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RESONANCES FOR AXIOM A FLOWS

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Abstract

Given an Axiom A flow on M and smooth functions $B, C: M \mapsto R$, we show that the time correlation function ρ_{BC} for a Gibbs state ρ has a Fourier transform $\hat{\rho}_{BC}$ meromorphic in a strip. This complements a result by Pollicott [7]. The residues of the poles of $\hat{\rho}_{BC}$ are investigated. In the simplest case, they have the form $\sigma^-(B)\sigma^+(C)$ where σ^-, σ^+ are Gibbs distributions, i.e., (Schwartz) distributions on M further specified in the paper. This is a companion to an earlier paper [9] where similar results have been obtained for Axiom A diffeomorphism.

0. Introduction

In an earlier paper [9] we have studied the time correlation functions for Axiom A diffeomorphisms. These correlation functions have Fourier transforms which are meromorphic in a strip, and we have identified the residues of the poles in that strip in terms of *Gibbs distributions*. In the present paper we obtain a similar result for Axiom A flows.

Let (f^t) be a $C^{1+\varepsilon}$ Axiom A flow on a compact manifold M (which we may take as C^∞). We assume that ρ is a Gibbs measure on a nontrivial[1] basic set Λ (see Bowen and Ruelle [4]) and let $\mathbf{B}$, $\mathbf{C}$ be smooth real functions on M. Define the correlation function

$$\rho_{\mathbf{BC}}(t) = \int \rho(dx)\mathbf{B}(f^t x)\mathbf{C}(x) - \left[\int \rho(dx)\mathbf{B}(x)\right]\left[\int \rho(dx)\mathbf{C}(x)\right]$$

and its Fourier transform

$$\hat{\rho}_{\mathbf{BC}}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t}\rho_{\mathbf{BC}}(t)\,dt$$

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[1] The basic set Λ is nontrivial if it is not a fixed point or a periodic orbit.

(called the *power spectrum* if $\mathbf{B} = \mathbf{C}$). Completing an argument of Pollicott [7] we shall show that the function $\hat{\rho}_{\mathbf{BC}}$ is meromorphic in a strip $|\mathrm{Im}\,\omega| < \delta^*$ (see Theorem 4.1). The poles of $\hat{\rho}_{\mathbf{BC}}$ are called *resonances*, and we shall study their residues. For simplicity we shall consider only simple poles and make a further nondegeneracy assumption which is generically satisfied. Under these conditions, the residues are of the form $\sigma^-(B)\sigma^+(C)$, where σ^- and σ^+ are *Gibbs distributions* (see Theorem 4.2). The Gibbs distributions are distributions in the sense of Schwartz on M, which will be further specified below.

We refer the reader to Smale [10] and Bowen [1] for a general discussion of Axiom A flows and their basic sets. For the purposes of the present paper we shall essentially use the existence of *symbolic dynamics* as proved by Bowen [2]. Roughly speaking, symbolic dynamics is obtained by placing in the manifold M a certain number of pieces of hypersurfaces Σ_j transversal to the flow; a point x of a basic set $\Lambda \subset M$ is then specified by the sequence of intersections of its orbit $(f^t x)$ with the Σ_j.

In the next sections we describe the formal structure of symbolic dynamics (insofar as is needed). This structure is given by the construction of a space $\Omega^\#$, with a flow (τ_Θ^t), and a map $\bar{\omega}:\Omega^\# \mapsto M$ such that $\bar{\omega}\Omega^\#$ is a basic set Λ for the flow (f^t) and $\bar{\omega}\tau_\Theta^t = f^t\bar{\omega}$.

1. Symbolic dynamics: the shift τ

Let J be a nonempty finite set, and (t_{ij}) a square matrix indexed by $J \times J$, with elements 0 or 1. (The elements j of J correspond to the indices of the pieces of hypersurfaces Σ_j mentioned in the introduction; $t_{ij} = 1$ if an orbit $(f^t x)$ may successively cross Σ_i and Σ_j.) We define Ω to be the space of sequences $(j_k)_{k \in \mathbf{Z}}$ of elements of J such that $t_{j_k j_{k+1}} = 1$ for all k. The space Ω is compact with respect to the topology of pointwise convergence. The *shift* $\tau:\Omega \mapsto \Omega$ is defined by $(\tau\xi)_k = \xi_{k+1}$; τ is a homeomorphism. The pair (Ω, τ) is called a *subshift of finite type*. We assume that all matrix elements of t^N are > 0 for sufficiently large N. (This means that τ is topologically mixing on Ω, which can always be achieved in the present situation.)

Given $X \subset \mathbb{Z}$ and $\xi \in \Omega$, we let $\pi_X\xi = (\xi_j)_{j \in X}$ be the sequence obtained by restriction of the index set $\mathbb{Z}$ to X. We also write $\pi_X\Omega = \Omega_X$.

If $A \in \mathscr{C}(\Omega, \mathbf{C})$ we define

$$\|A\|_\infty = \max\{|A(\xi)| : \xi \in \Omega\},$$
$$\mathrm{var}_n A = \sup\{|A(\xi) - A(\xi')| : \xi_k = \xi'_k \text{ for } |k| < n\},$$
$$\|A\|_\theta = \sup_{n \geq 1} \theta^{-n}\mathrm{var}_n A, \quad \text{where } 0 < \theta < 1,$$
$$|||A|||_\theta = \|A\|_\infty + \|A\|_\theta.$$

We let $\mathscr{C}_\theta$ be the Banach space of those A for which $\lim_{n\to\infty}\theta^{-n}\operatorname{var}_n A = 0$, with the norm $|||\cdot|||_\theta$. Note that $\mathscr{C}_\theta$ is a Banach algebra (i.e. $|||AB|||_\theta \leqslant |||A|||_\theta |||B|||_\theta$). If $X \subset \mathbb{Z}$, we let

$$\mathscr{C}_\theta(X) = \{A \in \mathscr{C}(\Omega_X, \mathbb{C}) : A \circ \pi_X \in \mathscr{C}_\theta\}.$$

This is a Banach space with respect to the induced norm $A \mapsto |||A \circ \pi_x|||_\theta$. We denote by $\mathscr{C}_\theta^*$, $\mathscr{C}_\theta(X)^*$ the duals of $\mathscr{C}_\theta$, $\mathscr{C}_\theta(X)$. For $\sigma \in \mathscr{C}_\theta^*$ or $\mathscr{C}_\theta(X)^*$ it will be convenient to write

$$\sigma(A) = \int \sigma(d\xi)A(x)$$

as if σ were a measure.

