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GEODESIC LENGTH FUNCTIONS AND THE NIELSEN PROBLEM

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0. Introduction

Fundamental to the synthetic geometry of a space is the behavior of geodesics. The geometry of a noncomplete metric can be rather disappointing: points may fail to be joined by geodesics and length minimizing curves may fail to exist. The Weil-Petersson metric for Teichmüller space is not complete [21]. Nevertheless we shall show that its synthetic geometry is quite similar to that of a complete metric of negative curvature. Our main result is that every pair of points is joined by a unique geodesic. Also we find that Teichmüller space has an exhaustion by compact Weil-Petersson convex sets. In particular the exponential map is a homeomorphism from its domain to Teichmüller space, the analogue of the Hadamard-Cartan theorem, and furthermore the exponential map is distance increasing, a standard result for complete negative curvature metrics. We also show that a finite group of isometries has a fixed point, the analogue of the Cartan center of mass result. A solution of the Nielsen problem is an immediate corollary (see §6): every finite subgroup of the mapping class group fixes a point of Teichmüller space [12].

For the study of geodesics completeness is used to bound sequences and arcs away from infinity in the one-point compactification of a space. Our approach is to substitute a natural class of proper convex functions for completeness. The functions are the geodesic length functions introduced by Fricke-Klein [7], later studied by Fenchel-Nielsen [6], Keen [11], Kerckhoff [12], [13], Thurston [19], the author [22], [23], [24], [27], as well as others. Our main result on this topic is that a geodesic length function is strictly convex along a Weil-Petersson geodesic. In fact our results for the Weil-Petersson metric are based on three observations: that proper geodesic length functions exist, their

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convexity along Weil-Petersson geodesics, and the negative sectional curvature of the Weil-Petersson metric [20], [26] (see §§5 and 6).

We wish to compare and contrast the current convexity result with Kerckhoff's original observation that the geodesic length functions are convex along earthquake paths [12]. By way of comparison a positive proper sum of length functions $\sum_j a_j l_{\alpha_j}$, $a_j > 0$, has a (unique) minimum, where by the earthquake convexity the second derivative is positive definite. Recall that the second derivative of a function at a critical point is coordinate independent (a symmetric, 2-tensor), i.e., independent of the introduction of earthquakes. Now heuristically by varying the a_j the minimum point for the sum varies in an open set and again by the earthquake convexity the second derivative is positive definite at each such minimum. Thus, the convexity of the length functions is partially independent of the use of earthquakes. Now by way of contrast we remind the reader that not even the signature of the second derivative is an intrinsic quantity on a smooth manifold. An example is in order: the function $f(x) = x^2$, $x \in \mathbf{R}^+$, is convex but becomes concave after the simple change of variables $x(t) = t^{1/4}$. There is some evidence of this phenomenon even in the current situation. A geodesic length function is constant along an earthquake path provided α does not transversely intersect the underlying lamination. By contrast a geodesic length function is strictly convex along all Weil-Petersson geodesics.

Now, to get an idea of the Weil-Petersson geometry, we follow a suggestion of Thurston for describing the metric at infinity in the complex dimension one case. The moduli space for once punctured tori is the classical quotient $H/\mathrm{PSL}(2; \mathbf{Z})$. By a result of Masur if a neighborhood of infinity is modeled by the punctured disc $\mathrm{PD} = \{z \mid 0 < |z| < 1\}$, then the Weil-Petersson metric is comparable to $|dz|^2/|z|^2(\log 1/|z|)^3$ at the origin [16]. By comparing longitudes and meridians one finds that the surface of revolution S of $\{y = x^3, x > 0\}$ in $\mathbf{R}^3$ has the same asymptotic behavior at the origin. A longitude of S has geodesic curvature $\sim x$ for small x while the meridian has radius x^3 and geodesic curvature $\sim x^{-3}$; thus the curvature of S behaves as $-x^{-2}$ for small x. Indeed S has negative curvature, is geodesically convex, and obviously not complete.

The manuscript is organized into six sections. The first is a review of Teichmüller theory, specifically the theory for first-order deformations: the tangent and cotangent space of Teichmüller space, the Weil-Petersson metric, and the construction of normal coordinates by harmonic Beltrami differentials. In §2 we present a new technique using line integrals for solving the Beltrami equation. The nth order differential geometry of Teichmüller space is given by calculating approximate solutions of the Beltrami equation valid to nth order. For instance the tangent and cotangent space are first order, while of course curvature is second order. There are many techniques valid for first order calculations. Unfortunately this is not true for higher order, even for second order. As a practical matter it is essential to have a technique with which one can actually compute. The basis for our approach is the observation that an analogy exists between Eichler integrals and the Beltrami equation. For a hyperbolic surface of finite area and harmonic differentials the analogy provides a solution of the Beltrami equation by a line integral. Thus the standard 2-dimensional integral is replaced by a much simpler 1-dimensional integral. An immediate consequence is that for a series expansion of a Beltrami differential, convergent uniformly on compact sets, we may solve the equations term-by-term. This was not possible by the previous techniques which would require L^∞ convergence of the series.

§3 contains the initial discussion of the geodesic length functions. A general formula for the second derivative of a geodesic length function is given. In §4 we calculate the Weil-Petersson Hessian of a geodesic length function. Our result is that the Hessian is positive definite. Given a Fuchsian group Γ we conjugate to ensure that a transformation $z \to \lambda z$ is contained in Γ. With this normalization a holomorphic, Γ invariant, quadratic differential has a series expansion

$$\frac{1}{z^2}\sum_n a_n z^{\varepsilon n}, \qquad \varepsilon = \frac{2\pi i}{\log \lambda} \text{ for } z \in H.$$

Observe that the individual terms are quite simple. A harmonic Beltrami differential has a similar expansion also with simple terms. Our solution of the second order Beltrami equation is given by integrating the individual terms, an elementary procedure. §§5 and 6 are of a more general nature; the reader could assume the convexity result and start with §5. For the main result, geodesic length functions are substituted for completeness in the proof of the Hadamard-Cartan theorem: the exponential map is a homeomorphism. We also find that the exponential map is distance increasing and that a length minimizing sequence converges to a geodesic. Finally in §6 applications are considered. For a Kähler metric, such as the Weil-Petersson metric, the Riemannian and complex Hessians are readily obtained from each other. A consequence is that a geodesic length function is also strictly plurisubharmonic. Since proper length functions exist, a new proof is obtained that Teichmüller space is a Stein manifold [9]. Finally we apply Cartan's center of mass argument: a finite group of Weil-Petersson isometries fixes a point.

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1. A Review of Teichmüller theory and the Weil-Petersson geometry

1.1. For the sake of clarification we shall give a sketch of the necessary background material. The first item is the solution of the Beltrami differential equation [3]. With this as a basis we recall the constructions of holomorphic coordinate charts, the Weil-Petersson metric, and the description of the Levi-Civita connection via harmonic Beltrami differentials [1].

