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THE LOCAL EQUIVALENCE PROBLEM FOR $d^2y/dx^2 = F(x, y, dy/dx)$ AND THE PAINLEVÉ TRANSCENDENTS

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Abstract

We solve the local equivalence problem for $d^2y/dx^2 = F(x, y, dy/dx)$ under the natural group of coordinate changes $\bar{x} = \phi(x)$, $\bar{y} = \psi(x, y)$. There are three basic invariants which vanish iff the equation is equivalent to $d^2y/dx^2 = 0$. We show that two of the invariants vanish for the six Painlevé transcendents and that the third can be used to produce a complete set of invariants. We give necessary and sufficient conditions for $d^2y/dx^2 = F(x, y, dy/dx)$ to be equivalent to either of the first two Painlevé transcendents and give simple algebraic formulas for the change of variable which puts an equivalent equation into the standard form.

1. Introduction

In this paper we present the solution of the local equivalence problem for the equation

$$\frac{d^2y}{dx^2} = F\left(x, y, \frac{dy}{dx}\right) \tag{1.1}$$

under the group of coordinate transformations defined by

$$\bar{x} = \phi(x), \qquad \bar{y} = \psi(x, y). \tag{1.2}$$

Equation (1.1) has, because of its connections with mechanics and geometry, often been studied in the past. Tresse devoted a monograph [8] to the determination of the invariants of (1.2) under point transformations

$$\bar{x} = \phi(x, y), \qquad \bar{y} = \psi(x, y). \tag{1.3}$$

In [4] Cartan gave a geometric interpretation of Tresse's results in terms of a torsion free projective connection in a "generalized space".

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From the geometric point of view (1.2) can be regarded as generalizing the equation governing geodesics on a surface coordinatized by x and y. Thus point transformations form the appropriate group in this context.

If we view (1.1) as a differential equation however, the transformations (1.3) are too general, as they mix up the role of independent and dependent variables. In this context (1.2) is the appropriate group.

Another classical study, one which motivated our initial calculations, is the work of Painlevé [6], [7]. He gave a partial classification, later completed by Gambier, of the equations (1.1) whose solutions have no essential singularities or branch points which are movable in the sense of depending on the initial conditions $y(x_0)$ and $y'(x_0)$. This classification produced fifty equations of which six determine new transcendental functions—the so-called Painlevé transcendents.

The fifty equations, which one may find in Ince [6], are of course just representatives of equivalence classes. The group up to which the classification was done is given by

$$\bar{x} = \phi(x), \qquad \bar{y} = \frac{\psi_1(x)y + \psi_2(x)}{\psi_3(x)y + \psi_4(x)}.$$

Now, as this is only a subgroup of (1.2), some questions suggest themselves.

How can one tell if a given equation (1.1) is actually one of the six Painlevé transcendents in disguise? Are all fifty, and more importantly all six Painlevé transcendents, actually distinct under the larger group? It has been observed that the Painlevé transcendents arise when one looks for similarity solutions of soliton equations [1] and it has been conjectured that this property is intimately related to complete integrability. Now as the work of Painlevé has found representatives for all equations (1.1) one can approach the problem of deciding if a given equation has the Painlevé property by asking if the equation obtained for a similarity solution is *equivalent to* one of the Painlevé equations. The usual approach is to repeat Painlevé analysis and in order to establish sufficient conditions in this way one essentially has to solve the equation in question.

It seems likely that the differential invariants provided by Cartan's equivalence method will prove useful in the study of these questions. Indeed, Cartan's method provides a solution in principle. One need only calculate a complete set of invariants for the Painlevé transcendents and use them to give algebraic necessary and sufficient conditions for equivalence. The real question of course is how effective this solution is. As we indicate in §3 it is indeed effective.

In §2 we present our solution of the equivalence problem for (1.1) and (1.2). This requires one prolongation, independent of the right-hand side of (1.1), and leads to an identity structure on $J^1(\mathbf{R}, \mathbf{R}) \times G$, where G is a three-dimensional subgroup of $\mathrm{GL}(3, \mathbf{R})$. There are three basic invariants I_1, I_2 and I_3.