The *pressure* of $A \in \mathscr{C}(\Omega, \mathbb{R})$ is

$$P(A) = \max\{h(\sigma) + \sigma(A) : \sigma \text{ is a } \tau\text{-invariant probability measure}\},$$

where $h(\sigma)$ is the *entropy* of σ (= Kolmogorov-Sinai invariant). If $A \in \mathscr{C}_\theta$, the maximum is reached for a unique measure ρ called the *Gibbs state* for A. The theory of Gibbs states is discussed in Bowen [3] and Ruelle [8]. In [9] an extension to *Gibbs distributions* is given (these are elements of $\mathscr{C}_\theta^*$, not necessarily measures). We shall quote results from the above references as needed. Here we reproduce some definitions of [9] with slightly different notation.[2]

If $A_\# \in \mathscr{C}_{\theta^2}$, we may introduce an *interaction* Φ such that

$$A_\#(\xi) = A_\Phi(\xi) \equiv -\Phi_0(\xi_0) - \sum_{n=1}^{\infty} \Phi_{2n}(\xi_{-n}, \cdots, \xi_n), \tag{1.1}$$

where $|\Phi_{2n}| <$ const θ^{2n} (we write $\Phi_k = 0$ if k is odd). We then define $A'_\Phi \in \mathscr{C}_\theta((-\infty, 0])$ by

$$A'_\Phi(\xi') = -\sum_{k=0}^{\infty} \Phi_k(\xi'_{-k}, \cdots, \xi'_0). \tag{1.2}$$

Finally we let $\mathscr{L}'_\Phi$ be the operator on $\mathscr{C}_\theta((-\infty, 0])$ such that

$$(\mathscr{L}'_\Phi \phi)(\xi') = \sum_{\eta \in J} t_{\xi_0 \eta}[\exp A'_\Phi(\tau\xi' \vee \eta)]\phi(\tau\xi' \vee \eta), \tag{1.3}$$

where $\tau\xi' \vee \eta = (\cdots, \xi'_{-1}, \xi'_0, \eta) \in \Omega_{(-\infty, 0]}$ when $t_{\xi_0\eta} = 1$ (otherwise $\tau\xi' \vee \eta$ is undefined). The adjoint $\mathscr{L}'^*_\Phi$ acts on $\mathscr{C}_\theta((-\infty, 0])^*$.

The spectrum of $\mathscr{L}'_\Phi$ and $\mathscr{L}'^*_\Phi$ is contained in the disk $\{z : |z| \leqslant \exp P(\operatorname{Re} A_\#)\}$, and the part in $\{z : |z| > \theta \exp P(\operatorname{Re} A_\#)\}$ is discrete, consisting of eigenvalues of finite multiplicity.

[2] In particular, it is convenient to write $A_\#$ instead of A for purpose of later reference.

If $A_\#$ is real, $\exp P(A_\#)$ is a simple eigenvalue of $\mathscr{L}'_\Phi$ and $\mathscr{L}'^*_\Phi$, and there is no other eigenvalue with the same modulus. Let S' and σ' be the eigenvectors of $\mathscr{L}'_\Phi$ and $\mathscr{L}'^*_\Phi$ corresponding to $\exp P(A_\#)$. Then $S'\sigma'$ is (up to normalization) the image by $\pi_{(-\infty,\ 0]}$ of the Gibbs state ρ.

For $A_\#$ not necessarily real, let λ, μ be any eigenvalues of $\mathscr{L}'_\Phi$ and $\mathscr{L}'^*_\Phi$ with modulus $> \theta \exp P(\operatorname{Re} A_\#)$, and let S'_λ, σ'_μ be in the corresponding generalized eigenspaces of $\mathscr{L}'_\Phi$, $\mathscr{L}'^*_\Phi$. Then the *Gibbs distributions* on Ω have images by $\pi_{(-\infty,\ 0]}$ of the form $S'_\mu\sigma'_\lambda$ or linear combinations of such products (a precise description is given in [9]).

Let us write

(1.4)

$$d(ze^{A_\#}) = \exp\left[-\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{\xi\,:\,\tau^n\xi=\xi}\exp\bigl(A_\#(\xi)+A_\#(\tau\xi)+\cdots+A_\#(\tau^{n-1}\xi)\bigr)\right].$$

Then this series converges when $|z|\bar{\theta}\exp P(\operatorname{Re} A_\#) < 1$, with $\bar{\theta} < 1$ as in [9]. In this region, the zeros of $z \mapsto d(ze^{A_\#})$ coincide with the inverses λ^{-1} of the eigenvalues of $\mathscr{L}_\Phi$, and have the same multiplicity.

2. Symbolic dynamics: the flow (τ_Θ^t)

Consider the compact set $\Omega \times [0,1]$ and identify $(\xi, 1)$ with $(\tau\xi, 0)$; we obtain a compact space $\Omega^\#$. Let $(\xi, u) \mapsto A(\xi, u)$ be a continuous function $\Omega^\# \mapsto \mathbf{C}$ such that $A(\cdot, u) \in \mathscr{C}_\theta$ and $u \to A(\cdot, u)$ is continuous from $[0,1]$ to $\mathscr{C}_\theta$. We call $\mathscr{C}_\theta^\#$ the Banach space of such functions with norm

$$|||A|||_\theta^\# = \max_u |||A(\cdot, u)|||_\theta. \tag{2.1}$$

We denote by $\mathscr{C}_\theta^{\#*}$ the dual of this space.

Let Θ be a real continuous and strictly positive function on $\Omega^\#$. The *suspended flow* with *speed function* Θ^{-1} is the flow (τ_Θ^t) defined on $\Omega^\#$ by

$$\tau_\Theta^t(\xi, u) = (\xi, u(t)), \qquad \frac{du(t)}{dt} = \frac{1}{\Theta(\xi, u)}$$

with appropriate identifications when $u(t) = 0$ or 1. This flow is *mixing* if there is no $A \in \mathscr{C}_{\mathbf{C}}(\Omega^\#)$ satisfying $A \circ \tau_\Theta^t = e^{i\alpha t}A$ with $A \neq 0$, $\alpha > 0$, and (τ_Θ^t) is nonmixing if and only if it is isomorphic to a flow with *constant* speed function.[3] The correspondence between Ω and $\Omega^{\#}$ extends to invariant (probability) measures, and to functions, as follows:

$$\begin{array}{ccc} \Omega^{\#} \circlearrowright \left(\tau_{\Theta}^{t}\right) & \sigma^{\#} \quad \sigma^{\times} & A \\ \uparrow & \uparrow \nearrow \quad & \downarrow \\ \Omega \circlearrowright \ \tau & \sigma \qquad & A_{\#} \end{array}$$

(see formulas (2.2), (2.3), and (2.6) which follow).

If σ is a τ-invariant measure on Ω, a (τ_{Θ}^{t})-invariant measure $\sigma^{\#}$ on $\Omega^{\#}$ is defined by

$$\sigma^{\#}(d\xi du) = \sigma(d\xi)\Theta(\xi, u)\, du, \tag{2.2}$$

where du denotes Lebesgue measure. If σ is a τ-invariant probability measure, then a (τ_{Θ}^{t})-invariant probability measure $\sigma^{\times}$ is given by

$$\sigma^{\times} = \sigma^{\#}\left(\int \sigma(d\xi) r(\xi)\right)^{-1}, \tag{2.3}$$

where we have written

$$r(\xi) = \int \Theta(\xi, u)\, du. \tag{2.4}$$

The map $\sigma \mapsto \sigma^{\times}$ is a bijection of the τ-invariant probability measures on Ω to the (τ_{Θ}^{t})-invariant probability measures on $\Omega^{\#}$. The entropy $h_{\Theta}(\sigma^{\times})$ with respect to (τ_{Θ}^{t}) is given by Abramov's formula:

$$h_{\Theta}(\sigma^{\times}) = h(\sigma)\left(\int \sigma(d\xi) r(\xi)\right)^{-1}.$$

The *pressure* of $A \in \mathscr{C}(\Omega^{\#}, \mathbb{R})$ is defined by

$$P^{\#}(A) = \max\left\{ h_{\Theta}(\sigma^{\times}) + \sigma^{\times}(A) : \sigma^{\times} \text{ is a } \left(\tau_{\Theta}^{t}\right)\text{-invariant probability measure}\right\}. \tag{2.5}$$

Write

$$A_{\#}(\xi) = \int_{0}^{1} A(\xi, u)\Theta(\xi, u)\, du. \tag{2.6}$$

Then $A_{\#} \in \mathscr{C}(\Omega, \mathbb{R})$. (Note that $1_{\#} = r$ by (2.4).) If $A \in \mathscr{C}_{\theta}^{\#}$, there is a unique measure $\rho^{\times}$ realizing the maximum in (2.5). This is called the *Gibbs state* for A. In fact $\rho^{\times}$ corresponds by (2.2), (2.3) to the τ-invariant probability measure ρ on Ω which is the Gibbs state for $A_{\#} - P^{\#}(A)r$. Furthermore $P(A_{\#} - P^{\#}(A)r) = 0$ and this equation determines $P^{\#}(A)$ (see Bowen and Ruelle [4]).