1.2. We shall recall the normalizations for solutions of the Beltrami equation and the dependence of solutions on parameters. Consider a hyperbolic surface R of finite area uniformized by a discrete group $\Gamma \subset \mathrm{PSL}(2;\mathbf{R})$ acting on the upper half-plane H. Denote by $B(\Gamma)$ the complex Banach space of Γ invariant tensors of type $\partial/\partial z \otimes d\bar{z}$ on H with measurable coefficients and finite L^∞ norm. Denote by $Q(\Gamma)$ the complex Banach space of Γ invariant holomorphic tensors of type $dz \otimes dz$ with finite L^1 norm on H/Γ. To $\mu \in B(\Gamma)$, $\|\mu\|_\infty < 1$, we associate two normalized solutions f^μ, w^μ of the Beltrami equation:

$$\begin{aligned} f_{\bar z} &= \mu f_z, && z \in H,\\ f_{\bar z} &= \overline{\mu(\bar z)}\, f_z, && z \in L, \\ f & && \text{fixes } 0, 1, \text{ and } \infty; \end{aligned} \tag{1.1}$$

$$\begin{aligned} w_{\bar z} &= \mu w_z, && z \in H,\\ w_{\bar z} &= 0, && z \in L, \\ w & && \text{fixes } 0, 1, \text{ and } \infty. \end{aligned} \tag{1.2}$$

The reader will recall that $f^\mu\Gamma(f^\mu)^{-1}$ is a Fuchsian group, while in general $w^\mu\Gamma(w^\mu)^{-1}$ is a quasi-Fuchsian group. The μ dependence of solutions is studied in [4] and the main result is most readily stated for solutions $\hat{f}^\mu$ on the unit disc D. In particular, define $\hat{f}^\mu = A^{-1}f^\mu A$ for A the conformal transformation of D to H carrying $(-i, 1, i)$ to $(0, 1, \infty)$. Ahlfors-Bers state their result in terms of the Banach space AB^p: consider those functions f continuous on D with distributional derivatives f_z and $f_{\bar z}$ and finite AB^p norm:

$$\|f\|_{\mathrm{AB}^p} = \sup_{z_1, z_2} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^{1-2/p}} + \|f_z\|_p + \|f_{\bar z}\|_p,$$

with $\|\ \|_p$ the standard L^p norm.

Theorem 1.1. *Given $p > 2$ there exists an $\varepsilon = \varepsilon(p)$, $\varepsilon > 0$, such that if $\mu(t)$ varies real analytically in L^∞ with $\|\mu(t)\|_\infty < \varepsilon$, then $\hat{f}^{\mu(t)}$ varies real analytically in* AB^p.

1.3. Let $T_{g,n}$ be the Teichmüller space of genus g, n punctured surfaces. The description of Teichmüller space by Beltrami differentials is based on the existence of an *exponential map* Φ: $B_1(\Gamma) \to T_{g,n}$ given by associating to $\mu \in B(\Gamma)$, $\|\mu\|_\infty < 1$, the (equivalence class of the marked) surface $H/f^\mu\Gamma(f^\mu)^{-1} \in T_{g,n}$. Φ is a holomorphic submersion at the origin and thus the complex tangent space of $T_{g,n}$ at H/Γ is characterized as $B(\Gamma)/\mathrm{Ker}\, d\Phi$. It should be no surprise to the reader that the explicit computation of $d\Phi$ is a fundamental question of Teichmüller theory. We now review the basic approach [1], [3].

There is an integral pairing of $B(\Gamma)$ and $Q(\Gamma)$: given $\mu \in B(\Gamma)$ and $\phi \in Q(\Gamma)$ associate $(\mu, \phi) = \int_{H/\Gamma} \mu\phi$. If we introduce $N(\Gamma) = Q(\Gamma)^\perp \subset B(\Gamma)$, then with the obvious notation the main result is the following

Theorem 1.2. $N(\Gamma) = \mathrm{Ker}\, d\Phi$.

Corollary 1.3. *$B(\Gamma)/N(\Gamma) \sim T^{1,0}T_{g,n}$, $Q(\Gamma) \sim (T^{1,0})^*T_{g,n}$ and the natural pairing of tangent and cotangent spaces becomes* $B(\Gamma)/N(\Gamma) \times Q(\Gamma) \xrightarrow{(\,,\,)} \mathbf{C}$.

It is clear from the above that in order to describe local coordinates on $T_{g,n}$ we start with $\mu_1, \cdots, \mu_d$ whose $N(\Gamma)$ cosets form a complex basis for $B(\Gamma)/N(\Gamma)$. By general principles a neighborhood of the origin in the span of $\{\mu_j\}$ is mapped by Φ biholomorphically to its image in $T_{g,n}$. We shall describe a holomorphic coordinate chart $\tilde{\Phi}$ mapping a neighborhood U of the origin $\mathbf{C}^d$ to an open set in $T_{g,n}$. First pick U sufficiently small to ensure that $\|\mu(t)\|_\infty < 1$ for $t = (t_1, \cdots, t_d) \in U$, where $\mu(t) = \sum_{j=1}^d t_j\mu_j$. A coordinate mapping $\tilde{\Phi}$: $U \to T_{g,n}$ is given simply by $\tilde{\Phi}(t) = [H/\Gamma^{\mu(t)}]$, that is the tuple t is mapped to the equivalence class of $H/f^{\mu(t)}\Gamma(f^{\mu(t)})^{-1}$.

An essential point for calculations is to have a description of the holomorphic coordinate vector fields for the chart $\tilde{\Phi}$. To obtain the description we consider for $\mu, \nu \in B(\Gamma)$, $\|\mu\|_\infty < 1$, the expression

$$L^\mu\nu = \left(\frac{\nu}{1 - |\mu|^2}\frac{f_z^\mu}{\overline{f_z^\mu}}\right) \circ (f^\mu)^{-1}.$$

The holomorphic coordinate fields of $\tilde{\Phi}$ are given as

$$\left.\frac{\partial}{\partial t_j}\right|_t = \left(L^{\mu(t)}\mu_j\right) \bmod N(\Gamma^{\mu(t)}) \in B(\Gamma^{\mu(t)})/N(\Gamma^{\mu(t)})$$

for $t \in U$. As a special case we observe that the tangent field of the curve $\Phi(\varepsilon\mu) \subset T_{g,n}$, ε small, $\mu \in B(\Gamma)$, is given by $L^{\varepsilon\mu}\mu$.

1.4. An obvious method for studying the quotient space $B(\Gamma)/N(\Gamma)$ is to choose a representative from each coset. A natural choice is to consider the harmonic Beltrami differentials. If $\mathrm{HB}(\Gamma)$ is the subspace of harmonic Beltrami differentials then the natural map i: $\mathrm{HB}(\Gamma) \to B(\Gamma)/N(\Gamma)$, induced by the inclusion, is a complex linear isomorphism. Consequently $\mathrm{HB}(\Gamma)$ provides an alternate model for the tangent space at $[H/\Gamma]$ of $T_{g,n}$. Furthermore the Weil-Petersson metric is easily described on $\mathrm{HB}(\Gamma)$.

A Beltrami differential $\mu \in B(\Gamma)$ on H is harmonic provided there exists a $\phi \in Q(\Gamma)$ such that $\mu = (z-\bar{z})^2\overline{\phi(z)}$. The Weil-Petersson Hermitian pairing on $\mathrm{HB}(\Gamma)$ is simply

$$\langle \mu, \nu \rangle = \int_{H/\Gamma} \mu \bar{\nu}\, dA$$

for $\mu, \nu \in \mathrm{HB}(\Gamma)$ and dA is the area element of the hyperbolic metric [1], [2]. Recall that the metric is Kähler: by definition $g_{\mathrm{WP}}(\mu, \nu) = 2\,\mathrm{Re}\langle \mu, \nu \rangle$ is the Riemannian pairing and $\omega_{\mathrm{WP}}(\mu, \nu) = -2\,\mathrm{Im}\langle \mu, \nu \rangle$ is the Kähler form. For the sake of background we recall that the metric is not complete [21], has negative sectional curvature [20], [26], and that the Kähler form defines a projective embedding of $\overline{\mathscr{M}}_g$, the moduli space of stable curves [25].

The starting point for us will be the local Riemannian geometry. Choose $\mu_1, \cdots, \mu_d \in \mathrm{HB}(\Gamma)$ to be a unitary frame and consider the coordinate chart $\tilde{\Phi}\colon \mathbf{C}^d \to T_{g,n}$ given by $\tilde{\Phi}(t) = [H/\Gamma^{\mu(t)}]$, $\mu(t) = \sum_{j=1}^{d} t_j \mu_j$. In $\tilde{\Phi}$ local coordinates the Weil-Petersson metric is given in a special form [1], [2]:

$$g_{\mathrm{WP}} = 2\sum_j |dt_j|^2 + O(|t|^2).$$

Harmonic Beltrami differentials define normal coordinates. In particular the first derivatives of g_{WP} and thus the Christoffel symbols vanish at $t = 0$, equivalently the Levi-Civita connection is Euclidean at the origin. The reader will recall that the curvature tensor involves the second derivatives of the metric and thus these terms are not zero by virtue of negative curvature. The benefit of choosing unitary normal coordinates is that an expression at $t = 0$ involving at most the first derivatives of the metric is an intrinsic quantity, i.e. independent of the coordinate chart. We shall consider four such quantities for the above defined t normal coordinates at $t = 0$.

