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THE GODBILLON MEASURE OF AMENABLE FOLIATIONS

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To the memory of J. Vey

1. Introduction

This paper studies how the Godbillon-Vey classes of a foliation $\mathscr{F}$ depend upon its transverse dynamics. The secondary classes are differential topological invariants of C^2-foliations; the Godbillon-Vey classes are the secondary classes which contain a factor of the Reeb, or modular, class η corresponding to the generator of $H^1(\mathscr{gl}_n, O_n)$. Sullivan [32], and also Moussu and Pelletier in [26], posed the question: must a codimension-one foliation $\mathscr{F}$ of a compact manifold M with nonzero Godbillon-Vey class have leaves of exponential growth? This problem was the focus of much research [25], [23], [3], [9] which led to G. Duminy's elegant, unpublished solution [8]. He proved that a codimension-one foliation $\mathscr{F}$ with nonzero Godbillon-Vey class must have a resilient leaf, and thus there is an open subset of M consisting of leaves with exponential growth. This result was extended to codimension-one foliations of open manifolds by Cantwell and Conlon in [5], which is also an excellent reference for Duminy's proof.

In this paper, we combine techniques which originated in the study of the codimension-one problem with recent methods of ergodic theory to prove a general result relating the Godbillon-Vey classes to the growth of the leaves of $\mathscr{F}$, in all codimensions. Recall the definition of the growth type of leaf: Choose a Riemannian metric, for M, and a base point x in a leaf $L \subset M$ of $\mathscr{F}$. Give L the induced Riemannian metric; then the growth function $g(L, x, r)$ is the volume of the ball in L with radius r centered at x. We say the growth type of L is *exponential* if there are constants A, B, $C > 0$ so that $g(L, x, r) \geqslant A \exp Br + C$ for all $r \geqslant 0$, *subexponential* if for all $B > 0$ there exists A, $C > 0$

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with $g(L, x, r) \leqslant A \exp Br + C$ for all $r \geqslant 0$, and *nonexponential* otherwise. For M compact, the growth type of L does not depend upon the choice of Riemannian metric on M or the base point x [27]. Subexponential growth type is sometimes called quasi-polynomial in the literature [12], [4].

Theorem 1. *Let $\mathscr{F}$ be a codimension-n, C^2-foliation of a manifold M without boundary. Suppose that for some Riemannian metric on M, almost every leaf of $\mathscr{F}$ has subexponential growth. Then all Godbillon-Vey classes $\Delta_*(y_1 c_J) \in H^{2n+1}(M)$ are zero. The generalized Godbillon-Vey classes $\Delta_*(y_1 y_I c_J)$ of degree greater than $2n+1$ must also vanish.*

The proof of Theorem 1 for M compact will occupy §§2–4 of this paper. The proof for M open is given in §5.

Let X and Y be closed manifolds with dimension X equal n. Let ρ: $\Gamma \equiv \pi_1(Y) \to \mathrm{Diff}^{(2)} X$ define a right action of Γ on X by C^2-diffeomorphisms. The universal cover $\tilde{Y}$ of Y has a right Γ-action by deck transformations, so Γ acts on the right on $\tilde{Y} \times X$ to define a compact manifold $M = (\tilde{Y} \times X)/\Gamma$. The product foliation on $\tilde{Y} \times X$ with leaves $\{\tilde{Y} \times \{x\} | x \in X\}$ descends to a C^2-foliation $\mathscr{F}$ on M. The growth rates of the leaves of $\mathscr{F}$ are no greater than the growth rate of Γ, and a group must have either exponential or subexponential growth type (cf. [22]). Thus, we conclude from Theorem 1:

Corollary 2. *Let $X \to M \xrightarrow{\pi} Y$ be a fibration with X and Y compact and suppose $\pi_1(Y)$ has nonexponential, hence subexponential, growth type. Let $\mathscr{F}$ be a codimension-n, C^2-foliation of M everywhere transverse to the fibers of π. Then all Godbillon-Vey and generalized Godbillon-Vey classes of $\mathscr{F}$ are zero.*

When $X = S^n$, Hirsch and Thurston [16] prove that the rational Euler class of the bundle $M \to Y$ is zero if $\pi_1(Y)$ is amenable. For a *linear* action of $\pi_1(Y)$ on S^n, one knows from Heitsch [13] that the Godbillon-Vey classes of $\mathscr{F}$ are proportional to the Euler class of $M \to Y$, so Corollary 2 is a consequence of these two results in this special case.

Note that given any closed orientable manifold X and any class $y_1 c_J \in H^{2n+1}(WO_n)$, it is possible to find Y and some ρ as above for which $\Delta_*(y_1 c_J) \in H^{2n+1}(M)$ is nonzero [18].

Theorem 1 is a consequence of a general program begun in [17] and continued in [14], [19] to relate the ergodic theory of a foliation with its differential topological invariants. In the present paper, we begin by showing the Godbillon measure of $\mathscr{F}$ is a special case of a construction which assigns to each integrable R-cocycle a on the groupoid Γ of $\mathscr{F}$ a measure g^a on the transverse space $M/\mathscr{F}$. The main result of §3 is that g^a depends only on the cocycle a up to measurable coboundaries, so g^a is an invariant of the measurable cohomology class $[a]$ of a. This sets up a correspondence:

$$G: H_{(1)}(\Gamma, R) \to \text{measures on } M/\mathscr{F},$$
$$[a] \to g^a.$$

The Godbillon measure $g = g^{d\nu}$ is obtained from the Radon-Nikodým cocycle $d\nu$: $\Gamma \to R$. The values of g determine all of the Godbillon-Vey classes of $\mathscr{F}$. Conversely, if a class $\Delta_*(y_1 c_J) \neq 0$, then the cocycle $d\nu$ is not cohomologous to zero.

One of the most effective, recently introduced tools for studying characteristic classes is the ε-tempering process (cf. [19], [21]), which transforms measurable geometric data with asymptotic estimates into bounded measurable data that is analytically useful. The proof of Theorem 1 is obtained by estimating the Radon-Nikodým cocycle $d\nu$ of $\mathscr{F}$ when restricted to the set of leaves with subexponential growth. Our main result (Theorem 4.3 and Proposition 4.5) is that $d\nu$ has moderate positive growth on this set. The proof of Theorem 1 concludes by applying the ε-tempering process (Lemmas 4.7 and 4.8) to the cocycle $d\nu$ in order to conclude $\mathscr{F}$ has transverse measures which are arbitrarily close to being invariant, and are supported on the set of leaves with subexponential growth. It follows from the methods of §§2 and 3 that the Godbillon-Vey classes vanish on this set. At the end of §4 we prove Theorem 4.10, a more general result than Proposition 4.5, which applies to measurable group actions and relates the orbit growth rate to the asymptotic growth rate of $d\nu$. Theorem 4.10 generalizes both Theorem 4.1 of Schmidt [31] and the main theorem of [28].

