J. DIFFERENTIAL GEOMETRY
24 (1986) 1–14

THE HOMOLOGY OF THE MAPPING CLASS GROUP

EDWARD Y. MILLER

1. Introduction

The mapping class group Γ_g is the group of components of the groups $\mathrm{Diff}^+(S_g)$ or orientation preserving diffeomorphisms of a Riemann surface S_g of genus g. Since each component is contractible, there are natural isomorphisms of integral cohomology groups:

$$H^*(B\,\mathrm{Diff}^+(S_g); Z) = H^*(B\Gamma_g; Z). \tag{1.1}$$

In the context of complex analysis, Γ_g is called the Teichmuller group. It acts properly and discontinuously on the Teichmuller space T^{3g-3} with finite isotropy groups. The quotient of this action is the module space $\mathbf{M}_g$ of smooth algebraic curves of genus g. Consequently, there is an isomorphism of rational cohomology:

$$H^*(B\Gamma_g: A) = H^*(\mathbf{M}_g: Q). \tag{1.2}$$

In this paper we will show that $\mathbf{M}_g$, $B\Gamma_g$, and $B\,\mathrm{Diff}^+(S_g)$ get more and more complicated as the genus g tends to infinity. More precisely, we will prove:

Theorem 1.1. *Let $Q[z_2, z_4, z_6, \cdots]$ denote the polynomial algebra of generators z_{2n} in dimension $2n$, $n = 1, 2, 3, \cdots$. There are classes $y_2, y_4, \cdots, y_{2n}, \cdots$ with y_{2n} in the $2n$th cohomology group $H^{2n}(B\,\mathrm{Diff}^+(S_g); Z)$ such that the homomorphism of algebras sending z_{2n} to y_{2n}*

$$Q[z_2, z_4, \cdots] \to H^*(\mathbf{M}_g; Q) \cong H^*(B\,\mathrm{Diff}^+(S_g); Q)$$

*is an **injection** in dimensions less than $(g/3)$.*

These classes y_{2n} were first introduced by D. Mumford [7]. In the topological context, they are defined as follows:

Let p: $E \to B\,\mathrm{Diff}^+(S_g)$ be the universal S_g bundle with fiber S_g. Let d be the first Chern class of T_*, the tangent bundle along the fibers of the

Received December 25, 1985.

fibration p, and p_* denote the "integration along the fibers" homomorphism in integral cohomology. The homomorphism p_* maps $H^{2n+2}(E; Z)$ to $H^{2n}(B\,\mathrm{Diff}^+(S_g); Z)$ (since the fibers are dimension 2). Define y_{2n} by

$$(1.3) \qquad y_{2n} = p_*(d^{n+1}),$$

where d^{n+1} is the $(n+1)$-fold cup product of d. (*Note*: D. Mumford in [7] defines analogous classes in $H^*(\mathbf{M}_g; Z)$ by a strictly algebraic process. His classes extend to the closure of the moduli space $\mathbf{M}_g$.)

It is useful to utilize $\mathrm{Diff}(S_g, D^2)$, the group of orientation preserving diffeomorphism of S_g fixing a chosen disk D^2 in S_g. By taking connected sums of the surfaces S_g and S_h (of genera g and h) along their fixed disks we obtain natural homomorphisms

$$(1.4) \qquad \mathrm{Diff}\left(S_g, D^2\right) \times \mathrm{Diff}\left(S_h, D^2\right) \to \mathrm{Diff}\left(S_{g+h}, D^2\right),$$

$$(1.5) \qquad \mathrm{Diff}\left(S_g, D^2\right) \times (\text{identity}) \to \mathrm{Diff}\left(S_{g+h}\right).$$

In these terms one of the basic results concerning the homology of the mapping class group is the following remarkable theorem of J. Harer [3].

Theorem 1.2 (*J. Harer*). *The induced maps of classifying spaces*

$$(1.6) \quad B\,\mathrm{Diff}^+\left(S_g, D^2\right) \to B\,\mathrm{Diff}^+\left(S_{g+h}\right), \quad B\,\mathrm{Diff}^+\left(S_g, D^2\right) \to B\,\mathrm{Diff}^+\left(S_g\right)$$

give rise to isomorphisms on integral homology in dimensions less than $(g/3)$.

Note. Since Theorem 1.2 does not appear in Harer's work in the form stated here we will show in §4 how it follows from his much stronger results [3].

Harer's theorem implies that the rational cohomology of the moduli space $\mathbf{M}_g$ stabilizes. Indeed this is true integrally since $\mathbf{M}_g$ is a V-manifold whose singularities have codimension that increases with g (see [7]). The algebraic analog of $B\,\mathrm{Diff}^+(S_g)$ is the moduli space of triples (C_p, p, v) where C_g is a smooth curve of genus g, p is a point on C_g, and v is a nonzero cotangent vector based at p.

By Theorem 1.2 the limit of homology groups

$$\mathbf{A} = \varinjlim H_*\left(B\,\mathrm{Diff}^+\left(S_g, D^2\right); Q\right)$$

is of finite type. The homomorphisms (1.4) induce maps of classifying spaces

$$(1.7) \quad F\colon B\,\mathrm{Diff}^+\left(S_g, D^2\right) \times B\,\mathrm{Diff}^+\left(S_h, D^2\right) \to B\,\mathrm{Diff}^+\left(S_{g+h}, D^2\right).$$

These induce a product F_* on the limit and so a Hopf algebra structure on the limit $\mathbf{A}$.

