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THE SIMPLE LOOP CONJECTURE

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1. Introduction

The main result of this paper is the proof of the so called Simple Loop Conjecture, Theorem 2.1. In §3 we prove analogous results for compact surfaces with boundary. In that setting simple arcs play the role of simple closed curves.

I wish to thank Allen Edmonds [see 2.2] for making me aware of the applicability of my work to this problem and to thank Joel Hass and Will Kazez for helpful conversations.

Notation. If $E \subset S$, then $N(E)$ denotes a tubular neighborhood of E in S, $\mathring{E}$ denotes interior of E, and $|E|$ denotes the number of components of E. See [1] or [3] for basic definitions regarding branched covers.

2. Closed surfaces

Theorem 2.1. *If f: $S \to T$ is a map of closed connected surfaces such that f_*: $\pi_1(S) \to \pi_1(T)$ is not injective, then there exists a non contractible simple closed curve $\alpha \subset S$ such that $f|\alpha$ is homotopically trivial.*

Proof. We will assume that $T \neq S^2$.

Step 1. Either there exists a noncontractible simple closed curve $\alpha \subset S$ such that $f|\alpha$ is homotopically trivial or f is homotopic to a simple branched cover (i.e., if f is a branched cover of degree d, then for every $x \in T$ $|f^{-1}(x)| \geqslant d-1$) or $T = \mathbf{RP}^2$ and there exists a simple branched cover f': $S \to T$ such that $\ker f_* = \ker f'_*$.

Proof of Step 1. Let D be a 2-disc in T. Let $\lambda_1, \cdots, \lambda_n$ be properly embedded arcs in $T - \mathring{D}$ such that $T - (D \mathring{\cup} N) = E$ is a 2-disc where N is a product neighborhood in $T - \mathring{D}$, of $\cup\lambda_i$.

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Let g be a map homotopic to f such that:

$$(*) \qquad \begin{aligned} &1.\ g\colon g^{-1}(D) \to D \text{ is an immersion;}\\ &2.\ g \text{ is transverse to } \bigcup \lambda_i; \end{aligned}$$

and which minimizes $c(g) = (|g^{-1}(D)|, |g^{-1}(\bigcup\lambda_i)|)$ where such pairs are lexicographically ordered.

Let $S' = S - g^{-1}(\mathring{D})$. Note that $g^{-1}(\bigcup\lambda_i)$ is a union of pairwise disjoint properly embedded simple arcs and simple closed curves in S'. If $T \neq \mathbf{RP}^2$ and some component C of $g^{-1}(\bigcup\lambda_i)$ is a simple closed curve, then C is noncontractible in S hence Step 1 holds. Otherwise C bounds a disc in S and one can find, using the fact $\pi_2(T) = 0$, a map g_1 homotopic to f with $c(g_1) < c(g)$, contradicting minimality. If $T = \mathbf{RP}^2$ and C bounds a disc F in S define $g'\colon S \to T$ so that

$$g'|S - \mathring{N}(F) = g|S - \mathring{N}(F) \text{ and } \left(g(N(F))\right) \cap \left(D \cup \left(\bigcup \lambda_i\right)\right) = \varnothing.$$

Observe that $\ker g'_* = \ker g_*$ and $c(g') < c(g)$. g' might not be homotopic to g. No component of $g^{-1}(\lambda_i)$ is an arc C such that $g|C$ does not map onto λ_i. Otherwise g is homotopic to a map g_1 satisfying $(*)$ such that $|g_1^{-1}(D)| = |g^{-1}(D)| - 2$, again contradicting minimality of g. We can therefore assume that either $f|f^{-1}(N \cup D) \to N \cup D$ is an immersion or we have found a simple loop in $\ker f_*$.

Let $H = f^{-1}(E)$ and $K = S - \mathring{H}$. $f|\partial H$ is an immersion into ∂E. Since $\pi_1(E) = 1$ and each component of K is nonplanar if $T \neq \mathbf{RP}^2$, H is a union of 2-discs or Step 1 holds. If $T = \mathbf{RP}^2$ and some component c of ∂H bounds a disc F in S but not in H, then we can find $f'\colon S \to T$ such that $\ker f'_* = \ker f_*$ but $c(f') < c(f)$. Two maps $h_1, h_2\colon (D^2, \partial D^2) \to (E, \partial E)$ are homotopic if and only if $\deg h_1 = \deg h_2$. In particular, if $\deg h_1 = p \neq 0$ then h_1 is homotopic to the branched cover h_3 defined by $z \to z^p$ (viewing F, E as unit discs in $\mathbf{C}$) and by perturbing h_3 slightly we can obtain a simple branched cover. It follows that if H is a union of 2-discs then $f|H$, hence f is homotopic to a simple branched cover.

