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NONSTANDARD LORENTZ SPACE FORMS

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In their recent paper [8], Kulharni and Raymond show that a closed 3-manifold which admits a complete Lorentz metric of constant curvature 1 (henceforth called a *complete Lorentz structure*) must be Seifert fibered over a hyperbolic base. Furthermore on every such Seifert fibered 3-manifold with nonzero Euler class they construct such a Lorentz metric. Moreover the Lorentz structure they construct has a rather strong additional property, which they call "standard": A Lorentz structure is *standard* if its causal double cover possesses a timelike Killing vector field. Equivalently, it possesses a Riemannian metric locally isometric to a left-invariant metric on $\mathrm{SL}(2,\mathbf{R})$. Kulkarni and Raymond asked if every closed 3-dimensional Lorentz structure is standard. This paper provides a negative answer to this question (Theorem 1) and a positive answer to the implicit question raised in [8, 7.1.1] (Theorem 3).

Theorem 1. *Let M^3 be a closed 3-manifold which admits a homogeneous Lorentz structure and satisfies $H^1(M; \mathbf{R}) \neq 0$. Then there exists a nonstandard complete Lorentz structure on M.*

In [8] it is shown that the unit tangent bundle of a closed surface admits a homogeneous Lorentz structure. Therefore we obtain:

Corollary 2. *There exists a complete Lorentz structure on the unit tangent bundle of any closed surface F of genus greater than one which is not standard.*

The homogeneous Lorentz structures are all classified in [8]. A circle bundle of Euler number j over a closed surface F, $\chi(F) < 0$, has a homogeneous structure if and only if $j | \chi(F)$ (an analogous statement holds when M has singular fibers, i.e. when F is an orbifold).

We also show:

Theorem 3. *Let M^3 be a 3-manifold which admits a complete Lorentz structure. Then M^3 is not covered by a product $F \times S^1$, F a closed surface.*

Theorem 3 implies that the Euler class of the Seifert fiber structure of M^3 is *nonzero*.

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Corollary 4. *If a closed 3-manifold M admits a complete Lorentz structure, then M admits a standard Lorentz structure.*

In [7] the deformation theory of standard Lorentz structures is extensively discussed.

A key idea in the proof of Theorem 1 is the notion of a (small) deformation of a complete Lorentz structure. It is convenient to think of a Lorentz structure as a "locally homogeneous" geometric structure, defined by an atlas of charts which are homeomorphisms of coordinate patches into a model space X such that the coordinate changes on the overlaps lie in a certain group G of transformations of X. (See [12].) In our case X will be a simply connected complete Lorentz manifold of curvature 1 and G will be the identity component of its group of Lorentz isometries. A convenient model for X is the universal cover $\widetilde{\mathrm{SL}}(2,\mathbf{R})$ of $\mathrm{SL}(2,\mathbf{R})$, with the Lorentz metric defined by the Killing form on the Lie algebra $\mathfrak{sl}(2,\mathbf{R})$. The group of all its isometries is a 4-fold extension of the quotient of $\widetilde{\mathrm{SL}}(2,\mathbf{R}) \times \widetilde{\mathrm{SL}}(2,\mathbf{R})$ by a diagonally embedded central $\mathbf{Z}$. See [8] for further details on the resulting geometry.

One basic example of such a structure arises as follows. Consider any discrete cocompact subgroup Γ of $\mathrm{PSL}(2,\mathbf{R})$. Then the quotient $\mathrm{PSL}(2,\mathbf{R})/\Gamma$ has an induced left-invariant complete Lorentz structure. Such manifolds have homogeneous Lorentz metrics (cf. Kulkarni-Raymond [8, 10]). If Γ is torsionfree, so that $\mathrm{PSO}(2)\backslash\mathrm{PSL}(2,\mathbf{R})/\Gamma$ is a smooth hyperbolic surface F, then $\mathrm{PSL}(2,\mathbf{R})/\Gamma$ is the unit tangent bundle of F. By taking fiberwise coverings, we obtain homogeneous complete Lorentz structures on other oriented circle bundles over F; these circle bundles are characterized by the property that their Euler class divides $\chi(F)$. The class of Seifert fibered 3-manifolds which can be obtained as coverings of such quotients of $\mathrm{PSL}(2,\mathbf{R})$ are precisely the Seifert fibered 3-manifolds which admit homogeneous Lorentz structures. The nonstandard complete Lorentz structures constructed here will be deformations of these homogeneous structures.