We note that these invariants all vanish in the case $F = 0$ and hence the condition $I_i = 0$, $i = 1, 2, 3$, holds iff the equation (1.1) is transformable under (1.2) into

$$d^2\bar{y}/d\bar{x}^2 = 0. \tag{1.4}$$

In this case the structure equations are those of a six-dimensional subgroup of $\mathrm{SL}(3, \mathbf{R})$ (isomorphic to the affine group on $\mathbf{R}^2$) acting as linear fractional transformations on $\mathbf{R}^2$. This is the symmetry group of $y'' = 0$ under (1.2). Unless all of the invariants vanish one may use the remaining group freedom to obtain a coframe on a lower dimensional space. We note that in the case where $I_2 = I_3 = 0$ the structure equations are those of a connection for the affine group on $\mathbf{R}^2$ with curvature given by I_1. In this case we have a generalized space in the sense of Cartan whose points are parametrized by the solution of (1.1).

In §3 we consider the application of the Painlevé transcendents. For these, although not all fifty in the Painlevé list, both I_1 and I_2 vanish. Thus the vanishing of I_1 and I_2 gives necessary conditions for an equation to belong to the six classes. It requires that $F(x, y, dy/ax)$ have the form

$$F\left(x, y, \frac{dy}{dx}\right) = \frac{1}{2}\left(\frac{dy}{dx}\right)^2 M_y + \frac{dy}{dx} M_x + N \tag{1.5}$$

for some functions M and N depending on x and y.

It is not difficult to see directly from the form of (1.2) that the condition of being quadratic in dy/dx (which is one of Painlevé necessary conditions) is invariant. It requires somewhat more work to show that the form (1.5) is also invariant. How one would guess that it should be, aside from carrying out the equivalence problem calculations, is not at all apparent.

We give necessary and sufficient conditions for an equation to be equivalent to the first two Painlevé transcendents

$$y'' = 6y^2 + ax, \tag{1.6}$$

$$y'' = 2y^3 + xy + b. \tag{1.7}$$

In what follows we will refer to these normal forms as P(I) and P(II) respectively. In both cases we give a simple algebraic formula in terms of two fundamental invariants for the change of variable which puts an equivalent equation into the normal form (1.6) or (1.7).

Finally, we wish to point out that, while some of the calculations involved in solving this equivalence problem are long, they need only be done once. The basic invariants discovered then provide algebraic criteria necessary and sufficient for equivalence. It is worth noting that all that is involved in the calculation is exterior differentiation and exterior multiplication. Thus the laborious work can be done by a computer. Indeed, we used a program developed on the University of Waterloo MAPLE symbolic manipulation system [2] to check our calculation of the invariants I_1, I_2 and I_3. While this calculation is very tedious to do by hand, it required only 80 seconds of CPU time on a VAX 780.

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2. The equivalence problem

We use the method given by Cartan [3] as described by Gardner [5]. Thus we take as our starting point an exterior differential system on $J^1(\mathbf{R},\mathbf{R})$ whose solutions are in one-to-one correspondence with the solutions of (1.1). With coordinates x, y, p on $J^1(\mathbf{R},\mathbf{R})$, we take the coframe given by

$$dx, \quad dy - pdx \quad \text{and} \quad dp - F(x,y,p)\,dx.$$

The solutions of (1.1) are the curves in $J^1(\mathbf{R},\mathbf{R})$ on which

$$dx \neq 0, \quad dy - pdx = 0, \quad dp - F(x,y,p)\,dx = 0. \tag{2.1}$$

The coframe we have chosen is not uniquely defined however and the solutions of (2.1) are the same as the solutions of

$$\begin{gathered} Adx \neq 0, \qquad B(dy - pdx) = 0, \\ BC(dy - pdx) + D(dp - Fdx) = 0, \end{gathered} \tag{2.2}$$

where A, B and D are nowhere vanishing functions and C is an arbitrary function on $J^1(\mathbf{R},\mathbf{R})$. Now it is easy to verify that the prolongation of (1.2) to give coordinate transformations on $J^1(\mathbf{R},\mathbf{R})$ is given by

$$\bar{x} = \phi(x), \quad \bar{y} = \psi(x,y), \quad \bar{p} = \frac{\psi_x + p\psi_y}{\phi_x} \tag{2.3}$$

so that $D = B/A$ under such changes of coordinates. Thus we are led to the problem of adapting the coframe,

$$\begin{bmatrix} w^1 \\ w^2 \\ w^3 \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & BC & B/A \end{bmatrix} \begin{bmatrix} dx \\ dy - pdx \\ dp - F(x,y,p)\,dx \end{bmatrix}. \tag{2.4}$$