Remark. It was pointed out to me that an almost identical version of Step 1 and its proof is contained in the unpublished work of Tucker [5].

Step 2. Construct $g\colon S \to T \times I$ such that the following 3 conditions hold.

1) The diagram

$$\begin{tikzcd} & T \times I \arrow[d, "p"] \\ S \arrow[ur, "g"] \arrow[r, "f"'] & T \end{tikzcd}$$

commutes where p = projection onto the first factor.

2) If $x \in T$ is a branch point there exists a disc $D_x \subset T$ such that $gf^{-1}(D_x)$ is a disjoint union of $n - 2$ horizontal (i.e., contained in Tx point) embedded discs and one nearly horizontal branched disc, as in Figure 2.1.

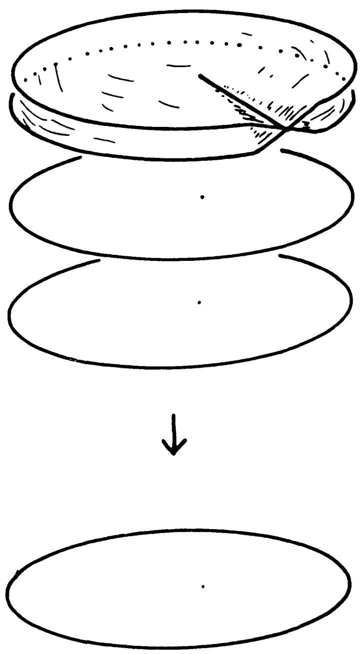

FIGURE 2.1

For each branch point x, let E_x be an open disc such that $E_x \subset D_x$. Let

$$S' = f^{-1}\Bigg(T - \bigcup_{x \text{ branch pts.}} E_x\Bigg).$$

3) $g|S'$: $S' \to N \times I$ is a general position immersion i.e., at most 3 distinct points of S' map to the same point of $N \times I$ and if D_1, D_2, D_3 are pairwise disjoint discs in S' such that $g|D_p$ $p = 1, 2, 3$ is an embedding, then $g(D_i)$ intersects $g(D_j)$, $g(D_j) \cap g(D_k)$ transversely for $i \neq j$ or k. q.e.d.

Observation. To each branch point x in $T \times I$ there exists an immersed double arc in $T \times I$ with one endpoint on x and another endpoint on y, y another branch point in $T \times I$.

Step 3. g can be homotoped to h: $S \to T \times I$ so that $h' = p \circ h$ is a branched cover. Step 2 holds with h, h' in place of g, f and each double arc (connecting branch points) is *embedded* in $T \times I$ and disjoint from all other double curves of $h(S)$ in $T \times I$.

Proof of Step 3. Induction on the number of triple points of $g(S)$ in $T \times I$. If J is an immersed double arc which is either not embedded or intersects other double curves, then J must pass through triple points. In particular there exists a double arc $J' \subset J$ such that one endpoint of J' is a branch point and the other end of J' is a triple point (Figure 2.2(a)). Now homotope g to g' as in Figure 2.2(b). Note that $p \circ g'$ is a branched cover, $g'(S)$ has one fewer triple point than $g(S)$ and after a small isotopy (to satisfy 2) of Step 2) g', $p \circ g'$ satisfy 1), 2), 3) of Step 2. Step 3 now follows by induction. q.e.d.

By homotoping g further so that the images, in T, of the double curves connecting branch points are very short and disjoint one can find pairwise disjoint discs $D_1, \cdots, D_r \subset T$ (where $2r$ = number of branch points) such that $gf^{-1}(D_i)$ appears as in Figure 2.3(a). See Figure 3.1 for a view of Figure 2.3 chopped in half.

