \cdots, V_\nu$ (called *primitive parts*) with the following properties (see [7]):

(a) ν is the greatest common divisor of $\{|c| \mid c \text{ is closed}\}$,

(b) if $v, w \in V_i$, there exists an integer $m(v, w) \geqslant 1$ such that for all integer $m \geqslant m(v, w)$ one can find a path c with $|c| = m\nu$, $v = o(c)$, the origin of c, and $w = t(c)$, the terminus of c,

(c) if c is a path with $o(c) \in V_i$ and ν divides $|c|$, then $t(c)$ also lies in V_i,

(d) for a closed path c in (V, E), at least one vertex on c is contained in V_1.

We set

$$\mathscr{C}_m(V, E) = \{\text{closed paths } c \text{ in } (V, E) \text{ with } |c| = m\nu\},$$

$$\mathscr{P}_m(V, E) = \{\text{prime closed paths } c \in \mathscr{C}_m(V, E)\}.$$

We regard (V, E) as a 1-dimensional CW-complex in a natural manner, and denote by $\pi_1(V, E)$ the fundamental group. The first homology group of (V, E) with coefficients in $\mathbb{Z}$ is denoted by $H_1(V, E)$. The following is fundamental in our discussion.

Lemma 1-1. *$\pi_1(V, E)$ is generated by homotopy classes of closed paths in (V, E).*

Proof. Let v_0 be a point in V, and let c be a loop with base point v_0. We may write $c = c_1 \cdots c_k$, where $\{c_1, c_2^{-1}, \cdots, c_k^{(-1)^{k-1}}\}$ or $\{c_1^{-1}, c_2, \cdots, c_k^{(-1)^k}\}$ are paths in (V, E). In the former case, choose paths $c_1', \cdots, c_{k-1}'$ such that if k is even (resp. odd),

$$o(c_1') = t(c_1), t(c_1') = v_0, o(c_2') = v_0, t(c_2') = o(c_2^{-1}), \cdots,$$
$$o(c_{k-1}') = t(c_k), t(c_{k-1}') = v_0$$

(resp. $o(c_{k-1}') = v_0$, $t(c_{k-1}') = o(c_k)$). Then

$$c_1 \cdot c_1', c_2' \cdot c_2^{-1} \cdot c_1', c_2' \cdot c_3 \cdot c_3', \cdots, c_k^{-1} \cdot c_{k-1}'$$

(resp $c_{k-1}' \cdot c_k$) are closed paths, and c is obviously homotopic to

$$(c_1 \cdot c_1') \cdot (c_2' \cdot c_2^{-1} \cdot c_1')^{-1} \cdot (c_2' \cdot c_3 \cdot c_3') \cdots (c_k^{-1} \cdot c_{k-1}')^{-1}$$

(resp $(c_1 \cdot c_1') \cdot (c_2' \cdot c_2^{-1} \cdot c_1')^{-1} \cdot (c_2' \cdot c_3 \cdot c_3') \cdots (c_{k-1}' \cdot c_k)$). Employing a similar argument in the latter case, we observe that c is homotopic to a composition of closed paths and reversed closed paths.

We denote by π: $(\hat{V}, \hat{E}) \to (V, E)$ the universal covering. Put $\hat{V}_1 = \pi^{-1}(V_1)$. Let $\mathcal{G}$ denote the vector space of all complex-valued functions on $\hat{V}_1$. Given a positive function f on E, define an operator $\mathcal{L}_f$: $\mathcal{G} \to \mathcal{G}$ by

$$\mathcal{L}_f g(v) = \sum_c f(\pi(c)) g(t(c)),$$

where, in the summation, $c = (v_0, \cdots, v_\nu)$ runs over all paths in $(\hat{V}, \hat{E})$ with $o(c) = v$ and $|c| = \nu$, and

$$f(\pi(c)) = f(\pi(v_0), \pi(v_1)) \cdot f(\pi(v_1), \pi(v_2)) \cdots f(\pi(v_{\nu-1}), \pi(v_\nu)).$$

For each character χ: $H_1(V, E) \to U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ (as usual we regard χ as a character of $\pi_1(V, E)$), we define a vector subspace $\mathcal{G}_\chi$ by

$$\mathcal{G}_\chi = \{g \in \mathcal{G} \mid g \circ \gamma = \chi(\gamma) g \text{ for any } \gamma \in \pi_1(V, E)\}$$

and define the operator $\mathcal{L}_{f,\chi}$: $\mathcal{G}_\chi \to \mathcal{G}_\chi$ to be the restriction of $\mathcal{L}_f$ to this space. In case of the trivial character $\mathbb{1}$, it follows from the Perron-Frobenius theorem for nonnegative aperiodic matrices [7] that

(1) the operator $\mathcal{L}_{f,\mathbb{1}}$: $\mathcal{G}_{\mathbb{1}} \to \mathcal{G}_{\mathbb{1}}$ has a simple positive eigenvalue $\lambda(f)$ with a positive eigenfunction,

(2) other eigenvalues μ satisfy $|\mu| < \lambda(f)$.

(Note that $\lambda(f)$ does not depend on the choice of a primitive part V_1).

Our interest is in the case of a general character χ.

Proposition 1-2 (*Twisted Perron-Frobenius theorem* [2]). (1) *Any eigenvalue* μ *of* $\mathscr{L}_{f,\chi}$ *satisfies* $|\mu| \leqslant \lambda(f)$.

(2) $\mathscr{L}_{f,\chi}$ *has an eigenvalue of the form* $\lambda(f)\exp(\sqrt{-1}\,\theta)$, $0 \leqslant \theta < 2\pi$, *if and only if for any close path c*

$$\chi([c]) = \exp\left(\sqrt{-1}\,\theta|c|/\nu\right),$$

where $[c]$ *denotes the homology class* $\in H_1(V, E)$ *represented by c.*

(3) $\operatorname{Trace}\mathscr{L}_{f,\chi}^m = \sum_{c \in \mathscr{C}_m(V,E), o(c) \in V_1} \chi([c])f(c)$.

In what follows, we suppose that the graph (V, E) possesses an orientation-reversing involution κ, that is, κ is a map of V onto itself such that $\kappa^2 = \mathrm{Id}$, and $(\kappa v, \kappa u) \in E$ whenever $(u, v) \in E$. For a path $c = (v_0, \cdots, v_m)$, we denote $\bar{c} = (v_m, \cdots, v_0)$. If c runs over all closed paths, then so does $\kappa(\bar{c}) = \overline{\kappa(c)}$. Note that κ yields a homeomorphism of the CW-complex (V, E).