Theorem 1.3. (a) $\mathbf{A} = \varinjlim H_*(B\,\mathrm{Diff}^+(S_g, D^2); Q)$ *under the F_*-product is a commutative, cocommutative Hopf algebra of finite type.*

(b) $\mathbf{A}$ *is the tensor product of a polynomial algebra on even dimensional generators and an exterior algebra on odd dimensional generators.*

(c) $\mathbf{A}$ *contains at least one generator* x_{2n} *in each even dimension* $2n$, $n = 1, 2, 3, \cdots$.

As explained in §2, Theorem 1.3 part (a) is implied by general considerations. Part (b) then follows from the general structure theory of Hopf algebras over Q of Milnor and Moore (see [6]). Part (c) is proved by explicitly constructing the desired classes x_{2n} and detecting them by means of the universal cohomology classes y_{2n} of Mumford.

It is presently an open question whether or not there are nontorsion classes in the *odd* dimensional homology of the mapping class groups Γ_g in dimensions less than $(g/3)$. Mumford has conjectured that $\mathbf{A}$ is the polynomial algebra on precisely the classes x_{2n}, $n = 1, 2, 3, \cdots$ [7] (i.e., one generator in each even-dimension). Quite possibly the number of even dimensional generators might increase exponentially with dimension.

From the definition (1.3) the universal classes y_{2n} restricted to $H^{2n}(B\,\mathrm{Diff}^+(S_g, D^2); \mathbf{Z})$ are compatible under the inclusions (1.6). Consequently, they define universal cohomology classes in the inverse limit $\varprojlim H^{2n}(B\,\mathrm{Diff}^+(S_g, D^2); Z)$.

The main properties of these universal classes y_{2n} are:

Lemma 1.4. *The classes* y_{2n} *vanish on the* F_**-decomposibles of the Hopf algebra* $\mathbf{A}$ *above.*

By Lemma 1.4 the classes y_{2n} may be used to detect polynomial generators of $\mathbf{A}$. That is, if we construct classes x_{2n} in $\mathbf{A}$ with nonzero evaluation by y_{2n} (i.e., $[y_{2n}, x_{2n}] \neq 0$), then the x_{2n}'s are the desired polynomial generators sought in Theorem 1.3, part (c). Dually (again using Harer's Theorem 1.2) Theorem 1.1 is proved.

In view of Harer's Theorem 1.2, $H_{2n}(B\,\mathrm{Diff}^+(S_g, D^2); Z)$ is isomorphic to $H_{2n}(B\,\mathrm{Diff}^+(S_g); Z)$ for g. large. Hence to prove Theorem 1.3, part (c) it suffices to construct explicit classes u_{2n} in $H_{2n}(B\,\mathrm{Diff}^+(S_g); Z)$ with $[y_{2n}, u_{2n}] \neq 0$ for g large.

The desired examples are provided by Theorem 1.5 below.

Theorem 1.5. *For each n there is a fibration of smooth projective algebraic varieties* p_n: $Z^{n+1} \to X^n$ *with fiber a smooth connected curve*, $\dim_{\mathbf{C}} X^n = n$, $[d^{n+1}, Z^{n+1}] \neq 0$. *Here d equals the first Chern class of the tangent bundle along the fibers* T_* *to* p_n. *The genus of fiber* Y^n *of* p_n *may be made as large as desired.*

The equality $[d^{n+1}, Z^{n+1}] = [(p_n)_*(d^{n+1}), X^n] = [y_{2n}, X^n]$ follows from the definition of the "integration over the fibers" map $(p_n)_*$. Hence, once Theorem 1.5 is proved, Theorem 1.3, part (c) and Theorem 1.1 are proved as explained above.

The construction of p_n: $Z^{n+1} \to X^n$ of Theorem 1.5 is modeled on the methods of Atiyah [1]. In that paper a more standard detection procedure is suggested. It may be described as follows.

The local coefficient system $[H^1(\text{Fiber}; Z)]$ with its symplectic form via cup product defines a classifying map

$$L\colon B\,\mathrm{Diff}^+\left(S_g, D^2\right) \to B\,\mathrm{Sp}(2g, Z).$$

Equivalently, L is the classifying map of the homomorphism $\mathrm{Diff}^+(S_g, D^2) \to \mathrm{Sp}(2g, Z)$ which records the symplectic homomorphism induced by a diffeomorphism of the Riemann surface S_g. It is natural to attempt to detect nonzero classes in $B\,\mathrm{Diff}^+(S_g, D^2)$ by pulling back classes from $B\,\mathrm{Sp}(2g, Z)$. This is Atiyah's approach in studying two dimensional classes.

The real symplectic group $\mathrm{Sp}(2g, R)$ has maximal compact subgroup $U(g)$, the unitary group. Thus, the inclusion $U(g) \to \mathrm{Sp}(2g, R)$ induces a homotopy equivalence J: $BU(g) \to B\,\mathrm{Sp}(2g, R)$ with inverse J^{-1}. Consequently, the inclusions and homomorphisms of groups $\mathrm{Diff}^+(S_g, D^2) \to \mathrm{Sp}(2g, Z) \to \mathrm{Sp}(2g, R) \leftarrow U(g)$ induce a map of classifying spaces

$$G\colon B\,\mathrm{Diff}^+\left(S_g, D^2\right) \to BU(g). \tag{1.8}$$

Recall that the homology of $BU = \varinjlim BU(g)$ is a polynomial algebra under the Whitney sum on generators z_n in dimension $2n$; and that the primative characteristic class $s_{(n)}(t) = n!ch_{(n)}(t)$ in $H^{2n}(BU; Z)$ vanishes on decomposibles with $[ch_{(n)}(t), z_n] \neq 0$. Here t is the universal bundle over BU. See [1].