A geometric structure modelled on the geometry of (G, X) is sometimes called a "(G, X)-structure". To every (G, X)-structure on a manifold M there are associated homomorphisms h from the fundamental group $\pi = \pi_1(M)$ to G such that for each "holonomy homomorphism" h there exists a local diffeomorphism (called the "developing map") from the universal covering $\tilde{M}$ of M to X which is equivariant respecting h. (For a given (G, X)-structure, the holonomy homomorphism and the developing map are respectively unique up to conjugation and composition with a transformation in G.) If G is a group of isometries of a pseudo-Riemannian metric on X, then there is a unique pseudo-Riemannian metric on M such that the developing map is a local isometry of the induced structure on $\tilde{M}$ with X. In the language of [8], a manifold with a complete Lorentz structure is a "Lorentz space form".

A (G, X)-structure is said to be *complete* if its developing map is a covering map onto X. We will always take X to be a simply connected homogeneous space of G, so that the developing map will represent a complete (G, X)-manifold as a quotient of X by a discrete subgroup of G acting properly and freely. When X has a complete G-invariant pseudo-Riemannian metric, completeness of a (G, X)-structure is equivalent to the usual notion of geodesic completeness of the corresponding pseudo-Riemannian metric. However, unless G acts properly on X no general criterion for a (G, X)-structure on a closed manifold to be complete is known. (Indeed there are many well-known geometries (G, X) (such as affine geometry) for which incomplete (G, X)-structures exist on closed manifolds, see e.g. [11].) It is not known whether a Lorentz structure on a closed manifold is necessarily complete.

A Lorentz structure is *standard* if it (or perhaps a double cover of it) possesses a timelike Killing vector field ξ. In terms of (G, X)-structures a standard complete Lorentz structure is a (G, X)-structure whose "holonomy group" $h(\pi)$ normalizes the isometric flow generated by ξ. Alternatively we say that a standard Lorentz structure is a (G_0, X)-structure, where G_0 is the normalizer of ξ. Every homogeneous Lorentz structure on a closed manifold is complete (since G_0 acts properly on X, standard implies complete for closed manifolds).

The space of homomorphisms $\pi \to G$ forms a real analytic variety $\mathrm{Hom}(\pi, G)$. Suppose M is a closed manifold with a (G, X)-structure (denoted M_0) with holonomy homomorphism h_0: $\pi \to G$. Then there exists a neighborhood U of h_0 in $\mathrm{Hom}(\pi, G)$ such that for each $h_t \in U$, there is a (G, X)-structure M_t with holonomy h_t. (In this generality, this fact was first observed by Thurston [12]; See Lok [9] for a detailed discussion.) (Indeed, it is possible to define a deformation space of (G, X)-structures with a natural topology in such a way that the (G, X)-structures M_t form a continuous family near M_0.)

Let M be a 3-manifold which admits a homogeneous Lorentz structure, e.g. the unit tangent bundle of a closed surface F. Let h_0 be the holonomy representation $\pi \to \widetilde{\mathrm{SL}}(2, \mathbf{R})$ corresponding to one of the homogeneous Lorentz strucures above. Let B be a one-parameter subgroup in $\widetilde{\mathrm{SL}}(2, \mathbf{R})$ acting by right-multiplication on $\widetilde{\mathrm{SL}}(2, \mathbf{R})$. We shall deform the homomorphism $h_0 \in \mathrm{Hom}(\pi, G)$ using a deformation of the trivial representation in $\mathrm{Hom}(\pi, B)$.