Lemma 1-3. *Suppose that a positive function f satisfies* $f(\overline{\kappa(c)}) = f(c)$ *for every closed path c, and that there exists a homomorphism* φ: $H_1(V, E) \to H$ *with* $\varphi \circ \kappa_* = \varphi$. *Then for each character* χ *of H*, $\operatorname{Trace}\mathscr{L}_{f,\chi\circ\varphi}^m$ *is a real number.*

Proof. From the assumption, we find that

$$\chi \circ \varphi\left(\left[\,\overline{\kappa(c)}\,\right]\right) = \chi \circ \varphi(-[\kappa(c)]) = \chi(-\varphi(\kappa_*[c])) = \overline{\chi \circ \varphi[c]},$$

so that, by Proposition 1-2,

$$\begin{aligned} \operatorname{Trace}\mathscr{L}_{f,\chi\circ\varphi}^m &= \sum_c \chi \circ \varphi\left[\,\overline{\kappa(c)}\,\right] f\left(\overline{\kappa(c)}\right) \\ &= \sum_c \overline{\chi \circ \varphi[c]}\, f(c) = \overline{\operatorname{Trace}\mathscr{L}_{f,\chi\circ\varphi}^m}. \end{aligned}$$

Lemma 1-4. *Let* $\lambda_1(\chi), \cdots, \lambda_d(\chi)$ $(d = \operatorname{Card}(V_1))$, $|\lambda_1(\chi)| \geqslant |\lambda_2(\chi)| \geqslant \cdots \geqslant |\lambda_d(\chi)|$ *denote the eigenvalues of the operator* $\mathscr{L}_{f,\chi\circ\varphi}$. *Then* $\lambda_1(\chi)$ *is real whenever* $|\lambda_1(\chi)| > |\lambda_2(\chi)|$.

Proof. Set $A = \mathscr{L}_{f,\chi\circ\varphi}$, $\lambda_1 = \lambda_1(\chi) = re^{\sqrt{-1}\,\theta}$. For large m, $|\operatorname{Trace}A^m| \neq 0$, and

$$\pm 1 = \frac{\operatorname{Trace}A^m}{|\operatorname{Trace}A^m|} = \frac{e^{\sqrt{-1}\,m\theta} + (\lambda_2/r)^m + \cdots + (\lambda_d/r)^m}{\left|1 + \left(\lambda_2 e^{-\sqrt{-1}\theta}/r\right)^m + \cdots + \left(\lambda_d e^{-\sqrt{-1}\theta}/r\right)^m\right|}.$$

This implies $\lim_{m\to\infty} e^{2\sqrt{-1}\,m\theta} = 1$, which happens if and only if $\theta \in \pi\mathbb{Z}$, or equivalently $\lambda_1 = \pm r$.

Lemma 1-5. *If* $\mathscr{L}_{f,\chi\circ\varphi}$ *has an eigenvalue of the form* $\lambda(f)\exp(\sqrt{-1}\,\theta)$, *then* $\theta = 0$ *or* π, *and hence* $\chi \circ \varphi = \mathbb{1}$ *or*

$$(\#) \qquad \chi \circ \varphi[c] = \exp\left(\sqrt{-1}\,\pi|c|/\nu\right) = \pm 1$$

for every closed path c.

(When φ is surjective, χ is uniquely determined by this relation in view of Lemma 1-1.)

Proof. According to (a), we can select closed paths $c_1, \cdots, c_k$ and integers $a_1, \cdots, a_k$ such that $a_1|c_1| + \cdots + a_k|c_k| = \nu$. On the other hand,

$$\chi \circ \varphi[c] = \exp(\sqrt{-1}\,\theta|c|/\nu) = \exp\left(\sqrt{-1}\,\theta|\overline{\kappa(c)}|/\nu\right)$$
$$= \chi \circ \varphi[\overline{\kappa(c)}] = \overline{\chi \circ \varphi[c]},$$

which implies that $\chi \circ \varphi[c] = \pm 1$ for all c, so that

$$\exp(\sqrt{-1}\,\theta) = \exp\left(\sqrt{-1}\,\theta a_1|c_1|/\nu\right) \cdots \exp\left(\sqrt{-1}\,\theta a_k|c_k|/\nu\right) = \pm 1.$$

This proves that $\theta = 0$ or π.

Let $\alpha \in H$. We set

$$C_m(\alpha) = \sum_{\substack{c \in \mathscr{C}_m(V, E) \\ o(c) \in V_1,\, \varphi[c] = \alpha}} f(c),$$

and $\mathbf{F}_m(\chi) = \operatorname{Trace} \mathscr{L}^m_{f, \chi \circ \varphi}$. Let $d\chi$ denote the normalized Haar measure on the character group $\hat{H}$ of H. The orthogonal relation of characters leads to

$$\int_{\hat{H}} \chi(-\alpha)\mathbf{F}_m(\chi)\, d\chi = \sum_c \int_{\hat{H}} \chi(-\alpha)\chi \circ \varphi[c] f(c)\, d\chi = C_m(\alpha),$$

so that we have

Lemma 1-6. $|C_m(\alpha)| \leqslant \lambda(f)^m \operatorname{Card}(V_1)$.

We want to estimate $C_m(\alpha)$ from below.

Proposition 1-7. *Suppose that φ is surjective and $\lambda(f) > 1$. Then at least one of the following holds*:

$$\liminf_{m \to \infty} \frac{1}{2m} \log C_{2m}(\alpha) \geqslant \log \lambda(f),$$

or

$$\liminf_{m \to \infty} \frac{1}{2m+1} \log C_{2m+1}(\alpha) \geqslant \log \lambda(f).$$

Proof. We distinguish between two cases: (1) there exists no character χ_0 obeying the relation (#), or $\chi_0(\alpha) = 1$ if it exists, and (2) $\chi_0(\alpha) = -1$. First we treat the case (1). Put

$$W = \{\chi \in \hat{H} \mid \operatorname{Re}\chi(\alpha) < 0\},$$
$$U_B = \left\{\chi \in \hat{H} \mid |\lambda_1(\chi)| \geqslant B\right\} \quad \text{for } B > 0.$$

From Lemma 1-5, we observe that U_B shrinks to the two points $\{\mathbb{1}, \chi_0\}$ as $B \uparrow \lambda(f)$, so that, by the perturbation theory of eigenvalues of matrices, there are positive B and C such that $C < B < \lambda(f)$, $W \cap U_B = \varnothing$, and $|\lambda_2(\chi)| \leqslant C$

if $\chi \in U_B$. Let A be an arbitrary number with $\max\{B, \sqrt{\lambda(f)}\} < A < \lambda(f)$. Put $L = \min\{\operatorname{Re}\chi(\alpha) \mid \chi \in U_A\}$. Since $\lambda_1(\chi_0) = -\lambda(f)$ and $\mathbf{F}_{2m}(\chi_0) = \lambda(f)^{2m} + \sum_{i=2}^{d} \lambda_i(\chi_0)^{2m} > 0$ for large m, we have, when $\chi \in U_B$,

$$\mathbf{F}_{2m}(\chi) = |\mathbf{F}_{2m}(\chi)| = \left| \sum_{i=1}^{d} \lambda_i(\chi)^{2m} \right| \geqslant B^{2m} - dC^{2m},$$

so for sufficiently large m we have

$$\begin{aligned} C_{2m}(\alpha) &= \int_{\hat{H}} \operatorname{Re}\chi(-\alpha)\mathbf{F}_{2m}(\chi)\, d\chi \\ &= \int_{U_B} \operatorname{Re}\chi(\alpha)\mathbf{F}_{2m}(\chi)\, d\chi + \int_{\hat{H}\setminus U_B} \operatorname{Re}\chi(\alpha)\mathbf{F}_{2m}(\chi)\, d\chi \\ &\geqslant \int_{U_A} \operatorname{Re}\chi(\alpha)\mathbf{F}_{2m}(\chi)\, d\chi + \int_{\hat{H}\setminus U_B} \operatorname{Re}\chi(\alpha)\mathbf{F}_{2m}(\chi)\, d\chi \\ &\geqslant L(A^{2m} - dC^{2m}) \operatorname{vol}(U_A) - dB^{2m}. \end{aligned}$$

