THE HEAT EQUATION
SHRINKING CONVEX PLANE CURVES

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1.

Let M and M' be Riemannian manifolds and $F: M \to M'$ a smooth map. The Laplacian of F is defined intrinsically as a section of the pull-back of the tangent bundle of M,

$$\Delta F \in \mathscr{C}^\infty(M, F^*TM'),$$

and is given by the trace of the vector-valued matrix of second derivatives in any two systems of normal coordinates on M and M'. When F is an isometric immersion, the Laplacian of F is given by $\Delta F = kN$, where k is the mean curvature (the trace of the second fundamental form) and N is the unit normal vector. We can deform the immersion F by the heat equation

$$\frac{\partial F}{\partial t} = \Delta F \quad \text{or} \quad \frac{\partial F}{\partial t} kN$$

always computing ΔF in the varying metric on M induced by the immersion F. It is a theorem (see [5]) that the solution always exists for a short time, and is unique and smooth. Moreover, the immersed submanifolds $M_t = F_t(M)$ are independent of the parametrization. If two immersions F and F^* differ by a diffeomorphism h at time $t = 0$, then the solutions continue to satisfy $F^* = F \circ h$ as long as they exist for the same fixed h independent of t.

The equation is clearly of significant geometrical interest. It has the following variational interpretation. The space $\mathscr{M}$ of all immersed submanifolds M of M' has the structure of an infinite-dimensional manifold modeled on a Fréchet space (see [4]). the tangent space $T_M\mathscr{M}$ to $\mathscr{M}$ at M is naturally identified with the space $C^\infty(M)$ of functions f on M, where the variation in M is given by moving infinitesimally a distance f in the normal direction. The

volume $V(M)$ of M gives a function on $\mathcal{M}$ whose derivative in the direction of a normal variation is

$$DV(M)f = -\int_M fk.$$

It follows that the heat equation $\partial F/\partial t = kN$ describes the gradient flow for the Morse function V.

We consider here the special case of a curve M in a surface M'. For a compact curve M in a surface M' which is either compact or else convex at ∞ (the union of compact subsets with convex boundaries) it is conjectured that if the initial curve is embedded, then it remains embedded, and shrinks either to a point or a geodesic. This would have important consequences for the existence of geodesics on surfaces, by Morse theory. It is not known for arbitrary curves even in the simple case where $M' = \mathbf{R}^2$, the euclidean plane, although several computer studies tend to confirm the conjecture.* In this paper we prove the following special case:

1.1. Main Theorem. *If M is a convex curve embedded in the plane $\mathbf{R}^2$, the heat equation shrinks M to a point. The curve remains convex and becomes circular as it shrinks, in the sense that*

(a) *the ratio of the inscribed radius to the circumscribed radius approaches* 1, *and*

(b) *the ratio of the maximum curvature to the minimum curvature approaches* 1;

(c) *the higher order derivatives of the curvature converge to* 0 *uniformly.*

We remark that some curves in the plane which are immersed but not imbedded will surely develop singularities. Consider, for example, the Limaçon of Pascal, given in polar coordinates by $r = 1 + 2\cos\theta$. The little loop will shrink faster than the big loop, and eventually turn into a cusp. This example is geometrically significant in view of the result we prove that when the curvature k goes to ∞, it must do so on an interval of the curve where the angle θ of the tangent line changes by π or more (such as in the little loop).

In the higher dimensional case, Gerhard Huisken (see [6]) has shown that a convex hypersurface M^n in $\mathbf{R}^{n+1}$ for $n \geqslant 2$ will shrink to a point. The two proofs are strikingly different in that our proof uses integral estimates, while his uses the maximum principle. Essential to the higher-dimensional proof is the use of the Codazzi-Mainardi equations, which are vacuous for curves in a surface. This is analogous to the case for the heat equation for deforming metrics by their Ricci curvature (see [5]), where in dimension 3 essential use

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*(*Added in proof*). This conjecture has now been proved by Matt Grayson.

was made of the second Bianachi identity, which is likewise vacuous for surfaces.

We note that in the higher dimensional case even embedded surfaces in space can develop singularities if they are not convex. It is pretty clear that for an hour-glass surface with a long thin neck the neck will pinch off and pop, changing the topological type of the surface.

This paper is organized as follows: §2 has the proof for short-time existence and uniqueness for the heat equation for any compact manifold M immersed in M'; §3 contains theorems which apply to any closed simple curve in the plane; §4 has the a priori estimates which prove long term existence for the heat equation for convex plane curves; and §5 uses geometric estimates to prove that the ratio of the maximum curvature to the minimum curvature approaches one and that the higher derivatives converge to zero.

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2.

In this section we prove short-time existence and uniqueness for the heat equation $\partial F/\partial t = \Delta F$ for a compact submanifold M immersed in M'. We appeal to the result in [5] for evolution equations with an integrability condition, which is proved using the Nash-Moser inverse function theorem. The maps F lie in the Fréchet manifold of all immersions of M to M'. Given a vector field H on M tangent to M', let $\pi_M H$ denote the orthogonal projection of H on the tangent space to M. Since $\Delta F = kN$, where k is the mean curvature and N the unit normal vector, we always have $\pi_M \Delta F = 0$. This is our integrability condition.

An evolution equation $\partial F/\partial t = E(F)$ is parabolic if $E(F)$ is a second order strongly elliptic nonlinear operator. This means that, at the F we consider, the derivative $DE(F)\tilde{F}$ in the direction $\tilde{F}$ is a strongly elliptic linear operator, which means that all of the eigenvalues of its symbol $\sigma DE(F)(\xi)$ have stricly positive imaginary part (where we form the symbol in the direction of a cotangent vector ξ by replacing $\partial/\partial x^i$ by ξ_i and throwing away the lower order terms). It is a standard result for parabolic equations that solutions exist for a short time and are unique. Our equation is only weakly parabolic, in that the eigenvalues of its symbol are nonnegative, but some are zero. This may be seen from the fact that $\pi_M \Delta F = 0$, so if $E(F) = \Delta F$, then $\pi_M \sigma DE(F)(\xi) = 0$. We shall compute the symbol explicitly. Choose local coordinates $\{x^i\}$ on M and $\{y^\alpha\}$ on M, and let $y^\alpha = F^\alpha(x^i)$ be the immersion. Then

$$\Delta F^\alpha = g^{ij}\left(\frac{\partial^2 F^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial F^\alpha}{\partial x^k}\right),$$

where g_{ij} is the induced metric on M and Γ_{ij}^k are the Christoffel symbols. Here

$$g_{ij} = h_{\alpha\beta} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j}$$

where $h_{\alpha\beta}$ is the metric on M', and,

$$\Gamma_{ij}^l = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right).$$

A variation $\tilde{F}$ in F produces a variation $DE(F)\tilde{F}$ in ΔF. To compute the symbol of $DE(F)$ we only need to keep track of the highest order terms. These arise in two ways. First there is $\Delta \tilde{F}$. Then there are the terms which come from the variation of Γ_{ij}^k. Note that g_{ij} depends on the first derivatives of F, and Γ_{ij}^k depends on the first derivatives of g_{ij}, so Γ_{ij}^k involves the second derivatives of F. Computing on the symbols

$$\tilde{g}_{ij} = h_{\alpha\beta} \left(\frac{\partial F^\alpha}{\partial x^i} \tilde{F}^\beta \xi_j + \frac{\partial F^\beta}{\partial x^j} \tilde{F}^\alpha \xi_i \right),$$

$$\tilde{\Gamma}_{ij}^l = \frac{1}{2} g^{kl} \left(\tilde{g}_{jk} \xi_i + \tilde{g}_{ik} \xi_j - \tilde{g}_{ij} \xi_k \right)$$

and combining we get

$$\tilde{\Gamma}_{ij}^l = g^{kl} h_{\alpha\beta} \frac{\partial F^\alpha}{\partial x^k} \tilde{F}^\beta \xi_i \xi_j.$$

It follows that at the symbol level

$$\tilde{\Delta} F^\alpha = |\xi|^2 \tilde{F}^\alpha - g^{kl} h_{\gamma\beta} \frac{\partial F^\gamma}{\partial x^k} |\xi|^2 \tilde{F}^\beta \frac{\partial F^\alpha}{\partial x^l}.$$

Now it is easy to recognize that $\tilde{\Delta} F^\alpha = |\xi|^2 (\pi_N \tilde{F})^\alpha$, where π_N is the projection of the tangent space to M' on the normal direction to M. This clearly shows the zeros in the symbol.

In the theorem in ([5], Theorem 5.1) we assume that there is an integrability condition $B(F)H$ which is linear in H of degree 0 (or 1) in H such that $B(F)E(F)$ is also of degree 0 (or 1) in F, due to a cancellation of derivatives. We assume also that all the eigenvectors of $\sigma E(F)(\xi)$ in the null space of $\sigma B(F)$ have strictly positive real part, the others being identically zero of necessity. Then it follows that the evolution equation $\partial F/\partial t = E(F)$ always has a solution for a short time with given initial data (satisfying this condition) and the solution is unique. In this application we take $B(F)H = \pi_M H$, which has degree zero. Then $B(F)E(F) = 0$, since $B(F) = \pi_M$ and $E(F) = \Delta F$ and

$\pi_M \Delta F = 0$. The null space of the symbol of $B(F)$ is the normal vectors, and on this eigenspace the symbol of $DE(F)$ acts by multiplication by $|\xi|^2$, which is positive for $\xi \neq 0$. Hence the condition is satisfied.