Geodesics are the first example. Since the connection is Euclidean at the origin, a line $\gamma(\tau) \subset \mathbf{C}^d$, $\tau \in \mathbf{R}$, through the t origin will have order 2 contact with a geodesic [17]. The second example is the Riemannian Hessian, Hess_f, of a function f on $\mathbf{C}^d$. The directional derivative $d^2 f(\gamma(\tau))/d\tau^2$ at $\tau = 0$ is the Hessian (a symmetric 2-tensor) evaluated as $\mathrm{Hess}_f(\dot{\gamma}(0), \dot{\gamma}(0))$ [17]. To see this recall that $\mathrm{Hess}_f(U, V) = UVf - (D_U V)f$ for D covariant differentiation and U, V vector fields. Since the t coordinates are normal, $D_U V = 0$ at 0 for U, V coordinate vector fields. The third example is the Laplacian. By definition the Laplacian is the metric trace of the Hessian and in as much as the metric is

Euclidean at $t = 0$ this is just the trace in local coordinates [17]. For the final example we consider a tensor defined from the complex structure. The complex Hessian of a function h is the exterior 2-form

$$\frac{i}{2}\partial\bar{\partial}h = \sum_j h_{t_j\bar{t}_j}\frac{i}{2}\, dt_j \wedge d\bar{t}_j.$$

For $t = u + iv$ certainly $h_{t\bar{t}} = h_{uu} + h_{vv}$, a sum of terms from the Riemannian Hessian.

Our plan is to show that the second directional derivative of a geodesic length function l_α is positive along a coordinate axis for coordinates given by harmonic Beltrami differentials. The two results, that a geodesic length function is convex along Weil-Petersson geodesics and that a geodesic length function is strictly plurisubharmonic, follow by the above discussion. In fact since a harmonic Beltrami differential of unit norm can be included in a unitary basis it will suffice to consider the directional derivative

$$\frac{d^2 l_\alpha(R^\varepsilon)}{d^2\varepsilon}, \qquad R^\varepsilon = H/f^{\varepsilon\mu}\Gamma(f^{\varepsilon\mu})^{-1}, \quad \text{at } \varepsilon = 0,\ \mu \in \mathrm{HB}(\Gamma).$$

The calculation starts in §3.3 and is completed in §4.

2. Solution of the Beltrami equation by line integrals

2.1. A holomorphic quadratic differential $\phi(z)$, $z \in H$, determines a harmonic Beltrami differential $\mu = (z - \bar{z})^2\bar{\phi}$ and the vector field F on H giving the infinitesimal deformation, where $F_{\bar{z}} = \mu$. If ϕ is expanded in an infinite series, one is naturally tempted to try to solve the potential equation $F_{\bar{z}} = \mu$ term-by-term. The standard convergence estimates are for μ in L^p and certainly do not permit such an approach. Our key observation is that for μ harmonic a solution of the above equation is analogous to the classical Eichler integral [15]. We exploit the analogy to give a solution by a line integral. Consequently, for a series expansion of ϕ we can solve the potential equation term-by-term. The formalism of Teichmüller theory requires the standard solutions (1.1) and (1.2) of the Beltrami equation. Before considering applications we must express the standard solutions in terms of the line integral. This will not be done by direct computation; rather the standard solutions are characterized by their formal properties. Accordingly we start with a review of the standard solutions [3].

2.2. Consider the solutions of the Beltrami equation in the infinitesimal case, namely $\dot{w}[\mu] = dw^{\varepsilon\mu}/d\varepsilon$ (see (1.2)) and $\dot{f}[\mu] = df^{\varepsilon\mu}/d\varepsilon$ (see (1.1)) at $\varepsilon = 0$ [3],

[4]. The solutions are uniquely characterized for $\dot{w}[\mu]$ by

$$\begin{array}{ll} \dot{w}_{\bar{z}} = \mu & \text{on } H, \\ \dot{w}_{\bar{z}} = 0 & \text{on } L, \\ \dot{w} & \text{vanishes at } 0, 1 \text{ and is} \\ & o\left(|z|^2\right) \text{ for } |z| \text{ large,} \end{array}$$

and for $\dot{f}[\mu]$ by

$$\begin{array}{ll} \dot{f}_{\bar{z}} = \mu & \text{on } H, \\ \dot{f}_{\bar{z}} = \overline{\mu(\bar{z})} & \text{on } L, \\ \dot{f} & \text{vanishes at } 0\ 1 \text{ and is} \\ & o\left(|z|^2\right) \text{ for } |z| \text{ large.} \end{array}$$

It follows immediately that $\dot{f}(z) = \dot{w}(z) + \overline{\dot{w}(\bar{z})}$. Similarly it is not difficult to verify that $(\dot{w}(\gamma)/\gamma' - \dot{w})$ is a quadratic polynomial for $\mu \in B(\Gamma)$ and $\gamma \in \Gamma$, where a $'$ will be used to denote the derivative of a holomorphic function. Also an important observation, related to the Bers embedding, is the equation $\dot{w}[(z-\bar{z})^2\bar{\phi}(z)]''' = 2\bar{\phi}(\bar{z})$, $z \in L$.

An alternative approach for infinitesimal deformations is to consider vector fields on the universal cover. Classically this approach becomes the study of the Eichler integral [15]. For $\psi(u)$ a holomorphic quadratic differential on L consider the integral

$$E(v) = \int_{v_0}^{v} (v-u)^2 \psi(u)\, du, \tag{2.1}$$

u, v, v_0 in L and v_0 fixed. It is immediate that $E''' = 2\psi$, a basic equation. E is to be considered as a holomorphic vector field. Provided ψ is Γ invariant the period $P_\gamma = E(\gamma)/\gamma' - E$, $\gamma \in \Gamma$, represents the infinitesimal perturbation of γ (the Lie algebra of SL(2; $\mathbf{C}$) is quadratic polynomials, accordingly p_γ is a quadratic polynomial). The period is given by an elementary integral as we now recall:

$$\begin{aligned} E(\gamma)/\gamma' - E &= \int_{v_0}^{\gamma v} \frac{(\gamma v - u)^2}{\gamma'(v)} \psi(u)\, du - \int_{v_0}^{v} (v-u)^2 \psi(u)\, du \\ &= \int_{\gamma^{-1}v_0}^{v} \frac{(\gamma v - \gamma u)^2}{\gamma'(v)\gamma'(u)} \psi(u)\, du - \int_{v_0}^{v} (v-u)^2 \psi(u)\, du \qquad (2.2) \\ &= \int_{\gamma^{-1}v_0}^{v_0} (v-u)^2 \gamma(u)\, du = P_\gamma, \end{aligned}$$

where the identity

$$\frac{\gamma'(u)\gamma'(v)}{(\gamma u - \gamma v)^2} = \frac{1}{(u-v)^2}$$

has been used. Certainly p_γ is a quadratic polynomial in v.

2.3. Now by analogy with the Eichler integral we introduce the integral

$$F(z) = \overline{\int_{z_0}^{z} (\bar{z} - t)^2 \phi(t)\, dt}, \tag{2.3}$$

t, z, z_0 in H, z_0 fixed, and $\phi \in Q(\Gamma)$. The potential equation $F_{\bar{z}} = (z - \bar{z})^2\overline{\phi(z)}$ is an immediate consequence of differentiation under the integral. Next we consider the periods of F.