A foliation $\mathscr{F}$ on M is amenable if the equivalence relation $\mathscr{R}$ on $M \times M$ defined by the leaves of $\mathscr{F}$ has a left-invariant mean on a.e. orbit [7]. A foliation with a.e. leaf of subexponential growth is amenable, as well as the weak-stable foliations on the unit tangent bundles over surfaces with constant negative curvature. The first types of foliations have zero Godbillon-Vey classes by Theorem 1, and the latter types have nonzero classes by explicit calculation. The measure theory of an amenable foliation is especially simple by [7]. It is thus natural to ask how the interaction of the geometry of $\mathscr{F}$ with its measure theory determines the secondary classes, and for a description of the most general class of amenable foliations which must have zero Godbillon-Vey classes. For example, does Theorem 1 remain valid when subexponential growth type is replaced by nonexponential? A related problem is to understand how the characteristic classes of $\mathscr{F}$ are related to the flow of weights of $\mathscr{F}$ (cf. [6], [24]). The only known result is a remarkable vanishing theorem for type III_0-foliations whose flow of weights has no invariant probability measure, which was proved by Connes [6] via index theory methods and techniques of C^*-algebra derivations.

The author owes a special debt and many thanks to Anatole Katok for numerous discussions on tempering processes and on the ergodic theory of group actions. This paper can be viewed as an application to problems in foliation theory of Katokis' general program to study the asymptotics and special representations of cocycles. The fruitfulness of this new point of view is evident in Theorem 1 and the results of [19]. Conversations with Jack Feldman and Arlan Ramsay also contributed to this work in many ways.

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2. Preliminaries

This section begins with a brief description of the Godbillon measure and Godbillon-Vey classes of a foliation. The holonomy groupoud and principal groupoid associated to $\mathscr{F}$ are then constructed, and cocylces over these groupoids are introduced. Our treatment of Godbillon-Vey classes follows Bott [1], that of groupoids follows Haefliger [10] and Bott [2], and the theory of cocycles is discussed in Ramsay [29], [30] and Moore [24].

Let M be a closed manifold of dimension m and $\mathscr{F}$ a C^2-foliation of codimension n on M. For this paper we assume that both TM and the normal bundle $Q \to M$ of $\mathscr{F}$ are oriented. By passing to appropriate double covers this assumption can always be realized. Let ω be a nonvanishing n-form on M which defines the orientation class for the dual bundle $Q^* \to M$, so ω is a defining n-form for $\mathscr{F}$. Let $v \in \Gamma(M, \Lambda^n Q)$ be an n-vector on M such that $\omega(v) = 1$. Define a smooth 1-form on M

$$\eta = d_{\mathscr{F}}\omega \equiv i(v)\, d\omega.$$

The integrability of $\mathscr{F}$ implies $d\omega = \omega \wedge \eta$, and η can be viewed as the infinitesimal modular form for the transverse smooth measure ω.

The form $\eta \wedge (d\eta)^n$ is closed and its cohomology class is independent of the choices made. The Godbillon-Vey class of $\mathscr{F}$ is

$$gv = \left[\eta \wedge (d\eta)^n\right] \in H^{2n+1}(M).$$

For codimension $n > 1$, there are additional secondary classes for $\mathscr{F}$ of degree $2n + 1$, also called Godbillon-Vey classes. Let c_i denote the ith Chern polynomial on $\mathfrak{gl}_n$, the $n \times n$ matrix algebra. Then c_i is an $\operatorname{Ad}\mathrm{GL}_n$-invariant homogeneous polynomial of weight i. Let

$$c_J = c_1^{j_1} \cdots c_n^{j_n}$$

be a polynomial of weight $|J| = j_1 + 2j_2 + \cdots + nj_n = n$. The Chern-Weil construction using a basic connection yields a closed form, also denoted by c_J, on M of degree $2n$ with the property that there is an n-form $\hat{c}_J$ on M so that $c_J = \hat{c}_J \wedge \omega$. Observe that $\eta \wedge c_J$ is a closed $2n + 1$ form on M. For each polynomial c_J of weight n, the cohomology class $[\eta \wedge c_J] \in H^{2n+1}(M)$ is independent of the choices made. For $c_J = c_1^n$ we have $[\eta \wedge c_1^n] = [\eta \wedge (d\eta)^n]$; the *Godbillon-Vey classes* of $\mathscr{F}$ are the classes in the collection

$$\left\{[\eta \wedge c_J] | \text{weight } c_J = n\right\} \subset H^{2n+1}(M).$$

These are invariants of the concordance class of $\mathscr{F}$. In the notation of [20], $[\eta \wedge c_J] = \Delta_*(y_1 c_J)$.

The Godbillon measure of $\mathscr{F}$ isolates from the above construction the role played by the form η. Note the restriction of η to a leaf $L \subset M$ of $\mathscr{F}$ is a closed 1-form, and its cohomology class $[\eta | L] \in H^1(L)$ is called the *Reeb class* of L, or the first flat class of the normal bundle to $\mathscr{F}$ along L. The Godbillon measure uses the Reeb class as an operator on a Hilbert bundle over M to define new invariants of $\mathscr{F}$. This was first introduced by Duminy to prove the Sullivan conjecture in codimension one [8], and further analyzed in Heitsch and Hurder [14].

Let $A(M, \mathscr{F})$ denote the differential ideal of forms on M which are multiples of ω. Let $H^*(M, \mathscr{F})$ denote the cohomology of this ideal. Note that for any closed k-form ψ on M and c_J of degree $2n$ we obtain a class $[c_J \wedge \psi] \in H^{2n+k}(M, \mathscr{F})$. The Godbillon operator is the map

$$g\colon H^p(M, \mathscr{F}) \to H^{p+1}(M),$$
$$[\phi] \to [\eta \wedge \phi].$$

For each c_J of degree $2n$, define the *Vey homomorphism* (cf. [8])

$$V(c_J)\colon H^k(M) \to H^{2n+k}(M, \mathscr{F}),$$
$$[\psi] \to [c_J \wedge \psi].$$

The Godbillon-Vey class $[\eta \wedge c_J] = g \circ V(c_J)(1)$. For the special case $p = m - 1$, we make the alternate but equivalent definition

$$g\colon H^{m-1}(M, \mathscr{F}) \to R,$$
$$[\phi] \to \int_M \eta \wedge \phi.$$

Given any $[\psi] \in H^{m-2n-1}(M)$ we obtain a real number $g[c_J \wedge \psi]$ depending only on $g[c_J]$ and $[\psi]$. Poincaré duality for $H^*(M)$ implies the class $[\eta \wedge c_J] = g[c_J]$ is completely determined by the values of g on the classes $[c_J \wedge \psi]$ as $[\psi]$ runs through a basis of $H^{m-2n-1}(M)$. Thus, the Godbillon operator completely determines the Godbillon-Vey classes of $\mathcal{F}$. In particular, if $g = 0$, then all the Godbillon-Vey classes of $\mathcal{F}$ are zero. Note this includes the generalized Godbillon-Vey classes $\Delta_*(y_1 y_I c_J)$, as discussed further in [14].

A set $B \subset M$ is *saturated* if for each $x \in B$ the leaf L_x through x is contained in B. Let $\mathcal{B} = \mathcal{B}(\mathcal{F})$ denote the Σ-algebra of measurable saturated subsets of M.