3.   Curves in the plane

We turn now to the special case of a curve (not necessarily convex) in the plane. We take $M = S^1$ with parameter u (modulo 2π) and write the curve as $(x, y) = F(u)$. We will derive evolution equations for the length of the curve, its curvature and for the area it encloses. We will also show that as long as the curvature remains bounded an embedded initial curve remains embedded during the evolution.

3.1. The arclength s along the curve is unique only up to a constant, but the derivative with respect to arclength $\partial / \partial s$ is uniquely defined. We can write the heat equation for the curve as the system

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial s^2}, \qquad \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial s^2}.$$

Here the derivative $\partial / \partial t$ is taken along fixed values of the parameter u. The operator $\partial / \partial s$ is given in terms of u by

$$\frac{\partial}{\partial s} = \frac{1}{v} \frac{\partial}{\partial u},$$

where $v = \sqrt{(\partial x / \partial u)^2 + (\partial y / \partial u)^2} = |\partial F / \partial u|$.

The arclength parameter is $ds = v du$. We let T and N be the unit tangent vector and the (inward pointing) unit normal vectors to the curve. The Frenet equations are

$$\frac{\partial T}{\partial u} = vkN, \qquad \frac{\partial N}{\partial u} = -vkT.$$

3.1.1. Lemma.   *The derivative of v with respect to t is $\partial v / \partial t = -k^2 v$.*

Proof.   Using the Frenet equations and the evolution equation, we calculate

$$\frac{\partial}{\partial t}(v^2) = \frac{\partial}{\partial t}\left\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u} \right\rangle = 2\left\langle \frac{\partial F}{\partial u}, \frac{\partial^2 F}{\partial t \partial u} \right\rangle = 2\left\langle \frac{\partial F}{\partial u}, \frac{\partial^2 F}{\partial u \partial t} \right\rangle$$

$$= 2\left\langle vT, \frac{\partial}{\partial u}(kN) \right\rangle = 2\left\langle vT, \frac{\partial k}{\partial u}N - vk^2 T \right\rangle = -2v^2 k^2$$

($\partial / \partial u$ and $\partial / \partial t$ commute since u and t are independent coordinates.)
The lemma follows immediately.

The evolution of the length of the curve is easily calculated:

3.1.2. Lemma. $\partial L / \partial t = -\int k^2 ds$.

Proof.

$$\frac{\partial L}{\partial t} = \int_0^{2\pi} \frac{\partial}{\partial t} v \, du = -\int_0^{2\pi} k^2 v \, du = -\int_0^L k^2 \, ds.$$

We also have this useful relation for the operators $\partial/\partial s$ and $\partial/\partial t$.

3.1.3. Lemma.

$$\frac{\partial}{\partial t} \frac{\partial}{\partial s} = \frac{\partial}{\partial s} \frac{\partial}{\partial t} + k^2 \frac{\partial}{\partial s}.$$

Proof.

$$\frac{\partial}{\partial t} \frac{\partial}{\partial s} = \frac{\partial}{\partial t} \frac{1}{v} \frac{\partial}{\partial u} = k^2 \frac{1}{v} \frac{\partial}{\partial u} + \frac{1}{v} \frac{\partial}{\partial u} \frac{\partial}{\partial t} = k^2 \frac{\partial}{\partial s} + \frac{\partial}{\partial s} \frac{\partial}{\partial t}.$$

The derivatives of T and N are given by

3.1.4. Lemma.

$$\frac{\partial T}{\partial t} = \frac{\partial k}{\partial s} N \quad and \quad \frac{\partial N}{\partial t} = -\frac{\partial k}{\partial s} T.$$

Proof.

$$\frac{\partial T}{\partial t} = \frac{\partial^2}{\partial t \partial s} F = \frac{\partial^2 F}{\partial s \partial t} + k^2 \frac{\partial F}{\partial s}$$

$$= \frac{\partial}{\partial s}(kN) + k^2 T = \frac{\partial k}{\partial s} N - k^2 T + k^2 T.$$

The second equation follows from

$$0 = \frac{\partial}{\partial t} \langle T, N \rangle = \left\langle \frac{\partial k}{\partial s} N, N \right\rangle + \left\langle T, \frac{\partial N}{\partial t} \right\rangle$$

since $\partial N/\partial t$ must be perpendicular to N.

Let θ be the angle between the tangent vector and the x axis. Then

3.1.5. Lemma.

$$\frac{\partial \theta}{\partial t} = \frac{\partial k}{\partial s} \quad and \quad \frac{\partial \theta}{\partial s} = k.$$

Proof. Since $T = (\cos\theta, \sin\theta)$ we use the formula in Lemma 3.1.4 to calculate

$$\frac{\partial T}{\partial t} = -\frac{\partial k}{\partial s} N = \frac{\partial k}{\partial s}(-\sin\theta, \cos\theta).$$

Comparing components proves the first equation of this lemma. Writing out the Frenet equation in this way proves the second relation.

The curvature evolves according to

3.1.6. Lemma.

$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k^3.$$

Proof.

$$\frac{\partial k}{\partial t} = \frac{\partial^2 \theta}{\partial t \partial s} = \frac{\partial^2 \theta}{\partial s \partial t} + k^2 \frac{\partial \theta}{\partial s} = \frac{\partial^2 k}{\partial s^2} + k^3.$$

Remark. *From the equation above and the strong maximum principle one can show that if the initial curve is convex, but contains straight line segments, then the straight line segments disappear immediately as the curve evolves and the succeeding curves are all strictly convex.*

Finally we compute the evolution equation for the area A enclosed by a simple closed curve.

3.1.7. Lemma. $\partial A / \partial t = -2\pi$.

Proof.

$$A = \frac{1}{2} \int_0^{2\pi} \left(x \frac{\partial y}{\partial u} - y \frac{\partial x}{\partial u} \right) du = -\frac{1}{2} \int_0^{2\pi} \langle F, vN \rangle \, du,$$

$$\frac{\partial A}{\partial t} = \frac{-1}{2} \int_0^{2\pi} \left\langle \frac{\partial F}{\partial t}, vN \right\rangle + \left\langle F, \frac{\partial v}{\partial t} N \right\rangle + \left\langle F, v \frac{\partial N}{\partial t} \right\rangle du$$

$$= \frac{-1}{2} \int_0^{2\pi} vk - \langle F, vk^2 N \rangle + \left\langle F, -\frac{\partial k}{\partial u} T \right\rangle du.$$

Integrating the last term by parts yields

$$\frac{\partial A}{\partial t} = \frac{-1}{2} \int_0^{2\pi} vk - \langle F, vk^2 N \rangle + \left\langle \frac{\partial F}{\partial u}, kT \right\rangle + \langle F, vk^2 N \rangle \, du$$

$$= -\int_0^{2\pi} vk \, du = -2\pi$$

since $\partial F / \partial u = vT$ and the total curvature of a simple closed curve is 2π.

3.1.8. Remark. *For a nonsimple closed curve the line integral in* (3.1.7) *defines the weighted area*

$$A = \int_{(x, y) \in \mathbf{R}^2} w(x, y) \, dx \, dy,$$

where $w(x, y)$ is the winding number of the curve with respect to the point (x, y). The rate of decrease of A will be -2π times the rotation index of the curve in this case.

3.2. We devote the remainder of this section to proving that an embedded curve remains embedded during its evolution provided the curvature remains bounded.

3.2.1. Theorem.   *Let F: $S^1 \times [0, T) \to \mathbf{R}^2$ represent a one parameter family of closed curves satisfying the evolution equation $\partial F/\partial t = \Delta F = kN$. If $|k(u, t)| < C$ and if the initial curve $F(\cdot, 0)$ is embedded, then $F(\cdot, t)$: $S^1 \to \mathbf{R}^2$ is an embedded curve for each t.*

The maximum principle will be the chief ingredient in the proof. We first prove several lemmas.

We introduce the function f: $S^1 \times S^1 \times [0, T) \to \mathbf{R}$ defined by $f(u_1, u_2, t) = |F(u_1, t) - F(u_2, t)|^2$.

3.2.2. Lemma.   *The function f satisfies the heat equation*

$$\frac{\partial f}{\partial t} = \Delta f - 4 = \frac{\partial^2 f}{\partial s_1^2} + \frac{\partial^2 f}{\partial s_2^2} - 4,$$

where Δf is computed using the product of the induced metric on S^1 with itself.

Proof.   From the evolution equation we find

$$\frac{\partial f}{\partial t} = 2\langle F(u_1, t) - F(u_2, t), kN(u_1, t) - kN(u_2, t)\rangle$$

while

$$\frac{\partial f}{\partial s_1} = 2\langle F(u_1, t) - F(u_2, t), T(u_1, t)\rangle,$$

$$\frac{\partial^2 f}{\partial s_1^2} = +2\langle T(u_1, t), T(u_1, t)\rangle + 2\langle F(u_1, t) - F(u_2, t), kN(u_1, t)\rangle,$$

$$\frac{\partial^2 f}{\partial s_2^2} = 2 - 2\langle F(u_1, t) - F(u_2, t), kN(u_2, t)\rangle.$$

Adding these last two equations and comparing with $\partial f/\partial t$ proves the result.