Claim 2.1. *For $\psi(z) = \overline{\phi(\bar{z})}$, $z \in H$, and $v_0 = \bar{z}_0$ the integrals E and F have identical periods.*

Proof. By a calculation analogous to (2.2) we have for $\gamma \in \Gamma$ that

$$F(\gamma)/\gamma' - F = \overline{\int_{\gamma^{-1}z_0}^{z_0} (\bar{z} - t)^2 \phi(t)\, dt}.$$

The conclusion follows on substituting $t = \bar{u}$, $\overline{\phi(\bar{u})} = \psi(u)$, and $\overline{\gamma(z_0)} = \gamma(\bar{z}_0)$ (note $\Gamma \subset \mathrm{PSL}(2; \mathbf{R})$).

Motivated by the above observation we introduce for $\phi \in Q(\Gamma)$ the function

$$\mathscr{F}(z) = \begin{cases} F(z), & z \in H \text{ for } \phi \in Q(\Gamma), \\ E(z), & z \in L \text{ for } \psi(z) = \overline{\phi(\bar{z})} \text{ and } v_0 = \bar{z}_0. \end{cases} \tag{2.4}$$

The next step is to show that $\mathscr{F}$ and $\dot{w}[(z - \bar{z})^2\bar{\phi}]$ differ by a quadratic polynomial. On L we have that $\mathscr{F}''' = \dot{w}''' = 2\overline{\phi(\bar{z})}$ and thus for the restriction to L certainly $\dot{w} - \mathscr{F} = q$, a quadratic polynomial.

Claim 2.2. *$G = \dot{w} - \mathscr{F} - q$ is a Γ-invariant, holomorphic vector field on $H \cup L$.*

Proof. $G(z)$, $z \in L$, is identically zero and thus it remains to consider $G(z)$, $z \in H$. By construction $\dot{w}_{\bar{z}} = \mathscr{F}_{\bar{z}} = F_{\bar{z}}$ on H and thus G is indeed holomorphic. Now for $\gamma \in \Gamma$ we consider the period $G(\gamma)/\gamma' - G$; recall that the γ period of each of $\dot{w}$, $\mathscr{F}$, and q is a quadratic polynomial on $H \cup L$. Thus the γ period of G is a polynomial; since G actually vanishes on L the γ period is trivial. This is the desired conclusion: G is a Γ-invariant, holomorphic vector field.

The final step of the argument is to use the Riemann-Roch theorem to conclude that G vanishes identically. Indeed by Riemann-Roch if G defines a vector field on H/Γ which vanishes at the punctures, then G is trivial.

Claim 2.3. *Given H/Γ of finite hyperbolic area, then G, as above, defines a vector field on H/Γ vanishing at the punctures.*

Proof. Of course our approach is to estimate the growth of G at a Γ cusp. The hyperbolic metric for the punctured disc PD, $\{0 < |\tau| < 1\}$, is $|d\tau|^2/|\tau|^2(\log 1/|\tau|)^2$. Thus a vector field, holomorphic on PD, extends to be holomorphic on the disc $\{|\tau| < 1\}$ and vanishing at the origin provided the (intrinsic) norm

$$\|V\| = \frac{|V|}{|\tau| \log 1/|\tau|}$$

is for instance $O(|\tau|^{-1/2})$ at 0. Passing to the universal cover H of PD by substituting $\tau = \exp(-i/(z-\alpha))$, $\alpha \in \mathbf{R}$, we obtain the equivalent condition

$$\|\tilde{V}\| = \frac{|\tilde{V}|}{\operatorname{Im} z} \quad \text{is} \quad O\left(\exp\left(\frac{1}{2}\,\frac{\operatorname{Im} z}{|z-\alpha|^2}\right)\right)$$

for $\tilde{V}$, the lift of V, and z approaching α nontangentially.

Now we assume α represents a cusp of Γ and estimate the norm of G. Since $\dot{w}$ and q are continuous on $\mathbf{C}$ the estimate is immediate for these terms. Only the integral F remains, $F = \overline{\int_{z_0}^{z} (\bar{z} - t)^2 \phi(t)\, dt}$. By hypothesis $|(z-\bar{z})^2\bar{\phi}|$ is bounded and thus it suffices to bound

$$\int_{z_0}^{z} \frac{|\bar{z} - t|^2}{(\operatorname{Im} t)^2}\, |dt|.$$

Choosing the line segment $\overline{zz_0}$ as the path of integration we may bound the integral by

$$|z - z_0| \max_t \frac{|\bar{z} - t|^2}{(\operatorname{Im} t)^2}$$

for t varying on $\overline{zz_0}$. That this last quantity is bounded by $\exp(\frac{1}{2}\operatorname{Im} z/|z-\alpha|^2)$ for z approaching α nontangentially is left for the reader to check. In conclusion $G = \dot{w} - \mathscr{F} - q$ satisfies the desired estimate and therefore extends to be holomorphic on H/Γ vanishing at the punctures of H/Γ, the desired result

The main result of the chapter now follows.

Theorem 2.4. *With the above notation, the potential $\dot{w}$ and the integral $\mathscr{F}$ differ on $H \cup L$ by a quadratic polynomial. In particular $\mathscr{F}$ extends to a continuous function on $\mathbf{C}$.*

Corollary 2.5. *With the above notation, the potential $\dot{f}$ and the integral $\mathscr{F}(z) + \overline{\mathscr{F}(\bar{z})}$ differ on $H \cup L$ by a quadratic polynomial with real coefficients.*

As an exercise we show that if Γ is conjugated such that $\gamma\colon z \to \lambda z$ is an element of the group, then it is easy to solve for the coefficients a and c of the polynomial $q(z) = az^2 + bz + c$. The same technique may be applied if it is known that 1 is a fixed point of an element of Γ.

First observe that since $\dot w$ fixes 0 and is $o(|z|^2)$ for $|z|$ large, it follows that the period (a vector field) $\dot w(\lambda z)/\lambda - \dot w(z)$ vanishes at 0 and ∞, thus is a multiple of z (in fact $\dot\lambda z$ for $\lambda^\varepsilon = \dot w(\lambda z)/\dot w(z) = \lambda(1 + \varepsilon\dot\lambda)$). By the above we have that

$$\begin{aligned}\dot w(\lambda z)/\lambda - \dot w(z) &= (\mathscr{F} + q)(\lambda z)/\lambda - (\mathscr{F} + q)(z)\\ &= a(\lambda - 1)z^2 + c(\lambda^{-1} - 1) + \overline{\int_{\lambda^{-1}z_0}^{z_0} (\bar z - t)^2 \phi(t)\,dt}\,.\end{aligned}$$

Equating coefficients

$$a = 1/(1-\lambda)\,\overline{\int_{\lambda^{-1}z_0}^{z_0} \phi(t)\,dt}\,, \qquad c = \lambda/(\lambda - 1)\,\overline{\int_{\lambda^{-1}z_0}^{z_0} t^2\phi(t)\,dt}\,,$$

and of course the known formula [10]

$$\dot\lambda = -2\,\overline{\int_{\lambda^{-1}z_0}^{z_0} t\phi(t)\,dt}\,.$$

The goal of §4 is to compute the second variation of the multiplier of the transformation $\gamma\colon z \to \lambda z$. The vector field $z\partial/\partial z$ commutes with γ and thus the deformation of γ induced by $z\partial/\partial z$ is trivial. It suffices to determine $\mathscr{F} + q$ modulo $z\partial/\partial z$ in order to calculate the variation of γ. In §4 we shall explicitly see that the variation is independent of the coefficient of $z\partial/\partial z$.