Theorem 2.1 [14]. *For each $B \in \mathcal{B}$, there is a well-defined functional*

$$g_B\colon H^{m-1}(M, \mathcal{F}) \to R,$$
$$[\phi] \to \int_B \eta \wedge \phi.$$

The correspondence $B \mapsto g_B \in \operatorname{Hom}_{\mathrm{cont}}(H^{m-1}(M, \mathcal{F}), R)$ is called the *Godbillon measure* on $\mathcal{B}$. It is a countably additive measure on the quotient space $M/\mathcal{F}$ which is continuous with respect to Lebesgue measure.

The foliation $\mathcal{F}$ is *ergodic* if whenever $M = M_1 \cup M_2$, where M_1, $M_2 \in \mathcal{B}$, then one of M_1 or M_2 has measure 0. If $\mathcal{F}$ is not ergodic, then let $M = \bigcup_{i=1}^{\infty} B_i$ be a countable decomposition into disjoint sets with each $B_i \in \mathcal{B}$. For any $[\phi] \in H^{m-1}(M, \mathcal{F})$, then

$$g[\phi] = \sum_{i=1}^{\infty} g_{B_i}[\phi].$$

In particular, for the Godbillon-Vey classes we can decompose $[\eta \wedge c_J] = g[c_J] = \sum_{i=1}^{\infty} g_{B_i}[c_J]$ in $H^{2n+1}(M)$ via this device and Poincaré duality. We write $[\eta \wedge c_J]|B$ for the contribution to $[\eta \wedge c_J]$ from the set $B \in \mathcal{B}$. The localized class $[\eta \wedge c_J]|B$ is determined by the operator g_B.

Let $I_c = (-c, c)$ be the open interval and I_c^p the p-fold product of I_c. Given $z \in M$, a *foliation chart* about z is a pair (U, Φ), where U is an open neighborhood of z and $\Phi\colon U \to I_c^{m-n} \times I_c^n$ is a diffeomorphism onto such that $\Phi(z) = (0,0)$, and for each $x \in I_c^n$, $\Phi^{-1}(I_c^{m-n} \times \{x\})$ is a leaf of $\mathcal{F}|U$. The chart (U, Φ) is *regular* if there is a foliation chart (W, Ψ) with $\overline{U} \subset W$ and Ψ: $W \to I_d^m$, $d > c$, extends Φ.

Choose and fix a finite open covering of M by regular foliation charts $\{(U_\alpha, \Phi_\alpha) | \alpha = 1, \cdots, d\}$. Set $\pi = \pi_c\colon I_c^{m-n} \times I_c^n \to I_c^n$, the projection onto the second factor. A pair $1 \leqslant \alpha$, $\beta \leqslant d$ is said to be *admissible* if $U_{\alpha\beta} \equiv U_\alpha \cap U_\beta$ is nonempty. Let $T_\alpha = \pi \circ \Phi_\alpha(U_\alpha) = I_c^n$, and for α, β admissible let $T_{\alpha\beta} = \pi \circ \Phi_\alpha(U_{\alpha\beta})$. Define

$$\gamma_{\alpha\beta}\colon T_{\alpha\beta} \to T_{\beta\alpha},$$
$$\gamma_{\beta\alpha}(x) = \pi \circ \Phi_\beta \circ \Phi_\alpha^{-1}(\{0\} \times \{x\}), \qquad x \in T_{\alpha\beta}.$$

We also insist that for each α the orientation of $Q|U_\alpha$ agrees with the orientation $d\bar{x} = dx_1 \wedge \cdots \wedge dx_n$ of I_c^n under the map $\pi \circ \Phi_\alpha$. This implies each $\gamma_{\beta\alpha}$ is orientation preserving.

The collection $\{U_\alpha, U_{\alpha\beta}, \gamma_{\beta\alpha} | 1 \leqslant \alpha, \ \beta \leqslant d\}$ determines a pseudo-group denoted by $\mathscr{G} = \mathscr{G}(\mathscr{F})$. A typical element $\gamma \in \mathscr{G}$ is a local diffeomorphism from some T_α into some T_β. The domain of γ is denoted V_γ, an open subset of T_α.

The topological groupoid associated to $\mathscr{F}$ is constructed from $\mathscr{G}$. The object space of Γ is the topological measure space $\Gamma^{(0)} \equiv T \equiv \dot{\bigcup}_{\alpha=1}^{d} T_\alpha$, the latter being a disjoint union. The measure μ on T is Lebesgue measure, associated to the smooth volume form $d\bar{x}$ on T_α, which is the restriction to T_α of $dx_1 \wedge \cdots \wedge dx_n$ on R^n. Given a local diffeomorphism $\gamma \in \mathscr{G}$ defined in a neighborhood of x, let Germ_x denote the germ of γ at x. Given $x, y \in T$ the morphisms in Γ from x to y are the elements of the set

$$\Gamma_x^y = \left\{ \mathrm{Germ}_x \gamma \,|\, \gamma \in \mathscr{G} \text{ with } x \in V_\gamma \text{ and } y = \gamma(x) \right\}.$$

Let $\Gamma_x = \bigcup_{y \in T} \Gamma_x^y$ and $\Gamma^y = \bigcup_{x \in T} \Gamma_x^y$. The composition in Γ is given by composition of germs. There are natural maps $s, r\colon \Gamma \to T$ where $s(\Gamma_x^y) = x$ and $r(\Gamma_x^y) = y$.

The equivalence relation on T given by $\mathscr{F}$ defines a principal groupoid denoted $\mathscr{R}$. The object space of $\mathscr{R}$ is the topological measure space T. For $x, y \in T$ the morphisms from x to y are

$$\mathscr{R}_x^y = \begin{cases} (x, y) & \text{if } x \text{ and } y \text{ are on the same leaf,} \\ \varnothing & \text{if not.} \end{cases}$$

The composition in $\mathscr{R}$ is given by $(x, y) \circ (y, z) = (x, z)$. For $x, y \in T$, we write $x \sim y$ to mean $(x, y) \in \mathscr{R}$. The *orbit* of $x \in T$ is the set $\mathscr{R}(x) = \{ y \in T | x \sim y \}$. A set $X \subset T$ is *saturated* if $x \in X$ implies $\mathscr{R}(x) \subset X$. For a set $B \in \mathscr{B}$, let $B \cap T$ denote the saturated subset of T whose points correspond to the leaves of B in M.

Given a saturated set $X \subset T$, the restricted groupoid $\Gamma|X$ is the subgroupoid of Γ with object space X and morphisms

$$\Gamma|X = \bigcup_{x, y \in X} \Gamma_x^y.$$

Similarly, the restricted groupoid $\mathscr{R}|X$ has object space X and morphisms

$$\mathscr{R}|X = \bigcup_{x, y \in X} \mathscr{R}_x^y.$$

There is a natural map $(s, r)\colon \Gamma \to \mathscr{R}$ of groupoids which is the identity on objects, and $(s, r)(\Gamma_x^y) = (x, y)$ if $x \sim y$. The fiber over the pair $(x, x) \in \mathscr{R}$ is precisely the holonomy group Γ_x^x of the leaf L_x through x. This is the reason that Γ is sometimes called the holonomy groupoid of $\mathscr{F}$ (cf. [11]).