That is, we consider the group G of matrices of the form

$$S = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & BC & B/A \end{bmatrix}.$$

Then we try to use the natural G-action to provide a coframe of the form w which is adapted to our problem. The first step is to compute dSS^{-1}, which gives the 0th order principal components. It is easy to verify that dSS^{-1} has the form

$$dSS^{-1} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & c & b-a \end{bmatrix}.$$

Thus we have

$$d\begin{bmatrix} w^1 \\ w^2 \\ w^3 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & b & 0 \\ 0 & c & b-a \end{bmatrix}\begin{bmatrix} w^1 \\ w^2 \\ w^3 \end{bmatrix} + \begin{bmatrix} \Gamma^1{}_{jk}w^jw^k \\ \Gamma^2{}_{jk}w^jw^k \\ \Gamma^3{}_{jk}w^jw^k \end{bmatrix},$$

where

$$dSS^{-1} \equiv \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & \gamma & \beta-\alpha \end{bmatrix} \quad \operatorname{mod} w^i,$$

and all products of forms are wedge products.

From a direct calculation using (2.4) we find that after absorbtion of torsion by redefining α, β and γ, we have

$$(2.5) \qquad d\begin{bmatrix} w^1 \\ w^2 \\ w^3 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & \gamma & \beta-\alpha \end{bmatrix}\begin{bmatrix} w^1 \\ w^2 \\ w^3 \end{bmatrix} + \begin{bmatrix} 0 \\ w^1w^3 \\ 0 \end{bmatrix}.$$

Thus there are not torsion terms to normalize and we must prolong the problem. Consider the coframe on $J^1(\mathbf{R}, \mathbf{R}) \times G$ given by ${}^t(w, \alpha, \beta, \gamma)$.

The remaining freedom in modifying the 1-forms α, β, γ so as to leave (2.5) invariant determines a group $G^{(1)}$ which acts on $\mathbf{R}^6$. It is easy to verify that the indeterminacy in our new coframe is given by

$$G^{(1)} = \left\{ \left(\begin{array}{ccc|c} & 1_3 & & 0 \\ \hline 0 & 0 & 0 & \\ 0 & f & 0 & 1_3 \\ 0 & g & f & \end{array} \right) \in \mathrm{GL}(6, \mathbf{R}) \right\}.$$

By calculating d^2w^i, $i = 1, 2, 3$, we find that

$$(2.6)\qquad \begin{aligned} d\alpha &= 2w^1\gamma + aw^1w^2 + bw^2w^3, \\ d\beta &= w^2\gamma + \rho w, \\ d\gamma &= \gamma\alpha + \rho w^3 + \delta w^2 - aw^1w^3, \end{aligned}$$

where a and b are unabsorbable torsion terms and ρ, δ give a basis for the right invariant 1-forms of $G^{(1)}$.

The $G^{(1)}$ action on a and b is calculated by computing $d^2\alpha$ and $d^2\gamma$. It is given by

$$(2.7)\qquad da + 2\delta \equiv 0, \quad db + 2\rho \equiv 0 \quad \operatorname{mod} w^i, \alpha, \gamma,$$

so we can always translate a and b to 0.

