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ON CARNOT-CARATHÉODORY METRICS

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1. Introduction

Consider a smooth Riemannian n-manifold (M, g) equipped with a smooth distribution of k-planes. Such a distribution Δ assigns to each point $m \in M$ a k-dimensional subspace of the tangent space $T_m M$. An absolutely continuous curve α in M is said to be horizontal if it is a.e. tangent to the distribution Δ. One may define a metric on M as follows.

Definition. The Carnot-Carathéodory distance between two points $p, q \in M$ is $d_c(p, q) = \inf_{\omega \in C_{p,q}}\{\text{length}(\omega)\}$, where $C_{p,q}$ is the set of all horizontal curves which join p to q. The metric d_c is finite provided that the distribution Δ satisfies Hörmander's condition (assuming that M is connected). To describe this condition, let $X_1, X_2, \cdots, X_k$ be a local basis of vector fields for the distribution near $m \in M$. If these vector fields, along with all their commutators, span $T_m M$, then the vector fields are said to satisfy Hörmander's condition at m. Denote by $V_i(m)$ the subspace of $T_m M$ spanned by all commutators of the X_j's of order $\leqslant i$ (including, of course, the X_j's). It is easy to see that $V_i(m)$ does not depend upon the choice of local basis $\{X_j\}$, so it makes sense to say that the distribution satisfies Hörmander's condition at m if $\dim V_i(m) = \dim(M)$ for some i. This infinitesimal transitivity implies local transitivity:

Theorem (*Chow*). *If a smooth distribution satisfies Hörmander's condition at $m \in M$, then any point $p \in M$ which is sufficiently close to m may be joined to m by a horizontal curve.*

Thus, if M is connected, the metric d_c is finite.

We will prove below the following two local theorems concerning the metric space (M, d_c) associated to a generic distribution Δ on M. (A distribution is said to be *generic* if, for each i, $\dim(V_i(m))$ is independent of the point

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$m \in M$.)

Theorem 1. *For a generic distribution Δ on M, the tangent cone of (M, d_c) at $m \in M$ is isometric to (G, d_c), where G is a nilpotent Lie group with a left-invariant Carnot-Carathéodory metric. (The tangent cone is defined in* §2, *Definition* 2.2.)

Theorem 2. *For a generic distribution Δ the Hausdorff dimension of the metric space (M, d_c) is*

$$Q = \sum_i i(\dim(V_i) - \dim(V_{i-1})).$$

See Hurewicz and Wallman [9] for a definition of Hausdorff dimension.

It should be pointed out, that Theorem 1 is a geometric version of the approximation procedure used by Rothschild-Stein and Goodman in their studies of hypoelliptic operators. Likewise, Theorem 2 may be viewed as a geometric analogue of Metivier's analytic results. A very nice discussion of the Rothschild-Stein approximation result and of the geometry associated to hypoelliptic operators may be found in Goodman [6]. More information concerning Carnot-Carathéodory metrics may also be found in Franchi & Lanconelli [14], Pansu [12].

2. Preliminaries

Carnot-Carathéodory metrics are closely related to nilpotent Lie groups. Consider, as an example, the Heisenberg group G, a simply connected, three-dimensional nilpotent Lie group (it is diffeomorphic to $\mathbf{R}^3$). Let X, Y generate the Lie algebra $\mathfrak{g}$, so that X, Y and $Z = [X, Y]$ are a vector space basis for $\mathfrak{g}$. There is a family of automorphisms $\{\delta_t\}$ of $\mathfrak{g}$, whose representation with respect to the basis X, Y, Z is

$$\begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^2 \end{pmatrix}.$$

Consider the left-invariant Riemannian metric g on G for which X, Y, Z are orthonormal. On $\mathfrak{g}$, this metric is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The metric $(1/t^2)g$ is isometric to $(1/t^2)\delta_t^*(g)$ (δ_t provides the isometry), which is easily seen to be represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^2 \end{pmatrix}.$$

Thus, as $t \to +\infty$, the lengths of vectors transverse to the distribution spanned by X and Y (thought of as left-invariant vector fields on G) become infinite, while the lengths of horizontal vectors remain unchanged. In the limit, only horizontal curves have finite lengths, and the sequence of metric spaces $(G, g/t^2)$ converges to the metric space (G, d_c). Thus, the global geometry of (G, g) is shrunk to the local geometry of (G, d_c). This phenomenon occurs for general nilpotent Lie groups:

Theorem (*Pansu*). *If G is a nilpotent Lie group with left-invariant Riemannian metric g, then*

$$\lim_{t \to +\infty} (G, g/t) = (\overline{G}, d_c),$$

where $\overline{G}$ is a nilpotent Lie group and d_c is a Carnot-Carathéodory metric on $\overline{G}$. If G is graded (*see Goodman*), *then $\overline{G} = G$; otherwise $\overline{G}$ is the graded nilpotent Lie group associated to G* (*see Pansu* [12]).

The limit used in the theorem above is the *Hausdorff limit* of a sequence of metric spaces, which we now define (see Gromov [7]).

Definition. *The Hausdorff distance* between two compact subsets A, B of metric space C is denoted by $H_C(A, B)$ and equals

$$\inf\{\,\varepsilon \mid B \subset N_\varepsilon(A), A \subset N_\varepsilon(B)\},$$

where N_ε denotes the ε-neighborhood.

The Hausdorff distance between two "abstract" compact metric spaces A, B is denoted $H(A, B)$ and equals $\inf_C H_C(A, B)$, where the infimum is taken over all isometric imbeddings of the pair A, B into all possible metric spaces C. (Note that such metric spaces exist; for example $C = A \times B$.)

A sequence $\{A_i\}$ of compact metric spaces is said to *converge in the sense of Hausdorff* to a metric space B if $\lim_{i\to\infty} H(A_i, B) = 0$. Note the following more practical definition (see Gromov [8]).

Theorem. *A sequence $\{A_i\}$ of compact metric spaces converges to B if and only if there is a sequence of positive real numbers $\varepsilon_i \to 0$ such that, for each i, there is an ε_i-dense net $\Gamma_i \subset A_i$ and an ε_i-dense net $\Gamma_i' \subset B$ which is ε_i-quasi-isometric to Γ_i.*

(An ε-dense net in a space A means a set of points with the property that each point of A is within distance ε of some point of the set. An ε-quasi-isometry between two metric spaces is a mapping which preserves distances up to a factor of $1 + \varepsilon$.)

If the spaces A_i are not compact, convergence will mean that for each $R > 0$, the balls of radius R about fixed base points in A_i converge to the ball of radius R about a fixed point in B.

Gromov has provided the following necessary and sufficient condition for existence of a convergent subsequence of a sequence of compact metric spaces.

Definition 2.1. The sequence $\{A_i\}$ is uniformly compact if

(i) the diameters, $\operatorname{diam}(A_i)$, are uniformly bounded.

(ii) For any $\varepsilon > 0$, the minimum number of ε-balls needed to cover A_i is bounded (uniformly in i).

One may use the notion of Hausdorff convergence to define the tangent cone of a metric space.

Definition 2.2. *The tangent cone of a metric space* (M, d) at a point $m \in M$ is $T_m M = \lim_{\lambda \to \infty}(M, \lambda \cdot d)$ if the limit exists. Of course, m is chosen as base point for all the spaces $(M, \lambda \cdot d)$.

Returning to the example of the Heisenberg group, it is easy to see that, in canonical coordinates,

$$d_c((0,0,0),(0,0,z)) \approx \sqrt{z},$$

for example. Thus d_c is, in general, not smooth so it is interesting to ask what its *Hausdorff dimension* (see Hurewicz & Wallman [9]) is. In this case, the answer is four. Theorem 2 answers this question in a more general setting.

3. Proofs of the theorems

Theorems 1 and 2 are based directly on the work of Rothschild-Stein, Goodman and Metivier, involving hypoelliptic operators. The theorem we need is stated below. It is due to Metivier and is based on techniques introduced by Goodman (see Goodman [6]).