3. Variations of a geodesic length function

3.1. Basic invariants of a hyperbolic metric are the lengths of the unique geodesic representatives of the free homotopy classes. Fricke-Klein observed that the geodesic lengths are a complete set of invariants; by choosing classes $\alpha_1, \cdots, \alpha_p$ appropriately the associated map from Teichmüller space $T_{g,n}$ to $\mathbf{R}^p$ is a smooth embedding. More recently Kerckhoff observed that the geodesic length functions are convex along Thurston's earthquake paths [12]. Following this lead we calculated the first and second Lie derivative of a geodesic length function along a simple earthquake path: the derivatives are evaluated in terms of the trigonometry of the corresponding geodesics on the surface [23]. Recall that on a differentiable manifold the second derivative of a function is not intrinsically defined; a normalization is required. Differentiation along unit speed earthquakes is such a normalization but the result is neither symmetric nor a tensor. An alternative is to consider unit speed Weil-Petersson geodesics. The resulting derivative, a symmetric 2-tensor, is the (Riemannian) Hessian. Recall further that the trace of the Hessian is the Laplacian and as mentioned in §1 it is easy to obtain the complex Hessian from the Riemannian Hessian.

This section is divided into two parts: the basic formula for the exterior derivative dl_α, l_α a geodesic length function, and the preliminary calculation of the second derivative of l_α.

3.2. A Beltrami differential determines a curve in Teichmüller space. Specifically consider a surface R uniformized by the upper half-plane H and a group Γ. Let $\mu \in L^\infty(\Gamma)$ be a Γ-invariant Beltrami differential and f^μ the solution of the Beltrami equation (1.1). The 1-parameter family $R^\varepsilon = H/f^\varepsilon\Gamma(f^\varepsilon)^{-1}$, $f^\varepsilon = f^{\varepsilon\mu}$, ε small, defines a curve in $T_{g,n}$. Our goal is to compute $d^2 l_\gamma(R^\varepsilon)/d\varepsilon^2$ at $\varepsilon = 0$; the first derivative $dl_\gamma(R^\varepsilon)/d\varepsilon$ is simply the pairing of the tangent direction determined by μ and the 1-form dl_γ. For the sake of normalization we again assume that γ lifts to the positive imaginary axis and that the associated deck transformation is $z \to \lambda z$. The following is a classical result [18].

Theorem 3.1. *With the above notation, at* $\varepsilon = 0$

$$\frac{dl_\gamma(R^\varepsilon)}{d\varepsilon} = \frac{2}{\pi} \operatorname{Re} \int_{1<|z|<\gamma} \frac{\mu}{z^2}\, dE,$$

where dE is Euclidean area.

3.3. We proceed and calculate the second derivative. The formula is elementary given the above result and the description of the tangent field of the curve $R^\varepsilon \in T_{g,n}$. At R^ε the tangent is represented by $L^{\varepsilon\mu}\mu \in B(R^\varepsilon)$,

$$L^{\varepsilon\mu}\mu = \left(\frac{\mu}{1-|\varepsilon\mu|^2}\frac{f_z^{\varepsilon\mu}}{\overline{f_z^{\varepsilon\mu}}}\right)\circ(f^{\varepsilon\mu})^{-1}, \tag{3.1}$$

and let $\dot{f} = df^{\varepsilon\mu}/d\varepsilon$ at $\varepsilon = 0$.

Theorem 3.2. *With the above notation, at* $\varepsilon = 0$

$$\frac{d^2 l_\gamma(R^\varepsilon)}{d\varepsilon^2} = \frac{4}{\pi} \operatorname{Re} \int_{1<|z|<\lambda} \mu\left(\frac{z\dot{f}_z - \dot{f}}{z^3}\right) dE.$$

Proof. For $w = f^\varepsilon(z)$, $f^\varepsilon = f^{\varepsilon\mu}$, by the previous theorem applied to the w structure, i.e. ε small but arbitrary,

$$\frac{dl_\gamma}{d\varepsilon} = \frac{2}{\pi} \operatorname{Re} \int_{1<|w|<\Lambda} \frac{1}{w^2}\left(\frac{\mu}{1-|\varepsilon\mu|^2}\frac{f_z^\varepsilon}{\overline{f_z^\varepsilon}}\right)\circ(f^\varepsilon)^{-1} dE,$$

where $f^\varepsilon(\lambda z) = \Lambda(\varepsilon) f^\varepsilon(z)$. The change of variables $w = f^\varepsilon(z)$ gives the formula

$$\frac{dl_\gamma}{d\varepsilon} = \frac{2}{\pi} \operatorname{Re} \int_{1<|z|<\lambda} \mu \left(\frac{f_z^\varepsilon}{f^\varepsilon} \right)^2 dE.$$

By Theorem 1.1 the solution $\hat{f}^\varepsilon$ varies real analytically in AB^p, $p > 2$, and hence for the sector $1 < |z| < \lambda$ we find that f^ε varies in L^∞ and f_z^ε varies in L^p, $p > 2$, both real analytically. We differentiate under the integral using that $f^\varepsilon = z + \varepsilon \dot{f} + O(\varepsilon^2)$ and obtain the desired formula.

4. Calculation of the second derivative of l_γ

4.1. The plan is straightforward. By normalization, Γ contains a transformation $z \to \lambda z$ and thus $\mu = (z - \bar{z})^2 \bar{\phi}$ admits a series expansion (μ is a $2\pi i/\log \lambda$ periodic function of $\log z$). Substituting the expansions into the integrand of Theorem 3.2, a quadratic form in μ, we integrate term-by-term. We find that in analogy to the orthogonality of the functions $\{\sin 2\pi n x\}$ in $L^2(0,1)$ that only the *diagonal terms* of the integral are nonzero and in fact each is positive. Of course this is the main result.

The discussion is divided into two parts. In §4.2 we introduce the series expansion of μ and use the method of §2 to obtain the expansion of $z\dot{f}_z - \dot{f}$. In §4.3 we substitute the expansions into the integral of Theorem 3.2 and calculate term-by-term.

4.2. Given $\phi \in Q(\Gamma)$, ϕ admits an expansion

$$\phi = \frac{1}{z^2} \sum_n a_n z^{\varepsilon n}, \quad \text{where } \varepsilon = \frac{2\pi i}{\log \lambda}.$$

A branch of $\log z$ holomorphic on H and real at 1 is now fixed and the convergence of the series is uniform on compact subsets of H. We proceed to calculate $z\dot{f}_z - \dot{f}$ term-by-term. Picking up the discussion of §2.3

$$\begin{aligned} \dot{w} &= \mathscr{F} + q \\ &= \overline{\int_{z_0}^{z} (\bar{z} - t)^2 \phi \, dt} + \frac{z^2}{1-\lambda} \overline{\int_{\lambda^{-1} z_0}^{z_0} \phi \, dt} + bz + \frac{\lambda}{\lambda - 1} \overline{\int_{\lambda^{-1} z_0}^{z_0} t^2 \phi \, dt}. \end{aligned}$$

The linear term bz of q is annihilated by the operator $(z\partial/\partial z - 1)$ and so does not contribute to $z\dot{f}_z - \dot{f}$. Thus for our purposes

$$\dot{f} \equiv z^2 2 \operatorname{Re} A + z 2 \operatorname{Re} B + 2 \operatorname{Re} C, \tag{4.1}$$

where

$$A(z) = \int_{z_0}^{z} \phi\,dt + \frac{1}{1-\lambda}\int_{\lambda^{-1}z_0}^{z_0} \phi\,dt, \qquad B(z) = -2\int_{z_0}^{z} t\phi\,dt,$$

$$C(z) = \int_{z_0}^{z} t^2\phi\,dt + \frac{\lambda}{\lambda-1}\int_{\lambda^{-1}z_0}^{z_0} t^2\phi\,dt.$$

Now recalling $\partial/\partial z \operatorname{Re} h = h'$ for h holomorphic the reader will check that

$$(4.2) \qquad z\dot{f}_z - \dot{f} = z^2 2\operatorname{Re} A - 2\operatorname{Re} C,$$

the remaining terms cancel.