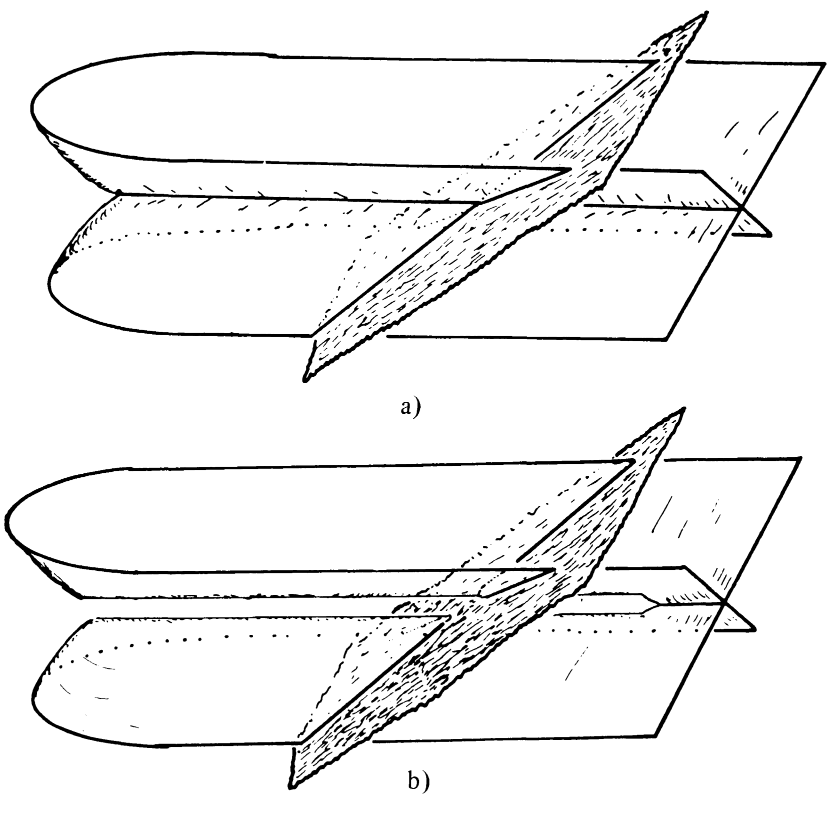

FIGURE 2.2

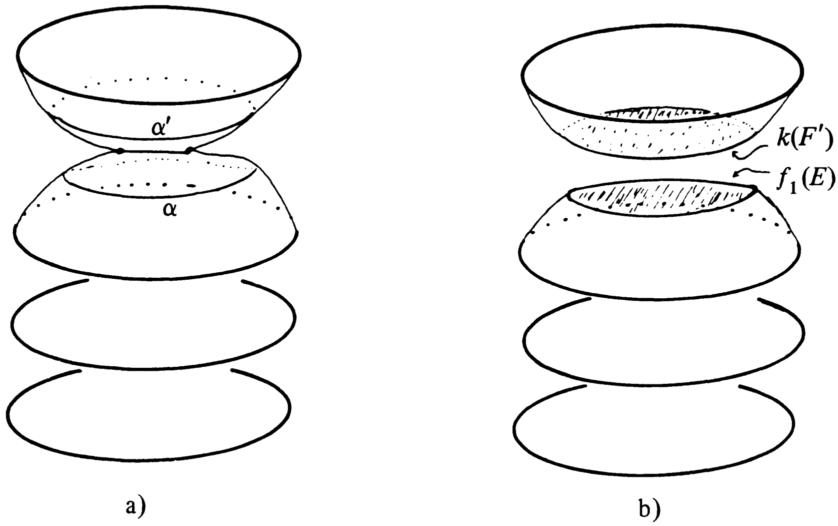

FIGURE 2.3

Since a branched cover without branch points is a covering map; hence, is injective on π_1, Theorem 2.1 follows by Step 4.

Step 4. α is homotopically trivial in T and $g^{-1}(\alpha) = \lambda$ is either a homotopically nontrivial simple closed curve in S or $T = \mathbf{RP}^2$ and there exists a map $f'\colon S \to T$ such that $\ker f'_* = \ker f_*$ and f' has fewer branch points than f.

Proof. α bounds a disc in $T \times I$ hence is homotopically trivial. We now suppose that λ is homotopically trivial in S for otherwise Step 4 has been completed.

λ and $\lambda' = g^{-1}(\alpha')$ (Figure 2.3(a)) bound an annulus in S and individually bound discs E, E' such that (after possibly changing the names of λ, λ') $E \supset E'$. It follows that there exist branched covers

$$k' = p \circ k\colon S^2 \to T, \qquad f' = p \circ f_1\colon S \to T,$$

such that

$$S^2 = E' \cup F', \quad F' \text{ a 2-disc} \quad \text{and} \quad k|E' = g|E'.$$

$k(F')$ is a horizontal disc (Figure 2.3(b)) such that $k(\partial F') = \alpha'$,

$$f_1|S - E = g|S - E,$$

$f_1(E)$ is a horizontal disc (Figure 2.3(b)) such that $f_1(\partial E) = \alpha$.