Note that the map G sends the F-product in $\varinjlim B\,\mathrm{Diff}^+(S_g, D^2)$ to the Whitney sum product of bundles in $BU = \varinjlim BU(g)$. Consequently, $G^*(ch_{(n)}(t))$ vanishes on the F_*-decomposibles of $\mathbf{A}$ and so may be used to detect possible polynomial generators.

The relationship between this detection procedure and the nonmultiplicativity of the signature has been elucidated by Atiyah [1]. He shows that the signature of the total space of a $(4k-2)$ dimensional family X^{2k-1} of Riemann surfaces can be expressed in terms of the classes $G^*(ch_{(n)}(t))$ evaluated against the characteristic classes of X^{2k-1}.

The relationship between these detection procedures was independently discovered by D. Mumford. It is:

Theorem 1.6. *There exist as classes in $H^*(B\,\mathrm{Diff}^+(S_g, D^2)\colon Q)$:*

$$G^*\left(ch_{(n)}(t)\right) = N_n(y_{2n}) + (\textit{decomposible}) \tag{1.9}$$

with $N_{2k} = 0$ and $N_{2k-1} = (-1)^{k-1}B_k/(2k)!$, where B_k is the kth Bernoulli number.

Combining the above Theorem 1.1 and 1.4 we have proved the result.

Theorem 1.7. *The map* G^*

(1.10)

$$H^*(BU; Q) \to H^*(B\,\mathrm{Sp}(2g, Z); Z) \to \varprojlim H^*\left(B\,\mathrm{Diff}^+\left(S_g, D^2\right); Q\right)$$

is an ***injection*** *of the polynomial algebra* $Q[c_n(t)/n \text{ odd}]$.

Recall that Borel [2] has proved that the cohomology of $B\,\mathrm{Sp}(2g, Z)$ stabilizes and the limit is a polynomial algebra on generators in dimensions 2, 6, 10, 14, $\cdots$. Thus we have proved:

Theorem 1.8. *The map* $H^*(B\,\mathrm{Sp}(Z); Q) \to \varprojlim H^*(B\,\mathrm{Diff}^+(S_g, D^2); Q)$ *is an injection.*

Results similar to those described here have been independently obtained by Morita.

It is a pleasure to acknowledge the help and encouragement which I received from John Harer in doing this work.

2. Proofs of the above results assuming Theorems 1.2 and 1.5

Proposition 2.1. (a) *There is an action of the little square operad of disjoint squares in* D^2 *on the disjoint union of the* $B\,\mathrm{Diff}^+(S_g, D^2)$*'s extending the F-product.*

(b) *The group completion of the disjoint union of the* $B\,\mathrm{Diff}^+(S_g, D^2)$*'s under F is a double loop space.*

(c) *F induces a commutative, cocommutative, associative, coassociative Hopf algebra structure on the limit* $\mathbf{A} = \varinjlim H_*(B\,\mathrm{Diff}^+(S_g, D^2); Q)$.

(d) $\mathbf{A}$ *is of finite type and is a tensor product of a polynomial algebra on even dimensional generators and an exterior algebra on odd dimensional generators.*

This proposition is easily proved. Part (a) is obtained by taking connected sums of the chosen fixed disks with the disjoint squares in the disk D^2 to get maps

$$\mathrm{Config}_j(D^2) \times \left[\mathrm{Diff}^+\left(S_g, D^2\right)\right]^j \to \mathrm{Diff}^+\left(S_g, D^2\right) \tag{2.1}$$

which when classified give the desired structural maps of part (a). Here $\mathrm{Config}_j(D^2)$ is the space of configurations of j disjoint squares in the disk (sides parallel to the x, y axes). General loop space theory (see May [5]) shows that part (a) implies (b). Harer's Theorem 1.2 above implies that $\mathbf{A}$ is of finite type. This combined with the structure theory of Hopf algebras over Q of Milnor and Moore [6] implies part (c). Note Proposition 2.1 subsumes Theorem 1.3, parts (a) and (b).

Proof of Lemma 1.4. The universal bundle E over $B\,\mathrm{Diff}^+(S_g, D^2) \times B\,\mathrm{Diff}^+(S_h, D^2)$ is a union of bundles $E = E_1 \cup E_2$, where E_1, E_2 are smooth surface bundles with fibers (S_g − (interior of D^2)), respectively (S_h − (interior of D^2)). The intersection $E_1 \cap E_2$ is equal to the common boundaries $\partial E_1 = \partial E_2$ which is a trivial circle $S^1 = \partial D^2$ bundle.

Form the bundles E_{j*} ($j = 1, 2$) by identifying two points x, y of E_j if they are in the same fiber and lie in the boundary circle. Equivalently, E_{1*}, E_{2*}, may be obtained from E by identifying two points x, y of E if they both lie in the same fiber and both lie in E_2, respectively E_1. These identifications define continuous maps

$$f_1: E \to E_{1*}, \qquad f_2: E \to E_{2*}. \tag{2.2}$$

Let p denote the bundle map for E and p_1: $C_1 \to B\,\mathrm{Diff}(S_g, D^2)$, p^2: $C_2 \to B\,\mathrm{Diff}^+(S_h, D^2)$ denote the universal (S_g, D^2), respectively (S_h, D^2), bundles. Thus the pullback bundles $(\mathrm{pr}_1)^*(C_1)$, $(\mathrm{pr}_2)^*(C_2)$ of C_1, respectively C_2, to the product $B\,\mathrm{Diff}^+(S_g, D^2) \times B\,\mathrm{Diff}^+(S_h, D^2)$ are precisely E_{1*}, E_{2*} respectively. Let d, d_1, d_2, denote the first Chern class of the tangent bundle along the fibers of the universal bundles p: $E \to B\,\mathrm{Diff}^+(S_g, D^2) \times B\,\mathrm{Diff}^+(S_h, D^2)$, p_1: $C_1 \to B\,\mathrm{Diff}^+(S_g, D^2)$, p_2: $C_2 \to B\,\mathrm{Diff}^+(S_h, D^2)$ respectively.