If we substitute the expansion $\phi = (1/z^2)\Sigma_n a_n z^{\varepsilon n}$ and integrate term-by-term we obtain

$$A(z) = \sum_n \frac{a_n}{\varepsilon n - 1}\left(t^{\varepsilon n - 1}\Big|_{z_0}^{z} + \frac{t^{\varepsilon n - 1}}{1-\lambda}\Big|_{\lambda^{-1}z_0}^{z_0}\right) = \sum_n \frac{a_n z^{\varepsilon n - 1}}{\varepsilon n - 1},$$

$$C(z) = \sum_n \frac{a_n}{\varepsilon n + 1}\left(t^{\varepsilon n + 1}\Big|_{z_0}^{z} + \frac{t^{\varepsilon n + 1}}{1-\lambda^{-1}}\Big|_{\lambda^{-1}z_0}^{z_0}\right) = \sum_n \frac{a_n}{\varepsilon n + 1} z^{\varepsilon n + 1},$$

where in each case the equation $(\lambda t)^{\varepsilon n} = t^{\varepsilon n}$ has been used to simplify the expression. The proof is now complete.

Lemma 4.1. *With the above notation, if* $\phi = (1/z^2)\Sigma_n a_n z^{\varepsilon n}$, *then*

$$z\dot{f}_z - \dot{f} = z^2 2\operatorname{Re}\sum_n \frac{a_n z^{\varepsilon n - 1}}{\varepsilon n - 1} - 2\operatorname{Re}\sum_n \frac{a_n z^{\varepsilon n + 1}}{\varepsilon n + 1},$$

converging uniformly on compact subsets of H.

4.3. We are ready to evaluate the integral

$$\frac{4}{\pi}\operatorname{Re}\int_{1<|z|<\lambda} \frac{\mu}{z^3}(z\dot{f}_z - \dot{f})\,dE;$$

the integrand is in L^1 and the series converge uniformly on compact sets. We integrate term-by-term and treat separately the terms for A and C. The computation for the general term of the expansion A is considered in the following lemma.

Lemma 4.2. *For* $\gamma, \delta \in \mathbf{C}$

$$\operatorname{Re}\int_{1<|z|<\lambda} (z-\bar{z})^2\left(\overline{\frac{\gamma z^{\varepsilon m}}{z^2}}\right)\frac{1}{z}\operatorname{Re}\frac{\delta z^{\varepsilon n - 1}}{\varepsilon n - 1}\,dE = 0 \quad \textit{if } m \neq \pm n,$$

$$\operatorname{Re}\int (z-\bar{z})^2\left(\overline{\frac{\gamma z^{\varepsilon n} + \delta z^{-\varepsilon n}}{z^2}}\right)\frac{1}{z}\operatorname{Re}\left(\frac{\gamma z^{\varepsilon n - 1}}{\varepsilon n - 1} + \frac{\delta z^{-\varepsilon n - 1}}{-\varepsilon n - 1}\right)dE \geqslant 0,$$

with equality only if $\gamma = \delta = 0$.

Proof. In polar coordinates $z = re^{i\theta}$ the first integral becomes

$$-4\int \operatorname{Re}\left(e^{i\theta}\overline{\gamma z^{\varepsilon m}}\right) \operatorname{Re}\left(e^{-i\theta}\frac{\delta z^{\varepsilon n}}{\varepsilon n - 1}\right) \sin^2\theta \frac{dr}{r}\, d\theta.$$

Now $z^{\varepsilon} = \exp(-2\pi\theta/\log\lambda)(\cos 2\pi\rho + i\sin 2\pi\rho)$ for $\rho = \log r/\log\lambda$ and the result follows from the ρ integration. To compute the second integral replace the real part of

$$\left(\frac{\gamma z^{\varepsilon n - 1}}{\varepsilon n - 1} + \frac{\delta z^{-\varepsilon n - 1}}{-\varepsilon n - 1}\right)$$

by the sum of the term and its conjugate. Integrating in ρ, $0 \leqslant \rho \leqslant 1$, we obtain

$$\frac{2}{1 + |\varepsilon n|^2}\int \left(|\gamma z^{\varepsilon n}|^2 + |\delta z^{-\varepsilon n}|^2 + 2\operatorname{Re}\left(e^{-2i\theta}\gamma\delta\right)\right) \sin^2\theta\, d\theta,$$

or completing the square and substituting $|z^{\varepsilon}| = e^{i\varepsilon\theta}$ we have

$$\frac{2}{1 + |\epsilon n|^2}\int \left|\gamma e^{i\theta(\varepsilon n - 1)} + \bar{\delta} e^{i\theta(1 - \varepsilon n)}\right|^2 \sin^2\theta\, d\theta.$$

The positivity is now clear and the proof is complete.

Corollary 4.3. *With the above notation, the integral*

$$\frac{4}{\pi}\operatorname{Re}\int (z - \bar{z})^2 \bar{\phi} \frac{2}{z} \operatorname{Re} A\, dE$$

is positive provided ϕ is nontrivial.

Proof. Apply the preceding lemma for the series expansion of ϕ and A. The result follows.

The computation for the general term of C is considered in the following.

Lemma 4.4. *For $\gamma, \delta \in \mathbf{C}$*

$$-\operatorname{Re}\int (z - \bar{z})^2 \left(\overline{\frac{\gamma z^{\varepsilon m}}{z^2}}\right) \frac{1}{z^3} \operatorname{Re}\frac{\delta z^{\varepsilon n + 1}}{\varepsilon n + 1}\, dE = 0 \quad \textit{if}\ m \neq \pm n,$$

$$-\operatorname{Re}\int (z - \bar{z})^2 \left(\overline{\frac{\gamma z^{\varepsilon n} + \delta z^{-\varepsilon n}}{z^2}}\right) \frac{1}{z^3} \operatorname{Re}\left(\frac{\gamma z^{\varepsilon n + 1}}{\varepsilon n + 1} + \frac{\delta z^{-\varepsilon n + 1}}{-\varepsilon n + 1}\right) dE \geqslant 0,$$

with equality only if $\gamma = \delta = 0$.

Proof. In polar coordinates the first integral becomes

$$4\int \operatorname{Re}\left(e^{-i\theta}\overline{\gamma z^{\varepsilon m}}\right) \operatorname{Re}\left(e^{i\theta}\frac{\delta z^{\varepsilon n}}{\varepsilon n + 1}\right) \sin^2\theta \frac{dr}{r}\, d\theta.$$

Again the result follows on integration of $\rho = \log r/\log\lambda$. By the method of Lemma 4.2 the ρ integration of the second integral leaves

$$\frac{2}{1 + |\varepsilon n|^2}\int \left(|\gamma z^{\varepsilon n}|^2 + |\delta z^{-\varepsilon n}|^2 + 2\operatorname{Re}\left(e^{2i\theta}\gamma\delta\right)\right) \sin^2\theta\, d\theta,$$

and on completing the square

$$\frac{2}{1+|\varepsilon n|^2}\int \left|\gamma e^{i\theta(\varepsilon n+1)}+\bar{\delta}e^{-i\theta(\varepsilon n+1)}\right|^2 \sin^2\theta\, d\theta.$$

The positivity is immediate.

Corollary 4.5. *With the above notation, the integral*

$$\frac{4}{\pi}\operatorname{Re}\int (z-\bar{z})^2\bar{\phi}\left(-\frac{2}{z^3}\right)\operatorname{Re} C\, dE$$

is positive provided ϕ is nontrivial.

Proof. Apply the preceding lemma for the series expansion of ϕ and C.

The main result follows immediately from (4.2), Corollaries 4.3 and 4.5, and the discussion of the Hessian in §1.4.

