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SINGULARITIES OF A SIMPLE ELLIPTIC OPERATOR

H. P. McKEAN

1. Introduction

Consider the operator A: $f \to -\Delta f + \Lambda(f)$ acting upon real functions of $x \in D$ with $f(x) = 0$ for $x \in \partial D$. $D \subset R^d$ is a d-dimensional domain. Λ: $R \to R$ is a convex function in the strict sense: $\Lambda''(f) > 0$. Let $0 < \lambda_1(D) < \lambda_2(D) \leqslant \lambda_3(D) \leqslant$ etc. $\uparrow \infty$ be the spectrum of $-\Delta \,|\, D$. Ambrosetti-Prodi [1] and Berger-Church [2] proved that if

$$-\infty < \Lambda'(-\infty) < \lambda_1(D) < \Lambda'(+\infty) < \lambda_2(D),$$

then A is a *global fold*, i.e., up to diffeomorphisms front and back, it has the form[1]

$$(f_1, f_2, f_3, \cdots) \to (f_1^2, f_2, f_3, \cdots).$$

This attractive result prompted McKean-Scovel [4] to study the simplest example in which $\Lambda'(R)$ crosses the whole spectrum of $-\Delta \,|\, D$: to wit, $d = 1$ and $D = (0, 1)$ with $\Lambda(f) = f^2/2$, in which case A is the simple operator

$$B\colon f \to -f'' + f^2/2 \quad \text{subject to} \quad f(0) = f(1) = 0.$$

The present paper completes the description of the singularities of B.

Singular locus. This is the locus where the differential $dB = -d^2/dx^2 + f \equiv F$ is not $1:1$; it consists of a countable number of *sheets* M_n $(n \geqslant 1)$ indicated in Figure 1, marked off in function space by the vanishing of one of the (necessarily simple) eigenvalues

$$-\infty < \lambda_1(f) < \lambda_2(f) < \text{etc.} \to \infty$$

of F; in other words, $f \in M_n$ if and only if $Fe = 0$ has a solution $e = e_n$ with $e(0) = e(1) = 0$ and $n - 1$ more interior roots $e(r) = 0$ $(0 < r < 1)$. M_n is a smooth connected surface of codimension 1 with normal vector $\nabla \lambda_n = e_n^2$, the

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[1] $f \to (f_1, f_2, f_3, \cdots)$ is an assignment of coordinates in function-space.

eigenfunction being standardized by $\int_0^1 e_n^2 = 1$ and $e_n'(0) > 0$. M_1 is convex, while for $n \geqslant 2$, M_n has $n - 1$ principal directions of negative curvature; in fact, the spectrum of the second fundamental form at a point of M_n comprises the reciprocals of the distances to the other sheets in the direction e_n^2, these being reckoned negative for the higher sheets and positive for the lower.

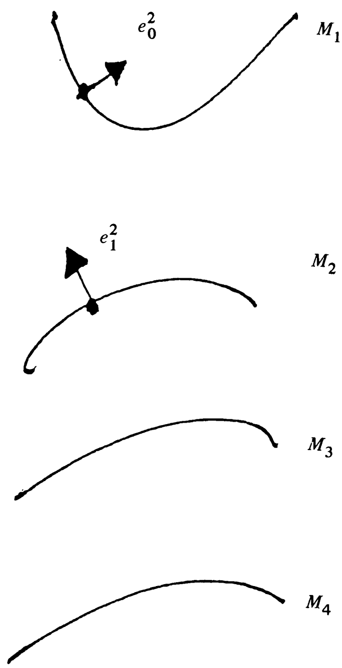

FIGURE 1

Local folds and cusps. F is of corank 1 at singular points $f \in M_n$ ($n \geqslant 1$), e_n being the null direction. For $n = 1$, e_1 is transverse to the sheet as it is of one signature which prevents the vanishing of

$$I_2 = \int_0^1 e_1^3 = \text{the inner product of } e_1 \text{ with the normal } e_1^2.$$

It follows that the map B is a *local fold*. More complicated singularities appear on M_2 already; in fact, M_2 and the higher sheets exhibit one and the same pattern of singularities, so we will deal with M_2 only. M_2 is most easily