By construction the equality of $d = (f_1)^*(\mathrm{pr}_1)^*(d_1) + (f_2)^*(\mathrm{pr}_2)^*(d_2)$. Also the two terms in this sum have disjoint supports. Thus, $c^{n+1} = (f_1)^*(\mathrm{pr}_1)^*(d_1)^{n+1} + (f_2)^*(\mathrm{pr}_2)^*(d_2)^{n+1}$ and so $F^*(y_{2n}) = (y_{2n} \times 1) + (y_{2n} \times 1)$ as claimed in Lemma 1.4.

As explained by Atiyah [1], Theorem 1.6 follows from the Grothendieck Riemann Roch theorem. Theorems 1.7 and 1.8 follow from this by combining Theorems 1.1, 1.2, 1.6 and the fact that G sends the F-product to the Whitney sum on BU.

As in §1, Theorem 1.1 follows from Theorems 1.2 and 1.5 and the above. The whole crux of this paper therefore rests on the construction of the examples of Theorem 1.5.

3. Construction of p_n: $Z^{n+1} \to X^n$

Our construction is modeled on that in Atiyah's paper [1]. There he produces a curve bundle over a curve with nonzero signature. Hence we review his methods.

Let C be a connected curve with free involution A and genus g. In other words, C is the double cover of a curve $C' = (C/A)$ of genus g'. These exist as soon as g' is at least 1 and we take g' at least 2. Note that $g = 2g' - 1$ and so is at least 3 and is odd.

Let X be the covering of C given by the homomorphism

$$\pi_1(C) \to H_1(C; Z) \to H_1(C; Z/2Z) = (Z/2Z)^{2g}. \tag{3.1}$$

It has the property that if f: $X \to C$ is the associated covering map, the induced homomorphism

$$f': H^1(C; Z/2Z) \to H^1(X; Z/2Z) \tag{3.2}$$

is zero.

Now consider in $X \times C$ the graphs G_f and G_{Af} of f and Af. Atiyah's choice of f was to ensure the following property of these graphs.

Lemma 3.1 [1, p. 75]. *The homology class of the sum* $(G_f + G_{Af})$ *in* $H_2(X \times C; Z)$ *is even* (*i.e., divisible by* 2).

By lemma 3.1 we may form the ramified double covering Z^2 of $X \times C$ along the divisor $(G_t + G_{Af})$. This gives Atiyah's example p_1: $Z^2 \to X \times C \xrightarrow{(\mathrm{pr}_1)} X = X^1$. Z^2 is a 4-manifold with nonzero signature which fibers over a Riemann surface. The fiber of the map p_1 in this example is Y^1, the ramified double covering of the curve C branched at two points. For his example Atiyah proves

$$[y_2, X^1] = [d^2, Z^2] = 3(\text{signature of } Z^2) = 3(g-1)2^{g-1}. \tag{3.3}$$

d is the first Chern class of the tangent bundle along the fibers of the map p_1.

To generalize the above construction it is convenient to form certain finite covers of C. For this purpose choose an epimorphism $H_1(C; Z) \to (Z + Z)$. Let G_n be the subgroup of $\pi_1(C)$ which is the kernel of the epimorphism

$$\pi_1(C) \to H_1(C; Z) \to (Z + Z) \to ((Z/2^2nZ) + (Z/2^nZ)), \tag{3.4}$$

and let $C_n \to C$ be the associated 4^n-fold covering of C with its free $(Z/2^nZ) + (Z/2^nZ)$ action.

The subgroups G_n of $\pi_1(C)$ fit into a descending sequence

$$\pi_1(C) = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_{n-1} \supset G_n \supset \cdots \tag{3.5}$$

with $(G_n/G_{n-1}) = (Z/2Z) + (Z/2Z)$. Equivalently, the finite covers C_n fit into a tower of coverings

$$C = C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow \cdots \leftarrow C_{n-1} \leftarrow C_n \to l, \tag{3.6}$$

where $C_n \to C_{n-1}$ is a 4-fold covering. Indeed, C_{n-1} is the quotient of C_n by a free $(Z/2Z) + (Z/2Z)$ group action. Note by construction C_n is a connected curve of genus $g(n) = 4^n(g-1)+1$.

Starting from Atiyah's example $p_1\colon Z^2 \to X^1$ we will inductively define smooth algebraic fibrations of smooth projective algebraic varieties $p_n\colon Z^{n+1} \to X^n$ with fiber Y_n such that:

$A(n)$: p_n has fiber Y_n a connected curve.

$B(n)$: There are maps $Z^{n+1} \to Z^n$ such that the composite map

$$Z^{n+1} \to Z^n \to Z^{n-1} \to \cdots \to Z^2 \to X \times C \to C \tag{3.7}$$

sends *both* $\pi_1(Y_n)$ and $\pi_1(Y_n)$ *onto* the subgroup G_{n-1} of $\pi_1(C)$.