Theorem 4.6. *The Weil-Petersson Hessian of a geodesic length function is positive definite.*

Corollary 4.7. *A geodesic length function is strictly convex along a Weil-Petersson geodesic.*

Corollary 4.8. *A geodesic length function is subharmonic as well as strictly plurisubharmonic.*

5. Convexity of the geodesic length sublevel sets

5.1. A basic property of a complete Riemannian manifold is that an arbitrary pair of points is joined by a geodesic. On the other hand starting with an arbitrary complete manifold and removing a closed set one can obtain a manifold, where points may not be joined by a geodesic. Given that the Weil-Petersson metric is not complete the convexity of $T_{g,n}$ is an open problem.

Our main result is that $T_{g,n}$ is geodesically convex: every pair of points is joined by a unique geodesic. In fact we show the analogue of the Hadamard-Cartan result: the exponential map is a homeomorphism. The main result of the previous section suggests an even stronger result. Recall that the normal curvature of the level surface of a function is given by its Hessian. The l_α level surfaces are locally strictly convex, relative to the inward normal—grad l_α. We find the following result: the sublevel sets $\mathrm{SL}(\alpha, M)=\{p \mid l_\alpha(p)<M\}$, α arbitrary, are geodesically convex. Pursuing the analogy with a complete, negative curvature manifold we consider the Weil-Petersson distance. Our result is that geodesics are uniquely length minimizing and thus the distance between points is measured along the unique geodesic connecting them.

The proof that $T_{g,n}$ is convex relies on three observations: special sums of geodesic length functions are proper [12], the Weil-Petersson metric has negative sectional curvature [20], [26], and the positivity of the Hessian of an l_α. An underlying principle is that starting with a convex set C the constructions of geometry give points and geodesics again in C. Since $T_{g,n}$ is exhausted by the convex sets $\mathrm{SL}(\alpha, M)$, completeness is not required.

5.2. We start by considering proper sums of the l_α.

Definition 5.1. A family of closed geodesics $\{\alpha_j\}_{j=1}^m$ fills up a hyperbolic surface R provided each component of $R - \bigcup_j \alpha_j$ is topologically a disc or a cylinder with one boundary contained in ∂R.

The following result is the first ingredient for our approach [12].

Lemma 5.2. *If $A = \{\alpha_j\}_{j=1}^m$ fills up R, $[R] \in T_{g,n}$, then the sum $L_A = \sum_j l_{\alpha_j}$ is a proper function.*

We cite an elementary application. By Theorem 4.6 the Hessian of L_A is everywhere positive definite. In particular at a critical point its index is zero. L_A is a Morse function on $T_{g,n}$. It follows immediately that $T_{g,n}$ as well as the sublevel sets $\mathrm{SL}(A, M) = \{p \mid L_A(p) < M\}$ are cells.

5.3. Negative curvature will be used to ensure the absence of conjugate points along a geodesic; a result independent of completeness [14]. Specifically consider a Riemannian manifold N and a geodesic γ from p to q such that $q = \exp_p(v)$, q is the exponential of v at p. Recall that a vector v is in the domain of the exponential map at p provided the geodesic with initial data (p, v) is defined on the interval $[0, 1]$. For nonpositive curvature the differential of $\exp_p$ at $v \in T_p(N)$, the p tangent space, is an isomorphism. The proof uses the description of the differential of exp by Jacobi fields; the curvature hypothesis is used to estimate from below the length of a Jacobi field.

We cite a simple consequence. On its domain, $\exp_p$ is a local homeomorphism. Equivalently, provided p is joined to q by a geodesic, every point of a neighborhood of q is also joined to p by a geodesic.

5.4. The uniqueness of geodesics is an easy consequence of negative curvature and thus the existence is the main question. Given a family A which fills up and $M > 0$ we consider the sublevel set $\mathrm{SL}(A, M) = \{p \mid L_A(p) < M\}$.

Theorem 5.3. *Every pair of points of* $\mathrm{SL}(A, M)$ *is joined by a geodesic in* $\mathrm{SL}(A, M)$.

Proof. $\mathrm{SL}(A, M)$ is open and connected; the proof is by connectedness. Fix $p \in \mathrm{SL}(A, M)$ and define J to be the set of points joined to p by a geodesic in $\mathrm{SL}(A, M)$. The exponential $\exp_p$ is a local homeomorphism at p; J is nonempty. Now we check that J is open. Given $q \in J$ by definition there exists a geodesic γ from p to q with $\gamma \subset \mathrm{SL}(A, M)$. As noted above there exists a neighborhood $U \subset \mathrm{SL}(A, M)$ of q such that a point r of U is joined to

p by a geodesic β. Now $L_A(r)$, $L_A(p) < M$ and L_A is convex along β. It follows that $\beta \subset \mathrm{SL}(A, M)$ and thus $U \subset J$.

It remains to show that J is closed. Consider a sequence $\{q_n\} \subset J$ with $q_n \to q_* \in \mathrm{SL}(A, M)$. By hypothesis each q_n is joined to p by a geodesic γ_n. L_A is proper thus its sublevel sets are precompact. By passing to a subsequence (same notation) we can assume that $\gamma_n \to \gamma_* \subset \overline{\mathrm{SL}(A, M)}$. By a standard argument γ_* is a geodesic from p to q_* and since $L_A(p)$, $L_A(q_*) < M$ the convexity ensures that $\gamma_* \subset \mathrm{SL}(A, M)$. We have that $q_* \subset J$; J is closed and the proof is complete.

Corollary 5.4. *The exponential map is a homeomorphism.*

Proof. Teichmüller space is a cell, thus it will suffice to show that the exponential map from an arbitrary point is a covering. To this end recall that a proper local homeomorphism of Hausdorff spaces is a covering. In the previous subsection we observed that the exponential map, given negative curvature, is a local homeomorphism. All that remains is to show that the exponential map is proper. Certainly it will suffice to show that the preimage of $\overline{\mathrm{SL}(A, M)}$ is compact.

The $\exp_p$ preimage of $\overline{\mathrm{SL}(A, M)}$ is described by considering geodesics leaving p. Given v a unit tangent vector at p, consider the ray $\gamma(t) = \exp_p(tv)$. If $\gamma(t)$ has maximal domain $[0, \infty)$, then by the strict convexity of $L_A(\gamma(t))$ there exists a unique t_0 such that $L_A(\gamma(t_0)) = M$. On the other hand if $\gamma(t)$ has maximal domain $[0, c)$, c finite, then since $\gamma(t)$ cannot be prolonged it must be proper. Since L_A is proper and strictly convex, again there exists a unique t_0 such that $L_A(\gamma(t_0)) = M$. We may consider the quantity $t_0(v)$ as a function on the unit tangent sphere and certainly the preimage of $\overline{\mathrm{SL}(A, M)}$ is the subset $\{w \mid \|w\| \leqslant t_0(w/\|w\|)\}$ of the p tangent space. Provided we show that t_0 is a continuous function of v it follows that the preimage is a compact subset of the tangent space and thus a compact subset of the domain of $\exp_p$. By the implicit function theorem t_0 is a smooth function of v ($\exp_p$ is a local diffeomorphism and we avoid the minimum of L_A, the unique point where dL_A vanishes). The proof is complete.

Corollary 5.5. *Let $B = \{\beta_j\}$ be an arbitrary family of closed geodesics and $L_B = \sum_j l_{\beta_j}$ its length function. The sublevel set* $\mathrm{SL}(B, M)$ *is convex.*

Proof. Given $p, q \in \mathrm{SL}(B, M)$, L_B is convex along the unique geodesic $\gamma(t)$, $0 \leqslant t \leqslant 1$, connecting the pair. Consequently $L_B(\gamma(t)) \leqslant \max\{L_B(p), L_B(q)\} < M$, $0 \leqslant t \leqslant 1$.

Corollary 5.6. *$T_{g,n}$ has an exhaustion by compact Weil-Petersson convex sets.*

Proof. If $A = \{\alpha_j\}_{j=1}^m$ fills up, then by Lemma 5.2 and Corollary 5.5 the sublevel sets $\overline{\mathrm{SL}(A, M)}$, $M \in \mathbf{Z}^+$, provide an exhaustion.