For $v \in \mathrm{Hom}(\pi, B)$ in a sufficiently small neighborhood of the trivial homomorphism, the homomorphism (h_0, v): $\pi \to \mathrm{Hom}(\pi, G)$ (where h_0 acts on the left and v acts on the right) will be the holonomy representation of a complete Lorentz structure near the homogeneous structure on M. In other words:

Proposition 5. *Let h_0: $\pi \to \widetilde{\mathrm{SL}}(2, \mathbf{R})$ be the holonomy of a homogeneous complete Lorentz structure as above. Then there exists a neighborhood U of the trivial representation* 0 *in* $\mathrm{Hom}(\pi, B)$ *such that for all $v \in U$, (h_0, v) is a free proper action of π on X with quotient a closed manifold.*

When B is either a hyperbolic or parabolic one-parameter subgroup, then the resulting quotient manifold has a nonstandard complete Lorentz structure. Thus Proposition 5 implies Theorem 1. Observe that we obtain two quite different families of nonstandard complete Lorentz structures, depending on whether B is parabolic (lightlike) or hyperbolic (spacelike). By small deformations of the holonomy, we construct "nearby" Lorentz structures with the deformed holonomy. Proposition 5 is proved by showing this deformed structure is complete.

We begin by describing one viewpoint on (G, X)-structures in which the existence of deformed (G, X)-structures is quite transparent. Let dev: $\tilde{M} \to X$ be a developing map which is equivariant with respect to a homomorphism $h \in \mathrm{Hom}(\pi, G)$. The equivariance of dev with respect to h implies that the graph of dev is a section of the trivial X-bundle $\tilde{M} \times X$ over $\tilde{M}$ which is invariant under the action of π on $\tilde{M} \times X$ defined by γ: $(u, x) \mapsto (\gamma u, h(\gamma)x)$. It follows that the graph of dev defines a section (the "developing section") f of the (G, X)-bundle $X(h)$ whose total space is the quotient $(\tilde{M} \times X)/\pi$.

The bundle $X(h)$ has a *flat structure*, i.e. a foliation transverse to the fibers. The leaves of this foliation are the images of the sets $\tilde{M} \times \{x_0\}$, where $x_0 \in X$. The nonsingularity of the developing map is equivalent to the transversality of f to the flat structure. Conversely, any section of a flat (G, X)-bundle which is transverse to the flat structure defines a (G, X)-structure: local charts for this structure are found by composing the submersive local charts of the foliation with the section. In this way every transverse section of the flat (G, X)-bundle $X(h)$ is a "developing section" of a (G, X)-structure with holonomy h. For more details on this picture of (G, X)-structures, the reader is referred to Goldman [3], Goldman-Hirsch [5], Kulkarni [6], or Sullivan-Thurston [11].

We can now understand the deformation theorem as follows. Fix a (G, X)-structure on M as well as a holonomy homomorphism h_0, developing section f_0 of $X(h_0)$, etc. We will prove that there is a neighborhood W of h_0 in $\mathrm{Hom}(\pi, G)$ such that every $h \in W$ is the holonomy of a "nearby" (G, X)-structure. First choose a contractible neighborhood W' of h_0 in $\mathrm{Hom}(\pi, G)$. Then there is a natural (G, X)-bundle over $\tilde{M} \times W'$ whose total space is the quotient of $\tilde{M} \times W' \times X$ by the action of π given by γ: $(u, h_t, x) \mapsto (\gamma u, h_t, h_t(\gamma)x)$. The covering homotopy property implies that this bundle is equivalent to the product $X(h_0) \times W'$, *as an X-bundle*. Fix a smooth trivialization of this bundle over W'. The foliation defining the flat structure on $X(h_t)$ varies continuously with respect to h_t in the C^1 topology. Using the trivialization over W', we find a smooth section f' of this bundle over M extending f_0. Since f_0 is transverse to the flat structure it follows that the restriction f_t of f' to $M \times \{h_t\}$ is also transverse, at least for h_t in a neighborhood W of h_0 in W. Thus for each $h_t \in W$ there is a (G, X)-structure with holonomy h_t. We shall refer to the new structures with holonomy h_t as structures "nearby" the original structure with holonomy h_0.