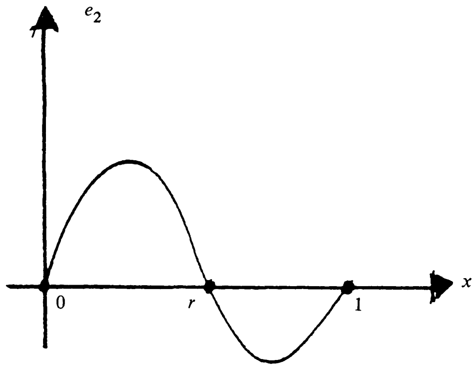

FIGURE 2

described as the class of functions $f = e_2''/e_2$ formed out of any realistic second eigenfunction, as in Figure 2. In particular, e_2, e_2'', and e_2''' but not e_2' vanish at the ends, $e_2(r) = 0$ at one more place $0 < r < 1$, $e_2'(r) < 0$, and $e_2''(r) = 0$. From this description, it is plain that $I_2 = \int_0^1 e_2^3$ is mostly different from 0 along M_2 so that e_2 is transverse to the sheet and the map B is a local fold, as for $n = 1$. I_2 *does* vanish on a sublocus $L_2 \subset M_2$ which is (1) *nonvoid* since e_2 may be taken antisymmetrical about $1/2$; (2) *smooth and of codimension* 1 *in* M_2 since,[2] with $e = e_2$, the gradient $\nabla I_2 = -3e(F - \lambda_2)^{-1}(e^2 - eI_2)$ is independent of the normal e^2 along L_2;[3] and (3) *connected*, by inspection. The type of singularity of B along L_2 differs according as $e = e_2$, which is tangent to M_2 along that locus, is or is not tangent to L_2 as well, i.e., according as

$$I_3 = -3\int_0^1 e_2^2 F^{-1} e_2^2 = \text{the inner product of } e_2 \text{ with } \nabla I_2$$

vanishes or not: if $I_3 \neq 0$, the map is a *local cusp*, i.e., up to diffeomorphisms front and back, it takes the form

$$(f_1, f_2, f_2, \cdots) \to (f_1 f_2 + f_1^3, f_2, f_3, \cdots),$$

while if $I_3 = 0$, which takes place along a sublocus L_3 of codimension 1 in L_2, then the singularity is more complicated. McKean-Scovel [4] left the matter there. The full description of these higher singularities comes next.

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[2] $(F - \lambda_2)^{-1}$ acts upon the annihilator of $e = e_2$. The computation of ∇I_2 rests upon the variational formula $(F - \lambda_2)e^\bullet = -\dot{f}e + \lambda_2^\bullet e$ in which $\lambda_2^\bullet = \int_0^1 e^2 \dot{f}$.

[3] $\nabla I_2 = -3eF^{-1}e^2$ along L_2 so that $\nabla I_2 = ce^2$ implies $-3e^2 = Fce = 0$.

Higher singularities. Let X be the vector field determined by $Xf = e_2$. Then, with $I_1 = \lambda_2$ and $e_2 = e$, you have

$$I_2 = \int_0^1 e^3 = \int_0^1 \nabla I_1 e = XI_1 \qquad \text{on } M_2,$$

$$I_3 = -3\int_0^1 e^2F^{-1}e^2 = \int_0^1 \nabla I_2 e = X^2 I_1 \quad \text{on } L_2,$$

and, more generally,

$$I_{n+1} \equiv X^n I_1 = XI_n = \int_0^1 \nabla I_n e \quad \text{by definition, for } n \geqslant 3;$$

see the table at the end of the paper for I_4 and I_5. Define $L_1 = M_2$, $L_2 = L_1 \cap (I_2 = 0)$, $L_3 = L_2 \cap (I_3 = 0)$, etc., so that, for any $n \geqslant 1$, L_{n+1} is the sublocus of L_n where $e = e_2$ is tangent to L_n.