5.5. An important property of a complete manifold of nonpositive curvature is that the exponential map is distance increasing. The natural analogue of this result is valid for the Weil-Petersson metric. As an application we have that Weil-Petersson geodesics are uniquely length minimizing.

It is important to clarify a distinction before starting the proofs. Associated to a Riemannian metric are two measures of displacement. The first, the infinitesimal metric, gives the length of tangent vectors. The second, the distance function, gives the distance between points, defined by the infimum over all paths joining the points. Finally we remind the reader that a Riemannian metric defines a Euclidean metric (in both senses) in each tangent space.

Lemma 5.7. *The Weil-Petersson exponential map is distance increasing for the infinitesimal metrics.*

Proof. Let L_A be a proper length function and $p \in T_{g,n}$. The standard proof may be applied to the set $\exp_p^{-1}(\mathrm{SL}(A, M)) \subset T_p(T_{g,n})$ [14].

Corollary 5.8. *A Weil-Petersson length minimizing sequence converges to the geodesic.*

Proof. Start with a minimizing sequence $\{\gamma_n(t) \mid 0 \leqslant t \leqslant d_n\}$ of unit speed arcs joining p to q such that $\lim_n \|\gamma_n\| = \lim_n d_n = d(p,q)$ and $\gamma(t)$, $0 \leqslant t \leqslant d_\gamma$, is the unique unit speed geodesic joining p to q. Furthermore choose L_A a proper length function and $M > L_A(p)$, $L_A(q)$. $\overline{\mathrm{SL}(A, M)}$ is compact and γ is a positive distance ε from $\partial\,\mathrm{SL}(A, M)$. For $\varepsilon_0 = \min\{\varepsilon, d(p,q)\}$ the arcs $\{\gamma_n(t) \mid 0 \leqslant t \leqslant \varepsilon_0\}$ are contained in $\overline{\mathrm{SL}(A, M)}$; at the very least the sequence has a subsequence (same notation) convergent to an arc $\gamma_*(t)$ on the interval $0 \leqslant t \leqslant \varepsilon_0$. We shall consider the lifts to $T_p(T_{g,n})$:

$$\tilde{q} = \exp_p^{-1}(q), \qquad \tilde{\gamma}_n(t) = \exp_p^{-1}(\gamma_n(t)),$$
$$\tilde{\gamma}(t) = \exp_p^{-1}(\gamma(t)), \qquad \tilde{\gamma}_*(t) = \exp_p^{-1}(\gamma_*(t)).$$

The first part of the argument is to show that $\tilde{\gamma}_*(t) = \tilde{\gamma}(t)$ for $0 \leqslant t \leqslant \varepsilon_0$. The approach is to use the Euclidean metric δ of the tangent space and argue by contradiction. If the claim is false then there exists a t_1, $0 < t_1 \leqslant \varepsilon_0$, such that $\tilde{\gamma}_*(t_1) \neq \tilde{\gamma}(t_1)$. Since $\tilde{\gamma}_*(t)$ has speed at most 1 (by Lemma 5.5) and $\tilde{\gamma}(t)$ is the line connecting 0 to $\tilde{q}$ we have that $\delta(\tilde{\gamma}_*(t_1), \tilde{q}) > \delta(\tilde{\gamma}(t_1), \tilde{q}) = d_\gamma - t_1$. Now the sequence $\{\tilde{\gamma}_n(t_1)\}$ converges to $\tilde{\gamma}_*(t_1)$ and thus the limit of the lengths of the segments $\{\tilde{\gamma}_n(t) \mid t_1 \leqslant t \leqslant d_n\}$ is at least $\delta(\tilde{\gamma}_*(t_1), \tilde{q}) > d_\gamma - t_1$. Applying Lemma 5.5 to the $\exp_p$ images of these arcs we have

$$\lim_n d_n - t_1 \geqslant \delta(\tilde{\gamma}_*(t_1), \tilde{q}) > d_\gamma - t_1,$$

hence $\lim_n d_n > d_\gamma$. This is a contradiction of the sequence $\{\gamma_n\}$ being length minimizing.

The remainder of the argument is essentially finite induction. By the claim the initial segments $\gamma_*(t)$ and $\gamma(t)$, $0 \leqslant t \leqslant \varepsilon_0$, coincide; repeat the argument starting at $\gamma_*(\varepsilon_0) = \gamma(\varepsilon_0)$. After a finite number of steps we have $\gamma_* = \gamma$. The proof is complete.

Corollary 5.9. *Weil-Petersson distance is measured along geodesics.*

Proof. The distance between a pair of points is given by a length minimizing sequence. By the above result the limiting length is simply the length along the geodesic.

6. $T_{g,n}$ is Stein and the Nielsen problem

6.1. As applications of the convexity of the Weil-Petersson geometry we give new proofs that $T_{g,n}$ is Stein and of the Nielsen realization problem. In fact the former result is now immediate. The length function of a family A that fills up is proper and plurisubharmonic (Lemma 5.2 and Corollary 4.8), the result follows [9]. For the second result we shall apply Cartan's original center of mass argument.

6.2. For a complete metric of nonpositive curvature the distance from a fixed point to a geodesic is a convex function along the geodesic [14]. This property carries over to the present situation, given the results of Theorem 5.3 and Corollary 5.6 on the behavior of geodesics.

$M_{g,n}$, the genus g, n puncture, mapping class group, acts on $T_{g,n}$ by Weil-Petersson isometries. The following result is sufficient for the Nielsen problem.

Theorem 6.1. *Let G be a finite group of Weil-Petersson isometries. G has a fixed point.*

Proof. Choose a point $p \in T_{g,n}$ and consider the G-invariant function $D(q) = \sum_{\gamma \in G} d(q, \gamma(p))$, D is the Weil-Petersson distance of q to the G orbit of p. The plan is to show that D has a unique minimum, the center of mass of the G orbit of p. Since D is clearly G-invariant its unique minimum will be a G fixed point.

The first step is to show that D has minima. Choose L_A the length function of a family that fills up and $M > L_A(\gamma(p))$ for all $\gamma \in G$. We shall show that D has minima and they are contained in $S = \{r \mid L_A(r) \leqslant M\}$. As discussed in the previous section $\partial S = L_A^{-1}(M)$ is a smooth hypersurface bounding a cell. To establish the existence of D minima it is sufficient to prove that grad D is strictly outward pointing on ∂S. We start by recalling the description of the gradient of the distance function: Let $f(x) = d(r, x)$ and let β be the unit speed geodesic connecting r and s; then $\operatorname{grad} f(s) = \dot{\beta}|_s$. Now consider that r

is an interior point of S and s is a boundary point. Since $L_A(s) > L_A(r)$, the strict convexity of L_A along β provides that L_A is strictly increasing at s:

$$\langle \operatorname{grad} L_A, \operatorname{grad} f\rangle_s = \langle \operatorname{grad} L_A, \dot\beta\rangle_s = \left.\frac{dL_A}{d\beta}\right|_s > 0$$

(relative to the Weil-Petersson metric). Certainly $\operatorname{grad} L_A$ is the normal direction on ∂S and thus $\operatorname{grad} d(r, x)$ is strictly outward pointing on ∂S. D is a sum of terms $d(r, x)$; $\operatorname{grad} D$ is strictly outward pointing. Minima of D exist and are contained in S.

The final step is to show that there exists a unique minimum. Proceeding by contradiction, connect q_1 and q_2, minima of D, by a geodesic β. By virtue of negative curvature D is strictly convex along β forcing $q_1 = q_2$. There is a unique minimum and the argument is complete.

As a last remark we point out that a variant of the Kerckhoff argument can be substituted for the Cartan result. The length function L_A of a family that fills up is G-invariant. By properness, L_A has at least one minimum and by Corollary 4.7, L_A would be strictly convex along a geodesic connecting two candidate minima, a contradiction.

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