Let $s(u_1, u_2, t) = |\int_{u_1}^{u_2} v(u, t)\, du|$ be the distance along the curve from u_1 to u_2 traversing the curve in the positive sense. Suppose now that we have an upper bound on the curvature $|k| \leqslant c$; we show that for $s(u_1, u_2, t) \leqslant 2/c$ we have $f(u_1, u_2, t) = 0$ if and only if $u_1 = u_2$. This means geometrically that the curves have no self intersections resulting from short kinks:

3.2.3. Lemma (*A. Schur and E. Schmidt*) [3]. (*See Figure 1.*) *Let g: $[0, L] \to \mathbf{R}^2$ be a curve parametrized by arclength from A to B such that g together with the chord connecting A to B forms a convex curve. Let f be a second curve of the same length L with endpoints C and D. We assume the curves have continuous tangents and piecewise continuous curvature and that the curve g is transversed in the counterclockwise sense so that its curvature is positive. If the curvature at each point of g is greater than the absolute value of the curvature at the corresponding point on f (i.e. $k_g(s) \geqslant |k_f(s)|$), then $\mathrm{dist}(A, B) \leqslant \mathrm{dist}(C, D)$.*

Proof.   Orient the curves so that AB and CD lie on the x axis. We will use the arclength parameter s for both curves since in this lemma we are not dealing with families of curves. Let $\theta_g(s)$ be the angle of the tangent vector of the curve g at s. There is exactly one point s_0 where the tangent to $g(s_0)$ is parallel to the x axis ($\theta_g(s_0) = 0$). Since

$$\frac{d\theta_g}{ds} = k_g \geqslant |k_f| = \left| \frac{d\theta_f}{ds} \right|$$

we have by integration that

$$\left| \theta_f(s) - \theta_f(s_0) \right| \leqslant \left| \theta_g(s) \right|.$$

$|\theta_g(s)| \leqslant \pi$ for $0 \leqslant s \leqslant L$ because g is convex. Therefore,

$$\int_0^L \cos\left(\theta_f(s) - \theta_f(s_0) \right) ds = \int_0^L \cos\left| \theta_f(s) - \theta_f(s_0) \right| ds$$

$$\geqslant \int_0^L \cos\left| \theta_g(s) \right| = \text{dist}(A, B).$$

The left-hand integral is the projection of the line segment CD onto the tangent to $f(s_0)$. Since the line segment is longer than the projection this proves the result.
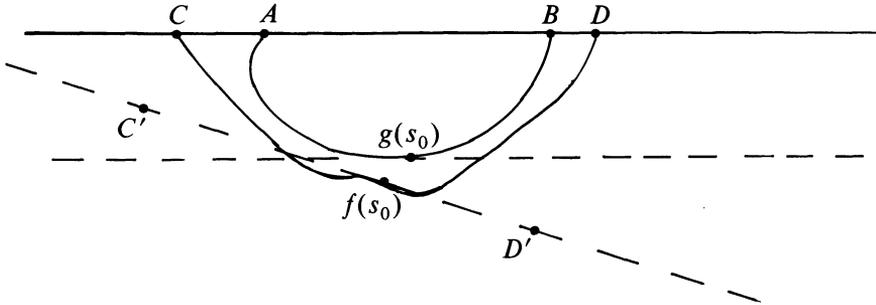

FIGURE 1

3.2.4. Corollary.   *If* $|k(u, t)| \leqslant c$, *then*

$$f(u_1, u_2, t) \geqslant \left\{ \frac{2}{c} \sin\left(\frac{c}{2} s(u_1, u_2, t) \right) \right\}^2.$$

Proof. Let g be the arc of length $s(u_1, u_2, t)$ of the circle of radius $1/c$ and apply the previous lemma.

Proof of Theorem 3.2.1. On the set $E = \{(u_1, u_2, t) | s(u_1, u_2, t) < \pi/c\}$, $f(u_1, u_2, t) = 0$ if and only if $u_1 = u_2$. This follows immediately from 3.2.4. Now restrict f to the complementary domain $D = (S^1 \times S^1 \times [0, T)) - E$. We will use a version of the maximum principle to show that on D, f has a positive minimum. This will complete the proof of the theorem.

The boundary of D is given by

$$\{(u_1, u_2, t) | s(u_1, u_2, t) = \pi/c, 0 \leqslant t \leqslant T\}$$

$$\cup \{(u_1, u_2, 0) | s(u_1, u_2, t) \geqslant \pi/c\}.$$

On the first set $f(s_1, s_2, t) \geqslant (2/c)^2$ by Corollary 3.2.4 while the second set has a positive minimum because the initial curve is embedded. Let m be the smaller of these two quantities.

We consider the function $g(u_1, u_2, t) = f(u_1, u_2, t) + \varepsilon t$. It satisfies the equation

(3.2.5a)
$$\frac{\partial g}{\partial t} = \Delta g - 4 + \varepsilon.$$

Let $0 < \delta < m$ and suppose that g achieves the value $m - \delta$ on $(S^1 \times S^1 \times [0, T)) - E$. Let $t_0 = \inf\{t | g(u_1, u_2, t) = m - \delta\}$. The continuity of g and the compactness of D together with the boundary estimate insures that the value $m - \delta$ is achieved for the first time at some interior point $(\bar{u}_1, \bar{u}_2, t)$. At this point $\partial g/\partial t \leqslant 0$ and

$$\frac{\partial^2 g}{\partial s_1^2} \cdot \frac{\partial^2 g}{\partial s_2^2} - \left(\frac{\partial^2 g}{\partial s_1 \partial s_2}\right)^2 \geqslant 0.$$

$(\partial/\partial s_1, \partial/\partial s_2$ are operators rather than partial derivatives but an easy calculation verifies this last inequality.) We calculate that

$$\frac{\partial^2 g}{\partial s_1 \partial s_2} = -2\langle T(u_2, t), T(u_1, t)\rangle = \pm 2$$

because at a minimum point the tangent lines to the curve at s_1 and s_2 must be parallel.

$$\Delta g = \frac{\partial^2 g}{\partial s_1^2} + \frac{\partial^2 g}{\partial s_2^2} \geqslant 2\sqrt{\frac{\partial^2 g}{\partial s_1^2} \cdot \frac{\partial^2 g}{\partial s_2^2}} \geqslant 2\left|\frac{\partial^2 g}{\partial s_1 \partial s_2}\right| \geqslant 4.$$

This contradicts the assumption that g satisfies (3.2.5a). Since δ is arbitrary, we have that $g(u_1, u_2, t) \geqslant m$ on D, which implies that $f(u_1, u_2, t) \geqslant m - \varepsilon T$. Letting ε go to zero we see that $f(u_1, u_2, t) \geqslant m > 0$ on D which completes the proof.

4. Convex curves in the plane

We now turn our attention to convex curves in the plane. In this case we find that the curve shortening problem is equivalent to the initial value problem for a certain nonlinear parabolic differential equation. We obtain the a priori estimates needed to show long term existence of this equation, proving that convex curves shrink to points.

4.1. For convex curves we can use the angle θ of the tangent line as a parameter. We write the curvature $k = k(\theta)$ in terms of this parameter and determine which positive, 2π periodic functions arise as the curvature function of convex curves.

4.1.1. Lemma. *A positive 2π periodic function represents the curvature function of a simple closed strictly convex C^2 plane curve if and only if*

$$\int_0^{2\pi} \frac{\cos\theta}{k(\theta)} \, d\theta = \int_0^{2\pi} \frac{\sin\theta}{k(\theta)} \, d\theta = 0.$$

Proof. If k is the curvature function of some curve, then this relation follows directly from the fact that the curve is closed, i.e. that $\int_0^L T \, ds = 0$. In the other direction, given an arbitrary k, the associated curve, up to translation, is defined by

$$x(\theta) = \int_0^\theta \frac{\cos\tau}{k(\tau)} \, d\tau, \qquad y(\theta) = \int_0^\theta \frac{\sin\tau}{k(\tau)} \, d\tau.$$

It is easy to check that this curve is closed, has the proper curvature, and has a one-to-one Gauss map (which insures that it is simple).

4.1.2. To determine the evolution equation for curvature when using θ as a parameter we take $\tau = t$ as the time parameter when using θ as the other coordinate; thus we change variables from (u, t) to (θ, τ). The point is that $\partial/\partial t \neq \partial/\partial\tau$, because $\partial/\partial t$ is the partial derivative with u fixed, and $\partial/\partial\tau$ is the partial derivative with θ fixed. We obtain the following equation for k in terms of θ and τ.

4.1.3. Lemma.

$$\frac{\partial k}{\partial\tau} = k^2 \frac{\partial^2 k}{\partial\theta^2} + k^3.$$

Proof.   By the chain rule:

$$\frac{\partial k}{\partial t} = \frac{\partial k}{\partial \tau} + \frac{\partial k}{\partial \theta}\frac{\partial \theta}{\partial t} = \frac{\partial k}{\partial \tau} + \frac{\partial k}{\partial \theta}\frac{\partial k}{\partial s} = \frac{\partial k}{\partial \tau} + k\left(\frac{\partial k}{\partial \theta}\right)^2$$

and

$$\frac{\partial^2 k}{\partial s^2} = \frac{\partial \theta}{\partial s}\frac{\partial}{\partial \theta}\left(\frac{\partial \theta}{\partial s}\frac{\partial k}{\partial \theta}\right) = k\left(\frac{\partial k}{\partial \theta}\right)^2 + k^2\frac{\partial^2 k}{\partial \theta^2},$$

where $k\partial/\partial\theta = \partial/\partial s$ and $\partial\theta/\partial t = \partial k/\partial s$ (Lemma 3.1.5) have been used to simplify expressions. Substituting these expressions in formula (3.1.6) proves the lemma.

In the rest of the paper we will deal with this equation only and for simplicity we replace τ by t.