$C(n)$: $\left[(d_n)^{n+1}, Z^{n+1}\right] \neq 0$, where d_n is the first Chern class of the tangent bundle along the fibers of p_n.

Atiyah's construction is the $n = 1$ case, $p_1\colon Z^2 \to X^1$. Such a construction will then provide the desired examples of Theorem 1.5.

Let us assume inductively that $p_i\colon Z^{i+1} \to X^i$ has been constructed satisfying properties $A(i)$, $B(i)$, $C(i)$ for $i \leqslant n$ $[n \geqslant 1]$.

In view of $B(n)$ we may lift the map (3.7) $Z^{n+1} \to C$ to the covering C_{n-1} thereby obtaining a map $Z^{n+1} \to C_{n-1}$. By $B(n)$ we conclude that this map sends both $\pi_1(Y_n)$ and $\pi_1(ZD^{n+1})$ onto $\pi_1(C_{n-1}) = G_{n-1}$.

Let Z' be the 4-fold covering $c\colon Z' \to Z^{n+1}$ induced from the 4-fold covering $C_n \to C_{n-1}$ by the map constructed above. By definition, Z' comes equipped with two commuting free involutions [say A_n, B_n] giving a free $(Z/2Z)+(Z/2Z)$ action on Z' and a map $Z' \to C_n$ which is $(Z/2Z) + (Z/2Z)$ equivariant. Also $c\colon Z' \to Z^{n+1}$ is the quotient map of the free action. Let Y' denote the fiber of $Z' \to Z^{n+1} \to X^n$. The fiber Y_n of p_n is then the quotient of Y' by the free action. Since both $\pi_1(Y_n)$ and $\Pi_1(Z^{n+1})$ map onto G_{n-1} and thence onto $(G_{n-1}/G_n) = (Z/2Z) + (Z/2Z)$, Y' is a connected curve. Moreover we have the property:

$$\pi_1(Z') \to \pi_1(Z^{n+1}) \to \pi_1(C) \text{ and } \pi_1(Y') \to \pi_1(Z^{n+1}) \to \pi_1(C) \tag{3.8}$$

both have image G_n.

Now consider the fiber product of Z' with Z' over X^n defined by the pullback diagram

$$\begin{tikzcd} (Z' \times_{X^n} Z') \arrow[r, "\mathrm{pr}_2"] \arrow[d, "(\mathrm{pr}_1)"'] & Z' \arrow[d, "r"] \\ Z' \arrow[r, "r"] & X^n \end{tikzcd} \tag{3.9}$$

with r the composite of c: $Z' \to Z^{n+1}$ with p_n: $Z^{n+1} \to X^n$. The common fiber is Y', the fiber of r.

Note. The fiber product of two smooth algebraic fibrations of smooth projective algebraic varieties (say f: $V \to W$, g: $V' \to W$) is a smooth projective algebraic variety. this is proved by showing that the fiber product is a Hodge manifold and appealing to the intrinsic characterization of smooth projective algebraic varieties of Kodiera [4].

Let A, B be the fiber preserving commuting free involutions on the fiber product of (3.9) defined by $A(x, y) = (x, A_n y)$, $B(x, y) = (x, B_n(y))$. These give a free $(Z/2Z) + (Z/2Z)$ action on the fiber product (3.9) which is fiber preserving for the projection pr_1 (projection on the first factor). Let S: $Z' \to (Z' \times_{X^n} Z')$ be the section $S(z) = (z, z)$ and consider the smooth divisor

(3.10) $$D = S(Z') + AS(Z').$$

This smooth divisor intersects each fiber Y' of pr_1 in precisely two points.

Corresponding to Lemma 3.1 we will later prove:

Lemma 3.2. *Let R: $\pi_1(Z') \to \mathrm{Aut}[H^1(Y'; Z/2Z)]$ be the representation of $\pi_1(Z')$ on the cohomology of the fiber of* pr_1 *above which records the monodromy of the fibration. Then the kernel K_n = (kernel of R) has finite index and so defines a finite covering $T^{n+1} \to Z'$. Let $X^{n+1} \to T^{n+1}$ be the finite covering associated to the epimorphism $\pi_1(T^{n+1}) \to H_1(T^{n+1}; Z/2Z)$. Then in the pull-back diagram which defines W^{n+2},*

(3.11) $$\begin{tikzcd} W^{n+2} \arrow[rr, "h"] \arrow[dr, "t"'] & & (Z' \times_{X^n} Z') \arrow[d, "(\mathrm{pr}_1)"] \\ & X^{n+1} \to T^{n+1} \arrow[r] & Z' \end{tikzcd}$$

the divisor $h^{-1}1(D)$ regarded as an element of $H^2(W^{n+2}; Z)$ is even (i.e., divisible by 2).

Given Lemma 3.2, we may form the ramified double covering Z^{n+2} of W^{n+2} along the divisor $h^{-1}(D)$. The composite p_{n+1}: $Z^{n+2} \xrightarrow{b} W^{n+2} \xrightarrow{t} X^{n+1}$ projective algebraic varieties. (See pp. 76–77 of [1].)

By construction Y_{n+1}, the fiber of p_{n+1}, is the ramified double covering of Y' (ramified at two points). Also Y' is a nontrivial 4-fold covering of Y_n, the fiber of p_n. Since Y_n is a connected curve, Y_{n+1} is a connected curve. This proves property $A(n + 1)$ of (3.7).