If $z \in \pi_1(S)$, then z can be represented by a curve $\gamma \subset S$ such that $\gamma \cap E = \varnothing$, therefore $f'_*(z) = f_*(z)$ hence $\ker f'_* = \ker f_*$. f' has at least 2 fewer branch points than f. Finally if $T \neq \mathbf{RP}^2$ the following Euler characteristic calculation yields, (where b is the number of branch points of k' and r is the degree of k')

$$2 = \chi(S^2) = r(\chi(T)) - b \leqslant 0.$$

Remarks. Partial results on this problem were obtained by Berstein and Edmonds in [3] and [2].

Acknowledgement 2.2. The author is grateful to Allen Edmonds for pointing out that the simple loop conjecture follows as a corollary from the remark stated without proof on page 502 of [4] (the remark claims that a stronger theorem than the one proven in [4] is in fact true).

The remark implies that if $f\colon S \to T$ is a continuous map of closed surfaces, then either there exists a simple loop in $\ker f_*\colon \pi_1(S) \to \pi_1(T)$, or one can find $g\colon S \to T \times I$ where g is an immersion, $p \circ g$ is homotopic to f and either $g(S)$ is transverse to the product fibration $T \times I$, except along saddle tangencies, or $g(S)$ is an immersion onto some fibre $T \times pt$. The latter implies that $p \circ g$ is a covering map, hence f is 1-1 on π_1 while the former could not occur. A point $x \in g(S)$ which is maximal in the I factor of $T \times I$ would correspond to a non saddle tangency between $g(S)$ and the fibration.

3. Surfaces with boundary

One cannot find in general noncontractible simple loops in the kernel of a map of surfaces with boundary. The following example is due to Tom Tucker. If $S = S^2 - 3$ discs, $T = S^1 \times I$ and $f\colon S \to T$ is the 2 fold branched cover, branched over a single point, then $\ker f_* \neq \varnothing$ but contains no simple loops.

For manifolds with boundary, simple non boundary parallel arcs play the role of simple loops.

Theorem 3.1. *If $f\colon S \to T$ is a map of bounded connected surfaces such that $f_*\colon \pi_1(S) \to \pi_1(T)$ is not injective, then there exists an essential simple arc $\alpha \subset S$ and a map g homotopic to f such that $g(\alpha)$ is a boundary parallel arc.*

This is an unpublished but known result. We indicate a proof along the lines of the proof of Theorem 2.1.

Proof. Apply the methods of Step 1 to conclude that either Theorem 3.1 holds or f is homotopic to a branched cover. Argue as in Steps 2 and 3 to homotope f so that double arcs in $T \times I$ emanating from branched points appear either in pairs (Figure 2.3) or as singles (Figure 3.1). By homotoping f a bit further we can assume that all such double arcs appear as in Figure 3.1. If

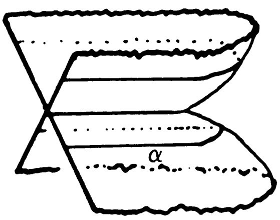

FIGURE 3.1

$T = D^2$ the result is trivial. If $T \neq D^2$ then arguing as in Step 4 shows that α (Figure 3.1) is the desired simple arc.

Question 3.2. Let $f\colon S \to T$ be a map between surfaces with boundary. When does there exist an essential simple closed curve $C \subset S$ such that $f|C$ is homotopically trivial?

References

[1] I. Berstein & A. Edmonds, *On the construction of branched coverings of low-dimensional manifolds*, Trans. Amer. Math. Soc. **247** (1979) 87–124.

[2] ______, *On the classification of generic branched coverings of surfaces*, Illinois J. Math., **28** (1984) 64–82.

[3] A. Edmonds, *Deformation of maps to branched coverings in dimension two*, Ann. of Math. (2) **110** (1979) 113–125.

[4] D. Gabai, *Foliations and the topology of 3-manifolds*, J. Differential Geometry **18** (1983) 445–503.

[5] T. Tucker, *On simple loops and surface maps*: A correction to *Boundary reducible 3-manifolds and Waldhausen's theorem*, preprint, 1975, unpublished.

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