[3] For a precise statement see [1].

Let us return to the original Axiom A flow (f^t) on the manifold M. The connection between the flow (τ_Θ^t) on $\Omega^\#$ and (f^t) restricted to a basic set Λ of M is by a map $\bar{\omega}:\Omega^\# \mapsto \Lambda$ (see Bowen [2]). The map $\bar{\omega}$ sends $(\xi, 0)$ to a point x_ξ of the hypersurface Σ_{ξ_0} such that its orbit successively intersects all Σ_{ξ_k} in the order given by the components ξ_k of ξ. The point $(\xi, u) = \tau_\Theta^t(\xi, 0)$ goes to $f^t x_\xi$. Using $\bar{\omega}$ one can send functions on Λ to functions on $\Omega^\#$ and measures on $\Omega^\#$ to measures on Λ. In this manner, the study of correlation functions for the Axiom A flow (f^t) translates into the study of correlation functions for the suspended flow (τ_Θ^t). This approach, called *symbolic dynamics*, has the disadvantage of a certain arbitrariness (the choice of $\Omega^\#$, (τ_Θ^t), $\bar{\omega}$ is nonunique) but we shall not further consider the question. (For Axiom A diffeomorphisms, the problem has been discussed in [9], and one could repeat the same remarks here, *mutatis mutandis*.)

Note that the positive function r on Ω defined by (2.4) expresses the time between crossing Σ_{ξ_0} and the next hypersurface Σ_{ξ_1} in terms of the symbol sequence. By suitably choosing the hypersurfaces Σ_i (they should be unions of *unstable manifolds*) one can assume that $r(\xi)$ depends only on the components ξ_k of ξ with $k \leqslant 1$.

We thus have

$$r \circ \tau^{-1} = \tilde{r} \circ \pi_{(-\infty,0]}, \tag{2.7}$$

where $\tilde{r}$ is a function on $\Omega_{(-\infty,0]}$ and $\pi_{(-\infty,0]}:\Omega \to \Omega_{(-\infty,0]}$ has been defined in §1. From the general theory of Axiom A flows (see Bowen [2], Bowen and Ruelle [4]), it follows that the time between crossings of the hypersurfaces Σ_i is a Hölder continuous function and, as a consequence, that $\tilde{r}$ belongs to $\mathscr{C}_\theta((-\infty, 0])$ for suitable θ. Similarly, if $\mathbf{A}$ is a smooth function on the manifold M, and we define $A = \mathbf{A} \circ \bar{\omega}$ and $A_\#$ by (2.6) we find $A_\# \in \mathscr{C}_{\theta^2}$ for suitable θ (for technical reasons we want θ^2 here rather than θ).

From now on we shall work with the symbolic dynamics, remembering from the differentiable setup only that $A, \Theta \in \mathscr{C}_{\theta^2}^\#$, so that

$$\tilde{r} \in \mathscr{C}_\theta((-\infty,0]), \qquad A_\# \in \mathscr{C}_{\theta^2}$$

follow from (2.4), (2.6).

Remember that an interaction Φ has been associated with $A_\#$ by (1.1). It is convenient to introduce also an interaction Ψ, associated with the function $\tilde{r}$ defined by (2.7), such that

$$\tilde{r}(\xi') = -\Psi_0(\xi_0') - \sum_{k=1}^{\infty} \Psi_k(\xi_{-k}', \cdots, \xi_0') \tag{2.8}$$

and $|\Psi_k| < \text{const}\, \theta^k$. Note that with the notation of (1.1) we have $\tilde{r} = A'_\Psi$.

For real $A \in \mathscr{C}_{\theta^2}^{\#}$, a *zeta function* is defined by

$$
\begin{aligned}
\zeta_A(s) &= \prod_\gamma \left[1 - \exp \int_0^{l(\gamma)} \left(A\left(\tau_\Theta^t x_\gamma\right) - s\right) dt\right]^{-1} \\
&= \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\xi : \tau^n \xi = \xi} \exp \sum_{k=0}^{n-1} \left[A_\#(\tau^k \xi) - s r(\tau^k \xi)\right] \qquad (2.9) \\
&= \left[d(\exp(A_\# - sr))\right]^{-1},
\end{aligned}
$$

where the product is over the periodic orbits γ for the flow, $x_\gamma \in \gamma$, and $l(\gamma)$ is the prime period of γ (Ruelle [8]); the functional d is defined by (1.4). The expressions (2.9) converge, and $\zeta_A(s)$ is analytic for $\operatorname{Re} s > P^{\#}(A)$.

2.1. Theorem. (a) (*Pollicott* [6]) *ζ_A extends to a meromorphic function in $\{s : \operatorname{Re} s > P^{\#}(A) - \delta\}$, where δ is determined by $P^{\#}(A_\# - (P^{\#}(A) - \delta)r) = \log \bar{\theta}^{-1}$. The poles of $\zeta_A(s)$ are located at the points s such that* 1 *is an eigenvalue of $\mathscr{L}'_{\Phi - s\Psi}$.*

(b) (*Ruelle* [8]) *ζ_A has a simple pole at $s = P^{\#}(A)$.*

(c) (*Parry and Pollicott* [5]) *If (τ_Θ^t) is mixing, ζ_A has no pole on the line $\{s : \operatorname{Re} s = P^{\#}(A)\}$ apart from the pole at $s = P^{\#}(A)$.*

By analogy with the proof of the prime number theorem one can, in view of (c), apply the Wiener-Ikehara Tauberian theorem to $\zeta_0(s)$ to study the distribution of the periods $l(\gamma)$ (Parry and Pollicott [5]).

It may be convenient to consider a functional defined with respect to the flow (τ_1^t) with unit speed. For $A \in \mathscr{C}(\Omega^{\#}, \mathbb{C})$, write

$$
\mathscr{D}(A) = \prod_\gamma \left[1 - \exp \int_0^{l_1(\gamma)} A\left(\tau_1^t x_\gamma\right) dt\right] = d(\exp A_1),
$$

where $A_1 = \int_0^1 A(\xi, u)\, du$, and $l_1(\gamma)$ is the (integer) period of γ with respect to (τ_1^t). With this definition, $\zeta_A(s) = [\mathscr{D}((A - s)\Theta)]^{-1}$. The function $A \mapsto \mathscr{D}(A)^{-1}$ is holomorphic on $\mathscr{C}(\Omega^{\#}, \mathbb{C})$ when $P(\operatorname{Re} A_1) < 0$; the function $A \mapsto \mathscr{D}(A)$ is holomorphic on $\mathscr{C}_{\theta^2}^{\#}$ when $P(\operatorname{Re} A_1 + \log \bar{\theta}) < 0$.