Theorem (*see Metivier* [10]). *Let ω be a neighborhood of $\rho \in M$. Suppose that $v_i = \dim(V_i(x))$ is constant for each i ($x \in \omega$) and that $\dim(V_r(x)) = n = \dim(M)$ for some r. (Assume r is minimal.) Then for any $x_0 \in \omega$, there exist neighborhoods $\omega_1 \subset\subset \omega_0 \subset\subset \omega$ of x_0, a neighborhood U_0 of the origin 0 in $\mathbf{R}^n$, and a C^∞ mapping $\theta\colon \bar{\omega}_1 \times \omega_0 \to \mathbf{R}^n$ such that:*

(i) *For each $x \in \bar{\omega}_1$ the map $\theta_x\colon y \Rightarrow \theta(x, y)$ is a C^∞ diffeomorphism from ω_0 to $\theta_x(\omega_0) = U_0$, and $\theta_x(x) = 0$.*

(ii) *For each $x \in \bar{\omega}_1$, the vector fields $X_{i,x} = (\theta_x)_* X_i$, $i = 1, \cdots, k$, are of degree $\leqslant 1$ at* 0.

(iii) *If $\hat{X}_{i,x}$ denotes the homogeneous part of degree one of $X_{i,x}$, then the vector fields $\hat{X}_{i,x}$ generate a nilpotent Lie algebra of dimension n. Furthermore, let $\hat{V}_i(\xi) = V_i(\xi, \hat{X}_{1,x}, \cdots, \hat{X}_{k,x})$. Then* $\dim \hat{V}_i(\xi) = v_i$ *for all $\xi \in \mathbf{R}^n$, $i = 1, \cdots, r$.*

(iv) *The vector fields $\hat{X}_{i,x}$ and $X_{i,x}$ depend smoothly on $x \in \omega_1$.*

It should be noted that Metivier's theorem is based directly on the work of Goodman (see [6]).

To prove Theorems 1 and 2 we will define a one-parameter group of dilations of M (locally). Let us denote by X_I the m-fold commutator $[X_{i_1}, \cdots, [X_{i_{m-1}}, X_{i_m}] \cdots]$ for a multi-index $I = \{i_1, \cdots, i_m\}$. We may choose from among the X_I's a subset $\{Y_j\}$, $j = 1, \cdots, n$, of vector fields such that $\{Y_i\}_i$, is a basis of T_xM for all $x \in \omega$. Thus, any point x in ω (or in a smaller neighborhood, again denoted by ω) may be uniquely written in the form

$$x = \exp\left(\sum_{i=1}^{n} a_i Y_i\right)$$

for some real numbers a_i. The a_i are the normal coordinates of x. Define the dilation γ_r in terms of normal coordinates as follows:

$$(\gamma_r x)_i = r^{[i]} a_i, \quad \text{where } [i] = k \text{ if } \dim(V_{k-1}) < i \leqslant \dim(V_k).$$

The $\hat{X}_{i,x}$ are homogeneous with respect to γ_r.

One may choose, for each k, $1 \leqslant k \leqslant r$, a subset $\{\hat{X}_{jk,x}\}$, $j = 1, 2, \cdots$, of the commutators of the $\hat{X}_{i,x}$'s which yields a basis for $V_k(x)/V_{k-1}(x)$. A vector field Y on $\mathbf{R}^n$ may be written

$$Y = \sum_{j,k} a_{jk} \hat{X}_{jk,x}, \qquad a_{jk} \in C^\infty(M).$$

If we expand the a_{jk}'s in their Taylor series about zero in normal coordinates, Y will be exhibited as a formal sum of homogeneous differential operators. Y is of degree $\leqslant \lambda$ if each term in this formal sum is homogeneous of degree $\leqslant \lambda$. For the definition of this last term, see Goodman [6].

Let Δ_r be the distribution spanned by $\{\gamma_{r_*}(X_i)\}$, and let d_r denote the associated Carnot-Carathéodory metric. Δ_∞ will denote the distribution spanned by $\{\hat{X}_i\}$ and d_∞ is its associated metric. $B_r(k)$ and $S_r(k)$ denote the ball and sphere of radius k in the metric d_r, $1 \leqslant r \leqslant \infty$.

The proof of Theorem 1 is based on the following two lemmas.

Lemma 3.1. *d_r converges, in the sense of Hausdorff, to d_∞ as $r \to \infty$.*

Lemma 3.2. *The quasi-isometric distance between (M, rd_1) and (M, d_r) tends to zero as $r \to \infty$.*

The quasi-isometric distance between two metric spaces (X, d_X) and (Y, d_Y) is denoted (X, Y) and is defined as the logarithm of the infimum of the metric distortion of all homeomorphisms f: $X \to Y$. If X and Y are not homeomorphic, then $(X, Y) = \infty$.