The morphisms $\gamma_{\beta\alpha}$: $T_{\alpha\beta} \to T_{\beta\alpha}$ of $\mathcal{G}$ are C^2 and thus preserve the smooth measure *class* of μ on T, so Γ, $\mathcal{R}$ and all of their restrictions are measured groupoids in the sense of Ramsay [29] or Moore [24].

The groupoids Γ and $\mathcal{R}$ inherit orbit-norms from the word metric on $\mathcal{G}$. Specifically, given $\mathrm{Germ}_x\gamma \in \Gamma_x^y$, we say $|\mathrm{Germ}_x\gamma| \leqslant N$ if there are admissible pairs $\{(\alpha_i, \beta_i)|i = 1,\cdots, N\}$ such that $\mathrm{Germ}_x\gamma = \mathrm{Germ}_x(\gamma_{\beta_N\alpha_N} \circ \cdots \circ \gamma_{\beta_1\alpha_1})$, where the right-hand composition is assumed to be well defined. Similarly, for $(x, y) \in \mathcal{R}$ we say $|x, y| = |(x, y)| \leqslant N$ if there are admissible pairs $\{(\alpha_i, \beta_i)|i = 1,\cdots, N\}$ such that $y = \gamma_{\beta_N\alpha_N} \circ \cdots \circ \gamma_{\beta_1\alpha_1}(x)$. The norm $|x, y|$ is just the minimum number of flow boxes needed to form a connected chain between x and y on the leaf containing both points.

Definition 2.2. A measurable R-cocycle on Γ is a map a: $\Gamma \to R$ satisfying

(a) $a(\mathrm{Germ}_x\gamma_2 \circ \gamma_1) = a(\mathrm{Germ}_y\gamma_2) + a(\mathrm{Germ}_x\gamma_1)$ for $\mathrm{Germ}_x\gamma_1 \in \Gamma_x^y$, $\mathrm{Germ}_y\gamma_2 \in \Gamma_y^z$, a.e. $x, y \in T$.

(b) For all $\gamma \in \mathcal{G}$ with domain V_γ, the map $V_\gamma \to R$, $x \mapsto a(\mathrm{Germ}_x\gamma)$, is measurable.

Given $\gamma \in \mathcal{G}$, for each $x \in V_\gamma$ let $J\gamma(x) \in \mathrm{GL}_n R$ denote the Jacobian matrix for $d\gamma_x$ with respect to the standard basis of $TI_c^n \subset TR^n$. Our constructions yield $\det(J\gamma(x)) > 0$ for all $x \in V_\gamma$.

Definition 2.3. The Radon-Nikodým cocycle of Γ is the additive measurable cocycle $d\nu$: $\Gamma \to R$,

$$d\nu(\mathrm{Germ}_x\gamma) = \log\det J\gamma(x).$$

Two measurable cocycles a, b: $\Gamma \to R$ are *cohomologous* if there is a measurable map f: $T \to R$, that is a collection of maps $\{f_\alpha: T_\alpha \to R|\alpha = 1,\cdots, d\}$, such that for all $\gamma \in \mathcal{G}$, $V_\gamma \subset U_\alpha$, range $\gamma \subset T_\beta$,

$$b(\mathrm{Germ}_x\gamma) = f_\beta \circ \gamma(x) + a(\mathrm{Germ}_x\gamma) - f_\alpha(x) \quad \text{a.e. } x \in V_\gamma.$$

We denote this by $a \sim_f b$, or simply $a \sim b$.

If $a \sim_f d\nu$, then a is the Radon-Nikodým cocycle of the maps $\gamma \in \mathcal{G}$ with respect to a new measure $\tilde{\mu}$ on T, where on T_α we have $d\tilde{\mu}_\alpha = e^{f_\alpha} \cdot d\bar{x}$. Thus, the study of the cohomology class of $d\nu$ is precisely the study of the divergences of the $\gamma \in \mathcal{G}$ with respect to all possible absolutely continuous measures of the form $e^f \cdot d\bar{x}$ on T.

Definition 2.4. Let a: $\Gamma \to R$ be a measurable cocycle. Then we say a is:

(a) *smooth* if for all $\gamma \in \mathcal{G}$ the map $x \mapsto a(\mathrm{Germ}_x\gamma)$ is smooth (i.e. C^2) at all $x \in V_\gamma$,

(b) *tempered*, or more precisely *c-tempered* for a constant c, if for all $\gamma \in \mathcal{G}$,

$$|a(\mathrm{Germ}_x\gamma)| \leqslant c \cdot |\mathrm{Germ}_x\gamma| \quad \text{for a.e. } x \in V_\gamma,$$

(c) *c-tempered on a set* $X \subset T$ if for all $\gamma \in \mathcal{G}$,

$$|a(\mathrm{Germ}_x\gamma)| \leqslant c|\mathrm{Germ}_x\gamma| \quad \text{for a.e. } x \in X \cap V_\gamma,$$

(d) *integrable* if for all $\gamma_{\beta\alpha}$, the integral

$$\int_{T_{\alpha\beta}} |a(\mathrm{Germ}_x\gamma_{\beta\alpha})|\, d\bar{x} < \infty.$$

For example, the cocycle $d\nu$ is obviously smooth, and it is bounded and hence integrable because the foliation charts covering M were assumed regular.

The above definitions and remarks all have corresponding versions for the groupoid $\mathcal{R}$.

A cocycle a: $\mathcal{R} \to R$ determines a cocycle a: $\Gamma \to R$ such that $a|\Gamma_x^y$ depends only on (x, y). Conversely, given a: $\Gamma \to R$ so that $a|\Gamma_x^x = 0$ for almost all $x \in T$, it is easy to see a descends to a cocycle on $\mathcal{R}$.

One can similarly define cocycles over Γ with range any group G. In particular, the linear holonomy cocycle dh: $\Gamma \to \mathrm{GL}_n R$ is defined by the rule $dh(\mathrm{Germ}_x\gamma) = J\gamma(x)$. Clearly, $d\nu = \log \circ \det \circ dh$. The following elementary result is proved in Proposition 7.2 of [19].

Proposition 2.5 [19]. *For almost every* $x \in T$, *the restriction* dh: $\Gamma_x^x \to \mathrm{GL}_n R$ *is the identity, and* $d\nu$: $\Gamma_x^x \to R$ *is zero.*

Proposition 2.5 implies the cocycle $d\nu$: $\Gamma \to R$ descends to a cocycle on $\mathcal{R}$, again denoted $d\nu$: $\mathcal{R} \to R$. Observe that if a: $\mathcal{R} \to R$ satisfies $a \sim d\nu$, then a lifts to a cocycle: a: $\Gamma \to R$ with $a \sim d\nu$ because the cocycle and coboundary conditions are only assumed to hold a.e. $x \in T$.

The space of measurable R-cocycles on Γ is denoted $Z(\Gamma, R)$, and the space of equivalence classes is denoted by $H(\Gamma, R) = Z(\Gamma, R)/\sim$.

We single out the subspace of integrable classes:

$$H_{(1)}(\Gamma, R) = \{[a] \in H(\Gamma, R) \,|\, \text{there exists } a \in [a] \text{ with } a \text{ integrable}\}.$$

Similarly, set $H(\mathcal{R}, R) = Z(\mathcal{R}, R)/\sim$ with the integrable subspace $H_{(1)}(\mathcal{R}, R) \subset H(\mathcal{R}, R)$.