3. Gibbs distributions for the flow $\left(\tau_\Theta^t\right)$

The concept of Gibbs distributions for a lattice system introduced in [9] was shown to be a natural extension of the concept of Gibbs state. If we want to study Axiom A flows rather than diffeomorphisms, we need another concept. The definition presented here is somewhat *ad hoc*, but appropriate for the discussion of correlation functions as we shall see later. It is in fact a natural *continuous time* version of the concept introduced earlier for discrete systems, but restricted to the simplest case (see Remark below).

For discrete systems, a space $\mathscr{G}_{\lambda\mu}$ of Gibbs distributions on Ω is defined as the span of elements of the form

$$\sigma'(d\xi')\sigma''(d\xi'')e^{-V_\Phi(\xi' \vee \xi'')}.$$

In this formula, σ', σ'' belong to the generalized eigenspaces to the eigenvalues λ, μ of operators $\mathscr{L}_\Phi'^*$, $\mathscr{L}_\Phi''^*$ acting on $\mathscr{C}_\theta^*((-\infty, 0])$ and $\mathscr{C}_\theta^*([1, \infty))$ respectively. The operator $\mathscr{L}_\Phi'^*$ is the dual of $\mathscr{L}_\Phi'$ defined by (1.3), and $\mathscr{L}_\Phi''^*$ is defined analogously. We have written

$$V_\Phi(\xi' \vee \xi'') = \sum_{l=0}^{\infty} \sum_{m=1}^{\infty} \Phi_{l+m}(\xi'_{-l}, \dots, \xi'_0, \xi''_1, \dots, \xi''_m). \tag{3.1}$$

Thus, all the ingredients of $\mathscr{G}_{\lambda\mu}$ are defined with respect to an interaction Φ, or equivalently a function A_Φ (see (1.1)). (Note that $A_\Phi \circ \tau^{-k}$ is, up to sign, the contribution to the *energy* of the *lattice site* k for the standard interpretation of the formalism we are describing.)

Our first step towards a definition of Gibbs distributions for continuous time systems will be to replace $A_\Phi \circ \tau^{-k}$ by different functions for $k \leqslant 0$ and $k \geqslant 1$. More precisely, we replace A_Φ by[4] $A_\# - vr \circ \tau^{-1}$ for $k \leqslant 0$ and $A_\# - wr \circ \tau^{-1}$ for $k \geqslant 1$. (We start from real Θ, $A \in \mathscr{C}_{\theta^2}^{\#}$, with $\Theta > 0$, and r, $A_\#$ are defined by (2.4), (2.6). The complex numbers v, w will be specified in a minute.) Using the interactions Φ, Ψ defined by (1.1), (2.8) and the defintions (1.2), (1.3) we see that the operator $\mathscr{L}'$ associated with $A_\# - vr \circ \tau^{-1}$ is $\mathscr{L}'_{\Phi - v\Psi}$ such that

$$(\mathscr{L}'_{\Phi - v\Psi}\phi)(\zeta') = \sum_{\eta \in J} t_{\xi_0 \eta}\left[\exp A'_{\Phi - v\Psi}(\tau\zeta' \vee \eta)\right]\phi(\tau\xi' \vee \eta).$$

There is an analogous definition for $\mathscr{L}''_{\Phi - w\Psi}$. In the function V_Φ defined by (3.1) we replace Φ by $\Phi - w\Psi$ (*not* $\Phi - v\Psi$, the reason for this asymmetric choice is that the difference between $A_\# - vr \circ \tau^{-1}$ and $A_\# - wr \circ \tau^{-1}$, viz. $-(v - w)r \circ \tau^{-1}$, depends only on arguments ξ_k with $k \in (-\infty, 0]$).

The numbers v, w are specified by the condition that 1 be an eigenvalue of $\mathscr{L}'_{\Phi - v\Psi}$ and $\mathscr{L}''_{\Phi - w\Psi}$, and that

$$P^\#(A) - \delta < \operatorname{Re} v, \operatorname{Re} w, \tag{3.2}$$

where δ is determined by Theorem 2.1(a). (We have also automatically $\operatorname{Re} v$, $\operatorname{Re} w \leqslant P^\#(A)$.)

[4] Equivalently, we might use $(A_\# - vr)\circ\tau^{-1}$ for $k \leqslant 0$ and $(A_\# - wr)\circ\tau^{-1}$ for $k \geqslant 1$; the final definitions would not change.

Let $F_v'^*$ and $F_w''^*$ be the eigenspaces to the eigenvalue 1 of $\mathscr{L}_{\Phi-v\Psi}'^*$ and $\mathscr{L}_{\Phi-w\Psi}''^*$. (Note: the strict, *not* the generalized eigenspaces.) We let $\mathscr{F}_{vw}$ be the finite dimensional subspace of $\mathscr{C}_\theta^*$ generated by the elements

$$\sigma_\#(d\xi' \vee d\xi'') = \sigma_{(v)}'(d\xi')\sigma_{(w)}''(d\zeta'')e^{-V_{\Phi-v\Psi}(\xi'\vee\xi'')}, \tag{3.3}$$

where $\sigma_{(v)}' \in F_v'^*$, $\sigma_{(w)}'' \in F_w''^*$. (It is not hard to see that $F_v'^* \otimes F_w''^* \mapsto \mathscr{F}_{vw}$ is bijective.)

The restriction of $\mathscr{L}_{\Phi-v\Psi}'^*$ to $F_v'^*$ is the identity operator; similarly for the restriction of $\mathscr{L}_{\Phi-w\Psi}''^*$ to $F_w''^*$. Using (3.3), it is now readily checked that

$$\begin{aligned}(\tau\sigma_\#)(d\xi' \vee d\xi'') = \sigma_{(v)}'(d\xi')\sigma_{(w)}''(d\xi'')&\\ \cdot\exp\left[(v-w)\tilde{r}(\xi') - V_{\Phi-w\Psi}(\xi\vee\xi'')\right]&\end{aligned}$$

so that, for all $\sigma_\# \in \mathscr{F}_{vw}$,

$$\tau\sigma_\# = \exp\left[(v-w)r\circ\tau^{-1}\right]\cdot\sigma_\#, \tag{3.4}$$

or equivalently

$$\tau^{-1}\sigma_\# = \exp[-(v-w)r]\cdot\sigma_\#.$$

Define now $\sigma^\# \in \mathscr{C}_\theta^{\#*}$ by

$$\begin{aligned}\sigma_\#(d\xi du) &= \sigma_\#(d\zeta)\cdot\exp[-(v-w)t(\xi,u)]\cdot\Theta(\xi,u)\,du\\ &= \sigma_\#(d\xi)\cdot\exp[-(v-w)t]\,dt,\end{aligned} \tag{3.5}$$

where $t = t(\xi,u)$ is the inverse of the function $t \mapsto u(t)$ such that $du/dt = \Theta(\xi,u)^{-1}$ and $u(0) = 0$, i.e., $t(\xi,u) = \int_0^u d\alpha\Theta(\xi,\alpha)$. (Note that $t(\xi,1) = r(\xi)$ and that $|||t|||_\theta^\# \leqslant |||\Theta|||_\theta^\#$ in view of (2.1).) We define the space $\mathscr{F}_{vw}^\#$ of *Gibbs distributions* to consist of the $\sigma^\#$ constructed above.