This yields the structure equations

$$(2.8)\qquad \begin{aligned} &dw^1 = \alpha w^1, \qquad dw^2 = \beta w^2 + w^1w^3, \\ &dw^3 = \gamma w^2 + (\beta - \alpha)w^3, \qquad d\alpha = 2w^1\gamma, \\ &d\beta = w^1\gamma + I_1w^2w^3 + I_2w^1w^2, \\ &d\gamma = \gamma\alpha + I_2w^1w^3 + I_3w^1w^2 \end{aligned}$$

with the three basic invariants I_1, I_2 and I_3 given by

$$(2.9)\qquad \begin{aligned} I_1 &= -\frac{A}{2B^2}F_{ppp}, \\ I_2 &= \frac{1}{2AB}\left(\frac{d}{dx}F_{pp} - F_{py}\right), \\ I_3 &= -CI_2 + \frac{1}{2A^2B}\left(\frac{d}{dx}F_{py} + F_{pp}F_y - F_{py}F_p - 2F_{yy}\right), \end{aligned}$$

where $d/dx = \partial/\partial x + p\partial/\partial y + F\partial/\partial p$ is the total derivative. In terms of our original variables A, B, and C,

$$(2.10)\qquad \begin{aligned} \alpha &= \frac{dA}{A} - \left(2C + \frac{Fp}{A}\right)w^1, \\ \beta &= \frac{dB}{B} - Cw^1 + \frac{Fpp}{2B}w^2, \\ \gamma &= dC + \frac{CdA}{A} + \left(\frac{Fy}{A^2} - \frac{CF}{A^p} - C^2\right)w^1 \\ &\quad + \left(\frac{Fpy}{2AB} - \frac{C}{2B}F_{pp}\right)w^2 + \frac{Fpp}{2B}w^3. \end{aligned}$$

Now it is clear from (2.9) that in the case $F = 0$, $I_1 = I_2 = I_3 = 0$ so the vanishing of all three invariants occurs iff (1.1) is transformable under (1.2) to $d^2\bar{y}/d\bar{x}^2 = 0$.

In the case where all three invariants vanish the structure equations are the Maurer-Cartan equations for the symmetry group of $y'' = 0$. This is the six-dimensional subgroup of SL(3, **R**) of matrices of the form

$$S = \begin{bmatrix} a_1 & 0 & a_2 \\ b_1 & b_2 & b_3 \\ c_1 & 0 & c_2 \end{bmatrix}, \qquad \det S = 1,$$

which acts as fractional linear transformation in the plane by

$$\bar{x} = \frac{a_1 x + a_2}{c_1 x + c_2}, \qquad \bar{y} = \frac{b_1 x + b_2 y + b_2}{c_1 x + c_2}. \tag{2.11}$$

If we let $w = {}^t(w^1, w^2, w^3)$, $\bar{w} = Sw$ and take $x = w^1/w^3$, $y = w^2/w^3$, then (2.11) gives

$$d\bar{x} = \varepsilon_2 + \bar{x}(2\varepsilon_1 + \varepsilon_4) + \bar{x}^2\varepsilon_6,$$
$$d\bar{y} = \varepsilon_5 + \bar{x}\varepsilon_3 + \bar{y}(\varepsilon_1 + 2\varepsilon_4) + \bar{x}\bar{y}\varepsilon_6,$$

where

$$dSS^{-1} \begin{bmatrix} \varepsilon_1 & 0 & \varepsilon_2 \\ \varepsilon_3 & \varepsilon_4 & \varepsilon_5 \\ -\varepsilon_6 & 0 & -(\varepsilon_1 + \varepsilon_4) \end{bmatrix}.$$

The equations $d^2\bar{x} = d^2\bar{y} = 0$ are just the structure equations (2.8) with $I_i = 0$, where

$$w^1 = \varepsilon_2, \quad w^2 = \varepsilon_5, \quad w^3 = -\varepsilon_3$$
$$\alpha = 2\varepsilon_1 + \varepsilon_4, \quad \beta = \varepsilon_1 + 2\varepsilon_4, \quad \gamma = \varepsilon_6.$$

These are also the structure equations for the affine group in the plane if we make the identification.

$$(\theta^1, \theta^2, \Theta_1^1, \Theta_2^1, \Theta_1^2, \Theta_2^2) = (w^2, w^3, -\beta, -w^1, -\gamma, \alpha - \beta).$$

Then (2.8) became

$$d\theta^1 = \theta^1\Theta_1^1 + \theta^2\Theta_2^1, \qquad d\theta^2 = \theta^1\Theta_1^2 + \theta^2\Theta_2^2,$$
$$d\Theta_1^1 = -\Theta_2^1\Theta_1^2, \qquad d\Theta_2^1 = -\Theta_1^1\Theta_2^1 + \Theta_2^1\Theta_2^2,$$
$$d\Theta_1^2 = -\Theta_1^2\Theta_2^1 - \Theta_2^2\Theta_1^2, \qquad d\Theta_2^2 = -\Theta_1^2\Theta_2^1.$$