3. The measure associated to a cocycle

In this section we construct a measure on $\mathcal{B}$ corresponding to each integrable cohomology class in $H_{(1)}(\Gamma, R)$.

Choose a partition of unity $\{\lambda_1, \cdots, \lambda_d\}$ subordinate to the open cover $\{U_1, \cdots, U_d\}$ of M by foliation charts.

Let a: $\Gamma \to R$ be an additive cocycle. For each admissible α, β define on $U_{\alpha\beta}$ the function

$$\zeta^a_{\beta\alpha}(z) = a\left(\mathrm{Germ}_{\pi \circ \Phi_\alpha(z)}\gamma_{\beta\alpha}\right)$$

and on U_α define the 1-form

$$(3.1) \qquad \eta^a_\alpha = \sum_{\beta=1}^{d} d\lambda_\beta \cdot \zeta^a_{\beta\alpha}.$$

Lemma 3.2. *For α, δ admissible, $\eta^a_\alpha = \eta^a_\delta$ on $U_{\alpha\delta}$.*

Proof.

$$\eta^a_\alpha = \left(\sum_\beta d\lambda_\beta \cdot \zeta^a_{\beta\alpha}\right) = \left(\sum_\beta d\lambda_\beta\left(\zeta^a_{\beta\alpha} + \zeta^a_{\alpha\delta}\right)\right) = \eta^a_\delta.$$

where we use that $\Sigma_\beta d\lambda_\beta \cdot \zeta^a_{\alpha\delta} = d(\Sigma_\beta \lambda_\beta) \cdot \zeta^a_{\alpha\delta} = 0$.

Proposition 3.3. *Let a: $\Gamma \to R$ be an integrable cocycle. Then there is a well-defined measure g^a on $\mathscr{B}$ associated to a, given by*

$$g^a_B[\phi] = \int_B \eta^a \wedge \phi$$

for $B \in \mathscr{B}$ and $[\phi] \in H^{m-1}(M, \mathscr{F})$.

Proof. On U_α set $\eta^a \wedge \phi | U_\alpha = \eta^a_\alpha \wedge \phi$. By 3.2 this gives a well-defined m-form $\eta^a \wedge \phi$ on M. The hypotheses a is integrable implies $\eta^a_\alpha \wedge \phi$ is integrable on U_α, so the integral in 3.3 exists. For $\tau \in A^{m-2}(M, \mathscr{F})$ the integral $\int_B \eta^a \wedge d\tau = 0$ by the leafwise Stokes' theorem [14], so the integral $\int_B \eta^a \wedge \phi$ depends only on the cohomology class of ϕ in $H^{m-1}(M, \mathscr{F})$. q.e.d.

For $a = d\nu$ by the Radon-Nikodým cocycle and $\zeta_{\beta\alpha}(x) = d\nu(\mathrm{Germ}_x\gamma_{\beta\alpha})$, (3.1) becomes the expression for the 1-form η of §2, calculated using the n-form ω on M whose restriction to U_α is $\exp(\Sigma_\beta \lambda_\beta \zeta_{\beta\alpha}) \wedge \pi^*_\alpha(d\bar{x})$, and thus $g^{d\nu} = g$ is the Godbillon measure of $\mathscr{F}$.

Lemma 2 of [8] introduced the critically important technique of calculating g via an unbounded sequence of transverse measures. Our next result is based upon this technique, and extends Proposition 3.4 of [5].

Theorem 3.4. *Let a, b: $\Gamma \to R$ be integrable cocycles and suppose $a \sim b$. Then the corresponding measures g^a and g^b are equal.*

Proof. For each α, let f_α: $T_\alpha \to R$ be a measurable function so that for α, β admissible

$$b(\mathrm{Germ}_x\gamma_{\beta\alpha}) = f_\beta \circ \gamma_{\beta\alpha}(x) + a(\mathrm{Germ}_x\gamma_{\beta\alpha}) - f_\alpha(x), \qquad x \in U_{\alpha\beta}.$$

For each positive integer N define

$$f_\alpha^N() = \begin{cases} f_\alpha(x) & \text{if } |f_\alpha(x)| \leqslant N, \\ N \cdot \operatorname{sign}\{f_\alpha(x)\} & \text{if } |f_\alpha(x)| > N. \end{cases}$$

Define a cocycle $b^N\colon \Gamma \to R$ by the rule

$$b^N(\operatorname{Germ}_x \gamma_{\beta\alpha}) = f_\beta^N \circ \gamma_{\beta\alpha}(x) + a(\operatorname{Germ}_x \gamma_{\beta\alpha}) - f_\alpha^N(x), \qquad x \in U_{\alpha\beta}.$$

The cocycle b^N is integrable, since a is integrable and $|f^N|$ is bounded by N.

For $\phi \in A^{m-1}(M, \mathscr{F})$ on U_α we have

$$\begin{aligned} \left(\eta^{b^N} - \eta^a\right) \wedge \phi &= \sum_\beta d\lambda_\beta \cdot \left(\zeta_{\beta\alpha}^{b^N} - \zeta_{\beta\alpha}^a\right) \wedge \phi \\ &= \sum_\beta d\lambda_\beta \cdot \left(f_\beta^N \circ \pi \circ \Phi_\beta - f_\alpha^N \circ \pi \circ \Phi_\alpha\right) \wedge \phi \\ &= d\left\{\sum_\beta \lambda_\beta \cdot f_\beta^N \circ \pi \circ \Phi_\beta \wedge \phi\right\} = d\tau, \end{aligned}$$

where τ is a bounded measurable $(m-1)$-form on M having ω as a factor. By the leafwise Stokes' theorem we obtain $\int_B (\eta^{b^N} - \eta^a) \wedge \phi = \int_B d\tau = 0$. This shows that $g_B^{b^N}[\phi] = g_B^a[\phi]$ for all N. It remains to show:

Proposition 3.5. $\lim_{N\to\infty} \int_B \eta^{b^N} \wedge \phi = \int_B \eta^b \wedge \phi$.

Proof. We know $\eta^{b^N} \to \eta^b$ pointwise, so it will suffice to dominate the collection $\{|\eta^{b^N}|\}$ by an integrable function and apply the dominated convergence theorem. For each $x \in T_\alpha$ recall that

$$|a|_x = \max_{\substack{(\alpha,\beta)\\ \text{admissible}}} \left|a(\operatorname{Germ}_x \gamma_{\beta\alpha})\right|$$

and by assumption $\int_{T_\alpha} |a|_x\, dx < \infty$ and $\int_{T_\alpha} |b|_x\, dx < \infty$. For $z \in U_\alpha$ with $x = \pi \circ \Phi_\alpha(z)$, consider

$$\begin{aligned} \left|\eta_\alpha^{b^N}\right|_z &= \left|\sum_\beta d\lambda_\beta \cdot \zeta_{\beta\alpha}^{b^N}\right|_z \leqslant \sum_\beta |d\lambda_\beta|_z \left|\zeta_{\beta\alpha}^{b^N}\right|_z \\ &\leqslant \left(\sum_\beta |d\lambda_\beta|_z\right) \cdot \max_{\substack{(\alpha,\beta)\\ \text{admissible}}} \left|b^N(\operatorname{Germ}_x \gamma_{\beta\alpha})\right| \\ &\leqslant c \cdot \max_{(\alpha,\beta)} \left|a(\operatorname{Germ}_x \gamma_{\beta\alpha}) + f_\beta^N \circ \gamma_{\beta\alpha}(x) - f_\alpha^N(x)\right| \\ &\leqslant c \cdot |a|_x + c \cdot \max_{(\alpha,\beta)} \left|f_\beta^N \circ \gamma_{\beta\alpha}(x) - f_\alpha^N(x)\right|. \end{aligned}$$