4.1.4. Theorem.   *The curve shortening process for convex curves is equivalent to this initial value PDE problem*:

Find k: $S^1 \times [0, T) \to \mathbf{R}$ satisfying
  (i) $k \in C^{2+\alpha, 1+\alpha}(S^1 \times [0, T - \varepsilon])$ *for all $\varepsilon > 0$.*
  (ii) $\partial k/\partial t = k^2\partial^2 k/\partial\theta^2 + k^3$.
  (iii) $k(\theta, 0) = \psi(\theta)$ *where ψ satisfies*:
      (a) $\psi \in C^{1+\alpha}(S^1)$;
      (b) $\psi(\theta) > 0$ *and*
      (c) $\int_0^{2\pi} (\cos\theta/\psi(\theta))\, d\theta = \int_0^{2\pi} (\sin\theta/\psi(\theta))\, d\theta = 0$.

Proof.   Lemma 4.1.3 shows that given a solution to the evolution equation for curves, the curvature function, expressed in θ coordinates, will satisfy 4.1.4(ii). Given a solution to 4.1.4 it is easy to check that the hypotheses of Lemma 4.1.1 are satisfied for each t and that the corresponding curves defined by the formulas in the proof of Lemma 4.1.1 satisfy the evolution equation $\partial X/\partial t = kN - (\partial k/\partial\theta)T$. The partial with respect to t is taken while keeping θ fixed. By changing the space variable one can transform away the tangential component, without changing the shape of the curves. Briefly, we require that the partial of X with respect to t, *holding u fixed*, solves $\partial X/\partial t = kN$ and writing θ as a function of t and u we find that $\partial\theta/\partial t = k(\theta(u, t), t) \cdot (\partial k/\partial\theta)(\theta(u, t), t)$ must be satisfied. Solving this with the initial conditions $\theta(u, 0) = 2\pi u$ yields a function satisfying $\theta(u + 1, t) = \theta(u, t)$ and $\partial\theta/\partial u \geqslant 0$, hence it can be inverted to write u as a function of t and θ.

4.1.5. Remark.   Notice that equation 4.1.4(ii) is a straightforward parabolic equation for k in terms of θ and t. It is possible to prove short term existence and uniqueness for this equation from the standard results on parabolic equations and work backwards using Lemma 4.1.1 to find the motion of the

curve. This would avoid the use of the Nash-Moser machinery in the special case of a convex curve.

4.2. It is important to show that curves which are strictly convex remain so. This follows from the evolution equation 4.1.4(ii) via the following

Lemma. *If k satisfies 4.1.4, then $k_{\mathrm{MIN}}(t) = \inf\{k(\theta, t)|0 \leqslant \theta \leqslant 2\pi\}$ is a nondecreasing function.*

Proof. The proof is by contradiction: Let ε satisfy $k_{\mathrm{MIN}}(0) > \varepsilon > 0$ and suppose that $k_{\mathrm{MIN}}(\mathrm{t}) = k_{\mathrm{MIN}}(0) - \varepsilon$ for some t. Let $t_0 = \inf\{t|k_{\mathrm{MIN}}(t) = k_{\mathrm{MIN}}(0) - \varepsilon\}$. Continuity of k assures that this minimum is achieved at some point (θ_0, t_0). At this point, however,

$$\frac{\partial k}{\partial t}(\theta_0, t_0) \leqslant 0, \quad \frac{\partial^2 k}{\partial \theta^2}(\theta_0, t_0) \geqslant 0 \quad \text{and} \quad k(\theta_0, t_0) > 0.$$

This contradicts the hypothesis that k satisfies 4.1.4(ii).

4.3. We turn now to the estimates for convex curves. We introduce the median curvature k^* for a curve, defined as

$$k^* = \sup\{b|k(\theta) > b \text{ on some interval of length } \pi\}.$$

We will obtain the following estimates for families of curves satisfying the evolution equation.

Geometric estimate (4.3.2). If $k(\theta, t)$ is the curvature of a convex closed plane curve which encloses an area A and has length L, then $k^*(t) < L/A$.

Integral estimate (4.3.4). If $k^*(t)$ is bounded on $[0, T)$, then $\int_0^{2\pi} \log k(\theta, t)\, d\theta$ is bounded on $[0, T)$.

Pointwise estimate (4.3.6). If $\int_0^{2\pi} \log k(\theta, t)\, d\theta$ is bounded on $[0, T)$, then $k(\theta, t)$ is uniformly bounded on $S^1 \times [0, T)$.

Combining these estimates we prove the main theorem of this section.

4.3.1. Theorem. *If $k: S^1 \times [0, T) \to \mathbf{R}$ satisfies 4.1.4 and the area enclosed by the associated curves is bounded away from 0, then the curvature k is uniformly bounded on $S^1 \times [0, T)$.*

Proof. The length of the curves decreases during the evolution so a uniform lower bound on area yields a uniform upper bound on $k^*(t)$ and, as a consequence of the integral and pointwise estimates, a uniform upper bound on $k(\theta, t)$.

4.3.2. *Proof of geometric estimate.* If $M < k^*(t)$, then $k(\theta, t) > M$ on some interval $(a, a + \pi)$. This implies that the convex curve lies between parallel lines whose distance is given by

$$\int_a^{a+\pi} \frac{\sin(\theta - a)}{k(\theta, t)}\, d\theta \leqslant \frac{2}{M}$$

(see Figure 2). The diameter is bounded by $L/2$ and the area is bounded by the width times the diameter. Since M can be chosen arbitrarily close to $k^*(t)$ we have $k^*(t) \leqslant L/A$ as desired.
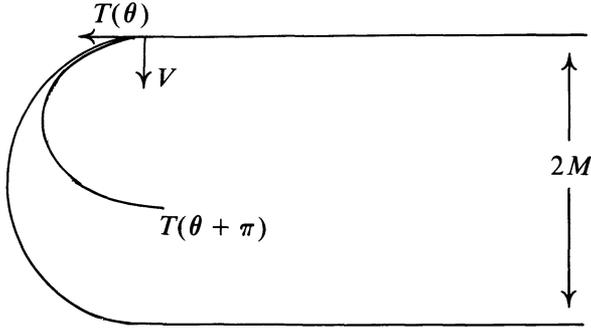

FIGURE 2.

4.3.3. (*Wirtinger's inequality* [7]).   If $f(a) = 0$ and $f(b) = 0$ with $b - a \leqslant \pi$, then

$$\int_a^b f^2 \, d\theta \leqslant \int_a^b \left(\frac{df}{d\theta} \right)^2 d\theta.$$

Proof. By translation we may take $a = 0$. We may also take $b = \pi$ by extending f to be zero for $b \leqslant \theta \leqslant \pi$. Then the result follows by expanding f in a Fourier series $f = \sum a_n \sin n\theta$. Equality holds only for $f(\theta) = \sin \theta$.

4.3.4. *Proof of integral estimate.*   Using the evolution equation 4.1.4(ii) and integration by parts we calculate

$$\frac{\partial}{\partial t} \int_0^{2\pi} \log k(\theta, t) \, d\theta = \int_0^L k^2 - \left(\frac{\partial k}{\partial \theta} \right)^2 d\theta.$$

Fix t and estimate the right-hand integral over the open set $U = \{ \theta | k(\theta, t) > k^*(t) \}$ and its complement $V = S^1 - U$. The definition of k^* implies that the open set U is the countable union of disjoint intervals I; each of length less than or equal to π. At the endpoints of the closures of these intervals, $k(\theta, t)$ equals $k^*(t)$ so that Wirtinger's inequality can be applied to the function $k(\theta, t) - k^*(t)$. Simplifying this inequality yields:

$$\int_{\bar{I}_i} k^2 - \left(\frac{\partial k}{\partial \theta} \right)^2 d\theta \leqslant 2k^*(t) \int_{\bar{I}_i} k(\theta, t) \, d\theta.$$

Summing this over the intervals comprising U we obtain

$$\int_U k^2 - \left(\frac{\partial k}{\partial \theta} \right)^2 d\theta \leqslant 2k^*(t) \int_U k(\theta, t) \, d\theta \leqslant 2k^*(t) \int_0^{2\pi} k(\theta, t) \, d\theta.$$

On the set V the estimate is simpler:

$$\int_V k^2 - \left(\frac{\partial k}{\partial \theta} \right)^2 d\theta \leqslant \int_V k^2 \, d\theta \leqslant 2\pi \left(k^*(t) \right)^2.$$

Adding these equations and recalling (Lemma 3.1.2) that $\partial L / \partial t = -\int k^2 \, ds = -\int k \, d\theta$ yields

$$\frac{\partial}{\partial t} \int_0^{2\pi} \log k(\theta, t) \, d\theta \leqslant 2k^*(t) \frac{\partial L}{\partial t} + 2\pi \left(k^*(t) \right)^2.$$

Finally assume that $k^*(t) < SM$ and integrate to obtain the desired estimate:

$$\int_0^{2\pi} \log k(\theta, t) \, d\theta \leqslant \int_0^{2\pi} \log k(\theta, 0) \, d\theta + 2M(L(0) - L(t)) + 2\pi M^2 t.$$

Lemma. *If $\int \log k(\theta, t) \, d\theta$ is bounded on $[0, T)$, then for any $\delta > 0$ we can find a constant C such that $k(\theta, t) \leqslant C$ except on intervals of length less than or equal to δ.*

Proof. If $k \geqslant C$ on $a \leqslant \theta \leqslant b$ and $b - a \geqslant \delta$, then

$$\int_0^{2\pi} \log k(\theta, t) \, d\theta \geqslant \delta \log C + (2\pi - \delta) \log k_{\text{MIN}}(0),$$

where $k_{\text{MIN}}(0)$ is a lower bound for k. (Recall that $k_{\text{MIN}}(t)$ does not decrease with time.) This gives a contradiction when C is large.