Writing $(\tau_\Theta^t\sigma^\#)(A) = \sigma^\#(A\circ\tau_\Theta^t)$ we find that

$$\frac{d}{dt}\tau_\Theta^t\sigma^\# = (v-w)\sigma^\#$$

for $\sigma^\# \in \mathscr{F}_{vw}^\#$, hence

$$\tau_\Theta^t\sigma^\# = \sigma^\#\cdot e^{(v-w)t}.$$

Returning to the space $\mathscr{F}_{vw}$, we note that the projection $\pi_{(-\infty,0]}\mathscr{F}_{vw}$ is readily characterized. We have indeed, from (3.3)

$$(\pi_{(-\infty,0]}\sigma_\#)(d\xi') = S_{(w)}'(\xi')\sigma_{(v)}'(d\xi'), \tag{3.6}$$

where

$$S_{(w)}'(\xi') = \int\sigma_{(w)}''(d\xi'')\exp\left[-V_{\Phi-w\Psi}(\xi'\vee\xi'')\right].$$

It is known (see [9]) that the functions $S'_{(w)}$ of this form (with $\sigma''_{(w)} \in F_w''^*$) constitute precisely the eigenspace F_w' to the eigenvalue 1 of the operator $\mathscr{L}'_{\Phi - w\Psi}$ acting on $\mathscr{C}_\theta(-\infty, 0]$. Thus $\pi_{(-\infty,0]}\mathscr{F}_{vw}$ is spanned by $F_w' F_v'^*$. Note that we also have

$$\pi_{(-\infty,0]}\tau\sigma_{\#} = \exp[(v - w)\tilde{r}] \cdot \pi_{(-\infty,0]}\sigma_{\#}.$$

Example. Since $P(A_{\#} - P^{\#}(A)r) = 0$ (§2), the operator $\mathscr{L}_{\Phi - P^{\#}(A)\Psi}$ has 1 as simple eigenvalue (see §1). Writing $P^{\#}(A) = P$, we see that $\mathscr{F}_{PP}$ is one-dimensional spanned by the Gibbs state $\sigma^{\#}$ on Ω for $A_{\#} - P^{\#}(A)r$. The space $\mathscr{F}^{\#}_{PP}$ is thus spanned by the Gibbs state $\rho^{\times}$ on $\Omega^{\#}$ for A, and $\rho^{\times}$ is therefore also a Gibbs distribution.

Remark. To avoid technical problems we have adopted a definition of Gibbs distributions which uses the *strict* eigenspaces of $\mathscr{L}_v'^*$, $\mathscr{L}_w''^*$. (This is no restriction as long as we consider simple eigenvalues.) The parallel study in [9] was based on a more comprehensive definition, using generalized eigenspaces. As a consequence we could identify in [9] all the coefficients of the poles of the Fourier transform of correlation functions. Here we identify the residues in terms of Gibbs distributions for the important case of simple poles. A more general analysis would of course be desirable.

Example. Let r be a constant function, say $r = T$. Then $\mathscr{L}'_{\Phi - v\Psi} = e^{-vT}\mathscr{L}'_{\Phi}$, $\mathscr{L}''_{\Phi - w\Psi} = e^{-wT}\mathscr{L}''_{\Phi}$. The eigenvalues and eigenvectors are thus readily determined. In particular, 1 is an eigenvalue of $\mathscr{L}'_{\Phi - v\Psi}$ if $\lambda \notin e^{-vT} = 1$, where λ is an eigenvalue of $\mathscr{L}'_{\Phi}$.

This gives

$$v = \frac{1}{T}(\log\lambda + 2k\pi i),$$

where the multivaluedness of the log has been made explicit. Writing similarly

$$w = \frac{1}{T}(\log\mu + 2k'\pi i)$$

we may identify $\sigma^{\#} \in \mathscr{F}_{vw}$ with an element of the space $\mathscr{G}_{\lambda\mu}$ of Gibbs distributions defined in [9]. The corresponding $\sigma^{\#} \in \mathscr{F}^{\#}_{vw}$ is given by

$$\sigma^{\#}(d\xi du) = \sigma^{\#}(d\xi)\exp\left[-(\log\lambda - \log\mu + 2(k - k')\pi i)\frac{t}{T}\right] dt$$

and we have

$$\tau^t_{\Theta}\sigma^{\#} = \sigma^{\#}\exp(\log\lambda - \log\mu + 2(k - k')\pi i)\frac{t}{T}.$$

Notice that $\mathscr{F}^{\#}_{v'w'} = \mathscr{F}^{\#}_{vw}$ when $v' - v = w' - w = l \cdot 2\pi i/T$, l an integer.

4. Correlation functions for the flow (τ^t_Θ)

We consider the suspended flow for a fixed speed function $\Theta^{-1} \in \mathscr{C}^{\#}_{\theta^2}$, and let $\rho^\times$ be the Gibbs state corresponding to the real function $A \in \mathscr{C}^{\#}_{\theta^2}$. If $B, C \in \mathscr{C}^{\#}_\theta$ we define

$$\rho^\times_{BC}(t) = \rho^\times\big((B \circ \tau^t_\Theta) \cdot C\big) - \rho^\times(B)\rho^\times(C)$$

and its Fourier transform

$$\hat{\rho}^\times_{BC}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t}\, dt \rho^\times_{BC}(t)$$

which has to be understood as a tempered distribution. We may express $\rho^\times$ in terms of the Gibbs state ρ for $A_\#$ on Ω (see (2.2), (2.3)). We write

$$\nu^{-1} = \int \rho(d\xi) \int \Theta(\xi, u)\, du = \int r(\xi)\rho(d\xi) \tag{4.1}$$

and

$$B' = B - \rho^\times(B), \qquad C' = C - \rho^\times(C).$$

The following manipulations then yield a correct result in the sense of distributions

$$\begin{aligned}
\hat{\rho}^\times_{BC}(\omega) &= \int_{-\infty}^{\infty} e^{i\omega t} dt \rho^\times\big((B' \circ \tau^t_\Theta) C'\big) \\
&= \nu \int_{-\infty}^{\infty} e^{i\omega t}\, dt \int_\Omega \rho(d\xi) \int_0^1 \Theta(\xi, u)\, du B'\big(\tau^t_\Theta(\xi, u)\big) C'(\xi, u) \\
&= \nu \int_{-\infty}^{\infty} e^{i\omega t}\, dt \int \rho(d\xi) \int_0^{r(\xi)} dt' B'\big(\tau^{t+t'}_\Theta(\xi, 0)\big) C'\big(\tau^{t'}_\Theta(\xi, 0)\big) \\
&= \nu \int \rho(d\xi) \int_{-\infty}^{\infty} e^{i\omega t''} dt'' B'\big(\tau^{t''}_\Theta(\xi, 0)\big) \int_0^{r(\xi)} e^{-i\omega t'} C'\big(\tau^{t'}_\Theta(\xi, 0)\big) \qquad (4.2) \\
&= \nu \int \rho(d\xi) \sum_{j=-\infty}^{\infty} \int_0^{r(\tau^j\xi)} dt \exp i\omega \left(\sum_{k=0}^{j-1} r(\tau^k \xi) + t \right) B'\big(\tau^t_\Theta(\tau^j \xi, 0)\big) \\
&\quad \cdot \int_0^{r(\xi)} e^{-i\omega t'}\, dt' C\big(\tau^{t'}_\Theta(\xi, 0)\big) \\
&= \nu \int \rho(d\xi) \sum_{j=-\infty}^{\infty} \exp i\omega \sum_{k=0}^{j-1} r(\tau^k \xi) \cdot \hat{B}(\tau^j \xi, \omega) \hat{C}(\xi, -\omega),
\end{aligned}$$

where

$$\begin{aligned}\hat{B}(\xi,\omega) &= \int_0^{r(\xi)} e^{i\omega t}\, dtB'\big(\tau_\Theta^t(\xi,0)\big)\\ &= \int_0^1 du\Theta(\xi,u)B'(\xi,u)\exp i\omega t(\xi,u),\end{aligned} \tag{4.3}$$

$$t(\xi,u) = \int_0^u d\alpha\Theta(\xi,\alpha). \tag{4.4}$$

The definition of $\hat{C}$ is similar.