Theorem. *Each of the loci $L_1 \supset L_2 \supset$ etc. is a nonvoid, smooth, connected submanifold of codimension* 1 *in the preceding locus. $L_\infty = \bigcap L_n$ is void so each point of M_2 belongs to $L_n - L_{n+1}$ for just one value of $n \geqslant 1$, the map B having at such points a local singularity of the type described by Morin* [5], *with normal form*

$$(f_1, f_2, f_3, \cdots) \to (f_1 f_2 + f_1^2 f_3 + \cdots + f_1^{n-1} f_n + f_1^{n+1}, f_2, f_3, \cdots),$$

i.e., for $n = 1$ a fold, for $n = 2$ a cusp, etc.

The important features of the singularities along L_n/L_{n+1} are as follows. $dB = F$ is of corank 1 so B may be brought into the local form

$$(1) \qquad (f_1, f_2, f_3, \cdots) \to (h(f_1, f_2, f_3, \cdots), f_2, f_3, \cdots)$$

by application of diffeomorphisms front and back, in which the first coordinate represents the singular direction $e = e_2$; in particular, $\partial/\partial f_1$ *represents the field X: $f \to e_2$ and $\partial h/\partial f_1$ is a smooth nonvanishing multiple of* $\lambda_2(f)$, as a routine comparison of differentials will confirm. Now $f \in L_n - L_{n+1}$ if and only if

(2) *$(\partial/\partial f_1)^m\, h$ vanishes for $m = 1, \cdots, n$ but not for $m = n + 1$, the successive vanishings taking place on smooth nested loci each of codimension* 1 *in the preceding.*

The final point is that

(3) *the gradients $\nabla[\partial h/\partial f_1], \cdots, \nabla[\partial^n h/\partial f_1^n]$ are independent on the locus indicated in* (2).

(1), (2), and (3) are precisely the conditions for B to have a singularity of the type displayed above (see Golubitsky-Guillemin [3, pp. 174–179]).

Summary. *B exhibits folds, cusps, and, more generally, Morin-type singularities of any degree $n + 1 \geqslant 2$ along nested subloci of codimension n on any singular sheet M_2, M_3, M_4, etc.*

Higher dimensions. (3) is equivalent to 2(1) below: *$\nabla I_1, \cdots, \nabla I_n$ are independent on L_n.* This, and its generalization 2(2), is all you need to extend the result to higher dimensions $d \geqslant 2$. A proof was not found, but see the end of §2 for comment.

2. The proof

The chief items to be checked are the following:

(1) $$\nabla I_1, \cdots, \nabla I_n \text{ are independent on } L_n.$$

(2) $$\nabla I_{n+1} \text{ is independent of } \nabla I_1, \cdots, \nabla I_n \text{ on } L_n.$$

(3) $$\begin{gathered} X\colon f \to e_2 \text{ can be integrated without obstruction and} \\ \lambda_2[\exp(tX)f] \text{ is analytic in } t \geqslant 0. \end{gathered}$$

(2) implies (1) but it is convenient to prove them in the stated order. This is postponed to the end.

The loci are nonvoid, connected, and smooth. (1) implies that each of $L_1 \subset L_2 \subset L_3$ etc. is a smooth submanifold of codimension 1 in the last, provided it is nonvoid. We assume that L_n *is nonvoid and connected* as is known already for $n = 1$, and prove that L_{n+1} *is the same.* (2) permits us to form the vector field

$$Zf = \frac{\text{the coprojection of } \nabla I_{n+1} \text{ upon } \nabla I_1, \cdots, \nabla I_n}{\text{its length in } L^2[0,1]},$$

and it is easy to see that you can integrate freely in L_n, i.e., $\partial f/\partial t = Zf$ can be solved backwards and forwards if $f_0 \in L_n$, the solution $f = \exp(tZ)f_0$ being confined to L_n in view of $ZI_1 = \cdots = ZI_n = 0$. Now $ZI_{n+1} = 1$, so I_{n+1} vanishes just once along each stream line of the flow; in particular, L_{n+1} *is not void.* Besides, any two points of L_{n+1} can be joined by a path in L_n, the latter being connected, and each point of this path can be swept along by the Z-flow until it hits L_{n+1}, producing a new path with the old end points but lying wholly in L_{n+1}: in brief, L_{n+1} *is connected.* A bit more detail about the flow may be welcome: $|Zf_2 - Zf_1|$ is majorized by a multiple of $|f_2 - f_1|_\infty$ in the small. This ensures local solvability. The global solvability follows easily from $|Zf|_2 = 1$ and (2).