We shall need an elementary property of this construction:

Lemma 6. *Suppose that M_0 is a closed (G, X)-manifold whose holonomy homomorphism h centralizes a connected subgroup H of G which acts properly and freely on X. Consider deformations M_t of M_0 induced as above by deformations h_t of h_0 which have the form $h_t(\gamma) = h(\gamma)\phi_t(\gamma)$, where ϕ_t is a deformation of the trivial representation in* $\mathrm{Hom}(\pi, H)$. *Let* dev_t *denote the corresponding developing maps of M_t, and let p_H denote the projection map $X \to X/H$. Then, as h_t varies, the composite map $p_H \circ \mathrm{dev}_t$ remains constant. In particular, if M_0 is complete, then $p_H \circ \mathrm{dev}_t$ is a fibration with fibers the orbits of the corresponding local H-action.*

Proof of Lemma 6. The actions of π on the quotient X/H defined by h_t are all equal. The family of associated flat X/H-bundles $(X/H)(h_t)$ over M is a bundle over W. Since W is contractible, this bundle is trivial. Furthermore there exists a trivialization over W of the family $X(h_t)$ of X-bundles over M which extends the trivialization of the associated flat X/H-bundles $(X/H)(h_t)$. Let p_t denote the bundle map $X(h_t) \to (X/H)(h_t)$ which on each fiber is given by the projection map p_H: $X \to X/H$. With respect to the trivialization the developing sections, f_t are all equal. Thus $p_t \circ f_t$ is constant in the t-parameter. Passing to the universal covering $\tilde{M}$ we see that $p_H \circ \mathrm{dev}_t$ is constant as well. q.e.d.

Proof of Proposition 5. Let M_0 be the (G, X)-manifold $X/h_0(\pi)$. Let U be a neighborhood of 0 in $\mathrm{Hom}(\pi, B)$ such that for each $v \in U$, every (h_0, v) is the holonomy of a nearby (G, X)-manifold M_v. We shall show that M_v is complete.

Let dev: $\tilde{M} \to X$ denote the developing map of M_v. We must show that dev is bijective. By the lemma, the composition $p_B \circ \mathrm{dev}$: $\tilde{M} \to X/B$ is equivalent to the composition of p_B with the developing map of M_0 and hence is a fibration. Let β_X be the Killing vector field on X which generates the action of

B. Let β_M be the Killing vector field on M which corresponds to β_X, i.e. $p^*(\beta_M) = \text{dev}^*(\beta_X)$, where p: $\tilde{M} \to M$ is the covering projection. Since M is compact, the vector field β_M is complete and hence $p^*(\beta_M)$ is also complete. Let $\{\phi_s\}_{\{s\in\mathbf{R}\}}$ be the flow on M generated by $p^*(\beta_M)$ and $\{\psi_s\}_{\{s\in\mathbf{R}\}}$ the flow on X generated by β_X. Clearly $\text{dev}\circ\phi_s = \psi_s\circ\text{dev}$.

dev *is surjective*: Let $v \in X$. Since $p_B\circ\text{dev}$ is surjective, there exists $u \in \tilde{M}$ such that $p_B(\text{dev}(u)) = p_B(v)$. Since the fibers of p_B are the orbits of B (i.e. the trajectories of $p^*(\beta_M)$), there exists $s \in \mathbf{R}$ such that $\phi_s(\text{dev}(u)) = v$. It follows that $\text{dev}(\psi_s(u)) = v$, as desired.

dev *is injective*: Suppose that $u_0, u_1 \in M$ satisfy $\text{dev}(u_0) = \text{dev}(u_1)$. Since $p_B\circ\text{dev}$ is a fibration with fibers the trajectories of $p^*(\beta_X)$, there exists $s \in \mathbf{R}$ such that $\phi_s(u_0) = u_1$. Thus $\psi_s(\text{dev}\, u_0) = \text{dev}(u_1) = \text{dev}(u_0)$. As B acts freely on X, it follows that $s = 0$, and $u_0 = u_1$. Thus dev is bijective and M is complete. q.e.d.