Lemma 3.6. *For all $x \in T_{\alpha\beta}$,*

$$\left|f_\beta^N \circ \gamma_{\beta\alpha}(x) - f_\alpha^N(x)\right| \leqslant |a|_x + |b|_x.$$

Proof. We have

$$\left| f_\beta^N \circ \gamma_{\beta\alpha}(x) - f_\alpha^N(x) \right| \leqslant \left| f_\beta \circ \gamma_{\beta\alpha}(x) - f_\alpha(x) \right| \leqslant |b|_x + |a_x|. \quad \text{q.e.d.}$$

By Lemma 3.6, for $z \in U_\alpha$ with $x = \pi \circ \Phi_\alpha(z)$, $|\eta_\alpha^{b^N}|_z \leqslant c \cdot (2|a|_x + |b|_x)$, which is integrable on U_α. The $\{U_\alpha\}$ form a finite cover of M, so the proof of 3.4 is complete.

Corollary 3.7. *For each class $[a] \in H_{(1)}(\Gamma, R)$, there is a well-defined measure g^a on $\mathscr{B}$ taking values in the continuous dual $H^{m-1}(M, \mathscr{F})^*$.*

Corollary 3.8. *Suppose $\mathscr{F}$ admits an absolutely continuous invariant transverse measure μ on T, whose Radon-Nikodým derivative $d\mu/d\bar{x}$ is almost everywhere positive on T. Then the Godbillon measure $g = 0$.*

Proof. The hypothesis implies the Radon-Nikodým cocycle $d\nu \sim 0$ on Γ, so $g = g^{d\nu} = g^0 = 0$.

4. Subexponential growth

A basic problem is to characterize the classes $[a] \in H_{(1)}(\Gamma, R)$ for which $g^a = 0$. Certainly if $a \sim 0$ this is true, but we seek more general criteria which imply $g^a = 0$.

Theorem 4.1. *Let a: $\Gamma \to R$ be an integrable cocycle, and $B \in \mathscr{B}$ a measurable saturated set. Suppose that for each $\varepsilon > 0$, there is a cocycle $b_\varepsilon \sim a$ such that b_ε is ε-tempered on B. Then $g_B^a = 0$.*

Proof. Given $[\phi] \in H^{m-1}(M, \mathscr{F})$,

$$|g_B^a[\phi]| = \left| \int_B \eta^a \wedge \phi \right| = \left| \int_B \eta^{b_\varepsilon} \wedge \phi \right| \leqslant \varepsilon \cdot \int_B \sum_\beta |d\lambda_\beta| \cdot |\phi|$$

and this tends to zero with ε. q.e.d.

We next consider a growth condition on $\mathscr{R}$ which will allow us to apply 4.1 to the Radon-Nikodým cocycle $d\nu$: $\mathscr{R} \to R$. For each $N > 0$ set

$$\mathscr{R}_N(x) = \{ y | (x, y) \in \mathscr{R} \text{ and } |x, y| \leqslant N \}.$$

Definition 4.2. Let $B \in \mathscr{B}$. The restricted groupoid $\mathscr{R}|B$ has a.e. subexponential growth if

$$\limsup_{N \to \infty} \frac{1}{N} \log \#\mathscr{R}_N(x) = 0 \quad \text{a.e. } x \in B \cap T.$$

Equivalently, for each $\varepsilon > 0$ there is a measurable function $c(\varepsilon, x)$: $T \cap B \to R$ such that

$$\#\mathscr{R}_N(x) \leqslant c(\varepsilon, x) \cdot \exp(N \cdot \varepsilon) \quad \text{a.e. } x \in B \cap T.$$

Theorem 4.3. *Let $B \in \mathscr{B}$. Suppose $\mathscr{R}|B$ has a.e. subexponential growth. Then for every $\varepsilon > 0$ there is a measurable cocycle a_ε: $\mathscr{R} \to R$ with $a_\varepsilon \sim d\nu$ and a_ε: $\mathscr{R}|B \to R$ is ε-tempered.*

Corollary 4.4. *Let $B \in \mathscr{B}$. Suppose almost every leaf of $\mathscr{F}$ in B has subexponential growth. Then $g_B = 0$.*

Proof. By Plante [27], $\mathscr{R}|B$ has a.e. subexponential growth and we apply Theorems 4.1 and 4.3. q.e.d.

Theorem 1 of §1 follows from 4.4 by taking $B = M$.

Proof of 4.3. Let $X \subset B \cap T$ be the conull saturated set on which the cocycle $d\nu$ depends only on the principal groupoid $\mathscr{R}|X$. Let μ denote Lebesgue measure on $T = \bigcup T_\alpha$, where each $T_\alpha \cong I_n^a \subset R^n$ has the Euclidean volume form $d\bar{x} = dx_1 \wedge \cdots \wedge dx_n$. Fix $\varepsilon > 0$. We will construct a coboundary g_ε: $T \to R$ for which $a_\varepsilon(x, y) = g_\varepsilon(y) + d\nu(x, y) - g_\varepsilon(x)$ is ε-tempered on X.

For each $\delta > 0$, define the *exceptional set*

$$X_\delta = \{x \in X | \forall N > 0, \text{ there exists } (x, y) \in \mathscr{R}|X \text{ with } |x, y| > N \text{ and } d\nu(x, y) > \delta \cdot |x, y|\}.$$

Observe that $X_\delta \subset X_\varepsilon$ for $\varepsilon < \delta$, so $X_0 \equiv \bigcup_{n=1}^\infty X_{1/n}$ is an increasing union. X_0 is the subset of X on which $d\nu$ has nonzero asymptotic positive growth. The key to the proof of 4.3 is then:

Proposition 4.5. *The set X_0 has zero Lebesgue measure.*

Proof. It will suffice to show $\mu(X_\delta) = 0$ for all $\delta > 0$. Let $c(\delta/4, x)$ be the function on X given in 4.2. Set $X(k) = \{x \in X \text{ such that } c(\delta/4, x) \leqslant k\}$. Since $X = \bigcup_{k=1}^\infty X(k)$ a.e., it suffices to show $\mu(X_\delta \cap X(k)) = 0$ for all k.