Note that, for each $n \geqslant 0$, $|\partial^n\hat{B}/\partial\omega^n|$, $|\partial^n\hat{C}/\partial\omega^n|$ are bounded uniformly with respect to ω and ξ. Using furthermore the fact that $\min r > 0$, we see that the right-hand side of (4.2) converges in Schwartz' space $\mathscr{S}'(\mathbb{R})$ of temperate distributions (with respect to ω) and thus also in the space $\mathscr{D}'(\mathbb{R})$ of all distributions. We shall next represent $\hat{B}$, $\hat{C}$ as series which converge uniformly on compacts, as well as their derivatives:

$$\begin{aligned}\hat{B} &= X_0 + X_1e^{i\omega r} + X_2e^{i\omega(r+r\circ\tau)} + \cdots + X_me^{i\omega(r+\cdots+r\circ\tau^{m-1})} + \cdots,\\ \hat{C} &= Y_0 + Y_1e^{-i\omega r} + Y_2e^{-i\omega(r+r\circ\tau)} + \cdots + Y_ne^{-i\omega(r+\cdots+r\circ\tau^{n-1})} + \cdots.\end{aligned} \tag{4.5}$$

We define successively $B_0 = \hat{B}$, X_0, B_1, $X_1,\cdots$ as follows:

(a) Treating ω as a parameter, which we now allow to be complex, we extract X_m as the part of B_m depending only on $\xi_{-m},\cdots,\xi_m$. This extraction is not unique, but can be achieved linearly, so that

$$\operatorname{var}_{m+1}X_m = 0,\quad \|X_m\|_\infty \leqslant \|B_m\|_\infty,\quad \|B_m - X_m\|_\infty \leqslant \operatorname{var}_{m+1}B_m. \tag{4.6}$$

(b) We define

$$B_{m+1} = (B_m - X_m)e^{-i\omega r\circ\tau^m} \tag{4.7}$$

We may assume that

$$\begin{aligned}&\operatorname{var}_k\hat{B} \leqslant K\theta^k, \qquad \|\hat{B}\|_\infty \leqslant K,\\ &\operatorname{var}_k r \leqslant L\theta^k \quad \text{for } k \geqslant 1.\end{aligned} \tag{4.8}$$

(K depends on ω, but is uniformly bounded on compacts; from (4.3) and (4.4) we see that we may take $K = |||\Theta|||_\theta^\# |||B'|||_0^\# \exp(|\omega|\,|||\Theta|||_0^\#)$. We also have $L = \|r\|_\theta \leqslant |||\Theta|||_\theta^\#$.) Note that by construction we have

$$\operatorname{var}_k(B_m - X_m) \leqslant \operatorname{var}_k B_m \quad \text{for } k > m. \tag{4.9}$$

In view of (4.6), (4.7), (4.8), (4.9), we obtain

$$\operatorname{var}_k B_m \leqslant K_m\theta^k \quad \text{for } k \geqslant m$$

provided the K_m satisfy $K_0 \geqslant K$ and

$$K_{m+1} \geqslant E\big(K_m + K_m|\omega|L\theta^{2m+1}\big)$$

with

$$E = \begin{cases} \exp(\operatorname{Im}\omega \cdot \max r) & \text{for } \operatorname{Im}\omega \geqslant 0, \\ \exp(\operatorname{Im}\omega \cdot \min r) & \text{for } \operatorname{Im}\,\omega \leqslant 0. \end{cases}$$

We take

$$K_m = \overline{K}E^m, \qquad \overline{K} = K\prod_{k=0}^{\infty}\left(1 + L|\omega|\theta^{2k+1}\right).$$

Thus

$$\operatorname{var}_k B_m \leqslant \overline{K}E^m\theta^k \quad \text{for } k \geqslant m,$$

$$\|B_m - X_m\|_\infty \leqslant \overline{K}E^m\theta^{m+1}, \qquad \|X_m\|_\infty \leqslant \|B_m\|_\infty \leqslant \overline{K}E^m\theta^m.$$

Similar estimates hold for the derivatives of the X_m with respect to ω. Therefore for ω real, and thus $E = 1$, the series (4.5) for $\hat{B}$, and the differentiated series converge exponentially fast on compact sets. In the sense of convergence in $\mathscr{D}'(\mathbb{R})$ we therefore have

$$\begin{aligned}
&\hat{\rho}^{\times}_{BC}(\omega) \\
&= \nu\int \rho(d\xi) \sum_{j=-\infty}^{\infty} \exp i\omega \sum_{k=0}^{j-1} r(\tau^k\xi) \\
&\quad \cdot \sum_{m=0}^{\infty} X_m(\tau^j\xi, \omega)\exp i\omega\left(r(\tau^j\xi) + \cdots + r(\tau^{j+m-1}\xi)\right) \\
&\quad \cdot \sum_{n=0}^{\infty} Y_n(\xi, -\omega)\exp - i\omega\left(r(\xi) + \cdots + r(r^{n-1}\xi)\right) \\
&= \nu\int \rho(d\xi) \sum_{m,n}\sum_{j=-\infty}^{\infty} X_m(\tau^j\xi, \omega)Y_n(\xi, -\omega)\exp i\omega \sum_{k=n}^{j+m-1} r(\tau^k\xi) \\
&= \nu\int \rho(d\xi) \sum_{m,n}\sum_{j=-\infty}^{\infty} X_m(\tau^{j-n}\xi, \omega)Y_n(\tau^{-n}\xi, -\omega)\exp i\omega \sum_{k=0}^{j+m-n-1} r(\tau^k\xi) \\
&= \nu\int \rho(d\xi) \sum_{m,n}\sum_{j=-\infty}^{\infty} X_m(\tau^{j-m}\xi, \omega)Y_n(\tau^{-n}\xi, -\omega)\exp i\omega \sum_{k=0}^{j-1} r(\tau^k\xi) \\
&= \nu\int \rho(d\xi)\Bigg[\sum_{l=0}^{\infty}\left(\exp - i\omega\sum_{k=1}^{l} r(\tau^{-k}\xi)\right)B''(\tau^{-l}\xi, \omega)C''(\xi, -\omega) \\
&\qquad + \sum_{l=0}^{\infty}\left(\exp i\omega \sum_{k=1}^{l} r(\tau^{-k}\xi)\right)B''(\xi, \omega)C''(\tau^{-l}\xi, -\omega) \\
&\qquad\qquad - B''(\xi, \omega)C''(\xi, -\omega)\Bigg],
\end{aligned}$$

where

$$
(4.10)\qquad
\begin{aligned}
B''(\xi,\omega) &= \sum_{m=0}^{\infty} X_m(\tau^{-m}\xi,\omega),\\
C''(\xi,-\omega) &= \sum_{n=0}^{\infty} Y_n(\tau^{-n}\xi,-\omega).
\end{aligned}
$$