In the second, we suppose $\chi_0(\alpha) = -1$. Define

$$\begin{aligned} U_B^+ &= \{\chi \in \hat{H} \mid \lambda_1(\chi) \in \mathbb{R} \text{ and } \lambda_1(\chi) \geqslant B\}, \\ U_B^- &= \{\chi \in \hat{H} \mid \lambda_1(\chi) \in \mathbb{R} \text{ and } \lambda_1(\chi) \leqslant -B\} \end{aligned}$$

for a positive B. There are positive B and C such that $C < B < \lambda(f)$, $W \cap U_B^+ = \varnothing$, $W \supset U_B^-$, and $|\lambda_2(\chi)| \leqslant C$ if $\chi \in U_B^+ \cup U_B^- = U_B$. Then for sufficiently large m and $\chi \in U_B$, $\operatorname{Re}\chi(\alpha)\mathbf{F}_{2m+1}(\chi)$ is positive. From this we have

$$\begin{aligned} C_{2m+1}(\alpha) &= \int_{\hat{H}} \operatorname{Re}\chi(-\alpha)\mathbf{F}_{2m+1}(\chi)\, d\chi \\ &\geqslant \int_{U_A^+} \operatorname{Re}\chi(\alpha)\mathbf{F}_{2m+1}(\chi)\, d\chi + \int_{\hat{H}\setminus U_B} \operatorname{Re}\chi(\alpha)\mathbf{F}_{2m+1}(\chi)\, d\chi \\ &\geqslant L^+(A^{2m+1} - dC^{2m+1}) \operatorname{vol}(U_A^+) - dB^{2m+1}, \end{aligned}$$

where A is as above and $L^+ = \min\{\operatorname{Re}\chi(\alpha) \mid \chi \in U_A^+\}$.

In any case, these estimates imply our assertion.

For later use, we consider several operations of graphs. Given an irreducible graph (V, E), we define the graph $(V^{(n)}, E^{(n)})$ of *n-step paths* as follows;

$$\begin{aligned} V^{(n)} &= \{(v^0, \cdots, v^n);\ (v^i, v^{i+1}) \in E\}, \\ E^{(n)} &= \{((v^0, \cdots, v^n), (w^0, \cdots, w^n)) \in V^{(n)} \times V^{(n)}; \\ &\qquad v^i = w^{i+1}, 0 \leqslant i \leqslant n-1\}. \end{aligned}$$

It should be noted that $(V^{(n)}, E^{(n)})$ is also irreducible, and if we put

$$V_j^{(n)} = \left\{ (v^0, \cdots, v^n) \in V^{(n)};\ v^0 \in V_j \right\},$$

then $V^{(n)} = V_1^{(n)} \cup \cdots \cup V_\nu^{(n)}$ is the decomposition into primitive parts of $V^{(n)}$. If (V, E) has an orientation-reversing involution κ, then the correspondence $\kappa^{(n)}$: $(v^0, \cdots, v^n) \to (\kappa(v^n), \cdots, \kappa(v^0))$ defines an orientation-reversing involution of $(V^{(n)}, E^{(n)})$.

It is useful to introduce an orientation reversing morphism $\omega^{(n)}$: $(V^{(n)}, E^{(n)}) \to (V, E)$ defined by $\omega^{(n)}(v^0, \cdots, v^n) = v^0$. Then the correspondence $c \mapsto \overline{\omega^{(n)}(c)}$ gives rise to a bijection between the set of cycles in $(V^{(n)}, E^{(n)})$ and that in (V, E). Note that $\kappa \circ \omega^{(n)} \neq \omega^{(n)} \circ \kappa^{(n)}$. But we have

Lemma 1-8. *The following diagram is commutative*:

$$\begin{array}{ccc} H_1(V^{(n)}, E^{(n)}) & \xrightarrow{(\kappa^{(n)})_*} & H_1(V^{(n)}, E^{(n)}) \\ \downarrow{\scriptstyle (\omega^{(n)})_*} & & \downarrow{\scriptstyle (\omega^{(n)})_*} \\ H_1(V, E) & \xrightarrow[\kappa_*]{} & H_1(V, E) \end{array}$$

Proof. Let $c = (v_0^{(n)}, \cdots, v_m^{(n)})$, $v_i^{(n)} = (v_i^0, \cdots, v_i^n) \in V^{(n)}$, be a closed path in $(V^{(n)}, E^{(n)})$. Then $\omega^{(n)}\kappa^{(n)}(c) = (\kappa(v_0^n), \cdots, \kappa(v_m^n))$, and $\kappa\omega^{(n)}(c) = (\kappa(v_0^0), \cdots, \kappa(v_m^0))$. Note that $v_{i+1}^j = v_i^{j-1}$, hence $(\kappa(v_0^n), \cdots, \kappa(v_m^n)) = (\kappa(v_{-n}^0), \cdots, \kappa(v_{m-n}^0))$, where by a convention $v_k^0 = v_{k+m}^0$, $k \in \mathbb{Z}$. This implies that $\omega^{(n)}\kappa^{(n)}(c)$ and $\kappa\omega^{(n)}(c)$ are identical as cycles. Since $H_1(V^{(n)}, E^{(n)})$ is generated by homology classes of closed paths, we get the conclusion.

Let (V, E) be a graph with a positive (length) function l on E. Given a path c, we define $l(c)$ to be $\Sigma l(e)$, where e runs over edges on c. A graph (V', E') with a positive function l' is called a *subdivision* of (V, E) if

(a) (V', E') is a subdivision of (V, E) as a CW-complex and the orientation of edges $\in E'$ are compatible with that of E, that is, each edge $e \in E$ is identified with a unique path $c(e)$ in (V', E'),

(b) $l(e) = l'(c(e))$, $e \in E$.

Note that there is a natural correspondence $\langle c \rangle \leftrightarrow \langle c' \rangle$ between the set of cycles in (V, E) and the set of cycles in a subdivision (V', E') such that $l(c) = l'(c')$.

Consider a function f of the form $f_s(e) = \exp(-sl(e))$, where l is a positive function on E and $s \in \mathbb{R}$. It is known that there is a unique positive constant $h = h(l)$ with $\lambda(f_h) = 1$. Since

$$\lim_{x \to \infty} \frac{1}{x} \log \operatorname{Card}\{\langle c \rangle;\ l(c) \leqslant x\} = h(l),$$

we have

Lemma 1-9. *If (V', E', l') is a subdivision, then* $h(l') = h(l)$.