The map $Z^{n+2} \to Z^{n+1}$ needed for property $B(n + 1)$ of (3.7) is provided by the composite

(3.12) $$Z^{n+2} \to W^{n+2}(Z' \times_{X^n} Z') \to Z' \to Z^{n+1}.$$

Now by (3.8) the images of $\pi_1(Z^{n+2})$ and $\pi_1(Y^{n+1})$ under the map $Z^{n+2} \to Z' \to Z^{n+1} \to C$ must be contained in G_n in $\pi_1(C)$. On the other hand, this composite maps the fiber Y_{n+1} via

$$
\begin{aligned}
Y_{n+1} &= (\text{fiber of } P_{n+1}) \to (\text{fiber of } W^{n+2} \to X^{n+1}) \\
&= \bigl(\text{fiber } (Z' \times_{X^n} Z') \to Z'\bigr) = (\text{fiber of } Z' \to X^n) \qquad (3.13)\\
&= Y' \to (\text{fiber of } Z^{n+1} \to X^n) \to C.
\end{aligned}
$$

By (3.8) the image of $\pi_1(Y')$ in $\pi_1(C)$ is G_n. Hence the image of $\pi_1(Y_{n+1})$ is G_n because the first map is a nontrivial branched covering and the next three maps are homeomorphisms. Here we use the geometric fact that any nontrivial ramified branched covering $A \to B$ with A, B connected curves induces an epimorphism of fundamental groups. Thus the image of the fundamental groups of both Y_{n+1} and Z^{n+2} equal G_n. This proves property $B(n+1)$ of (3.7).

To calculate $[d^{n+2}, Z^{n+2}]$ we follow Atiyah's analysis [1]. Note that the map (3.12) sends the fibers as indicated in (3.13). Thus if we consider the composite

$$Z^{n+2} \to Z^{n+2} \to Z^n \to \cdots \to Z^2 \to X \times C \to C, \qquad (3.14)$$

then we may pull back a holomorphic differential w on C to obtain forms $w(n+2)$, $w(n+1)$ on Z^{n+2}, Z^{n+1} respectively. These forms are holomorphic sections of the duals to the tangent bundle along the fibers of p_{n+2}, p_{n+1} respectively. Let $c_1(\ )$ denote the first Chern class and (form) denote the divisor class of zeros of a holomorphic form. We have equalities:

$$-(w(n+2)) = c_1(\text{Tangent bundle along the fibers of } p_{n+1}) = d_{n+1}, \qquad (3.15)$$

$$-(w(n+1)) = c_1(\text{Tangent bundle along the fibers of } p_n) = d_n. \qquad (3.16)$$

The relationship between the divisors $(w(n+2))$, $(w(n+1))$, has been explicated by Atiyah [1]. Denote by p the map $Z^{n+2} \to Z^{n+1}$ of (3.12). Since Z^{n+2} is constructed by taking the double branched covering along the ramification divisor $h^{-1}(D)$ in W^{n+2} and the map $W^{n+2} \to Z^{n+1}$ of (3.12) induces an isomorphism of fibers (see (3.13)) we obtain the equation:

$$(w(n+1)) = p^*(w(n+1)) + \bigl[(h^{-1}(D))\bigr]. \qquad (3.17)$$

The use of brackets here means that we regard the ramification divisor to be in Z^{n+2}. Combining (3.15)–(3.17) we have the equality:

$$d_{n+1} = p^*(d_n) - \bigl[(h^{-1}(D))\bigr]. \qquad (3.18)$$

For notational convenience let E_n equal $(Z' \times_{X^n} Z')$ and F_n equal $(Z^{n+1} \times_{X^n} Z^{n+1})$ in the following calculations.

Using formula (3.18) the following sequence of equalities shows that $[(d_{n+1})^{n+2}, Z^{n+2}]$ is nonzero, thereby proving property $C(n+1)$ of (3.7).

$$
\begin{aligned}
(3.19)\qquad (\#1)\quad & \left[(d_{n+1})^{n+2}, Z^{n+2}\right] \\
& = \left[\sum \binom{n+2}{i} p^*\left((d_n)^{n+2-i}\right)\left(-[(h^{-1}(D))]\right)^i, Z^{n+2}\right] \\
(\#2)\quad & = \left[\sum \binom{n+2}{i} (p')^*\left((d_n)^{n+2-i}\right)(2/2^i)\left(-(h^{-1}(D))\right)^i, W^{n+2}\right] \\
(\#3)\quad & = \left[\sum N\binom{n+2}{i} q^*\left((d_n)^{n+2-i}\right)(2/2^i)(-D)^i, E_n\right] \\
(\#4)\quad & = \left[\sum N\binom{n+2}{i} q^*\left((d_n)^{n+2-i}\right)(4/2^i)(-S(Z'))^i, E_n\right] \\
(\#5)\quad & = \left[\sum N\binom{n+2}{i} q^*\left((d_n)^{n+2-i}\right)(1/2^i)\left(-(c\times c)^{-1}S'(Z^{n+1})\right)^i, E_n\right] \\
(\#6)\quad & = \left[\sum N\binom{n+2}{i} (\mathrm{pr}_2)^*\left((d_n)^{n+2-i}\right)(16/2^i)\left(-S'(Z^{n+1})\right)^i, F_n\right] \\
(\#7)\quad & = \left[\sum N\binom{n+2}{i} \left(16/(-2)^i\right)(d_n)^{n+1}, Z^{n+1}\right] \\
(\#8, \#9)\quad & = 16N\left((1/2)^{n+2} - 1\right)\left[(d_n)^{n+1}, Z^{n+1}\right] \neq 0.
\end{aligned}
$$

The equalities in (3.19) are justified as follows:

Equality #1 by (3.18) and the binomial expansion. All the sums in (3.19) range over indices $i = 1$ to $n + 2$.