4.3.5. Lemma. *We can find a constant D such that*

$$\int_0^{2\pi} \left(\frac{\partial k}{\partial \theta} \right)^2 d\theta \leqslant \int_0^{2\pi} k^2 \, d\theta + D$$

holds for $0 \leqslant t < T$.

Proof. We have

$$\frac{\partial}{\partial t} \int k^2 - \left(\frac{\partial k}{\partial \theta} \right)^2 d\theta = 2 \int \left(k \frac{\partial k}{\partial t} - \frac{\partial k}{\partial \theta} \frac{\partial^2 k}{\partial \theta \partial t} \right) d\theta$$

$$= 2 \int \left(\frac{\partial^2 k}{\partial \theta^2} + k \right) \frac{\partial k}{\partial t} = 2 \int k^2 \left(\frac{\partial^2 k}{\partial \theta^2} + k \right)^2 \geqslant 0.$$

Integrating this inequality completes the proof.

4.3.6. Proof of the pointwise estimate. Since $\int_0^{2\pi} \log k(\theta, t) \, d\theta$ is bounded we have $k \leqslant C$ except on intervals $[a, b]$ of length less than δ. On such an interval

$$k(\varphi) = k(a) + \int_a^{\varphi} \frac{\partial k}{\partial \theta} \, d\theta \leqslant C + \sqrt{\delta} \left(\int \left(\frac{\partial k}{\partial \theta} \right)^2 d\theta \right)^{1/2}$$

$$\leqslant C + \sqrt{\delta} \left(\int k^2 \, d\theta + D \right)^{1/2}.$$

This shows that if k_{MAX} is the maximum value of k, then

$$k_{\text{MAX}} \leqslant C + \sqrt{\delta} \left(2\pi k_{\text{MAX}}^2 + D\right)^{1/2} \leqslant C + 2\pi\sqrt{\delta}\, k_{\text{MAX}} + \sqrt{\delta}\, D.$$

Choosing δ small we have $k_{\text{MAX}} \leqslant 2C$.

4.4. Using the assumption that k is bounded, we find bounds for the higher derivatives of k.

4.4.1. Lemma. *If k is bounded, then $\partial k/\partial\theta$ is bounded.*

Proof. We calculate

$$\frac{\partial^2 k}{\partial t\, \partial\theta} = k^2 \frac{\partial^3 k}{\partial\theta^3} + 2k \frac{\partial k}{\partial\theta} \frac{\partial^2 k}{\partial\theta^2} + 3k^2 \frac{\partial k}{\partial\theta}.$$

This implies that $\partial k/\partial\theta$ grows at most exponentially as can be seen by considering the PDE satisfied by $e^{\alpha t}\partial k/\partial\theta$ and choosing α so that the maximum principle can be used. On a finite time interval $\partial k/\partial\theta$ remains bounded.

To simplify notation we use k' to denote the partial derivative with respect to θ in what follows.

4.4.2. Lemma. *If k and k' are bounded, then $\int_0^{2\pi} (k'')^4$ is bounded.*

Proof. We calculate using the evolution equation:

$$\frac{\partial}{\partial t} \int_0^{2\pi} (k'')^4 = 4 \int_0^{2\pi} (k'')^3 (k^2 k'' + k^3)''$$

$$= -12 \int_0^{2\pi} (k'')^2 (k''')(k^2 k''' + 2kk'k'' + 3k^2 k')$$

$$= -12 \int_0^{2\pi} k^2 (k'')^2 (k''')^2 + 2kk'(k'')^3 (k''') + 3k^2 k'(k'')^2 k'''.$$

We use the inequality $ab \leqslant 4a^2/\varepsilon + \varepsilon b^2$ to bound the second and third terms by the first term and some additional penalty terms to obtain

$$\frac{\partial}{\partial t} \int_0^{2\pi} (k'')^4 \leqslant \int_0^{2\pi} C_1 k'^2 (k'')^4 + C_2 k^2 k'^2 (k'')^2.$$

Using the uniform bounds on k and k' and the fact that

$$\int_0^{2\pi} (k'')^2 \leqslant \left(\int_0^{2\pi} (k'')^4\right)^{1/2} \sqrt{2\pi}$$

we see that $\int_0^{2\pi}(k'')^4$ grows at most exponentially and therefore remains finite on a finite time interval.

4.4.3. Lemma. *If k, k' and $\int (k'')^4$ are bounded, so is $\int (k''')^2$.*

Proof. We compute

$$\frac{\partial}{\partial t} \int (k''')^2 = 2\int (k''')(k^2k'' + k^3)''' = 2\int k''''(k^2k'' + k^3)''$$

$$= -2\int k^2(k'''')^2 + 4kk'k'''k'''' + 2k(k'')^2k''''$$

$$+ 2(k')^2k''k'''' + 3k^2k''k'''' = 6(k')^2k''''.$$

We use the same trick as in the previous lemma to bound the last five terms by the first term and some additional penalty terms. This yields

$$\frac{\partial}{\partial t} \int (k''')^2 \leqslant C_1 \int k'^2(k''')^2 + C_2 \int (k'')^4 + C_3 \int \frac{(k')^4}{k^2}(k'')^2$$

$$+ C_4 \int k^2(k'')^2 + C_5 \int (k')^4,$$

where each term except the first is bounded by a constant. (Recall again that $k \geqslant k_{\text{MIN}}(t) \geqslant k_{\text{MIN}}(0)$.)

The first term is bounded by a constant times $\int (k''')^2$ which proves that the growth rate is at worst exponential and that the quantity is finite on finite intervals.

4.4.4. Corollary. *Under the same hypothesis k'' is bounded.*

Proof. In one dimension

$$\max|f|^2 \leqslant C\int |f'|^2 + f^2$$

and we apply this to k''.

4.4.5. Lemma. *If k, k' and k'' are uniformly bounded, then so are k''' and all the higher derivatives.*

Proof. This follows from the maximum principle. We compute

$$\frac{\partial}{\partial t}k''' = (k^2k'' + k^3)'''$$

$$= k^2k^{(v)} + 6kk'k^{(iv)} + (8kk'' + 6k'^2 + 3k^2)k'''$$

$$+ (6k'(k'')^2 + 18kk'k'' + 6(k')^3)$$

If k, k' and k'' are bounded then the maximum principle can be applied to $k'''e^{\alpha t}$ for suitable α. On a finite interval, this implies that $|k'''|$ is bounded. In general if $k, k', \cdots, k^{(n-1)}$ are bounded, then

$$\frac{\partial}{\partial t}k^{(n)} \leqslant k^2k^{(n+2)} + 2nkk'k^{(n+1)} + Ck^{(n)} + C$$

shows that $k^{(n)}$ is bounded on finite intervals.

4.5. Theorem. *The solution to* 4.1.4 *continues until the area goes to zero.*

Proof. As long as the area is bounded away from zero, we get bounds on k and all of its derivatives. Using the evolution equation we can bound the time derivatives also. Suppose the solution exists on the interval $[0, T)$ and the area does not go to zero ($\lim_{t \to T} A(t) > 0$). Then k has a limit as t goes to T which is C^∞ and we can extend the solution past T. The solution k to 4.1.4 can be converted to a solution of the heat equation for curves using Lemma 4.1.1.

5.

Once we know that the area goes to zero, the fact that the curve becomes circular follows from two articles by M. Gage. The first [1] shows that the isoperimetric ratio L^2/A decreases, so that if $A \to 0$, then $L \to 0$ and the curve shrinks to a point. The second [2] shows that the isoperimetric ratio approaches its optimum value of 4π and as a consequence the ratio r_{out}/r_{in} of the circumscribed radius to the inscribed radius goes to 1.

In the remainder of this paper we use similar geometric techniques to show that the ratio of the maximum curvature to the minimum curvature of the curve goes to 1. This can be considered "C^2" convergence to the circle as opposed to the "C^0" convergence obtained in [2]. Finally, we use a priori estimates similar to 4.4.2 and 4.4.3 to show that the higher derivatives of k converge to 0 and therefore the curves converge to circles in the "C^∞" sense as well.

5.1. The first lemma refines the geometric estimate. Let

$$k_w^* = \sup\{ b | k(\theta) > b \text{ on some interval of length } w \}.$$

Lemma.

$$k_w^*(t) r_{in}(t) < \frac{1}{1 - K(w)(r_{out}/r_{in} - 1)},$$

where r_{in} and r_{out} are respectively the radii of the largest inscribed circle and the smallest circumscribed circle of the curve defined by $k(\cdot, t)$. K is a positive decreasing function of w with $K(0) = \infty$ and $K(\pi) = 0$.