When I_2 and I_3 both vanish the equations (2.8) can therefore be interpreted as the structure equations for a connection for the affine group on $\mathbf{R}^2$. The curvature 2-form is

$$\Omega = \begin{bmatrix} -I_1\theta^1\theta^2 & 0 \\ 0 & -I_1\theta^1\theta^2 \end{bmatrix}.$$

We consider applications of these cases elsewhere. In what follows we consider the case $I_1 = I_2 = 0$, $I_3 \neq 0$ as this case contains the Painlevé transcendents.

3. The Painlevé transcendents

It is apparent from (2.9) that when $I_1 = 0$, F is quadratic in p. If we use $I_2 = 0$ for quadratic F we find that the quadratic and linear terms are related:

$$F = \frac{1}{2}p^2 M_y + pM_x + N \tag{3.1}$$

for some functions $M(x, y)$ and $N(x, y)$. The relation between the quadratic and linear terms is actually invariant under (1.2) as it follows from $I_1 = I_2 = 0$; however, it is not immediately obvious that this condition is preserved under transformations of the form (1.2). It is not difficult to check that the vanishing of I_1 and I_2 is necessary and sufficient for equation (1.1) to be the Euler-Lagrange equation for a "particle-type" Langrangian $L = \frac{1}{2}g(x, y)(dy/dx)^2 - V(x, y)$.

Now the structure equations with $I_1 = I_2 = 0$ become

$$\begin{aligned} dw^1 &= \alpha w^1, \quad dw^2 = \beta w^2 + w^1 w^3, \quad dw^3 = \gamma w^2 + (\beta - \alpha) w^3, \\ d\alpha &= 2w^1\gamma, \quad d\beta = w^1\gamma, \quad d\gamma = \gamma\alpha + I_3 w^1 w^2. \end{aligned} \tag{3.2}$$

From $d^2\gamma = 0$ we find that

$$dI_3 + I_3(2\gamma + \beta) \equiv 0 \mod w^1 w^2$$

and thus when $I_3 \neq 0$ we can scale it to 1 to obtain

$$2\alpha + \beta \equiv 0 \mod w^1 w^2. \tag{3.3}$$

In the case where F has the form (3.1) it follows from (2.9) that

$$I_3 = \frac{1}{2A^2B}\frac{\partial}{\partial y}\left(M_{xx} + NM_y - 2N_y - \frac{1}{2}M_x^2\right).$$

Thus the normalization of I_3 to 1 gives

$$2A^2B = \frac{\partial}{\partial y}\left[M_{xx} + NM_y - 2N_y - \frac{1}{2}M_x^2\right]$$

and we define $G(x, y)$ by

$$G(x, y) = \frac{\partial}{\partial y}\left[M_{xx} + NM_y - 2N_y - \frac{1}{2}M_x^2\right]. \tag{3.4}$$

It follows from (3.3) that $\beta - -2\alpha + aw^1 + bw^2$, and it is easy to verify from $0 = d^2w^2$ and $0 = d^2w^3$ that

$$da + a\alpha + 5\gamma \equiv 0, \quad db - 2b\alpha \equiv 0 \mod w^1 w^2 w^3. \tag{3.5}$$

Thus we can always translate a to 0 and, if b is nonzero, scale it to ± 1 depending on its sign.

One may check from (2.10) that

$$a = \frac{1}{AG}\frac{dG}{dx} - \frac{2}{A}\left(\frac{d}{dx}M\right) - 5C,$$

$$b = \frac{1}{B}\left(\frac{G_y}{G} + \frac{M_y}{2}\right).$$

If $b \neq 0$ we have reduced the group to the identity, as equations (3.4) become, with $a = 0$, $b = \pm 1$,

$$\gamma \equiv 0, \quad \alpha \equiv 0 \mod w^1 w^2 w^3.$$

If $b = 0$ we have only $\gamma \equiv 0$ so we must perform another group reduction. Now among the six Painlevé transcendents this occurs only for Painlevé (I): $d^2y/dx^2 = 6y^2 + ax$.