L_∞ is void by reason of analyticity. The same reasoning shows that X: $f \to e_2$ can be integrated without obstruction. This is part of (3) and the rest of (3) is easy, too. Let e_0 be the solution of $Fe_0 = \lambda e_0$ with $e(0) = 0$, $e'(0) = 1$, and adjustable λ. Then

$$e_0(x) = x + \sum_{n=1}^{\infty} \text{the integral operator } \int_0^x da \int_0^a [f(b) - \lambda]\, db$$
$$\text{applied } n\text{-fold to } x,$$

and this is analytic in λ and f, by inspection. The number λ_2 is the second, necessarily real and simple, root of $e_0(1) = 0$ for fixed f; as such, it is analytic in f, and this feature is inherited by

$$e_2 = \frac{e_0 \text{ computed for } \lambda = \lambda_2}{\sqrt{\int_0^1 e_0^2}} = Xf.$$

Now it is routine to check that $\lambda_2[\exp(tX)f_0]$ is analytic in $t \geqslant 0$, finishing the proof of (3). The application to L_∞ is this: $f_0 \in L_\infty$ means $X^n\lambda_2 = 0$ for $n \geqslant 0$. Then $\lambda_2[\exp(tX)f_0] = 0$ for $t \geqslant 0$ by analyticity, and this is impossible: with $f = \exp(tX)f_0$, you find $(Bf)^\bullet = dB\, f^\bullet = Fe_2 = \lambda_2 e_2 = 0$, violating the fact that $Bf = Bf_0$ has only a finite number of distinct solutions [4]: in short L_∞ *is void*. The discussion is finished except for the proofs of (1) and (2).

Proof of (1). $I_{n+1} = XI_n = \int_0^1 \nabla I_n e_2$ for $n \geqslant 1$, whence

$$\nabla I_{n+1} = (e_2 \cdot \nabla)\nabla I_n - e_2(F - \lambda_2)^{-1}(\nabla I_n - e_2 I_{n+1})$$

in view of the variational formula

$$\begin{aligned} I_{n+1}^\bullet &= \int_0^1 \left[\nabla I_n^\bullet e_2 + \nabla I_n e_2^\bullet\right] \\ &= \int_0^1 \left[e_2 \nabla^2 I_n f^\bullet - \nabla I_n (F - \lambda_2)^{-1}\left(e_2^\bullet f - e_2 \int_0^1 e_2^2 f^\bullet\right)\right] \\ &= \int_0^1 f^\bullet\left[\nabla^2 I_n e_2 - e_2\left(F - \lambda_2^{-1}\right)\left(\nabla I_n - e_2 \int_0^1 e_2 \nabla I_n\right)\right]. \end{aligned}$$

Now $e_2(x) > 0$ for $0 < x < r$, $r < 1$ being its only interior root, and with $e = e_2$ for brevity, a brief computation confirms that

$$(F - \lambda_2)^{-1}\colon g \to -e\int_0^x e^{-2} \int_0 eg \quad \text{for } x < r, \text{ modulo } e.$$

Besides,

$$(e \cdot \nabla)e = -(F - \lambda_2)^{-1}\left(e^2 - e\int_0^1 e^3\right) = -F^{-1}e^2 \quad \text{for } f \in M_2 \text{ and } I_2 = 0,$$

so you can evaluate ∇I_1, ∇I_2, etc. as follows:

$$\nabla I_1 = e^2;$$

$$\nabla I_2 = (e \cdot \nabla - eF^{-1})\nabla I_1 \quad \text{if } I_1 = I_2 = 0$$

$$= -3eF^{-1}e^2 = 3e^2\int_0^x e^{-2}\int_0 e^3 \quad \text{for } x < r \text{ modulo } \nabla I_1;$$