Lemma 1-10. *Let (V, E) be a graph with an orientation-reversing involution κ such that $l(\overline{\kappa(c)}) = l(c)$. Let $\varepsilon > 0$. There exist a subdivision (V', E', l') satisfying*

(1) *(V', E') has an orientation-reversing involution κ' such that $l'(\overline{\kappa'(c)}) = l'(c)$, and $\kappa'(c(e)) = c(\overline{\kappa(e)})$,*

(2) $\max l'/\min l' \leqslant 1 + \varepsilon$.

Proof. Without loss of generality, we may assume that the action $e \mapsto \overline{\kappa(e)}$ on E is free. We select a set $\{e_1, \cdots, e_n\}$ of edges such that

$$E = \{e_1, \cdots, e_n\} \cup \{\overline{\kappa(e_1)}, \cdots, \overline{\kappa(e_n)}\} \quad \text{(disjoint)}.$$

We shall subdivide each edge e_k and $\overline{\kappa(e_k)}$ in such a way that

$$\begin{aligned} e_k &= (e_{k,1}, \cdots, e_{k,n_k}), & l'(e_{k,i}) &= l(e_k)/n_k, \\ \overline{\kappa(e_k)} &= (\dot{e}_{k,1}, \cdots, \dot{e}_{k,n_k}), & l'(\dot{e}_{k,i}) &= l(e_k)/n_k. \end{aligned}$$

Assume $l(e_1) = \min l$. By Kronecker's approximation of real numbers by rationals, we may choose positive integers $p_1, \cdots, p_k$, $q_1, \cdots, q_k$ such that

$$\left| q_j - p_j \frac{l(e_j)}{l(e_1)} \right| < \varepsilon/2.$$

Then, putting $n_1 = \prod_1^n p_j$, $n_i = q_i \prod_{j \neq i} p_j$, we find

$$\begin{aligned} \left| \frac{l(e_1)}{n_1} - \frac{l(e_i)}{n_i} \right| &= \Big(q_i \prod_j p_j \Big)^{-1} |q_i l(e_1) - p_i l(e_i)| \\ &< \frac{\varepsilon}{2} \Big(q_i \prod_j p_j \Big)^{-1} l(e_1) \leqslant \frac{\varepsilon}{2} \min_j \frac{l(e_j)}{n_j}, \end{aligned}$$

which implies that

$$\max\{l(e_i)/n_i\}/\min\{l(e_i)/n_i\} < 1 + \varepsilon.$$

Defining the involution κ on (V', E') by $\kappa(e_{k,i}) = \dot{e}_{k, n_k - i + 1}$, we get the desired subdivision.

2. Proof of Theorem 1

We denote by h the *topological entropy* of the geodesic flow on the unit tangent bundle of a compact Riemannian manifold M, and by $l(\mathfrak{p})$ the length of a closed geodesic $\mathfrak{p}$.

The following proposition is crucial in our discussion.

Proposition 2-1. *Suppose that the geodesic flow is of Anosov type. Then for any positive* ε, *there exist a positive constant* J, *a finite family of oriented irreducible finite graphs* $(\mathbf{V}^a, \mathbf{E}^a)$, $a = 0, 1, \cdots, N$, *with positive functions* f_a *on* $\mathbf{E}^a$, *and a family of maps* Φ_a *of the set of cycles in* $(\mathbf{V}^a, \mathbf{E}^a)$ *into the set of closed geodesics in* M *satisfying the following properties.*

(1) Φ_0 *is surjective and bounded-to-one.*

(2) $l(\Phi_0(\langle c \rangle)) \leqslant |c| J$ *for every closed path* c.

(3) *There exists a surjective homomorphism* φ: $H_1(\mathbf{V}^0, \mathbf{E}^0) \to H_1(M, \mathbb{Z})$ *such that* $[\Phi_0(\langle c \rangle)] = \varphi([c])$

(4) $f_0 < exp\{-(h - \varepsilon)J\}$, $\lambda(f_0) > 1$, *and* $\lambda(f_a) < 1$, $a \geqslant 1$.

(5) $(\mathbf{V}^0, \mathbf{E}^0)$ *has an orientation-reversing involution* κ *such that* $f_0(\overline{\kappa(c)}) = f_0(c)$ *and* $\varphi \circ \kappa_* = \varphi$.

(6) *There exists a positive* Θ *such that if* c *is a prime closed path in* $(\mathbf{V}^0, \mathbf{E}^0)$ *with nonprime* $\Phi_0(\langle c \rangle)$, *one can find a prime closed path* c' *in* $(\mathbf{V}^a, \mathbf{E}^a)$ *for some* $a \geqslant 1$ *satisfying*

$$\Phi_a(\langle c' \rangle) = \Phi_0(\langle c \rangle), \quad |c'| \leqslant \Theta |c|, \quad \textit{and} \quad f_a(c') \geqslant f_0(c).$$

We defer the lengthy proof of this proposition to §4, and proceed to the proof of the main theorem.

Let ν_a denote the number of primitive parts of $(\mathbf{V}^a, \mathbf{E}^a)$, and let $\mathbf{V}_1^a$ be a fixed primitive part in $\mathbf{V}^a$. We put $d_a = \mathrm{Card}(\mathbf{V}_1^a)$. For simplicity, we set $(V, E) = (\mathbf{V}^0, \mathbf{E}^0)$, $f = f_0$, $\lambda = \lambda(f)$, $\Phi = \Phi_0$, $d = d_0$, $V_1 = \mathbf{V}_1^0$, and $H = H_1(M, \mathbb{Z})$.

By properties (3) and (5), we are in a position to apply Proposition 1-7. Put

$$G_m(\alpha) = \sum_{c \in \mathscr{P}_m(V, E) \text{ with prime } \Phi(\langle c \rangle),\, o(c) \in V_1, \text{ and } \varphi[c] = \alpha} f(c),$$

$$P_m(\alpha) = \sum_{c \in \mathscr{P}_m(V, E) \text{ with } o(c) \in V_1,\, \varphi[c] = \alpha} f(c).$$

Lemma 2-2. *There exists a positive constant* C_1 *such that for every* m

$$P_m(\alpha) \geqslant C_m(\alpha) - C_1 m^2 \lambda(f)^{m/2}.$$

Proof. Let $H = H^f \oplus H^t$ be a decomposition into the direct product of free part and torsion part. Put $\alpha = i\alpha_p + \alpha_t$, $\alpha_p \in H^f$ being primitive, $\alpha_t \in H^t$, and i being a nonnegative integer. Here an element in H^f is said to be primitive if it is not a nontrivial multiple of another element in H^f. Note that if a closed path c in (V, E) is not prime, then there exists a unique prime closed path c' with $c = (c')^s$ for some integer $s \geqslant 2$. The correspondence $c \to c'$ is at most m-to-1. Since $\varphi[c'] = i\alpha_p/s + \beta$ for some β with $s\beta = \alpha_t$, and $f(c) \leqslant f(c')$ in view of Proposition 2-1(4), we have

$$P_m(\alpha) = C_m(\alpha) - \sum_{c:\ \text{nonprime}} f(c) \geqslant C_m(\alpha) - \sum_{c} f(c')$$
$$\geqslant C_m(\alpha) - m \sum_{\substack{s=2 \\ s \mid m}}^{m} \sum_{\beta \in H'} C_{m/s}\left(\frac{i}{s}\alpha_p + \beta\right)$$
$$\geqslant C_m(\alpha) - m^2 \lambda(f)^{m/2} d \operatorname{Card}(H'),$$

where we have used Lemma 1-6.