In any case we can translate a to 0 which gives

$$C = \frac{1}{5A}\frac{d}{dx}(\ln G - 2M). \tag{3.6}$$

To test whether an equation is P(I) we can proceed as follows. We have as necessary conditions $I_1 = I_2 = 0$, $I_3 \neq 0$ and we normalize I_3 to 1 to get

$$A^2B = \tfrac{1}{2}G(x, y).$$

This gives $\beta = -2\alpha + aw^1 + bw^2$ and $b = 0$ is a necessary condition for equivalence. By normalizing a to 0 we get C as given by (3.6) and it follows from (3.5) and (2.10) that γ can now be expressed in terms of w^1, w^2 and w^3. As a further necessary condition we find from (3.2) that

$$\gamma \equiv cw^1 \mod w^2$$

and, as γ appears only in the equation for $dw^3 = \gamma w^2 - 3\alpha w^3$, we need only know $\gamma \bmod w^2$. The coefficient c is easily calculated from (2.10) and (3.5) as

$$c = \frac{1}{A^2}\left[\frac{1}{5}\frac{dH}{dx} - \frac{1}{25}H^2 - \frac{H}{5}(pM_y + M_x) + \frac{p^2}{2}M_{yy} + pM_{xy} + N_y\right], \tag{3.7}$$

where

$$H = \frac{d}{dx}[\ln G - 2M]. \tag{3.8}$$

It follows from (3.2) that $c \neq 0$ and that we can scale c to ± 1 depending on its sign. This determines A, and makes $\alpha \equiv 0 \mod w^1 w^2 w^3$. From (2.10) we have

$$\alpha = \frac{dA}{A} - \frac{1}{A}\left[\frac{2}{5}\frac{d}{dx}(\ln G + M)\right]w^1 \tag{3.9}$$

so we can calculate

$$\alpha = Iw^1 + J_0w^2 + K_0w^3.$$

By making use of (3.2) we find that $J_0 = -1/2$ and $K_0 = 0$. The invariant I is given for P(I) by

$$I = \frac{1}{4\sqrt{3}}\frac{p}{y^{3/2}} \tag{3.10}$$

If all of the necessary conditions are satisfied we calculate covariant derivatives of I to obtain further necessary conditions and, after two differentiations, sufficient ones. Thus we calculate dI and obtain

$$dI = \left(\frac{1}{4} - 3I^2 + J\right)w^1 + \frac{3}{2}Iw^2 + \frac{1}{2}w^3 \tag{3.11}$$

which defines J. For P(I), $J = ax/24y^2$ and we see that we have $dI \wedge dJ \neq 0$ iff $a \neq 0$. In the case $a = 0$ the equation has only one independent invariant and hence has a 2-parameter symmetry group given in these coordinates by $\bar{x} = sx + t$, $\bar{y} = s^{-2}y$. If $a \neq 0$ we can use I and J as coordinates. We calculate dJ which determines a third invariant K by

$$dJ = (K - 4IJ)w^1 + 2Jw^2. \tag{3.12}$$

In the case of P(I), $K = a/48\sqrt{3}y^{5/2}$. The final necessary conditions are given by

$$dK = -5IKw^1 + \frac{5}{2}Kw^2. \tag{3.13}$$

The test for equivalence is thus to calculate $\bar{I}\bar{J}$ and $\bar{K}$ and check that $d\bar{I}$, $d\bar{J}$ and $d\bar{K}$ are given in terms of $\bar{I}$, $\bar{J}$ and $\bar{K}$ by the same formulas as we have obtained for dI, dJ and dK. These conditions are also sufficient [3] and the change of variable which gives the equivalence is given by

$$I = \bar{I}, \quad J = \bar{J}, \quad K = \bar{K}.$$

Now we can make this explicit for P(I). If we define $\hat{J}$ and $\hat{K}$ by $\hat{J} = 24\bar{J}$ and $\hat{K} = 48\sqrt{3}\,\bar{K}$, we have

$$x = \hat{J}\hat{K}^{-4/5}a^{-1/5}, \qquad y = \hat{K}^{-2/5}a^{2/5}. \tag{3.14}$$

In particular it is obvious that all of the equations with parameter $a \neq 0$ are equivalent by scaling x and y.