Proof of Lemma. Let $M < k_w^*(t)$. The set of $\{ \theta | k(\theta, t) > M \}$ contains an interval of length at least w, which by changing the parametrization we can assume to contain the interval $(-w/2, w/2)$. If we construct the circular arc with the curvature M, and angle w tangent to the curve at $\theta = 0$ we see that the convex curve must lie in the region bounded by the arc and the rays tangent to the ends of the arc (see Figure 3).
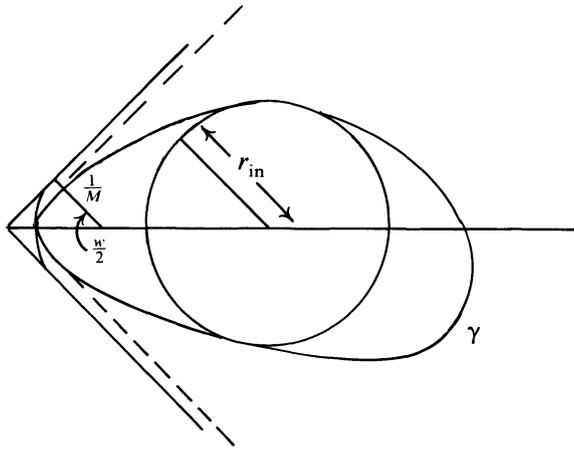
FIGURE 3

The convexity assumption insures that γ lies within the dotted lines, while the estimate $k(\theta) > M$ on $(-\frac{w}{2}, \frac{w}{2})$ insures that the dotted lines lie within the 'cone' formed by the circular arc and solid straight lines.

Since the inscribed circle lies within the cone and the circumscribed circle must encircle every point on the curve we see that for a given r_{in} the smallest r_{out} is obtained for the configuration shown in Figure 4.
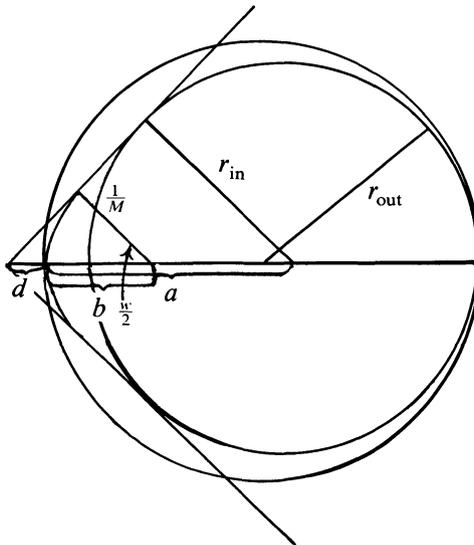

FIGURE 4

From Figure 4 and trigonometry, we determine that $|b| = 1/M$ and that

(5.1.1)
$$\cos\left(\frac{w}{2}\right) = \frac{1/M}{1/M + |d|} = \frac{r_{\text{in}}}{|a| + |d|},$$

(5.1.2)
$$2r_{\text{out}} \geq r_{\text{in}} + |a|.$$

From (5.1.2) we have

(5.1.3)
$$\left(\frac{r_{\text{out}}}{r_{\text{in}}} - 1\right) \geq -\frac{1}{2} + \frac{|a|}{2r_{\text{in}}}$$

and solving (5.1.1) for $|a|$ yields:

(5.1.4)
$$|a| = \frac{r_{\text{in}}}{\cos(w/2)} - \frac{1}{M}\left(\frac{1}{\cos(w/2)} - 1\right).$$

Substituting (5.1.4) in (5.1.3) and rearranging yields

$$Mr_{\text{in}} \leq \frac{1}{1 - K(w)(r_{\text{out}}/r_{\text{in}} - 1)},$$

where

$$K(w) = \left(\frac{1}{2\cos(w/2)} - \frac{1}{2}\right)^{-1} = \frac{2\cos(w/2)}{1 - \cos(w/2)}.$$

Since M can be chosen arbitrarily close to $k_w^*(t)$ this proves the lemma.

5.2. Corollary.

$$k_{\text{MAX}}(t)r_{\text{in}} \leq \left(\frac{1}{1-\varepsilon}\right)\frac{1}{1 - C_2(\varepsilon)(r_{\text{out}}/r_{\text{in}} - 1)},$$

where ε is any small positive number.

Proof. From the proof of the pointwise estimate we see that for any ε, if $w/2 < \delta$ then $k(\theta, t) > (1 - \varepsilon)k_{\text{MAX}}(t)$ for all $\theta \in (\theta_0 - w/2, \ \theta_0 + w/2)$ (where $k_{\text{MAX}}(t) = k(\theta_0, t)$). Hence $k_w^*(t) \geq k_{\text{MAX}}(t)(1 - \varepsilon)$ for all t. The choice of δ depends only on the initial curve.

5.3. Proposition. *For any positive ε*

$$k_{\text{MAX}}(t)r_{\text{in}}(t) \leq \left(\frac{1}{1-\varepsilon}\right)^2$$

for all t sufficiently close to T.

Proof. From the Bonnesen inequality [8] one has the estimate

$$\frac{L^2}{A} - 4\pi \geq \frac{\pi^2}{A}(r_{\text{out}} - r_{\text{in}})^2 \geq \left(1 - \frac{r_{\text{in}}}{r_{\text{out}}}\right)^2.$$

[2] contains the proof that L^2/A converges to 4π during the curve shortening process. Hence $r_{\text{out}}/r_{\text{in}}$ converges to 1. Combining this fact with the previous estimate proves the result.

5.4. Theorem. $k(\theta, t)r_{\text{in}}(t)$ *converges uniformly to* 1.

Proof. Using the techniques used to prove the pointwise estimate it is easily shown that the family $k(\theta, t)r_{\text{in}}(t)$ is equicontinuous. That a subsequence converges uniformly to a function $f(\theta)$ and $f(\theta) \leq 1$ follows from the estimate above. Now $(k(\theta, t_i)r_{\text{in}}(t_i))^{-1}$ converges pointwise to $f(\theta)^{-1}$ in the extended reals and from Fatou's lemma we have

$$\int \frac{1}{f(\theta)} \, d\theta \leq \liminf \int \frac{d\theta}{k(\theta, t_i)r_{\text{in}}(t_i)} = \liminf \frac{L(t_i)}{r_{\text{in}}(t_i)} = 2\pi.$$

On the other hand, $2\pi \leq \int d\theta / f(\theta)$; hence $f(\theta) \equiv 1$.

Since every convergent subsequence converges uniformly to 1, $k(\theta, t)r_{\text{in}}(t)$ converges uniformly to 1.

5.5. Corollary. $k_{\text{MIN}}(t)/k_{\text{MAX}}(t)$ *converges to* 1.

5.6. Corollary. $k(\theta, t)\sqrt{2T - 2t}$ *converges uniformly to* 1. (*At T the enclosed area is* 0.)

Proof. From the Bonnesen inequality and 3.1.7 we have

$$\frac{L^2}{A} - 4\pi \geq \frac{(L - 2\pi r_{\text{in}})^2}{A} = \left(\frac{L}{\sqrt{A}} - \frac{2\pi r_{\text{in}}}{\sqrt{2\pi(T - t)}} \right)^2.$$

From $L/\sqrt{A} \to 2\sqrt{\pi}$ it follows that $r_{\text{in}}/\sqrt{T - t}$ converges to $\sqrt{2}$, which, together with Theorem 5.4, proves the corollary.

5.7. To determine the rate of convergence of the derivatives of k it turns out to be most productive to consider the evolution of the "normalized" curvature κ where the normalization is chosen so that the related convex curve encloses an area π. κ is defined by

$$\kappa(\theta, t) = k(\theta, t)\sqrt{2T - 2t}.$$

It is also convenient to change the time parameter to $\tau = \frac{1}{2}\log((T - t)/T)$. The evolution equation for κ is

$$\frac{\partial}{\partial \tau}\kappa(\theta, \tau) = \frac{\partial \kappa}{\partial t} \cdot \frac{\partial t}{\partial \tau} = 2(T - t) \cdot \left(-\frac{1}{\sqrt{2T - 2t}}k(\theta, t) + \sqrt{2T - 2t}\frac{\partial k}{\partial t} \right)$$

$$= 2(T - t)\left[-\frac{1}{(2T - 2t)}\kappa + \sqrt{2T - 2t}\left(k^2\frac{\partial^2 k}{\partial \theta^2} + k^3 \right) \right],$$

$$= 2(T - t)\left[-\frac{1}{(2T - 2t)}\kappa + \frac{1}{2T - 2t}\left(\kappa^2\frac{\partial^2 \kappa}{\partial \theta^2} + \kappa^3 \right) \right],$$

$$\frac{\partial \kappa}{\partial \tau} = \left\{ \kappa^2\frac{\partial^2 \kappa}{\partial \theta^2} + \kappa^3 - \kappa \right\}. \tag{5.7.0}$$

We also have that κ converges to 1 uniformly as τ goes to infinity.

The aim of this section is to prove the

5.7.1. Theorem. $\|\partial^l \kappa / \partial \theta^l\|_\infty \leqslant C(l) e^{-2\alpha\tau}$ for $l \geqslant 1, 0 < \alpha < 1$.

5.7.2. Corollary. $\|\partial^l k / \partial \theta^l\|_\infty \leqslant \bar{C}(l)(T - t)^{\alpha - 1/2}$ for all $l \geqslant 1$ and $0 < \alpha < 1$. This implies that the derivatives of the original, nonnormalized curvature converge to 0 uniformly.

Proof. One makes the appropriate substitutions in 5.7.1.

5.7.3. Corollary. $k_{\text{MAX}}(t) - k_{\text{MIN}}(t) \to 0$ as $t \to T$.

To simplify notation we use $f', f'', \cdots$ to indicate partial differentiation by θ. We also use $\|f^{(l)}\|_p \equiv [\int (\partial^l f / \partial \theta^l)^p]^{1/p}$. The following facts are used repeatedly.

Peter-Paul Inequality. (*Pay Peter for the privilege of robbing Paul* [9].) *For all positive* ε, $ab \leqslant \varepsilon a^2 + b^2/4\varepsilon$.

Wirtinger inequality. *If* $\int_0^{2\pi} f = 0$, *then* $\int_0^{2\pi} f^2 \leqslant \int_0^{2\pi} (f')^2$.