Lemma 2-3. *There exists a positive constant C_2 such that for every m*

$$G_m(\alpha) \geqslant P_m(\alpha) - C_2 m.$$

Proof. We use properties (4) and (6) in Proposition 2-1. For a prime closed path c with nonprime $\Phi(\langle c \rangle)$, we let c' be a prime closed path given in (6). Then

$$G_m(\alpha) = P_m(\alpha) - \sum_{\substack{c \in \mathscr{P}_m(V,E) \text{ with nonprime } \Phi(\langle c \rangle), \\ o(c) \in V_1,\ \varphi[c] = \alpha}} f(c)$$
$$\geqslant P_m(\alpha) - \sum_{c} f_a(c') \geqslant P_m(\alpha) - mK \sum_{a=1}^{N} \sum_{c_a} f_a(c_a),$$

where c_a runs over closed paths in $(\mathbf{V}^a, \mathbf{E}^a)$ with $|c_a| \leqslant [n\nu\Theta/\nu_a] + 1$, $o(c_a) \in \mathbf{V}_1^a$ and $K = \sup \operatorname{Card}\{\Phi_0^{-1}(\Phi_0(\langle c \rangle))\}$. Since

$$\sum_{c_a} f_a(c_a) \leqslant d_a \sum_{k=1}^{\infty} \lambda(f_a)^k,$$

we find

$$G_m(\alpha) \geqslant P_m(\alpha) - mKN \cdot \max_a \left\{\lambda(f_a)(1 - \lambda(f_a))^{-1} d_a\right\}.$$

Combining Lemmas 2-2 and 2-3 with Proposition 1-7, we have

Proposition 2-4. *At least one of the following holds*:

$$\liminf \frac{1}{2m} \log G_{2m}(\alpha) \geqslant \log \lambda(f) > 0,$$

or

$$\liminf \frac{1}{2m+1} \log G_{2m+1}(\alpha) \geqslant \log \lambda(f) > 0.$$

Proof of Theorem 1. Let x be an arbitrary positive number. In view of (1) and (2) in Proposition 2-1, if $m\nu J \leqslant x$, then the correspondence Φ_0 gives rise to a map of the set $\{c \in \mathscr{P}_m(V, E) \mid o(c) \in V_1, \Phi_0(\langle c \rangle) \text{ is prime, and } \varphi[c] = \alpha\}$

into the set {prime closed geodesics $\mathfrak{p}$ in M; $[\mathfrak{p}] = \alpha$, $l(\mathfrak{p}) \leqslant x$}, which is at most mK-to-1. Hence, according to (4), we have

$$\begin{aligned} N(x, \alpha) & \\ &\geqslant \frac{1}{mK} \operatorname{Card}\{c \in \mathscr{P}_m(V, E);\ o(c) \in V_1,\ \Phi_0(\langle c \rangle) \text{ is prime},\ \varphi[c] = \alpha\} \\ &\geqslant \frac{1}{mK} G_m(\alpha) \exp\{(h - \varepsilon) m\nu J\}. \end{aligned}$$

Put $m(x) = [x/(2\nu J)]$. For sufficiently large x, one gets

$$\begin{aligned} N(x, \alpha) &\geqslant \frac{1}{2m(x)K} G_{2m(x)}(\alpha) \exp\{(h - \varepsilon) 2\nu m(x) J\} \\ &\geqslant \frac{1}{2m(x)K} G_{2m(x)}(\alpha) \exp\{(h - \varepsilon)(x - 2\nu J)\}, \\ N(x, \alpha) &\geqslant \frac{1}{(2m(x) - 1)K} G_{2m(x)-1}(\alpha) \exp\{(h - \varepsilon)(x - 3\nu J)\}. \end{aligned}$$

Since $\liminf x^{-1} \log G_{2m(x)}(\alpha) \geqslant 0$, or $\liminf x^{-1} \log G_{2m(x)-1}(\alpha) \geqslant 0$, we have

$$\liminf_{x \to \infty} \frac{1}{x} \log N(x;\ \alpha) \geqslant (h - \varepsilon).$$

On the other hand, it follows from Parry-Pollicott [12] that

$$\lim_{x \to \infty} \frac{1}{x} \log \operatorname{Card}\{\mathfrak{p} \mid \operatorname{length}(\mathfrak{p}) < x\} = h.$$

Since ε is arbitrary, this completes the proof of Theorem 1.

Remark 1. From [2], [12], we find

$$\lim_{x \to \infty} \frac{\operatorname{Card}\{\mathfrak{p} \mid [\mathfrak{p}] = \alpha \text{ and } \operatorname{length}(\mathfrak{p}) < x\}}{\operatorname{Card}\{\mathfrak{p} \mid \operatorname{length}(\mathfrak{p}) < x\}} = \begin{cases} (\operatorname{Card} H_1(M, \mathbb{Z}))^{-1}, & \text{if } H_1(M, \mathbb{Z}) \text{ is of finite order}, \\ 0, & \text{otherwise}. \end{cases}$$

We conjecture that the above ratio is asymptotically equal to $C/x^{b_1(M)}$ as x goes to infinity, where $b_1(M)$ denotes the first Betti number and C is a positive constant not depending on α.

Remark 2. Let ϕ be an Anosov diffeomorphism on a compact manifold X, and let $(\Sigma X, \phi_t)$ be the associated suspension flow, which is known to be of Anosov type. Consider the canonical map π: $\Sigma X \to \mathbb{R}/\mathbb{Z}$, and the induced homomorphism π_*: $H_1(\Sigma X, \mathbb{Z}) \to \mathbb{Z}$. From the definition, $\pi_*[\mathfrak{p}] = l(\mathfrak{p})$ for every closed orbit $\mathfrak{p}$, so that each $\alpha \in H_1(\Sigma X, \mathbb{Z})$ contains only finitely many $\mathfrak{p}$, and if $\pi_*(\alpha) < 0$, then there exists no closed orbit $\mathfrak{p}$ with $[\mathfrak{p}] = \alpha$.