We may write $\tilde\rho = \pi_{(-\infty,0]}\rho$, and

$$
r\circ\tau^{-1} = \tilde r\circ\pi_{(-\infty,0]}, \qquad \tau' = \pi_{(-\infty,0]}\tau^{-1},
$$

$$
(4.11)\qquad
\left.\begin{aligned}
B''(\xi,\omega) &= \tilde B_\omega\circ\pi_{(-\infty,0]}\,\xi,\\
C''(\xi,\omega) &= \tilde C_{-\omega}\circ\pi_{(-\infty,0]}\,\xi
\end{aligned}\right\}
$$

so that

$$
\begin{aligned}
\hat\rho^{\times}_{BC}(\omega) = \nu\int\tilde\rho(d\xi')\Bigg[&\sum_{l=0}^{\infty}\left(\exp i\omega\sum_{k=0}^{l-1}\tilde r(\tau'^k\xi')\right)\tilde B_\omega(\tau'^l\xi')\tilde C_{-\omega}(\xi')\\
&+\sum_{l=0}^{\infty}\left(\exp i\omega\sum_{k=0}^{l-1}\tilde r(\tau'^k\xi')\right)\tilde B_\omega(\xi')\tilde C_{-\omega}(\tau'^l\xi')\\
&-\tilde B_\omega(\xi')\tilde C_{-\omega}(\xi')\Bigg].
\end{aligned}
$$

We have (see (3.6))

$$
\tilde\rho(d\xi') = S'_{(P)}(\xi')\sigma'_{(P)}(d\xi'),
$$

where $P = P^{\#}(A)$ and $S'_{(P)}$ and $\sigma'_{(P)}$ are the eigenvectors corresponding to the eigenvalue 1 of the operators $\mathscr{L}'_{\Phi-P\Psi}$, $\mathscr{L}'^{*}_{\Phi-P\Psi}$ acting on $\mathscr{C}_\theta(-\infty,0]$, $\mathscr{C}_\theta^{\#}(-\infty,0]$. (These eigenvectors are unique up to normalization.) Thus

$$
\begin{aligned}
\hat\rho^{\times}_{BC}(\omega) &= \nu\sigma'_{(P)}\Bigg[\tilde B_\omega\sum_{l=0}^{\infty}\mathscr{L}'^{l}_{\Phi-(P+i\omega)\Psi}\left(S'_{(P)}C_{-\omega}\right)\\
&\qquad+\tilde C_{-\omega}\sum_{l=0}^{\infty}\mathscr{L}'^{l}_{\Phi-(P-i\omega)\Psi}\left(S'_{(P)}\tilde B_\omega\right)\Bigg]-\nu\tilde\rho(\tilde B_\omega\tilde C_{-\omega})\\
(4.12)\qquad &= \nu\tilde\rho\left[\tilde B_\omega\left(1-S'^{-1}_{(P)}\mathscr{L}'_{\Phi-(P+i\omega)\Psi}S'_{(P)}\right)^{-1}\tilde C_{-\omega}\right]\\
&\qquad+\nu\tilde\rho\left[\tilde C_{-\omega}\left(1-S'^{-1}_{(P)}\mathscr{L}'_{\Phi-(P-i\omega)\Psi}S'_{(P)}\right)^{-1}\tilde B_\omega\right]-\nu\tilde\rho(\tilde B_\omega\tilde C_\omega)\\
&= \nu\tilde\rho\left[\tilde B_\omega\left(\left(1-S'^{-1}_{(P)}\mathscr{L}'_{\Phi-(P+i\omega)\Psi}S'_{(P)}\right)^{-1}-\frac12\right)\tilde C_{-\omega}\right]\\
&\qquad+\nu\tilde\rho\left[\tilde C_{-\omega}\left(\left(1-S'^{-1}_{(P)}\mathscr{L}'_{\Phi-(P-i\omega)\Psi}S'_{(P)}\right)^{-1}-\frac12\right)\tilde B_\omega\right].
\end{aligned}
$$

Note that the two terms in the right-hand side are permuted by the interchange of B and C, and the replacement of ω by $-\omega$.

4.1. Theorem. *If B, $C \in \mathscr{C}_\theta^\#$, the function $\hat{\rho}_{BC}^\times$ extends to a meromorphic function in the strip*

$$|\mathrm{Im}\,\omega| < \delta^*, \tag{4.13}$$

where

$$\delta^* = \frac{|\log\theta|}{2\max r - \min r}.$$

If we also have $|\mathrm{Im}\,\omega| < \delta$, we may write

$$\begin{aligned} \hat{\rho}_{BC}^\times(\omega) = {} & \frac{N_{BC}(\omega)}{d\big(\exp\big(A_\# - \big(P^\#(A) + i\omega\big)r\big)\big)} \\ & + \frac{N_{CB}(-\omega)}{d\big(\exp\big(A_\# - \big(P^\#(A) - i\omega\big)r\big)\big)}, \end{aligned} \tag{4.14}$$

where N_{BC} is holomorphic in (4.13) *and d is as in* (1.4).

Note that, in view of (2.9), we may rewrite (4.14) as

$$\hat{\rho}_{BC}^\times(\omega) = N_{BC}(\omega)\xi\big(P^\#(A) + i\omega\big) + N_{CB}(-\omega)\zeta\big(P^\#(A) - i\omega\big).$$

The position of the poles of $\hat{\rho}_{BC}^\times$ is thus simply related to that of the poles of ζ. (They are of the form $\pm i(P^\#(A) - s)$, where s is such that 1 is an eigenvalue of $\mathscr{L}'_{\Phi - s\Psi}$.)

A partial proof of the above proposition has been obtained earlier by Pollicott [7].

Let $\theta < \theta^* < 1$; then $\omega \mapsto \tilde{B}_\omega$ is holomorphic with values in $\mathscr{C}_{\theta^*}(-\infty,\ 0]$ in the region defined by $\theta^{*-1}E\theta < 1$, i.e.,

$$2|\log\theta^*| < \begin{cases} |\log\theta| - \mathrm{Im}\,\omega \cdot \max r & \text{if } \mathrm{Im}\,\omega \geqslant 0, \\ |\log\theta| - \mathrm{Im}\,\omega \cdot \min r & \text{if } \mathrm{Im}\,\omega \leqslant 0. \end{cases}$$

On the other hand, $(1 - S'^{-1}_{(P)}\mathscr{L}'_{\Phi-(P-i\omega)}S'_{(P)})^{-1}$ is meromorphic as an operator on $\mathscr{C}_{\theta^*}(-\infty,\ 0]$ provided

$$\theta^* \exp P\big(\mathrm{Re}\big(A_\# - \big(P^\#(A) - i\omega\big)r\big)\big) < 1,$$

i.e.,

$$P\big(A_\# - \big(P^\#(A) + \mathrm{Im}\,\omega\big)r\big) < |\log\theta^*|.$$

Since $P(A_{\#} - P^{\#}(A)r) = 0$, this condition is implied by

$$-\operatorname{Im}\omega \cdot \max r < |\log \theta^*|. \tag{4.15}$$

Therefore $\omega \mapsto (1 - S_{(P)}'^{-1}\mathscr{L}'_{\Phi-(P-i\omega)\Psi}S'_{(P)})^{-1}\tilde{B}_\omega$ is meromorphic if

$$-\frac{|\log\theta|}{2\max r - \min r} < \frac{|\log\theta|}{\max r}$$

and $\omega \to \tilde{\rho}[\tilde{C}_{-\omega}(1 - S_{(P)}'^{-1}\mathscr{L}'_{\Phi-(P-i\omega)\Psi}S'_{(P)})^{-1}\tilde{B}_\omega]$ is also meromorphic. Interchanging B and C, we obtain from (4.12) the meromorphy of $\hat{\rho}^{\#}_{BC}$ in (4.13).