For each integer $N > 0$, set

$$X(\delta, N) = \{x \in X | \exists (x, y) \in \mathscr{R}, |x, y| \leqslant N, d\nu(x, y) > \delta N\},$$
$$X(\delta, N, k) = X(\delta, N) \cap X(k).$$

Given $x \in X(\delta, N)$, let $y \sim x$ have $|x, y| \leqslant N$ and $d\nu(x, y) > \delta N$. By the definition of the norm on $\mathscr{R}(x)$, there is a $\gamma \in \mathscr{G}$ with $|\gamma| = |x, y|$, $x \in U_\gamma$, $y = \gamma(x)$ and $d\nu(x, y) = d\nu(\mathrm{Germ}_x \gamma)$. It follows that there is an open neighborhood U_x of x such that for $z \in U_x$, $d\nu(\mathrm{Germ}_z \gamma) > \delta N$, and $|z, \gamma(z)| \leqslant N$. Thus, $X \cap U_x \subset X(\delta, N)$ which shows the set $X(\delta, N)$ is open in X.

Next choose a finite number of disjoint measurable sets $\{V_1, \cdots, V_s\}$ and maps $\{\gamma_1, \cdots, \gamma_s\} \subset \mathscr{G}$ where V_i is in the domain of γ_i, and such that

$$\mu\left(X(\delta, N) - \bigcup_{i=1}^s V_i\right) = 0, \qquad |\gamma_i| \leqslant N,$$
$$d\nu(x, \gamma_i(x)) = d\nu(\mathrm{Germ}_x \gamma_i) > \delta N \quad \text{all } x \in U_i \cap X.$$

This is possible because the number of maximal elements in $\mathscr{G}$ of length $\leqslant N$ is finite.

Now, observe that for $x \in U_i$ and $y \in U_j$, if $\gamma_i(x) = \gamma_j(y)$ and $i \neq j$, then $x \neq y$ and $(x, y) \in \mathscr{R}_{2N}(x)$. This implies that the number of $y \in X(\delta, N)$ for which $\gamma_i(x) = \gamma_j(y)$, $i \sim j$, is bounded by the number $\#\mathscr{R}_{2N}(x)$. For $x \in X(\delta, N, k)$ we have the estimate

$$\#\mathscr{R}_{2N}(x) \leqslant k \cdot \exp(N\delta/2).$$

Thus, there is a measurable function

$$A\colon X(\delta, N, k) \to \{1, 2, \cdots, k \cdot \exp(N\delta/2)\}$$

so that if $x \in U_i \cap X(\delta, N, k)$ and $y \in U_j \cap X(\delta, N, k)$ with $i \neq j$, and $\gamma_i(x) = \gamma_j(y)$, then $A(x) \neq A(y)$.

Use A and the collection $\{\gamma_1, \cdots, \gamma_s\}$ to define an embedding of $X(\delta, N, k)$ into the disjoint union of $(k \cdot \exp \delta N/2)$-copies of X by the rule

$$\gamma_A\colon X(\delta, N, k) \to \bigcup_{l=1}^{k \exp(\delta N/2)} X_l,$$

$$\gamma_A(x) = \gamma_i(x) \in X_{A(x)} \quad \text{for } x \in U_i \cap X(\delta, N, k).$$

Lemma 4.6. $\mu(X(\delta, N, k)) \leqslant k \cdot \mu(X) \cdot \exp(-\delta N/2)$.

Proof.

$$\begin{aligned} k \cdot \exp(\delta N/2) \cdot \mu(X) &\geqslant \int_{\gamma_A(X(\delta, N, k))} d\bar{x} \geqslant \int_{X(\delta, N, k)} \gamma_A^* \, d\bar{x} \\ &\geqslant \int_{X(\delta, N, k)} \exp \delta N \, d\bar{x} = \exp \delta N \cdot \mu(X(\delta, N, k)). \end{aligned}$$

q.e.d.

Now observe that for all $N_0 > 0$,

$$X_\delta \cap X(k) \subset \bigcup_{N=N_0}^{\infty} X(\delta, N, k)$$

so for all $N_0 > 0$,

$$\begin{aligned} \mu(X_\delta \cap X(k)) &\leqslant \sum_{N=N_0}^{\infty} \mu(X(\delta, N, k)) \\ &\leqslant \sum_{N=N_0}^{\infty} k \cdot \mu(X) \cdot \exp(-\delta N/2) \end{aligned}$$

which tends to zero as N_0 goes to infinity. q.e.d.

Let $\overline{X}_0$ be the saturation of X_0. Since $\mathscr{R}$ is discrete, Proposition 4.5 implies $\mu(\overline{X}_0) = 0$.

We have now proven that on the saturated conull subset $X - \overline{X}_0$ of X, the cocycle $d\nu$ has zero asymptotic positive growth. By Theorem 3.1 of [19], for all $\varepsilon > 0$ there is an ε-tempered cocycle $a_\varepsilon \sim d\nu$. Therefore, by Theorem 4.1, $g_B = g_B^{d\nu} = 0$, proving 4.3. For the sake of completeness, we include the elementary construction of the cocycle a_ε.

Given $\varepsilon > 0$, define a measurable function f_ε on T by the rule:

$$f_\varepsilon(x) = \begin{cases} \sum_{y \in \mathscr{R}(x)} e^{-\varepsilon|x,y|} \exp d\nu(x,y), & x \in X - \overline{X}_0, \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 4.7. *For* $x \in X - \overline{X}_0$, $f_\varepsilon(x) < \infty$.

Proof. Set $k = c(\varepsilon/4, x)$. The assumption $x \notin X_0$ implies there exists N_0 so that for $|x, y| > N_0$, $d\nu(x,y) \leqslant |x,y|\varepsilon/2$. Thus

$$\begin{aligned} f_\varepsilon(x) &= \sum_{|x,y| \leqslant N_0} e^{-\varepsilon|x,y|} \exp d\nu(x,y) + \sum_{|x,y| > N_0} e^{-\varepsilon|x,y|} \exp d\nu(x,y) \\ &\leqslant \sum_{|x,y| \leqslant N_0} e^{-\varepsilon|x,y|} \exp d\nu(x,y) + k \cdot \sum_{N > N_0} \exp(-\varepsilon N/4) < \infty. \end{aligned}$$

Lemma 4.8. *Let* $x \in X - \overline{X}_0$, $(x, y) \in \mathscr{R}$ *and* $|x, y| = 1$. *Then*

$$\log f_\varepsilon(x) - \varepsilon \leqslant \log f_\varepsilon(y) + d\nu(x,y) \leqslant \log f_\varepsilon(x) + \varepsilon.$$

Proof.

$$\begin{aligned} f_\varepsilon(y) &= \sum_{z \in \mathscr{R}(y)} e^{-\varepsilon|y,z|} \exp d\nu(y,z) \\ &= \sum_{z \in \mathscr{R}(y)} e^{-\varepsilon|y,z|} \exp d\nu(x,z) \cdot \exp d\nu(y,x). \end{aligned}$$

Now $|y, z| - 1 \leqslant |x, z| \leqslant |y, z| + 1$ implies $f_\varepsilon(x)e^{-\varepsilon} \leqslant f_\varepsilon(y) \cdot \exp d\nu(x,y) \leqslant f_\varepsilon(x)e^{\varepsilon}$. Apply log to this to get the result. q.e.d.

Now set $g_\varepsilon = \log f_\varepsilon \colon X \to R$, and set $a_\varepsilon(x,y) = g_\varepsilon(y) + d\nu(x,y) - g_\varepsilon(x)$.

Lemma 4.9. a_ε *is* ε*-tempered on* $X - \overline{X}_0$.