3. Review on symbolic dynamics

We review some basic notions and facts on symbolic dynamics which are required in a modified form. Given an oriented irreducible finite graph (V, E), we set

$$\Sigma(V,E)=\left\{\xi=(\xi^i)\in\prod_{-\infty}^{\infty}V\,|\,(\xi^i,\xi^{i+1})\in E\text{ for any }i\right\},$$

$$\Sigma^+(V,E)=\left\{\xi=(\xi^i)\in\prod_{0}^{\infty}V\,|\,(\xi^i,\xi^{i+1})\in E\text{ for any }i\geqslant 0\right\},$$

on which the shift operators σ and σ_+ are defined in the usual way. Note that if $V=V_1\cup\cdots\cup V_\nu$ is the decomposition into the primitive parts, then

$$\Sigma^+(V,E)=\bigcup_{i=1}^{\nu}\Sigma_i^+(V,E)\quad\text{(disjoint)},$$

$$\sigma_+^\nu(\Sigma_i^+(V,E))\subset\Sigma_i^+(V,E),$$

where $\Sigma_i^+(V,E)=\{\xi\in\Sigma^+(V,E)\,|\,\xi^0\in V_i\}$. For each $\theta\in(0,1)$, define a distance function d_θ on $\Sigma(V,E)$ (resp. on $\Sigma^+(V,E)$) by

$$d_\theta(\xi,\eta)=\theta^{\sup\{n\,|\,\xi^i=\eta^i\text{ for }|i|<n\}}$$

(resp. $d_\theta(\xi,\eta)=\theta^{\sup\{n\,|\,\xi^i=\eta^i\text{ for }0\leqslant i<n\}}$). For a complex valued function g on $\Sigma(V,E)$ (or on $\Sigma^+(V,E)$, $\Sigma_1^+(V,E)$), we set

$$\|g\|_\infty=\sup|g(\xi)|,\qquad \|g\|_\theta=\|g\|_\infty+\mathrm{Lip}_\theta(g),$$
$$\mathrm{Lip}_\theta(g)=\sup\left\{|g(\xi)-g(\eta)|/d_\theta(\xi,\eta)\,|\,\xi\neq\eta\right\}.$$

The space $\mathscr{G}_\theta=\{g\colon\ \Sigma_1^+(V,E)\to\mathbb{C}\,|\,\|g\|_\theta<\infty\}$ is a Banach space with respect to the norm $\|\ \|_\theta$.

Given a positive valued function F on $\Sigma^+(V,E)$ with $\|F\|_\theta<\infty$, define an operator $R_F\colon\mathscr{G}_\theta\to\mathscr{G}_\theta$ by

$$R_Fg(\xi)=\sum_{\eta:\ \sigma_+^\nu\eta=\xi}F(\eta)F(\sigma_+\eta)\cdots F(\sigma_+^{\nu-1}\eta)g(\eta).$$

The following proposition is a consequence of an infinite-dimensional version of the Perron-Frobenius theorem established by Ruelle [15].

Proposition 3-1. *R_F has simple positive eigenvalue $\Lambda(F)$. The rest of the spectrum is contained in a disc of radius strictly smaller than $\Lambda(F)$.*

Let l^+ be a positive valued function on $\Sigma^+(V,E)$ with $\|l^+\|_\theta<\infty$. Then $\Lambda(\exp(-sl^+))$ is a monotone decreasing function of $s\in\mathbb{R}$, and there exist a unique $h>0$ such that $\Lambda(\exp(-hl^+))=1$. The number h has a dynamical explanation. Suppose that a continuous function l on $\Sigma(V,E)$ is cohomologous to a continuous function l^+ depending only on the future $(\xi^i)_{i\geqslant 0}$, in the sense that there is a continuous function ψ on $\Sigma(V, E)$ with $l = l^+ - \psi + \psi \circ \sigma$. We regard l^+ as a function $\Sigma^+(V, E)$. Then the number h with $\Lambda(\exp(-hl^+)) = 1$ coincides with topological entropy of the suspension flow $(\Sigma(V, E, l), \sigma(l)_t)$ associated with the function l. Here

$$\Sigma(V, E, l) = \{(\xi, t) \mid \xi \in \Sigma(V, E), 0 \leqslant t \leqslant l(\xi)\},$$

$$\sigma(l)_t(\xi, s) = (\xi, s + t)$$

with appropriate identifications. (Note that the operator R_F is the restriction to the space $\mathscr{G}_\theta$ of the ν-iteration of the Ruelle operator introduced in Ruelle [15] and Parry-Pollicott [12], so that $\Lambda(F)$ is the νth power of the maximal positive eigenvalue of the Ruelle operator.)

The graph $(V^{(n)}, E^{(n)})$ of n-step paths is regarded as an approximation of $\Sigma^+(V, E)$ in the following sense.

Proposition 3-2 [14]. *Let $\{F_n\}_{n=1}^\infty$ be a sequence of positive valued functions on $\Sigma^+(V, E)$ with $\|F_n\|_\theta < \infty$ converging to a positive valued function F with respect to the $\| \ \|_\theta$-norm. Suppose that $F_n(\xi)$ depends only on $(\xi^i)_{i=0}^n$. If we define a function f_n on $E^{(n)}$ by*

$$f_n\big((v^0, \cdots, v^n), (w^0, \cdots, w^n)\big) = F_n(\xi),$$

where $\xi^0 = w^0, \cdots, \xi^n = w^n$, then $\lambda(f_n) = \Lambda(F_n)$, and $\lim_{n \to \infty} \lambda(f_n) = \Lambda(F)$. Here $\lambda(f_n)$ is the maximal positive eigenvalue of the operator $\mathscr{L}_{f_n, \mathbf{1}}$ acting on the space $\mathscr{G}_\mathbf{1} = \{g \colon V_1^{(n)} \to \mathbf{C}\}$.

4. Proof of Proposition 2-1

Let UM be the unit tangent bundle of M, and $\phi_t \colon UM \to UM$ be the geodesic flow of Anosov type. If we define an involution $\mu \colon UM \to UM$ by $\mu(x) = -x$, then we have $\mu \circ \phi_t = \phi_{-t} \circ \mu$.

First of all, we construct finite graphs derived from a Markov family of the geodesic flow. Roughly speaking, a Markov family V_0 of size α is a finite family of disjoint local cross-sections to ϕ_t with $X = \bigcup_{T \in V_0} \phi_{[0, \alpha]}(T)$. We define E_0 to be the set of all $(T, S) \in V_0 \times V_0$ such that there exists an "interior" point in T which firstly goes into S along the flow ϕ_t. The recurrence property guarantees that this graph is irreducible. From the reversible property $\mu \circ \phi_t = \phi_{-t} \circ \mu$ we may take a Markov family V_0, of small size, with an orientation-reversing involution κ and a continuous surjective map $\rho \colon \Sigma(V_0, E_0, l) \to X$ such that

(B-1) l is a strictly positive function satisfying $l = l \circ \bar{\kappa} \circ \sigma$, where $\bar{\kappa} \colon \Sigma(V_0, E_0) \to \Sigma(V_0, E_0)$ is defined by $\bar{\kappa}(\xi)^i = \kappa(\xi^{-i})$,

(B-2) $\rho \circ \sigma(l)_t = \phi_t \circ \rho$.

(B-3) ρ is a bounded-to-one map,

(B-4) $\rho(\xi, 0) \in \xi^0$ for any $\xi \in \Sigma(V_0, E_0)$,

(B-5) if we define $\bar{\mu}$: $\Sigma(V_0, E_0, l) \to \Sigma(V_0, E_0, l)$ by $\bar{\mu}(\xi, s) = (\bar{\kappa} \circ \sigma(\xi), l(\bar{\kappa} \circ \sigma(\xi)) - s)$, then $\rho \circ \bar{\mu} = \mu \circ \rho$ (see [1]). We refer the suspension $\Sigma(V_0, E_0, l)$ as the principal suspension of ϕ_t.