Sobolev inequality. *If* $\|f\|_2 \leqslant C$ *and* $\|f'\|_2 \leqslant C$, *then* $\|f\|_\infty \leqslant (1/\sqrt{2\pi} + \sqrt{2\pi})C$, *where* $\| \ \|_2$ *is the* L_2 *norm and* $\| \ \|_\infty$ *is the sup norm for functions on* S^1.

These last two inequalities imply that an exponential bound on the L_2 norm of the lth derivative guarantees an exponential bound, with the same exponential constant, for the L_∞ norms of all the lower derivatives.

We also make use of the following inequalities.

5.7.4. Lemma. *Let* $f: \mathbf{R}^+ \to \mathbf{R}^+$ *satisfy* $df/d\tau \leqslant cf^{1-1/p} - 2pf$. *Then* $f(\tau)^{1/p} \leqslant (c/2p + De^{-2\tau}) \leqslant \tilde{c}(p)$.

Proof. We calculate

$$\frac{d}{d\tau}\left(e^{2\tau} f^{1/p}\right) = 2e^{2\tau} f^{1/p} + \frac{1}{p} f^{1/p-1} \frac{\partial f}{\partial \tau} e^{4\tau}$$

$$\leqslant 2e^{2\tau} f^{1/p} + \left(\frac{c}{p} - 2f^{2/p}\right)e^{4\tau} \leqslant \frac{c}{p} e^{2\tau}.$$

Hence $e^{2\tau} f^{1/p} \leqslant D + ce^{2\tau}/2p$ and the result follows.

Similarly we have

5.7.5. Lemma. *If* $df/d\tau \leqslant -\alpha f + Ce^{-\beta\tau}$, *then*

$$f(\tau) \leqslant De^{-\alpha\tau} + \frac{C}{\alpha - \beta} e^{-\beta\tau} \quad \text{if } \alpha \neq \beta, \text{ or}$$

$$f(\tau) \leqslant De^{-\alpha\tau} + C\tau e^{-\alpha\tau} \quad \text{if } \alpha = \beta.$$

5.7.6. Lemma. $\|\kappa'\|_2$ *and* $\|\kappa'\|_4$ *are bounded by constants independent of* τ.

Proof. We calculate using equation (5.7.0) and integration by parts:

$$\frac{\partial}{\partial\tau} \int (\kappa')^4 = 4\int (\kappa')^3 (\kappa^2 \kappa'' + \kappa^3 - \kappa)'$$

$$= \int -4(\kappa')^4 - 12\kappa^2(\kappa')^2(\kappa'')^2 - 12\kappa^3(\kappa')^2\kappa''.$$

The last term is bounded using the Peter-Paul inequality by $12k^2(\kappa')^2(\kappa'')^2 + 3(\kappa')^2\kappa^4$. Observing that κ converges to 1 we use the Hölder inequality to obtain

$$\frac{\partial f}{\partial \tau} \leqslant -4f + Cf^{1/2},$$

where $f(\tau) = \int (\kappa')^4$. The conclusion for $\|\kappa'\|_4$ follows from Lemma 5.7.4 with $p = 2$. The boundedness of $\|\kappa'\|_2$ follows immediately from the Hölder inequality.

5.7.7. Lemma. $\|\kappa''\|_2$ *is bounded by a constant which does not depend on* τ.

Proof. We calculate as before using integration by parts once.

$$\frac{\partial}{\partial \tau} \int (\kappa'')^2 = \int -2(\kappa'')^2 - 2\kappa(\kappa''')^2 - 4\kappa\kappa'\kappa''\kappa''' - 6\kappa^2\kappa'\kappa'''.$$

We use the Peter-Paul inequality to bound the last two terms and obtain

(5.7.7a) $\quad \dfrac{\partial}{\partial \tau} \displaystyle\int (\kappa'')^2 \leqslant c_1 \int (\kappa')^2(\kappa'')^2 + c_2 \int \kappa^2(\kappa')^2 - 2\int (\kappa'')^2.$

We rearrange terms in equation (5.7.6) to obtain a bound for $(\kappa')^2(\kappa'')^2$:

(5.7.7b) $\quad \displaystyle\int 12\kappa^2(\kappa')^2(\kappa'')^2 \leqslant -\frac{\partial}{\partial \tau} \int (\kappa')^4 + \int -4(\kappa')^4 - 12\kappa^3(\kappa')^2(\kappa'').$

Let M be a lower bound for κ, and use the Peter-Paul inequality to bound the last term:

(5.7.7c) $\quad 12M^2 \displaystyle\int (\kappa')^2(\kappa'')^2 \leqslant -\frac{\partial}{\partial \tau} \int (\kappa')^4 + 12\int \varepsilon(\kappa'')^2 + \frac{1}{4\varepsilon}(\kappa')^4\kappa^6.$

We let $f = \int(\kappa'')^2$ and using (5.7.7a) and (5.7.7c) with ε small enough we obtain

$$\frac{\partial f}{\partial \tau} \leqslant -c_3\frac{\partial}{\partial \tau} \int (\kappa')^4 + c_4 - f,$$

where we have used the bounds in (5.7.6). Multiplying this by e^τ and integrating over τ yields

$$\int_0^A \frac{\partial}{\partial \tau}(e^\tau f)\, d\tau \leqslant -c_3\int_0^A e^\tau \frac{\partial}{\partial \tau}\left(\int_0^{2\pi} (\kappa')^4\right) d\tau + c_4 e^A$$

(5.7.7f) $\qquad\qquad \leqslant c_3\displaystyle\int_0^A e^\tau\left(\int_0^{2\pi} (\kappa')^4\right) d\tau - \left\{c_3 e^\tau \int_0^{2\pi} (\kappa')^4\right\}\Big|_0^A + c_4 e^A.$

Using the bounds in (5.7.6) again we obtain

(5.7.7g) $\qquad\qquad e^A f(A) \leqslant c_5 e^A + c_6 + c_4 e^A$

from which the conclusion follows immediately.

5.7.8. Lemma. $\|\kappa'\|_\infty$ *converges to zero as* $\tau \to \infty$.

Proof. From the Sobolev inequality and Lemmas 5.7.6 and 5.7.7 we conclude that $\|\kappa'\|_\infty < \infty$. The bound on $\|\kappa''\|_2$ implies that κ' is equicontinuous, hence a subsequence converges uniformly to $g(\theta)$. The anti-derivative of κ must converge to the anti-derivative of g. But κ converges uniformly to 1, hence $g \equiv 0$. Since every subsequence converges to zero, $\kappa'(\theta, \tau)$ converges uniformly to zero.

5.7.9. Now we wish to obtain good exponential decay bounds on the low order derivatives.

Lemma. *For any* $0 < \alpha < 1$ *we can choose* A *so that for* $\tau > A$

$$\int (\kappa'')^2 \geq 4\alpha \int (\kappa')^2.$$

Proof. We observe that since $\int \cos\theta/\kappa = \int \sin\theta/\kappa = 0$ we have

$$\int \frac{\kappa'}{\kappa^2}\cos\theta = \int \frac{\kappa'}{\kappa^2}\sin\theta = \int \frac{\kappa'}{\kappa^2} = 0.$$

Since κ'/κ^2 is perpendicular to the first eigenvectors, the Wirtinger inequality becomes

$$\int \left[\left(\frac{\kappa'}{\kappa^2}\right)'\right]^2 \geq 4\int \left(\frac{\kappa'}{\kappa^2}\right)^2,$$

the left-hand side of which expands to

$$\int \left(\frac{\kappa''}{\kappa^2} - 2\frac{(\kappa')^2}{\kappa^3}\right)^2 = \int \frac{(\kappa'')^2}{\kappa^4} - 4\frac{(\kappa')^2\kappa''}{\kappa^5} + 4\frac{(\kappa')^4}{\kappa^6}$$

$$\leq \int \left(\frac{\kappa''}{\kappa^2}\right)^2 + 4\frac{(\kappa')^4}{\kappa^6} + 4\varepsilon\frac{(\kappa'')^2}{\kappa^4} + \frac{1}{\varepsilon}\frac{(\kappa')^4}{\kappa^6}$$

$$\leq (1 + 4\varepsilon)\int \left(\frac{\kappa''}{\kappa^2}\right)^2 + o(\tau) \cdot \int \left(\frac{\kappa'}{\kappa^2}\right)^2$$

since $\|\kappa'\|_\infty \to 0$. For A sufficiently large we can assume that $\kappa \sim 1$; hence for any α, $0 < \alpha < 1$, we have $4\alpha\int(\kappa')^2 \leq \int(\kappa'')^2$ as desired.

5.7.10. Lemma. *For any* α, $0 < \alpha < 1$, *there is a constant* C *such that* $\|\kappa'\|_2 \leq Ce^{-2\alpha\tau}$.

Proof. We reconsider

$$\frac{\partial}{\partial\tau}\int (\kappa')^2 = 2\int \kappa'(\kappa^2\kappa'' + \kappa^3 - \kappa)' = \int -2\kappa^2(\kappa'')^2 + 6\kappa^2(\kappa')^2 - 2(\kappa')^2.$$

For any α, $0 < \alpha < 1$, we can choose A such that for $\tau > A$

$$\frac{\partial}{\partial \tau} \int (\kappa')^2 \leqslant -8\alpha \int (\kappa')^2 + 6\alpha \int (\kappa')^2 - 2\int (\kappa')^2$$

or

$$\frac{\partial}{\partial \tau} \int (\kappa')^2 \leqslant -4\alpha \int (\kappa')^2.$$

Hence $\|\kappa'\|_2^2 \leqslant Ce^{-4\alpha\tau}$ and the lemma follows immediately.