If $|\operatorname{Im}\omega| < \delta$, we may also write

$$\left(1 - S_{(P)}'^{-1}\mathscr{L}'_{\Phi-(P-i\omega)\Psi}S'_{(P)}\right)^{-1} = \frac{\mathscr{N}}{d\left(\exp\left(A_{\#} - \left(P^{\#}(A) - i\omega\right)r\right)\right)},$$

where the numerator is holomorphic in (4.15) (see [9, Proposition 3.3]); from this (4.14) follows readily.

4.2. Theorem. *Suppose that* 1 *is a simple eigenvalue of* $\mathscr{L}'_{\Phi-s\Psi}$. *There is thus a simple eigenvalue* $\lambda(z)$ *of* $\mathscr{L}'_{\Phi-z\Psi}$ *depending analytically on* z *for* z *close to* s. *Assume that the derivative* $\lambda'(s) \neq 0$. *Then* $\hat{\rho}^{\times}_{BC}$ *has simple poles at* $\pm i(P^{\#}(A) - s)$. *Their residues are*

$$\frac{i}{K}\sigma^{\#}_{Ps}(B)\sigma^{\#}_{sP}(C) \quad \textit{and} \quad -\frac{i}{K}\sigma^{\#}_{Ps}(C)\sigma^{\#}_{sP}(B)$$

respectively, with $\sigma^{\#}_{Ps} \in \mathscr{F}^{\#}_{Ps}$, $\sigma^{\#}_{sP} \in \mathscr{F}^{\#}_{sP}$, *and* K *a constant.*

(The normalization of $\sigma^{\#}_{Ps}$, $\sigma^{\#}_{sP}$, and the value of K are discussed in the Remark to follow).

The two poles come from the two terms in the right-hand side of (4.12). It suffices to discuss the first term, which we rewrite

$$\nu\sigma'_{(P)}\left[\tilde{B}_\omega\left(\left(1 - \mathscr{L}'_{\Phi-(P+i\omega)\Psi}\right)^{-1} - \tfrac{1}{2}\right)S'_{(P)}\tilde{C}_{-\omega}\right].$$

Up to a contribution regular at $i(P^{\#}(A) - s)$ this is

$$\nu\sigma'_{(P)}\left(\tilde{B}_\omega S'_{(s)}\right)\frac{1}{1 - \lambda\left(P^{\#}(A) + i\omega\right)}\sigma'_{(s)}\left(S'_{(P)}\tilde{C}_{-\omega}\right)$$

or, again up to a regular contribution,

$$\begin{aligned} &\nu\sigma'_{(P)}\left(S'_{(s)}\tilde{B}_{i(P^{\#}(A)-s)}\right)\sigma'_{(s)}\left(S'_{(P)}\tilde{C}_{-i(P^{\#}(A)-s)}\right)\left(\lambda'(s)\left(s - P^{\#}(A) - i\omega\right)\right)^{-1} \\ &= \frac{\nu\sigma_{Ps}\left(B''\left(\cdot, i\left(P^{\#}(A) - s\right)\right)\right)\sigma_{sP}\left(C''\left(\cdot, -i\left(P^{\#}(A) - s\right)\right)\right)}{\lambda'(s)\left(s - P^{\#}(A) - i\omega\right)}, \end{aligned} \tag{4.16}$$

where we have used (3.6), (4.11), and $\sigma_{Ps} \in \mathscr{F}_{Ps}$, $\sigma_{sP} \in \mathscr{F}_{sP}$. We have, using successively (4.10), (3.4), (4.5), (4.3), and (3.5),

$$\begin{aligned}
&\sigma_{Ps}\big(B''(\cdot, i(P^{\#}(A) - s))\big) \\
&\quad = \sigma_{Ps}\left(\sum_{m=0}^{\infty} X_m\big(\tau^{-m}\cdot, i(P^{\#}(A) - s)\big)\right) \\
(4.17)\quad &\quad = \sum_{m=0}^{\infty} (\tau^{-m}\sigma_{Ps})\big[X_m(\cdot, i(P^{\#}(A) - s))\big] \\
&\quad = \sigma_{Ps}\left[\sum_{m=0}^{\infty} X_m(\cdot, i(P^{\#}(A) - s))\exp\left(-(P^{\#}(A) - s)\sum_{k=0}^{m-1} r \circ \tau^k(\cdot)\right)\right] \\
&\quad = \sigma_{Ps}\big[\hat{B}(\cdot, i(P^{\#}(A) - s))\big] = \sigma^{\#}_{Ps}(B') = \sigma^{\#}_{Ps}(B)
\end{aligned}$$

with $\sigma^{\#}_{Ps} \in \mathscr{F}^{\#}_{Ps}$. Similarly

$$\sigma_{sP}\big(C''(\cdot, -i(P^{\#}(A) - s))\big) = \sigma^{\#}_{sP}(C) \tag{4.18}$$

with $\sigma^{\#}_{sP} \in \mathscr{F}^{\#}_{sP}$. Inserting (4.17) and (4.18) in (4.16), we obtain

$$i\nu\lambda'(s)^{-1}\sigma^{\#}_{Ps}(B)\sigma^{\#}_{sP}(C)\big(\omega - i(P^{\#}(A) - s)\big)^{-1}$$

which is the form of the residue announced in the theorem, with $K = \nu^{-1}\lambda'(s)$.

Remark. The product $\sigma^{\#}_{Ps}(B)\sigma^{\#}_{sP}(C)$ is unambiguously normalized in view of the formulas

$$\begin{aligned}
\sigma^{\#}_{Ps}(d\xi\, du) &= \sigma_{Ps}(d\xi)\exp[-(P - s)t]\, dt, \\
\sigma^{\#}_{sP}(d\xi\, du) &= \sigma_{sP}(d\xi)\exp[(P - s)t]\, dt, \\
(\pi_{(-\infty,0]}\sigma_{Ps})(d\xi') &= S'_{(s)}(\xi')\sigma'_{(P)}(d\xi'), \\
(\pi_{(-\infty,0]}\sigma_{sP})(d\xi') &= S'_{(P)}(\xi')\sigma'_{(s)}(d\xi'), \\
\sigma'_{(P)}(S'_{(P)}) = 1, &\qquad \sigma'_{(s)}(S'_{(s)}) = 1.
\end{aligned}$$

The constant K is given by

$$K = \nu^{-1}\lambda'(s) = \left[\int \sigma_{PP}(d\xi)r(\xi)\right]\left[\int \sigma_{ss}(d\xi)r(\xi)\right], \tag{4.19}$$

where σ_{PP} is the Gibbs state $\rho \in \mathscr{F}_{PP}$ and $\sigma_{ss} \in \mathscr{F}_{ss}$, with

$$\begin{aligned}
(\pi_{(-\infty,0]}\sigma_{PP})(d\xi') &= S'_{(P)}(\xi')\sigma'_{(P)}(d\xi'), \\
(\pi_{(-\infty,0]}\sigma_{ss})(d\xi') &= S'_{(s)}(\xi')\sigma'_{(s)}(d\xi').
\end{aligned}$$

We obtained (4.19) from (4.1), and formula (3.2) of [9]. Note that $K \neq 0$ by one of the assumptions of Theorem 4.2.

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