The following lemma allows one to use Lemma 3.2 to obtain a bound on the Hausdorff distance $H((M, r \cdot d_1), (M, d_r))$. Together with Lemma 3.1, this will show that $(M, r \cdot d_1)$ is Hausdorff close to (M, d_∞) for large r.

Lemma 3.3. *If X and Y are two metric spaces with finite diameters, then*

$$\frac{H(X,Y)}{\operatorname{diam}(X)+\operatorname{diam}(Y)} \leqslant (X,Y).$$

Theorem 2 may be obtained from an estimate of $\operatorname{vol}(B_1(\varepsilon))$ (vol = Riemannian volume):

$$(*) \qquad C^{-1}\varepsilon^Q \leqslant \operatorname{vol}(B_1(\varepsilon)) \leqslant C\varepsilon^Q$$

for some $C > 1$ and all small ε, where Q is as in Theorem 2.

This in turn follows from the fact that, for large r, γ_r multiplies volumes of regions contained in $\gamma_{1/r}(B_1(1))$ by r^Q, up to a bounded factor, together with the following estimate.

Lemma 3.4. $B_1(1/cr) \subset \gamma_{1/r}(B_1(1)) \subset B_1(c/r)$ *for some* $c > 1$ *and all large* r.

Lemmas 3.2 and 3.4 are similar in content and will be proved simultaneously later.

Proof of Lemma 3.1. In order to demonstrate that the Hausdorff distance $H((M, d_r), (M, d_\infty))$ tends to zero as $r \to \infty$ we must, for any compact "ball" $B \subset M$, produce a metric space C and a family of isometric imbeddings F_r: $(B, d_r) \to C$ such that for all sufficiently large r the images $F_r(B, d_r)$ and $F_\infty(B, d_\infty)$ are close as subsets of C. The space C may be taken to be the space of continuous functions on B with metric δ induced by the supreme norm. The imbeddings are defined as follows.

For $m \in B$ define $F_r(m) = d_r(m, \cdot)|_B$; that is, a point $m \in B$ is sent to the distance function based at m, restricted to B. The images $F_r(B)$ and $F_\infty(B)$ will be close in C provided that $\delta(F_r(m), F_\infty(m))$ is small for each $m \in B$. Thus we wish to show that

$$\delta(d_r(m,\cdot), d_\infty(m,\cdot)) = \sup_{x\in B} |d_r(m,x) - d_\infty(m,x)| \leqslant E(r)$$

for all $m \in B$, where $E(r) \to 0$ as $r \to \infty$. This is done as follows. For any r_1 and r_2 and for each piecewise-smooth curve joining m to x which is tangent to Δ_{r_1} a.e. we produce a curve of the same length which is tangent to Δ_{r_2} a.e. and which joins m to a point x'. If r_1 and r_2 are large, x' will be close to x with respect to d_1, and so, by Lemma 3.5 below, x' will also be close to x with respect to the metric d_r for any large r.

Lemma 3.5. *There is a function* $F(\rho) > 0$ *defined for* $\rho > 0$ *such that* $F(\rho) \to 0$ *as* $\rho \to 0$ *and* $d_1(p,q) < \rho$ *implies* $d_r(p,q) < F(\rho)$ *for all* $r \geqslant R$ *and for any* $p, q \in B$. *This R may depend on* ρ *but not on p and q.*

Proof. We recall the main idea in the proof of Chow's theorem (see, Chow [1], Pansu [12]). First, one chooses a linearly independent set from among the

X_I's which spans $T_m M$. Let us denote the multi-index subscripts appearing in this set by $I_1, I_2, \cdots, I_n$. To each multi-index I we associate a flow ϕ on M as follows: If $I = i$, set $\phi_I(t) = \exp(tX_i)(m)$, and if $I = (i, J)$, set $\phi_I(t) = \phi_J(-\sqrt{t}) \circ \phi_i(-\sqrt{t}) \circ \phi_J(\sqrt{t}) \circ \phi_i(\sqrt{t})$. (Here (i, J) denotes the multi-index obtained by appending an i to the beginning of the multi-index J.) Now define a map ϕ: $\mathbf{R}^n \to M$ as

$$\phi(t_1, \cdots, t_n) = \phi_{I_n}(t_n) \circ \phi_{I_{n-1}}(t_{n-1}) \circ \cdots \circ \phi_{I_1}(t_1).$$