Remarks. (i) It seems plausible to conjecture that for every representation $v \in \text{Hom}(\pi, \widetilde{\text{SL}}(2,\mathbf{R}))$ sufficiently near a standard representation, the homomorphism $(h_0, v) \in \text{Hom}(\pi, G)$ defines a properly discontinuous free action on X. It would also be interesting to know explicitly, for given h_0, which $v \in \text{Hom}(\pi, B)$ define properly discontinuous actions.

(ii) By taking B to be a parabolic one-parameter group, we note that the deformation space for complete Lorentz structures is *not Hausdorff*. Let (h_0, v) be a holonomy homomorphism for a nonstandard complete Lorentz structure as above, where v: $\pi \to B$. Let N be a hyperbolic one-parameter subgroup normalizing B; then the orbit of (h_0, v) under conjugation by N on the second factor contains the original homomorphism $(h_0, 1)$ in its closure. Thus the space of equivalence classes of holonomy representations, and hence the deformation space of complete (G, X)-structures, is not Hausdorff.

(iii) In a similar way, when B is parabolic *every* homomorphism v: $\pi \to B$ is realized as the second component of the holonomy of a nonstandard Lorentz structure on M. For the deformation arguments above realize an open neighborhood U of 1 in $\text{Hom}(\pi, B)$ and every homomorphism $\pi \to B$ is N-conjugate to one in U.

Proof of Theorem 3. Let M be a closed 3-manifold which is a product $F \times S^1$, where S is a closed surface and $\chi(F) < 0$. By [8] the holonomy representation h: $\pi \to G$ composed with the projection

$$p'\colon G \to G' = G/\text{center}(G) = \text{PSL}(2,\mathbf{R}) \times \text{PSL}(2,\mathbf{R})$$

is of the form (h_1, h_2), where either h_1 or h_2 is a Fuchsian representation. We may assume that h_1 is Fuchsian. Suppose the genus of F is g and that

$\langle A_1, B_1, \cdots, A_g, B_g | [A_1, B_1] \cdots [A_g, B_g] = I \rangle$ is the standard presentation for $\pi' = \pi_1(F) = \pi/\text{center}(\pi)$. (Compare [8, 7.1.1].)

Let μ be the element of π corresponding to the fiber; since μ is central in π, $h_1(\mu)$ centralizes $h_1(\pi)$ and $h_2(\mu)$ centralizes $h_2(\pi)$. Since $h_1(\pi)$ is Fuchsian, $h_1(\pi)$ must lie in the center of PSL(2, **R**), i.e. $h_1(\mu) = 1$. If $h_2(\pi)$ is nonabelian, then its centralizer is trivial and $h_2(\pi) = 1$. Otherwise $h_2(\pi)$ is abelian, in which case some power of $h_2(\mu)$ (which is a product of commutators in $h_2(\pi)$), is the identity element of PSL(2, **R**). Thus some power of $h(\mu)$ must lie in the center of G. By passing to a finite covering of M we may assume that $h(\mu) = 1$ and that h factors through a homomorphism h': $\pi' \to G$. Let h_1' and h_2' be the two components of the composition $p' \circ h'$: $\pi \to \text{PSL}(2, \mathbf{R}) \times \text{PSL}(2, \mathbf{R})$.