$$\nabla I_3 = (e \cdot \nabla - eF^{-1})\nabla I_2 \quad \text{if } I_1 = I_2 = I_3 = 0$$

$$= -6e(F^{-1}e^2)\int_0^x \dot{e}^{-2}\int_0 e^3 + 6e^2\int_0^x e^{-3}(F^{-1}e^2)\int_0 e^3$$

$$-9e^2\int_0^x e^{-2}\int_0 e^2F^{-1}e^2$$

$$-3eF^{-1}e^2\int_0 e^{-2}\int_0 e^3 \quad \text{for } x < r \text{ modulo } \nabla I_1 \text{ and } \nabla I_2$$

$$= 12e^2\int_0^x e^{-2}\int_0 e^3\int_0 e^{-2}\int_0 e^3$$

$$+3e^2\int_0^x e^{-2}\int_0 e^3 \cdot \int_0^x e^{-2}\int_0 e^3 \quad \text{with the same proviso.}$$

The pattern has been set: if $f \in L_n$ and if $x < r$, then, modulo lower gradients, ∇I_n is a sum of iterated and/or multiplied double integrals of the form

$$e^2\int_0 e^{-2}\int_0 e^3\int_0 e^{-2}\int_0 e^3 \cdots \int_0 e^{-2}\int_0 e^3 \qquad (n-1) \text{ fold};$$

see the table for I_n. The proof is by induction, starting from $n = 1$. ∇I_n is taken in the stated form on L_n:

$$\nabla I_n = \sum e^2\int_0 e^{-2}\int_0 e^3 \cdots \int_0 e^{-2}\int_0 e^3 \ \ (n-1) \text{ fold} + \sum_{j<n} c_j \nabla I_j.$$

$e \cdot \nabla - eF^{-1}$ is now applied at $L_{n+1} \subset L_n$ to produce the desired expression

$$\nabla I_{n+1} = \sum e^2\int e^{-2}\int_0 e^3 \cdots \int_0 e^{-2}\int_0 e^3 \ n\text{-fold} + \sum_{j \leqslant n} c_j' \nabla I_j$$

by virtue of the following remarks:

(a) Modulo lower gradients, $e \cdot \nabla$ produces from the sum of $(n-1)$ fold integrals a new sum of n fold integrals plus a fixed multiple of the old sum, which is nothing but ∇I_n.

(b) $-eF^{-1}$ does the same except the added term is simply a multiple of $e^2 = \nabla I_1$.

(c) $(e \cdot \nabla - eF^{-1})\nabla I_j = \nabla I_{j+1}$ for $j < n$.

(d) The numbers c_j ($j < n$) are smooth functions of f, as you will check inductively, and these contribute, under the action of $e \cdot \nabla$, the quantity $(\int_0^1 \nabla c_j e)\nabla I_j$ to the expression for ∇I_{n+1}.

The proof of (1) can now be finished: if ∇I_{n+1} is dependent upon $\nabla I_1, \cdots, \nabla I_n$ at a point of L_{n+1}, then with new coefficients c_j ($j \leqslant n$),

$$\sum e^2 \int_0 e^{-2} \int_0 e^3 \cdots \int_0 e^{-2} \int_0 e^3 \; n \text{ fold}$$

$$= \sum_{1 \leqslant j \leqslant n} c_j \sum e^2 \int_0 e^{-2} \int_0 e^3 \cdots \int_0 e^{-2} \int_0 e^3 \; (j-1) \text{ fold}$$

for $x < r$. But the product of e^2 and a $(j-1)$ fold integral is of the form $[e'(0)]^{j+1} x^d [1 + o(1)]$ at $x = 0$ with $d = 2 + 3(j-1) = 3j - 1$, and as no cancellation can occur between such quantities, so $c_1, c_2, c_3, \cdots, c_n$ are seen, in that order, to vanish, and a contradiction is obtained.