Proof. Let $x \in X - \overline{X}_0$ and $(x, y) \in \mathscr{R}$ with $|x, y| = 1$. Then

$$|a_\varepsilon(x,y)| = |\log f_\varepsilon(y) + d\nu(x,y) - \log f_\varepsilon(x)| \leqslant \varepsilon. \quad \text{q.e.d.}$$

This completes the proof of Theorem 4.3.

The main idea in the proof of 4.3 is to show that cocycle $d\nu$ has zero positive exponents, and then to apply the smoothing technique of Hurder-Katok [19] to construct f_ε. The delicate part of Proposition 4.5 is that we must squeeze the sets X_δ into unions of sets with exponentially decreasing size. If the number of these sets grows exponentially fast, then we are in trouble. The natural indexing set for the image is the pseudo-group $\mathcal{G}$; however, $\mathcal{G}$ may have exponential growth even though each orbit of $\mathcal{R}$ has subexponential growth. (There are simple examples to show this can happen.) By taking the summation process in 4.5 over the individual orbits, which have subexponential growth, we circumvent the difficulties that arise from trying to work with $\mathcal{G}$. This is perhaps the main difference between our approach and the previous works in codimension one.

The proof of 4.5 can be adapted to yield a general result of independent interest in ergodic theory. Let (X, μ) be a standard measure space with μ a σ-finite measure. Let G be a finitely generated group acting on X by measure-class preserving maps. Define a groupoid $G \times X$ and equivalence relation $\mathcal{R} \subset X \times X$ from the action, and define orbit norms on $\mathcal{R}$ using a work metric on G (cf. [19]). Let $d\nu$: $G \times X \to \mathbf{R}$ be the additive Radon-Nikodým cocycle. The orbit growth rate of $x \in X$ is defined as

$$g(x) = \limsup_{N \to \infty} \frac{1}{N} \log \#\mathcal{R}_N(x).$$

Theorem 4.10. *Let $G \times X \to X$ be a measure-class preserving action of G on X such that $d\nu$ is tempered. Suppose there exists $A > 0$ such that a.e. $x \in X$ has orbit growth rate $g(x) \leqslant A$. Then for all $\varepsilon > 0$, there exists a $(3A + \varepsilon)$-tempered cocycle a_ε: $G \times X \to R$ with $a_\varepsilon \sim d\nu$.*

Proof. This is a variation on the proof of 4.5, so only the main points are given. First, the sets X_δ for $\delta > 2A$ have measure 0, which follows by the same argument given for the $A = 0$ case. Then define $X_{2A} = \bigcup_{n=1}^{\infty} X_{2A+1/n}$, and note the saturation $\overline{X}_{2A}$ has measure zero. For $\varepsilon > 0$, define f_ε on the conull set $X - \overline{X}_{2A}$ by

$$f_\varepsilon(x) = \sum_{y \sim x} e^{-(3A+\varepsilon)|x,y|} \cdot \exp d\nu(x, y)$$

which converges on $X - \overline{X}_{2A}$ by estimates identical to those used in 4.7. Then $a_\varepsilon(x, y) = f_\varepsilon(y) + d\nu(x, y) - f_\varepsilon(x)$ is $(3A + \varepsilon)$-tempered. q.e.d.

We draw one more corollary from the proof of Theorem 4.3.

Corollary 4.11. *Let $\mathcal{F}$ be a C^2-foliation of a compact manifold M. Suppose that no closed transversal to $\mathcal{F}$ is null-homotopic, and $d\nu$ is not moderately growing; i.e.,*

$$\limsup_{|x,y| \to \infty} \frac{d\nu(x, y)}{|x, y|} > 0.$$

Then $\pi_1 M$ has exponential growth.

Proof. From the proof of 4.3, we conclude that $\mathscr{F}$ must have a leaf L which is not of subexponential growth. Thus, there is a sequence $\{r_j\}$ with $r_j \to \infty$ for which the balls $B(x, r_j)$ in L centered at $x \in L$ have exponentially growing volume. Then for some foliation chart U with transversal T and some subsequence of $\{r_j\}$, the number of points in $T \cap B(x, r_j)$ increases exponentially in $\{r_j\}$. It is then a standard construction to define, for each point $y \in T \cap B(x, r_j)$, a closed transversal γ_y to $\mathscr{F}$ based at x with length $\gamma_y \leqslant 2r_j +$ constant (cf. [28]). The paths $\{\gamma_y\}$ are all pairwise nonhomotopic, for otherwise we can construct a null-homotopic closed transversal to $\mathscr{F}$. Thus, $\pi_1(M, x)$ does not have subexponential growth, hence it must be of exponential type. q.e.d.

A codimension-one Anosov foliation of M satisfies both conditions of Corollary 4.11, so this yields the main result of [28]. Of course, in this case the only new aspect of the above proof is the method by which we conclude that not all leaves of $\mathscr{F}$ have subexponential growth. The point of Corollary 4.11 is that it applies to any $\mathscr{F}$ with the "no null-homotopic closed transversals" condition and for which $d\nu$ is not moderately growing; it is not necessary to have the much stronger condition

$$\liminf_{|x,y| \to \infty} \frac{|d\nu(x, y)|}{|x, y|} > 0$$

which is associated to Anosov foliations.

5. The open manifold case

The proof of Theorem 1 given in §§2–4 for M compact also works for M open with slight modifications:

Theorem 5.1. *Let $\mathscr{F}$ be a C^2-foliation of an open manifold M, and suppose that for some metric on M almost every leaf has subexponential growth. Then all Godbillon-Vey classes are zero.*

Proof. It suffices to show the cohomology class $\Delta_*(y_1 y_I c_J) \in H^p(M)$ evaluates to zero on every cycle in $H_p(M)$. Passing to an appropriate double cover of M, we can assume M and $Q \to M$ are orientable. The homology $H_p(M)$ is dual to the compactly supported cohomology $H_c^{m-p}(M)$, so it suffices to show $\int_M \eta \wedge \Delta(y_I c_J) \wedge \phi = 0$ for every compactly supported closed $(m-p)$-form ϕ on M. The support of ϕ can be covered by a finite number of regular foliation charts; we let U denote the union of their domains and K be the closure of U. Then K is compact, so the transversal T to $\mathscr{F}$ in U, defined by the finite covering by flow boxes, has finite volume.

Let h be a Riemannian metric on M for which a.e. leaf of $\mathscr{F}$ has subexponential growth. Let $\mathscr{F}|U$ denote the restriction of $\mathscr{F}$ to U, L a leaf of $\mathscr{F}|U$ and $L' \subset M$ the leaf of $\mathscr{F}$ containing L. The restriction of h to L defines a distance function on L, and it is easy to see the inclusion $L \to L'$ is distance decreasing. Thus, if L' has subexponential growth, so must L. We can therefore assume a.e. leaf of $\mathscr{F}|U$ has subexponential growth. This implies a.e. orbit of the equivalence relation $\mathscr{R}|T \subset T \times T$ induced by $\mathscr{F}|U$ has subexponential growth, with respect to the given finite cover of U by regular foliation charts. Since U has finite transverse volume, the proof of Theorem 4.3 shows $\int_U \eta \wedge \Delta(y_I c_J) \wedge \phi = 0$ exactly as in the case M is compact.

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