Now consider lifts $\widetilde{h(A_i)}$ of $h(A_i)$ (respectively $h(B_i)$ of $h(B_i)$) to the universal covering $G = \widetilde{\text{SL}}(2, \mathbf{R}) \times \widetilde{\text{SL}}(2, \mathbf{R})$ of G. Let $S = [h(A_1), h(B_1)] \cdots [h(A_g), h(B_g)]$. Since h_1 is Fuchsian, the projection of s on the first factor must be z^{2-2g}. Since h factors through π', the projection of s on the second factor of this element is also z^{2-2g}. Thus the Euler class of each representation h_1', h_2' equals $2 - 2g$. (Compare the proof of Theorem 7.2 in [8], as well as 7.1.1.)

We claim that this implies that the Lorentz volume of M is zero. For the G-invariant volume form on X determines a continuous 3-dimensional cohomology class $\omega \in H^3(BG^\delta)$ such that if f: $M \to BG^\delta$ is the classifying map of the flat G-structure on the tangent bundle, then $f^*\omega = \text{vol}(M)[M]$. (Here G^δ denotes G with the discrete topology. See [1], [2], [3] and [4] for more information on such classes.) Now the continuous cohomology of G can easily be computed from the extensions $\mathbf{Z} \to \tilde{G} \to G$ and $\mathbf{Z} \to G \to \text{PSL}(2, \mathbf{R}) \times \text{PSL}(2, \mathbf{R})$, in terms of the continuous cohomology of PSL(2, **R**) and its universal cover $\widetilde{\text{SL}}(2, \mathbf{R})$. The continuous cohomology of PSL(2, **R**) has one generator a in dimension 2 corresponding to the Euler class, and the continuous cohomology of $\widetilde{\text{SL}}(2, \mathbf{R})$ has one generator b in dimension 3 corresponding to its bi-invariant volume form. If $\mathbf{Z} \to S \to T$ is an extension of groups there is an exact Gysin sequence

$$\cdots \to H^i(T) \to H^i(S) \to H^{i+2}(S) \to H^{i+1}(T) \to \cdots$$

(all the H^i denoting continuous cohomology), where the first map $H^i(T) \to H^i(S)$ is induced from the homomorphism $S \to T$ and the second map $H^i(S) \to H^{i+2}(S)$ is given by cup product with the characteristic class in $H^2(S)$ corresponding to the extension $\mathbf{Z} \to S \to T$. In the Gysin sequence for the extension $\mathbf{Z} \to \widetilde{\mathrm{SL}}(2,\mathbf{R}) \to \mathrm{PSL}(2,\mathbf{R})$ the generator of $H^3(\widetilde{\mathrm{SL}}(2,\mathbf{R}))$ corresponding to the invariant volume form maps to the class in $H^2(\mathrm{PSL}(2,\mathbf{R}))$ corresponding to the Euler class [2].

Now let $j\colon \tilde{G} \to \tilde{G}$ be the involution given by $(x, y) \to (y, x)$; on X, j is represented by an orientation-reversing Lorentz isometry (thinking of X as $\widetilde{\mathrm{SL}}(2,\mathbf{R})$, this isometry is just $x \to x^{-1}$). Let b_1, b_2 be the generators of the continuous cohomology of G coming from the volume forms on each factor. Because j preserves the image of $\mathbf{Z} \to G$ and takes the class $\omega \in H^3(G)$ corresponding to Lorentz volume to $-\omega$, we see that the image of ω under $H^3(G) \to H^3(\tilde{G})$ is $b_1 - b_2$.

Now consider the extension $\mathbf{Z} \to G \to \mathrm{PSL}(2,\mathbf{R}) \times \mathrm{PSL}(2,\mathbf{R})$. One sees that the image of $\omega \in H^3(G)$ under the map $H^3(G) \to H^2(\mathrm{PSL}(2,\mathbf{R}) \times \mathrm{PSL}(2,\mathbf{R}))$ is the class $a_1 - a_2$. It follows that the volume $\omega(h)$ is given (up to a normalizing constant) by the difference of the Euler classes $e(h_1) - e(h_2)$. Thus if $e(h_1) = e(h_2)$, then $\mathrm{vol}(M) = \omega(h) = 0$.

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