In order to cancel out the overcounting of closed orbits of (UM, ϕ_t), we need to introduce finite auxiliary graphs (V_a, E_a), $a = 1, \cdots, N$, which correspond to the auxiliary suspensions $\Sigma(V_a, E_a, l_a)$ defined by Bowen [4]. The auxiliary suspension admits a canonical map ρ_a: $\Sigma(V_a, E_a, l_a) \to UM$ with $\rho_a \circ \sigma(l_a)_t = \phi_t \circ \rho_a$. Applying the counting lemma due to Manning [10] and Bowen [4] we observe that

(B-6) for each ϕ_t-closed orbit $\mathfrak{q}$, there is a $\sigma(l)_t$-closed orbit $\mathfrak{p}$ with $\rho(\mathfrak{p}) = \mathfrak{q}$. If $\mathfrak{p}$ is a $\sigma(l)_t$-closed orbit with nonprime $\rho(\mathfrak{p})$, then there exists a $\sigma(l_a)_t$-closed orbit $\mathfrak{p}'$ with $\rho(\mathfrak{p}) = \rho_a(\mathfrak{p}')$ and $\tau(\mathfrak{p}) = \tau(\mathfrak{p}')$ for some a.

It should be noted that the topological entropy of ϕ_t is equal to that of $\sigma(l)_t$, and is greater than that of $\sigma(l_a)_t$.

Using the above result we show the following.

Proposition 4-1. *The oriented irreducible finite graphs (V_a, E_a), $a = 0, \cdots, N$ satisfy the following conditions.*

(1) *There is a continuous map ι_0: $(V_0, E_0) \to UM$.*

(2) *There are bounded-to-one correspondences Ψ_a of the set of cycles in (V_a, E_a) into the set of ϕ_t-orbit cycles.*

(3) *Ψ_0 is surjective, and for a closed path c in (V_0, E_0), $\Psi_0(\langle c \rangle)$ is free homotopic to $\iota_0(c)$.*

(4) *For each closed path c in (V_0, E_0), $\Psi_0(\langle \overline{\kappa c} \rangle) = \mu\Psi_0(\langle c \rangle)$.*

(5) *If c is a prime closed path in (V_0, E_0) with nonprime $\Psi_0(\langle c \rangle)$, one can find a prime closed path c' in (V_a, E_a) with $\Psi_0(\langle c \rangle) = \Psi_a(\langle c' \rangle)$ for some $a \geqslant 1$.*

Proof. For each $v \in V_0$ choose a point $x(v) \in v$. Whenever $(v, w) \in E_0$, the distance from $x(v)$ to $x(w)$ is sufficiently small, hence we can join them by a minimal geodesic $\gamma(v, w)$. Define the map ι_0: $(V_0, E_0) \to X$ by $\iota_0(v) = x(v)$ and $\iota_0(v, w) = \gamma(v, w)$.

Let $c = (v_0, \cdots, v_m)$ be a closed path in the graph (V_0, E_0). Define $\xi(c) \in \Sigma(V_0, E_0)$ by $\xi(c)^i = v^k$, where $i \equiv k \bmod m$. Then $(\xi(c), 0) \in \Sigma(V_0, E_0, l)$ is a periodic point of $\sigma(l)_t$, and

$$\mathfrak{p}(c) = \{\sigma(l)_t(\xi(c), 0)\}_{0 \leqslant t \leqslant \tau(\mathfrak{p}(c))}, \qquad \tau(\mathfrak{p}(c)) = \sum_{j=0}^{m-1} l(\sigma^j \xi(c)),$$

is an orbit cycle in $\Sigma(V_0, E_0, l)$ (note that the correspondence $c \leftrightarrow \mathfrak{p}(c)$ yields a bijection of the set of cycles in (V, E) onto the set of orbit cycles in $\Sigma(V_0, E_0, l)$, under which the set of prime cycles goes onto the set of closed orbits, and that $\mathfrak{p}(\overline{\kappa(c)}) = \bar{\mu}(\mathfrak{p}(c))$). Since $\rho(\sigma^k \xi(c), 0)$ and $x(v_k)$ lie in v_k

and the size of the Markov family is small, it is clear that if we define Ψ_0 by $\Psi_0(\langle c\rangle) = \rho(\mathfrak{p}(c))$, then $\iota_0(c) = \gamma(v_0, v_1)\cdot\gamma(v_1, v_2)\cdots\gamma(v_{m-1}, v_m)$ and $\Psi_0(\langle c\rangle)$ are free homotopic. Properties (2), (3), and (4) come from (B-3), (B-6), and (B-5) respectively.

For auxiliary graphs (V_a, E_a) we define the correspondence Ψ_a in a similar manner. Property (5) is a direct consequence of (B-6).

It is known that $\|l\|_\theta < \infty$, $\|l_a\|_\theta < \infty$ for some $\theta \in (0,1)$. We now construct positive functions $\{l_n^+\}$ and $\{l_{a,n}^+\}$ depending on finite coordinates $(\xi^i)_{i=0}^n$ such that l_n^+ and $l_{a,n}^+$ converge to positive functions l^+ and l_a^+ cohomologous to l and l_a respectively. A major subtle point is to impose a condition on l_n^+ which inherits the relation $l = l\circ\bar{\kappa}\circ\sigma$.

For each $n > 0$, define a positive function l_n on $\Sigma(V_0, E_0)$ by

$$l_n(\xi) = \operatorname{Sup}\left\{ l(\eta) \mid \eta \in \Sigma(V_0, E_0) \text{ with } \eta^i = \xi^i \text{ for } -n \leqslant i \leqslant n+1 \right\}.$$

It is obvious that $l_n = l_n\circ\bar{\kappa}\circ\sigma$. Imitating the arguments in Bowen [4], we find that l_n converges to l with respect to $\|\ \|_{\theta'}$ for $\theta' > \theta$. To l and l_n, we associate strictly positive functions l^+ and l_n^+ on $\Sigma^+(V_0, E_0)$ such that (a) l^+ and l_n^+ are cohomologous to l and l_n respectively as functions on $\Sigma(V_0, E_0)$, and (b) $l_n^+(\xi)$ depends only on $(\xi^i)_{0\leqslant i\leqslant n}$, and l_n^+ converges to l^+ with respect to the $\|\ \|_{\theta'}$-norm for some $\theta < \theta' < 1$.

Lemma 4-2. *If $\xi \in \Sigma(V_0, E_0)$ satisfies $\sigma^m\xi = \xi$, then*

$$\sum_{j=1}^{m-1} l_n^+(\sigma^j\xi) = \sum_{j=1}^{m-1} l_n^+(\sigma^j\bar{\kappa}\xi).$$

Proof.

$$\begin{aligned}\sum_{j=1}^{m-1} l_n^+(\sigma^j\bar{\kappa}\xi) &= \sum_{j=1}^{m-1} l_n(\sigma^j\bar{\kappa}\xi) = \sum_{j=1}^{m-1} l_n(\bar{\kappa}\sigma^{-j}\xi) = \sum_{j=1}^{m-1} l_n(\sigma^{-j-1}\xi)\\ &= \sum_{j=1}^{m-1} l_n^+(\sigma^{-j-1}\xi) = \sum_{j=1}^{m-1} l_n^+(\sigma^j\xi).\end{aligned}$$

As for approximation of the suspending functon l_a of auxiliary suspensions, we simply define

$$l_{a,n}^+(\xi) = \inf\left\{ l_a^+(\eta) \mid \eta^i = \xi^i, 0 \leqslant i \leqslant n \right\},$$

where l_a^+ is a strictly positive function on $\Sigma(V_a, E_a)$ depending only on the future, cohomologous to l_a. Then $l_{a,n}^+$ converges to l_a^+ with respect to $\|\ \|_{\theta'}$ for some $\theta < \theta' < 1$.