Note that $\phi(\vec{0}) = m$. It is easy to check that ϕ is as C^1 mapping and that $\phi_*(\partial/\partial t_j)|_{\vec{t}=\vec{0}} = X_{I_j}$ for $j = 1, \cdots, n$. The inverse function theorem implies that ϕ is a C^1 diffeomorphism near the origin. Moreover, by the construction of ϕ, $\phi(\vec{t})$ is the endpoint of a horizontal curve, so any point near $m \in M$ may be reached by a horizontal curve.

If we apply this construction to a local basis of vector fields for Δ_∞, we see that some Riemannian ball $B_m(\varepsilon)$ about $m \in M$ is contained in the image under ϕ of some ball $\mathbf{B}(\delta)$ in $\mathbf{R}^n$. Now it is clear that we may choose a local orthonormal basis $\{X_i^r\}$ for Δ_r which depends continuously on r for $1 \leqslant r \leqslant \infty$. We may then construct a map ϕ^r: $\mathbf{R}^n \to M$ associated to each basis $\{X_i^r\}$, and it is clear that $\phi^r|_B$ depends continuously on the vector fields used to define it, so $\phi^r|_B$ depends continuously on r. Thus, for large r, $\phi^r(B)$ contains $\mathbf{B}(\varepsilon/2, m)$, for example. With $\rho = \varepsilon/2$ and $F(\rho) = \delta$ we see that

$$d(q, m) < \rho \Rightarrow d_r(q, m) < F(\rho)$$

for large r. Clearly, we may take $\delta \to 0$ as $\varepsilon \to 0$ and the estimate is obviously uniform on compact sets in M, so Lemma 1.5 is proved.

To return to the proof of Lemma 3.1 we associate to any piecewise-smooth curve c_1 joining m to x which is tangent a.e. to Δ_{r_1} a curve c_2 of the same length which joins m to a point x' and which is tangent a.e. to Δ_{r_2}. If r_1 and r_2 are large, x' will be close to x. The procedure is as follows.

The curve c_1 satisfies

$$\dot{c}_1(t) = \sum_{i=1}^{n} a_i(t) X_i^{r_1}(c_1(t)), \qquad c_1(0) = m, c_1(T) = x,$$

for a.e. t, $0 \leqslant t \leqslant T$. Define c_2 by the conditions

$$\dot{c}_2(t) = \sum_{i=1}^{n} a_i(t) X_i^{r_2}(c_2(t)), \qquad c_2(0) = m,$$

for $0 \leqslant t \leqslant T$. Since we may assume that $\{X_i^r\}$ is an orthonormal set for all r, we have $\|\dot{c}_1(t)\| = \|\dot{c}_2(t)\|$ and therefore length(c_1) = length(c_2). An elementary estimate based on the Gronwall lemma (see [11]) shows that x' is

Riemannian close to x if r_1 and r_2 are sufficiently large. There is thus, by the previous lemma, a d_{r_2}-short path from x to x', and so

$$d_{r_2}(m, x) \leqslant d_{r_1}(m, x) + \varepsilon(R) \quad \text{for } r_1, r_2 \geqslant R,$$

where $\varepsilon(R) \to 0$ as $R \to \infty$. Similarly we see that

$$d_{r_1}(m, x) \leqslant d_{r_2}(m, x) + \varepsilon(R).$$

Again, the estimates are clearly uniform for all $m, x \in B$ if B is compact, so $H((B, d_{r_1}), (B, d_{r_2})) \to 0$ as r_1 and $r_2 \to \infty$. In particular, letting $r_1 = \infty$ we have

$$\lim_{r\to\infty} H((B, d_r), (B, d_\infty)) = 0.$$

This completes the proof of Lemma 3.1.