For the second Painlevé transcendent,

$$y'' = 2y^3 + xy + a,$$

one proceeds in exactly the same way. In this case the invariant b is nonzero so by (3.5) we can normalize b to reduce the group G to the identity. The normalizations give

$$A^2B = \frac{1}{2}G, \quad B = -\partial_y\left(\ln G + \frac{M}{2}\right), \quad C = \frac{H}{5A}, \tag{3.15}$$

where

$$H = \frac{d}{dx}(\ln G - 2M)$$

and G is given by (3.4). In the case of P(II) we have

$$A^2 = 12y^2, \quad B = -1/y, \quad C = \frac{1}{5A}p/y. \tag{3.16}$$

The basic invariants are now the w^1 components of α and γ respectively, which we can calculate by substituting from (3.15) in (2.10). These invariants are given by

$$\begin{aligned} I &= \frac{1}{2A}\frac{d}{dx}\ln\left(\frac{G}{B}\right) - \left(2C + \frac{F}{A^p}\right), \\ J_0 &= \frac{1}{A^2}\left[\frac{1}{5}\frac{dH}{dx} - \frac{1}{25}H^2 - \frac{1}{5}HF_p + F_y\right]. \end{aligned} \tag{3.17}$$

For P(II)

$$\begin{aligned} &\text{(a)} \quad I = \frac{\sqrt{3}}{10}p/y^2, \\ &\text{(b)} \quad J_0 = \frac{8}{15} - \frac{2}{3}I^2 + \frac{J}{60}, \end{aligned} \tag{3.18}$$

where $J = (6xy + a)/y^3$. For convenience we use I and J which is defined (invariantly) by (3.18)(b).

In order to obtain further necessary conditions we take covariant derivatives of I and J. We find

$$dI = \left(\frac{1}{10} - \frac{10}{3}I^2 + \frac{K}{20}\right)w^1 + \frac{11}{5}Iw^2 - \frac{3}{5}w^3,$$

which defines a third invariant K. Now for P(II), $K = (xy + a)/y^3$ so we have $dJ \wedge dK = 0$ iff $a = 0$. Thus the case $a = 0$ is in a different equivalence class from the case $a \neq 0$. It is easy to verify, however, that there are three independent invariants even when $a = 0$ and thus that there is no Lie group of symmetries for the equation $y'' = 2y^3 + xy$. We continue with the case $a \neq 0$ in which we have $dI \wedge dJ \vee K \neq 0$ and can use I, J and K as coordinates. The remaining necessary conditions for equivalence are that dJ and dK have the same form in terms of $\bar{I}$, $\bar{J}$, $\bar{K}$, $\bar{w}^1$, $\bar{w}^2$ and $\bar{w}^3$ as they do in the case of P(II) where we find

$$\begin{aligned} dJ &= \left[(6K - J)\frac{\sqrt{3}}{5a} - \frac{I}{6}(3J + 2K)\right]w^1 + \frac{3}{5}(3J + 2K)w^2, \\ dK &= \left[(6K - J)\frac{1}{10\sqrt{3}\,a} - \frac{I}{3}(16K - J)\right]w^1 + \frac{1}{5}(16K - J)w^2. \end{aligned}$$

These conditions are also sufficient and we have the following formula for the change of coordinates, obtained from $\bar{I} = I$, $\bar{J} = J$, $\bar{K} = K$:

$$(3.19) \qquad x = \frac{1}{5}(\bar{J} - \bar{K})\left(\frac{5a}{6\bar{K} - \bar{J}}\right)^{2/3}, \qquad y = \left(\frac{5a}{6\bar{K} - \bar{J}}\right)^{1/3}.$$

In particular all of the equations with parameter $a \neq 0$ are equivalent by scaling x and y:

$$x = \left(\frac{\bar{a}}{a}\right)^{2/3} \bar{x}, \qquad y = \left(\frac{\bar{a}}{a}\right)^{1/3} \bar{y}.$$

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