Proof of (2). ∇I_{n+1} is now computed, not as before on L_{n+1}, but at any point of L_n. The resulting expression

$$\nabla I_{n+1} = (e \cdot \nabla)\nabla I_n - eF^{-1}(\nabla I_n - eI_{n+1})$$

contains just one novel term $I_{n+1} \times e^2 \int_0^1 e^{-2} \int_0 e^2$ not seen in the former computation; luckily, it is of degree 4 at $x = 0$ and 4 is not of the form $3j - 1 = 2, 5, \cdots$ so that the former conclusion (to wit, the independence of ∇I_{n+1}) is unchanged.

Higher dimensions. The present proof of (1) is hopeless in higher dimensions: it depends upon expressing the operator $F^{-1}e^2$, modulo e, in the form $-e\int_0^x e^{-2} \int_0 e^3$ near $x = 0$ in case $\int_0^1 e^3 = 0$, and this has no counterpart.

Example. $D = (0,1) \times (0, 1/2) \subset R^2$. The operator $F = -\nabla - 8\pi^2$ has ground state $\lambda_{11} = -3\pi^2$, simple 2nd eigenvalue $\lambda_{21} = 0$ with eigenfunction $e = e_{21} = 2\sqrt{2}\,\sin(2\pi x_1)\sin(2\pi x_2)$ subject to $I_2 = \int e^3 = 0$, and

$$F^{-1}e^2 = \sum_{(m,n) \neq (2,1)} \pi^{-2}(m^2 + 4n^2 - 8)^{-1} e_{mn} \otimes e^2\, e_{mn}$$

with $e_{mn} = 2\sqrt{2}\,\sin(\pi m x_1)\sin(2\pi n x_2)$ for general $m, n \geqslant 1$, from which it is easy to see that $F^{-1}e^2 = ce + o(e)$ is not possible near any of the sides of D: for example, if this happens along $x_1 = 0$, then $\partial F^{-1}e^2/\partial x_1 = 2\pi c\,\sin(2\pi x_2)$ for $x_1 = 0$ and $0 < x_2 < 1/2$, and explicit computation reveals that this is not so.

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3. Table

The functionals $I_1 = \lambda_2$, $I_2 = \int_0 e^3$, etc. and their gradients are recorded up to I_5.

n	I_n on L_{n-1}	∇I_n on L_n	∇I_n on L_n for $x < r$, modulo lower gradients
1	λ_2	e^2	e^2
2	$\int_0^1 e^3$	$-3eF^{-1}e^2$	$3e^2 \int_0^x e^{-2} \int_0 e^3$
3	$-3\int_0^1 e^2F^{-1}e^2$	$3 \times 4(eF^{-1})^2e^2$ $+ (F^{-1}e^2)^2$	$3e^2 \times 4\int_1^x e^{-2}\int_0 e^3 \int_0 e^{-2} \int_0 e^3$ $+(\int_0^x e^{-2}\int_0 e^3)^2$
4	$15\int_0^1 e(eF^{-1})^2e^2$	$-15 \times 4(eF^{-1})^3e^2$ $+ 2(F^{-1}e^2)(F^{-1}eF^{-1}e^2)$ $+ eF^{-1}(F^{-1}e^2)^2$	$15e^2 \times 4\int_0^x e^{-2}\int_0 e^3 \int_0 e^{-2} \int_0 e^3 \int_0 e^{-2} \int_0 e^3$ $+ 2(\int_0^x e^{-2}\int_0 e^3)(\int_0^x e^{-2}\int_0 e^3\int_0 e^{-2}\int_0 e^3)$ $+\int_0^x e^{-2}\int_0 e^3(\int_0 e^{-2}\int_0 e^3)^2$
5	-15 $\times 6\int_0^1 e(eF^{-1})^3e^2$ $+\int_0^1 (F^{-1}e^2)^3$	$45 \times 8(eF^{-1})^4e^2$ $+4(F^{-1}e^2)(F^{-1}(eF^{-1})^2e^2)$ $+4eF^{-1}(F^{-1}e^2)(F^{-1}eF^{-1}e^2)$ $+2eF^{-1}eF^{-1}(F^{-1}e^2)^2$ $+2(F^{-1}eF^{-1}e^2)^2$ $+(F^{-1}e^2)F^{-1}(F^{-1}e^2)^2$	

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