As for equality #2, note that the divisor $[(h^{-1}(D))]$ in Z^{n+2} is $b^*(c_1(L))$ for some complex line bundle over W^{n+2} with $c_1(L^2)$ dual to the ramification divisor $h^{-1}(D)$ in W^{n+2}. Therefore in rational cohomology we have $[(h^{-1}(D))] = b^*(c_1(L))^1 = (1/2)^i b^*(h^{-1}(D))^i$ where on the right $h^{-1}(D)$ is regarded as a divisor and dually a cohomology class on W^{n+2}. Atiyah gives a thorough discussion of this poiint in [1]. Since $b\colon Z^{n+2} \to W^{n+2}$ is of degree 2, equality #2 follows with $p' = C \cdot (\mathrm{pr}_2) \cdot h$ and $p = p'b$.

Next note that $h\colon W^{n+2} \to E_n = (Z' \times_{X^n} Z')$ is an N-fold unbranched covering. N is the degree of the finite covering $X^{n+1} \to Z'$ (see (3.11)). Hence, equality #3 holds with $q = c \cdot (\mathrm{pr}_2)$ and $p' = qh$.

Recall that the divisor D is $S(Z') + AS(Z')$ for disjoint sections X, AS of pr_1 (see (3.9)). Thus $(D)^i = (S(Z'))^i + (AS(Z'))^i$. The automorphism A sends $S(Z')^i$ into $A(S(Z')^i$ and in #3 these classes are evaluated against terms in the image of q^*. Consequently, the term involving $AS(Z')^i$ may be replaced by one involving $S(Z')^i$ instead. This shows equality #4.

Let $S'\colon Z^{n+1} \to F_n = (Z^{n+1} \times_{X^n} Z^{n+1})$ be the section $S'(a) = (a, a)$. Recall that the section $S\colon Z' \to E_n = (Z' \times_{X^n} Z')$ is given by $S(z) = (z, z)$. There is a commutative diagram:

(3.20)
$$\begin{tikzcd} E_n \arrow[r, "(c \times c)"] \arrow[d, "(\mathrm{pr}_2)"'] & F_n \arrow[d, "(\mathrm{pr}_2)"] \\ Z' \arrow[r, "c"] & Z^{n+1} \end{tikzcd}$$

Recall that (id, A, B, AB) gives a free $(Z/2Z) \times (Z/2Z)$ action on the space $F_n \cdot Z^{n+}$ is the quotient of the free action of (id, A, B, AB) on Z'. The quotient map is $c\colon Z' \to Z^{n+\cdot}$ Consequently, we may replace $(S(Z'))^i$ in #4 by $((c \times c)^{-1}S'(Z))^i = S(Z')^i + AS(Z')^i + BS(Z')^i + ABS(Z')^i$ at the cost of dividing by 4. Equality #5 follows. Since $(c \times c)$ is an unbranched covering of degree 16, equality #6 holds.

The normal bundle of the diagonal embedding of a manifold M in $M \times M$ is canonically identified with the tangent bundle of M. Similarly, the normal bundle of the section $S'(a) = (a, a)$ in F_n is precisely the tangent bundle along the fibers to $Z^{n+1} \to X^n$. Let U denote the Thom class of the normal bundle T. Hence we may replace $(-S'(Z^{n+1}))^i$ by $(-U)^i$ in #6. Since

$$\begin{aligned}\left[(\mathrm{pr}_1)^*\left((d_n)^{n+2-i}\right)(-U)^i, F_n\right] &= \left[(\mathrm{pr}_1)^*\left((d_n)^{n+1}\right)(-1)^i(U), F_n\right] \\ &= \left[(d_n)^{n+1}, Z^{n+1}\right]\end{aligned}$$

equality #7 holds. Here we used the facts that $U^2 = (\mathrm{pr}_1)^*(c_1(T))U$, $d_n = c_1(T)$, and U restricted to a fiber of pr_1 is the generator (since the section S' intersects each fiber precisely once).

Equality #8 holds by arithmetic while inequality #9 is true by the induction hypothesis.

This completes the induction step in the proof of Theorem 1.5 assuming Lemma 3.1.

Proof of Lemma 3.2. We use the notation of Lemma 3.2. Let $V \to T^{n+1}$ be defined by the pullback diagram:

(3.21)
$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d] & (Z' \times_{X^n} Z') \arrow[d] \\ T^{n+1} \arrow[r] & Z' \end{tikzcd}$$

By definition $T^{n+1} \to Z'$ is a finite covering arranged so that $\pi_1(T^{n+1})$ acts trivially on the cohomology $H^1(Y'; Z)$ of the fibration $V \to T^{n+1}$. (D) interests each fiber of $(Z' \times_{X^n} Z') \to Z'$ in two points. That is, the restriction of the cohomology class (D) to the fiber Y' is zero. Consequently, $j^*(D)$ (mod 2) in the spectral sequence for $V \to T^{n+1}$ lies in the sum of $E^{1,1} = H^1(T^{n+1}; Z/2Z) \otimes H^1(Y'; Z/2Z)$ and $E^{0,2} = H^2(T^{n+1}; Z/2Z)$.

The map $X^{n+1} \to T^{n+1}$ is prearranged to induce the zero map of $H^1(\ ; Z/2Z)$, the first cohomology with $Z/2Z$ coefficients. Hence the bundle t: $W^{n+2} \to X^{n+1}$ induced over X^{n+1} will have an associated map h: $W^{n+1} \to (Z' \times_{X''} Z')$ such that $h^*(D)$ lies in the image of $E^{0,2} = H^2(X^{n+1}, Z/2Z)$. Thus, $h^*(D) = t^*(e)$ for some e in $H^2(X^{n+1}; Z/2Z)$.