5.7.11. Lemma. *For any α, $0 < \alpha < 1$, we can find a constant C such that $\|\kappa''\|_2 \leqslant Ce^{-2\alpha\tau}$.*

Proof.

$$\frac{\partial}{\partial \tau} \int (\kappa'')^2 = \int -2(\kappa'')^2 - 2\kappa^2(\kappa''')^2 - 4\kappa\kappa'\kappa''\kappa''' - 6\kappa^2\kappa'\kappa'''$$

$$\leqslant \int -2(\kappa'')^2 - (2)\kappa^2(\kappa''')^2 + 4\varepsilon(\kappa\kappa''')^2 + \frac{1}{\varepsilon}(\kappa')^2(\kappa'')^2$$

$$+ 6\varepsilon\kappa^2(\kappa''')^2 + \frac{3}{2\varepsilon}\kappa^2(\kappa')^2.$$

We choose ε small, and then choose A large enough so that for $\tau > A$, $\|\kappa'\|_\infty$ is very small:

$$\frac{\partial}{\partial \tau} \int (\kappa'')^2 \leqslant \int -2\bar{\alpha}(\kappa'')^2 - 2\bar{\alpha}\kappa^2(\kappa''')^2 + Ce^{-4\bar{\alpha}\tau}.$$

Using the standard Wirtinger inequality and that κ converges uniformly to 1 we obtain

$$\frac{\partial}{\partial \tau} \int (\kappa'')^2 \leqslant -4\bar{\alpha} \int (\kappa'')^2 + Ce^{-4\bar{\alpha}\tau}.$$

From Lemma 5.7.5 we have

$$\frac{\partial}{\partial \tau} \int (\kappa'')^2 \leqslant \tilde{C}e^{-4\alpha\tau}$$

for any $\alpha < \bar{\alpha}$. This proves the lemma.

5.7.12. Corollary. *For any α, $0 < \alpha < 1$, there exists a constant C such that $\|\kappa'\|_\infty \leqslant ce^{-2\alpha\tau}$.*

To bound the third derivatives we will need the following

5.7.13. Lemma. *For any α, $0 < \alpha < 1$, we can find a constant C such that $\|\kappa''\|_4 \leqslant Ce^{-\alpha\tau}$.*

Proof.   Using the standard tricks we calculate

$$\frac{\partial}{\partial \tau} \int (\kappa'')^4 = 4\int (\kappa'')^3(\kappa^2\kappa'' + \kappa^3 - \kappa)''$$

$$= \int -12(\kappa'')^2\kappa'''(\kappa^2\kappa''' + 2\kappa\kappa'\kappa'' + 3\kappa^2\kappa') - 4(\kappa'')^4$$

$$= \int -12\kappa^2(\kappa'')^2(\kappa''')^2 - 24\kappa\kappa'(\kappa'')^3\kappa''' - 36\kappa^2\kappa'(\kappa'')^2\kappa''' - 4(\kappa'')^4.$$

The standard trick yields

$$\frac{\partial}{\partial \tau}\int (\kappa'')^4 \leqslant \int -12\kappa^2(\kappa'')^2(\kappa''')^2 + 24\varepsilon\kappa^2(\kappa''')^2 + \frac{24}{4\varepsilon}(\kappa')^2(\kappa'')^4$$

$$+ 36\varepsilon\kappa^2(\kappa''\kappa''')^2 + \frac{36}{4\varepsilon}\kappa^2(\kappa')^2(\kappa'')^2 - 4(\kappa'')^4.$$

We choose ε small and use the previous bounds on $\|\kappa'\|_\infty$ and $\|\kappa'\|_2$ to obtain, for large τ,

$$\frac{\partial}{\partial \tau}\int (\kappa'')^4 \leqslant -4\bar{\alpha}\int (\kappa'')^4 + Ce^{-8\bar{\alpha}\tau}.$$

The lemma follows from Lemma 5.7.5.

5.7.14. Lemma.   *For any α, $0 < \alpha < 1$, there is some constant C such that $\|\kappa'''\|_2 \leqslant Ce^{-2\alpha\tau}$.*

Proof.

$$\frac{\partial}{\partial \tau}\int (\kappa''')^2 = -2\int \left\{ k''''\left(\kappa^2\kappa'''' + 2\kappa\kappa'\kappa''' + 2(\kappa')^2\kappa'' + 2\kappa(\kappa'')^2 \right.\right.$$

$$\left.\left. + 2\kappa\kappa'\kappa''' + 6\kappa(\kappa')^2 + 3\kappa^2\kappa'' + (\kappa''')^2 \right) \right\}$$

$$\leqslant \int \left\{ -2\kappa^2(\kappa'''')^2 + C_1\varepsilon\kappa^2(\kappa'''')^2 + \frac{C_2}{\varepsilon}(\kappa')^2(\kappa''')^2 \right.$$

$$\left. + \frac{C_3}{\varepsilon}\frac{(\kappa')^4(\kappa'')^2}{\kappa^2} + \frac{C_4}{\varepsilon}(\kappa'')^4 + \frac{C_5}{\varepsilon}(\kappa')^4 + \frac{C_6}{\varepsilon}\kappa^2(\kappa'')^2 - 2(\kappa''')^2 \right\}$$

$$\leqslant -4\alpha\int (\kappa''')^2 + Ce^{-4\alpha\tau}.$$

The last inequality holds for τ sufficiently large so that the C_2 term can be absorbed. The Peter-Paul inequality and the Wirtinger inequality are used as before, as are Lemmas 5.7.12 and 5.7.11. Applying Lemma 5.7.5 yields the result.

The induction step is routine, but tedious.

5.7.15. Lemma.   *Assume that $l \geqslant 4$, $0 < \bar{\alpha} < 1$, $\|\kappa^{(l-1)}\|_2 \leqslant C_1e^{-2\alpha\tau}$ and $\|\kappa^{(j)}\|_\infty \leqslant C_2e^{-2\alpha\tau}$, $j = 1, 2, \cdots, l-2$. Then $\|\kappa^{(l)}\|_2 \leqslant C_3e^{-2\alpha\tau}$ and $\|\kappa^{(l-1)}\|_\infty \leqslant C_4e^{-2\alpha\tau}$.*

Proof.   As before we calculate

$$\frac{\partial}{\partial \tau} \int \left(\kappa^{(l)}\right)^2 d\theta = 2\int \kappa^{(l)}\left(\kappa^2\kappa'' + \kappa^3 - \kappa\right)^{(l)}$$

$$= \int -2\kappa^{(l+1)}\left(\kappa^2\kappa'' + \kappa^3 - \kappa\right)^{(l-1)} - 2\left(\kappa^{(l)}\right)^2$$

(a) $$= \int \left\{ -2\kappa^2\left(\kappa^{(l+1)}\right)^2 \right.$$

(b) $$-2\kappa^{(l+1)}\left(\kappa^2\right)'\kappa^{(l)}$$

(c) $$-2\kappa^{(l+1)}\left(\kappa^2\right)''\kappa^{(l-1)}$$

(d) $$-2\kappa^{(l+1)}\left(\kappa^2\right)^{(j)}\kappa^{(l+1-j)} \qquad (2 \leqslant j \leqslant l-2)$$

(e) $$-2\kappa^{(l+1)}\left(\kappa^2\right)^{(l-1)}\kappa''$$

(f) $$-2\kappa^{(l+1)}\left(\kappa^3\right)^{(l-1)}$$

(g) $$\left. -2\left(\kappa^{(l)}\right)^2 \right\}.$$

The middle terms can all be bounded in terms of an arbitrarily small fraction of term (a) and the following terms:

(b) $$C_1 \cdot \frac{\left(\left(\kappa^2\right)'\right)^2\left(\kappa^{(l)}\right)^2}{\kappa^2},$$

(c) $$C_2 \cdot \frac{\left(\left(\kappa^2\right)''\right)^2\left(\kappa^{(l-1)}\right)^2}{\kappa^2},$$

(d) $$C_3 \cdot \frac{\left(\left(\kappa^2\right)^{(j)}\right)^2\left(\kappa^{(l+1-j)}\right)^2}{\kappa^2},$$

(e) $$C_4 \cdot \frac{\left(\left(\kappa^2\right)^{(l-1)}\right)^2\left(\kappa''\right)^2}{\kappa^2},$$

(f) $$C_5 \cdot \frac{\left(\kappa^3\right)^{(l-1)}}{\kappa^2}.$$

Term (b) can be bounded by an arbitrarily small fraction of (g) if τ is chosen large enough. Let $\bar{\alpha} \in (\alpha, 1)$. Terms (c) and (e) are bounded by $Ce^{-4\bar{\alpha}\tau}$ (or better) using L_∞ norms on the low derivatives and the L_2 norm on the $l-1$ derivative. The terms in line (d) can be bounded by $Ce^{-4\bar{\alpha}\tau}$ using only

the L_∞ norm. Finally we bound (a) by $-\int (\kappa^{(l)})^2$ using the Wirtinger inequality and the fact that $\kappa \to 1$ uniformly. We obtain

$$\frac{\partial}{\partial \tau} \int \left(\kappa^{(l)}\right)^2 \leqslant -4\alpha \int \left(\kappa^{(l)}\right)^2 + Ce^{-4\bar{\alpha}\tau}$$

and use Lemma 5.7.5 to complete the proof.

The proof of Theorem 5.7.1 follows by induction from the previous lemmas.

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