Proof of Lemma 3.4. We may identify a neighborhood in M with a neighborhood of $0 \in \mathbf{R}^n$ via θ. Let $B_1(1)$ denote the Carnot-Carathéodory ball centered at 0. The estimate in Lemma 3.4 may be paraphrased as follows: Up to bounded distortion, γ_r, applied to curves or vectors in $\gamma_{1/r}(B_1(1))$ which are tangent to Δ, multiplies length by r. For the proof, let $x_0 \in S_1(1)$. To estimate the Carnot-Carathéodory distance of $\gamma_{1/r}(x_0)$ from 0, we need to estimate how γ_r acts on vectors in Δ whose base points lie in $\gamma_{1/r}(B(1))$. Let $y \in B(1)$ and let $V \in \Delta(\gamma_{1/r}(y))$. Then

$$V = \sum_1 v_i \hat{X}_{i,x}|\gamma_{1/r}(y) + \sum_i v_i R_i|\gamma_{1/r}(y), \qquad v_i \in \mathbf{R},$$

where $R_i = X_{i,x} - \hat{X}_{i,x}$ is a vector field of degree $\leqslant 0$. Thus

$$\gamma_{r_*}(v) = r\sum_i v_i \hat{X}_{i,x} + \sum_i v_i \gamma_{r_*}\big(R_i(\gamma_{1/r}(y))\big)$$

since $\gamma_{r_*}(\hat{X}_{i,x}) = r \cdot \hat{X}_{i,x}$. Now the definition of local degree (see Goodman, Rothschild & Stein) implies that if R_i has degree $\leqslant 0$, then the length of $\gamma_{r_*}(R_i(\gamma_{1/r}(y)))$ remains bounded as $r \to \infty$.

(*Proof*. The homogeneous terms in the formal expansion of R_i as a sum of homogeneous operators (with respect to γ_r) look like $a_{jk,l}\hat{X}_{jk,x}$ if a_{jk} has the formal expansion $a_{jk} = \sum_{l=0}^{\infty} a_{jk,l}$, where $a_{jk,l}$ is a function homogeneous of degree l. Since

$$a_{jk,l}\big(\gamma_{1/r}(y)\big) = r^{-1} a_{jk,l}(y) \quad \text{and} \quad \gamma_{r_*}\big(\hat{X}_{jk,x}\big(\gamma_{1/r}(y)\big)\big) = r^k \hat{X}_{jk,x'},$$

we have

$$\gamma_{r_*}\big(a_{jk,l}\hat{X}_{jk,x}\big(\gamma_{1/r}(y)\big)\big) = r^{k-1} a_{jk,l}\hat{X}_{jk}(y).$$

"R_i is of local degree $\leqslant 0$" means $k - 1 \leqslant 0$, so such a term remains bounded as $r \to \infty$. This implies the result.)

Also, $\|R_i(\gamma_{1/r}(y))\| \to 0$ as $r \to \infty$ ("$\|\ \|$" denotes Riemannian length) since $R_i(0) = 0$. Therefore

$$\frac{1}{r}\frac{\|\gamma_{r*}(V)\|}{\|V\|} = \frac{1}{r}\frac{\|r\Sigma_i v_i \hat{X}_{i,x|y} + \Sigma_i v_i \gamma_{r*}(R_i(\gamma_{1/r}(y)))\|}{\|\Sigma_i v_i \hat{X}_{i,x}|\gamma_{1/r}(y) + \Sigma_i v_i R_i(\gamma_{1/r}(y))\|} \to 1$$

as $r \to \infty$, and so this expression is bounded above and below by $1/c$ and c respectively for some $c > 1$, for all sufficiently large r.

From this estimate on vectors we get the estimate on curves. If p: $[0,1] \to \mathbf{R}^n$ is a path joining 0 to $\gamma_{1/r}(x_0)$ which is tangent to the distribution Δ a.e. (recall that M is identified with $\mathbf{R}^n$ locally, via θ) and which lies in $\gamma_{1/r}(B(1))$, then $\gamma_r(p)$ is a path joining 0 to x_0. Its length is therefore bounded below by a positive constant, and with the inequality on vectors proved above, we see that

$$\text{const} \leqslant \text{length}(\gamma_r(p)) \leqslant r\,\text{length}(p),$$

which gives the left side of the inequality in Lemma 3.4.