The section BS induces via pullback a section B' of t: $W^{n+2} \to X^{n+1}$ which is *disjoint* from $h^*(D) = h^*(S(Z') + AS(Z'))$. Hence, $e = (B')^*t^*(e) = (B')^*(h^*(D)) = 0$ and so $h^*(D) = t^*(e) = 0$ in $H^2(W^{n+2}; Z/2Z)$. This proves Lemma 3.2.

4. Harer's results

The mapping class group of a Riemann surface $F_{g,r}$ of genus g with r boundary components is $\Gamma_{g,r} = \pi_0(\Lambda_{g,r})$, where $\Lambda_{g,r}$ is the topological group of orientation preserving diffeomorphisms of $F_{g,r}$ which are the identity on the boundary of $F_{g,r}$.

Let A: $F_{g,r} \to F_{g,r+1}$ $(r \geqslant 1)$ and B: $F_{g,r} \to F_{g+1,r-1}$ $(r \geqslant 2)$ be the inclusions defined by adding a pair of pants (a copy of $F_{0,3}$) sewn along one boundary component for A and two boundary components for B. Also define C: $F_{g,r} \to F_{g+1,r-2}$ $(r \geqslant 2)$ by gluing two boundary components together.

Harer's theorem is:

Theorem 4.1 (*Harer* [3]). *The associated homomorphisms of mapping class groups defined by the maps A, B, C induce isomorphisms of integral homology*:

$$A_*\colon H_k(\Gamma_{g,r}) \to H_k(\Lambda_{g,r+1})$$

for $k > 1$ when $g \geqslant 3k - 2$, $r \geqslant 1$, and for $k = 1$, when $g \geqslant 2$, $r \geqslant 1$,

$$B_*\colon H_k(\Lambda_{g,r}) \to H_k(\Lambda_{g+1,r-1})$$

for $k > 1$, when $g \geqslant 3k - 1$, $r \geqslant 2$, and for $k = 1$, when $g \geqslant 3$, $r \geqslant 2$,

$$C_*\colon H_k(\Lambda_{g,r}) \to H_k(\Lambda_{g+1,r-2})$$

when $g \geqslant 3k$, $r \geqslant 2$.

Note that the homomorphisms $\Lambda_{g,1} \to \Lambda_{g+1,1}$ considered in Theorem 1.2 arise from the mapping A: $F_{g,1} \to F_{g,2}$ composed with B: $F_{g,2} \to F_{g+1,1}$. By Harer's result 4.1 the induced mapping $H_k(\Lambda_{g,1}) \to H_k(\Lambda_{g+1,1})$ is an isomorphism for k less than $(g/3)$.

Let $D(r)$: $F_{g,r} \to F_{g,r-1}$ ($g \geqslant 1$) be the inclusion obtained by filling in one of the boundary disks of $F_{g,r}$. The homomorphism of mapping class groups $\Lambda_{g,1} \to \Lambda_{g,0}$ induced by $D(1)$ is the homomorphism appearing in Theorem 1.2.

In the commutative diagram

$$
\begin{tikzcd}
F_{g-1,2} \arrow[r, "A"] & F_{g-1,3} \arrow[r, "C"] \arrow[d, "H(3)"] & F_{g,1} \arrow[d, "H(1)"] \\
& F_{g-1,2} \arrow[r, "C"] & F_{g,0}
\end{tikzcd} \tag{4.1}
$$

the inclusion $H(3) \cdot A$ induces the identity map on $\Lambda_{g-1,2}$. Thus the induced homomorphisms on the integral homology of the associated mapping class groups give a commutative diagram:

$$
\begin{tikzcd}
H_k(\Lambda_{g-1,2}) \arrow[r, "A_*"] \arrow[d, "\text{(identity)}"'] & H_k(\Lambda_{g-1,3}) \arrow[r, "C_*"] & H_k(\Lambda_{g,1}) \arrow[dl, "H(1)_*"] \\
H_k(\Lambda_{g-1,2}) \arrow[r, "C_*"] & H_k(\Lambda_{g,0}) &
\end{tikzcd} \tag{4.2}
$$

By Harer's Theorem 4.1 the maps A_*, C_* are isomorphisms if $k \leqslant ((g-1)/3)$. Hence, $D(1)_*$ is an isomorphism in this range also.

This completes the proof of Theorem 1.2.

References

[1] M. F. Atiyah, *The signature of fibre bundles*, in *Global Analysis*, Kodaira Volume, Princeton University Press, Princeton, NJ, 1969.

[2] A. Borel, *Stable real cohomology of arithmetic groups*, Ann. Sci. Ecole Norm. Sup. **7** (1974) 235–272.

[3] J. Harer, *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math. (2) **121** (1985) 215–251.

[4] K. Kodaira, *On Kahler varieties of restricted type (an intrinsic characterization of algebraic varieties)*, Ann. of Math. (2) **60** (1954) 28–48.

[5] J. P. May, *The geometry of iterated loop spaces*, Lecture Notes in Math. Vol. 271, Springer, Berlin, 1972.

[6] J. Milnor & J. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965) 211–264.

[7] D. Mumford, *Towards and enumerative geometry of the moduli space of curves*, Shafarevitch volume, Birkhauser Press, pp. 271–328.

Polytechnic Institute of New York