The following lemma is an immediate consequence of the definitions of l_n^+ and $l_{a,n}^+$.

Lemma 4-3. *Let $\xi \in \Sigma(V_0, E_0)$ and $\zeta \in \Sigma(V_a, E_a)$ be periodic points with minimal period m and m′ with respect to the shift operators, and let $\mathfrak{p}(\xi)$ and $\mathfrak{p}(\zeta)$ be the corresponding periodic orbits in the suspension flows. If $\rho(\mathfrak{p}(\xi)) = \rho_a(\mathfrak{p}(\zeta))$, then we have*

$$\sum_{j=1}^{m-1} l_n^+(\sigma^j\xi) \geqslant \tau(\mathfrak{p}(\xi)) = \tau(\mathfrak{p}(\zeta)) \geqslant \sum_{j=1}^{m'-1} l_{a,n}^+(\sigma^j\zeta).$$

We are now in position to prove Proposition 2-1. Denote by h and h_a the topological entropy of the principal suspension $(\Sigma(V, E, l), \sigma(l)_t)$ and the auxiliary suspension $(\Sigma(V_a, E_a, l_a), \sigma(l_a)_t)$. Let ε be an arbitrary small positive number. From the fact $h_a < h$, it follows that

$$\Lambda(\exp(-hl_a^+)) < \Lambda(\exp(-h_a l_a^+)) = 1 = \Lambda(\exp(-hl^+)),$$

so that for a sufficiently small $\varepsilon' > 0$,

$$\Lambda\big(\exp((\varepsilon' - h)l_a^+)\big) < 1 < \Lambda\big(\exp((\varepsilon' - h)l^+)\big) \quad \text{for all } a.$$

For later purpose, we take ε' with $(h - \varepsilon')(1 + \varepsilon')^{-1} > h - \varepsilon$. By Proposition 3-2,

$$\Lambda\big(\exp((\varepsilon' - h)l_{a,n}^+)\big) < 1 < \Lambda\big(\exp((\varepsilon' - h)l_n^+)\big)$$

for sufficiently large $n \geqslant n(\varepsilon')$.

The following lemma is an easy consequence of Proposition 3-2.

Lemma 4-4. *Let h_n be a positive number with $\lambda(e^{-h_n l_n^+}) = 1$. Then $\lim_{n\to\infty} h_n = h$.*

Take a large $n \geqslant n(\varepsilon')$ with $h - \varepsilon' < h_n$, and set

$$(\mathbf{V}^a, \mathbf{E}^a) = \big(V_a^{(n)}, E_a^{(n)}\big), \qquad a \geqslant 1.$$

We regard l_n^+ and $l_{a,n}^+$ as functions on $E_0^{(n)}$ and $\mathbf{E}^a$ respectively. By Lemma 1-10, we may take a subdivision $(\mathbf{V}^0, \mathbf{E}^0, (l_n^+)')$ with an involution κ of $(V_0^{(n)}, E_0^{(n)}, l_n^+)$ such that

$$\max(l_n^+)'/\min(l_n^+)' \leqslant 1 + \varepsilon'.$$

Define the correspondences Φ_a to be the compositions

$$\begin{aligned}\Phi_0\colon\quad &\{\text{cyles in } (\mathbf{V}^0, \mathbf{E}^0)\} \xrightarrow{\cong} \{\text{cycles in } (V_0^{(n)}, E_0^{(n)})\}\\ &\xrightarrow{\cong} \{\text{cycles in } (V_0, E_0)\} \xrightarrow{\Psi_0} \{\text{orbit cycles in } (UM, \phi_t)\}\\ &\xrightarrow{\cong} \{\text{closed geodesics in } M\},\end{aligned}$$

$$\begin{aligned}\Phi_a\colon\quad &\{\text{cycles in } (\mathbf{V}^a, \mathbf{E}^a)\} \xrightarrow{\cong} \{\text{cycles in } (V_a, E_a)\}\\ &\xrightarrow{\Psi_a} \{\text{orbit cycles in } (UM, \phi_t)\} \xrightarrow{\cong} \{\text{closed geodesics}\} \qquad (a \geqslant 1),\end{aligned}$$

and define a homomorphism φ: $H_1(\mathbf{V}^0, \mathbf{E}^0) \to H_1(M, \mathbb{Z})$ by the composition

$$H_1(\mathbf{V}^0, \mathbf{E}^0) \overset{\cong}{\to} H_1\left(V_0^{(n)}, E_0^{(n)}\right) \overset{-\omega_*^{(n)}}{\to} H_1(V_0, E_0)$$

$$\overset{(\iota_0)_*}{\to} H_1(UM, \mathbb{Z}) \to H_1(M, \mathbb{Z}).$$

Since Ψ_0 is bounded-to-one, and surjective, we get (1) in Proposition 2-1. Recalling that $\omega^{(n)}$ is an orientation-reversing morphism, we find that $[\Phi_0(\langle c \rangle)] = \varphi([c])$. From Proposition 4-1 and Lemma 1-8, it follows that $\varphi \circ \kappa_* = \varphi$. Define f_0 and f_a by

$$f_0 = \exp(\varepsilon' - h)(l_n^+)', \qquad f_a = \exp(\varepsilon' - h) l_{a,n}^+, \quad a \geqslant 1.$$

By Lemma 4-2, $f_0(\overline{\kappa(c)}) = f_0(c)$ for every closed path c. If we put $J = (1 + \varepsilon')\min(l_n^+)'$, then by Lemma 4-3, we find

$$l(\Phi_0(\langle c \rangle)) \leqslant l_n^+(c) = (l_n^+)'(c) \leqslant \left\{\max(l_n^+)'\right\}|c| \leqslant J|c|,$$

which is condition (2). Moreover,

$$f_0 < \exp\left\{(\varepsilon' - h)\min(l_n^+)'\right\} < \exp\left\{-(h - \varepsilon')(1 + \varepsilon')^{-1} J\right\}$$
$$< \exp\{-(h - \varepsilon) J\},$$

$$\lambda(f_0) > \lambda\left(e^{-h_n (l_n^+)'}\right) = \lambda\left(e^{-h_n l_n^+}\right) = 1$$

(Lemma 1-9), so that $\{f_a\}$ satisfies condition (4). Condition (6) comes from Lemma 4-3 if we put $\Theta = \max_a\{\max(l)/\min(l_a)\}$. This completes the proof of Proposition 2-1.

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