Lemma 3.1 implies that $B_\infty(k) \subset B_r(k + \delta)$ for all large r and some δ. Also, it is clear that $B_1(1) \subset B_\infty(p)$ for some k, so $B(1) \subset B_r(k + \delta)$ for all large r. This shows that we may choose a piecewise-smooth path $\tilde{p}$ tangent to Δ_r and joining 0 to x_0, of length $\leqslant k + \delta = $ constant. Then $\tilde{p} = \gamma_{1/r}(p)$ is tangent to Δ, joins 0 to $\gamma_{1/r}(x_0)$ and satisfies

$$\text{length}(p) \leqslant \frac{\text{const}}{r} \quad \text{for some constant.}$$

This gives the right side of the inequality in Lemma 3.4. Note that we have proven that

$$\lim_{r\to\infty} \frac{\text{length}(\gamma_r(p))}{r\,\text{length}(p)} = 1,$$

which is precisely the meaning of Lemma 3.2.

Proof of Lemma 3.3. Suppose that (X, d_1) and (Y, d_2) are two metric spaces with finite diameters. If $(X, Y) < \infty$, then there is a homeomorphism f: $X \to Y$ whose distortion is arbitrarily close to $e^{(X,Y)}$. Identify Y with X via f, to obtain a single X with two metrics d_1 and d_2. We may imbed each of these metric spaces isometrically into a third metric space; namely, $C^0(X) =$ continuous functions on X with metric induced by the sup norm. A point $x \in X$ is sent to the point $F_i(x) = d_i(x, \cdot) \in C^0(X)$, $i = 1, 2$. For any x_1, x_2 $\in X$,

$$\left|\log\left(\frac{d_1(x_1, x_2)}{d_2(x_1, x_2)}\right)\right| \leqslant (X, Y)$$

and

$$\max\{d_1(x_1, x_2), d_2(x_1, x_2)\} \leqslant \operatorname{diam}(X) + \operatorname{diam}(Y).$$

It follows that

$$|d_1(x_1, x_2) - d_2(x_1, x_2)| \leqslant (1 - e^{-(X,Y)})(\operatorname{diam}(X) + \operatorname{diam}(Y)).$$

Thus $H(X, Y) \leqslant (\operatorname{diam}(X) + \operatorname{diam}(Y))(X, Y)$. q.e.d.

Theorem 1 now follows from Lemmas 3.1, 3.2 and 3.3.

Theorem 2 follows easily from the volume estimate (*) appearing below Lemma 3.3: choose a maximal set of disjoint balls (in the Carnot-Carathéodory metric) of radius ε which cover the unit ball $B_1(1)$. The number N_ε of such balls does not exceed $\operatorname{vol}(B_1(1))/C^{-1}\varepsilon^Q$. The set of concentric balls of radius 2ε cover $B_1(1)$. Each of these balls has diameter $\leqslant 4\varepsilon$, so the Hausdorff δ-measure of $B_1(1)$ is at most

$$\lim_{\varepsilon\to 0}\left[\frac{\operatorname{vol}(B_1(1))}{C^{-1}\varepsilon^Q}\cdot\varepsilon^\delta\right] = 0 \quad \text{if } \delta > Q.$$

Thus $\dim \leqslant Q$. Conversely, given any covering of $B_1(1)$ by sets of diameter $\leqslant \varepsilon$, there is an associated covering by balls of radius ε, so the number N_ε of sets in the covering satisfies

$$N_\varepsilon \cdot C \cdot \varepsilon^Q \geqslant \sum_{i=1}^{N_\varepsilon} \operatorname{vol}(i\text{th ball}) \geqslant \operatorname{vol}(B_1(1)).$$

Thus

$$\sum_{\text{covering}} \varepsilon^\delta \geqslant \frac{\operatorname{vol}(B_1(1))}{C\cdot\varepsilon^Q}\varepsilon^\delta.$$

Taking the infimum over all coverings by sets of diameter $\leqslant \varepsilon$, then taking the limit as $\varepsilon \to 0$, gives Hausdorff δ-measure of $B_1(1) = \infty$ if $\delta < Q$. Thus $\dim \geqslant Q$. This proves Theorem 2.

Remark. These estimates show that, in fact, μ^Q = Hausdorff Q-dimensional measure is commensurate with Lebesgue measure (on $B_1(1)$):

$$\left(\frac{V_Q}{C\cdot 2^Q}\right)\mu \leqslant \mu^Q \leqslant (C\cdot V_Q)\mu,$$

where μ = Lebesgue measure and V_Q = volume of unit ball in $\mathbf{R}^Q$.

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