J. DIFFERENTIAL GEOMETRY
20 (1984) 343–387

THE HEAT EQUATION ON A CR MANIFOLD

RICHARD BEALS, PETER C. GREINER
& NANCY K. STANTON

0. Introduction

The trace of the heat semigroup for the Laplacian on a compact oriented Riemannian manifold has an asymptotic expansion in powers of the time t for small positive t whose coefficients are integrals of local geometric invariants (see [1], [6], [11] and their references). This expansion and its generalizations to other elliptic operators have been powerful tools in the study of the relationship between analysis and geometry on the manifold (see the surveys in [4], [12] and [16]).

In this paper we prove analogous results for the sublaplacian $\square_b$ on a compact CR manifold. The classical pseudodifferential calculus is not adequate for this purpose because $\square_b$ is not elliptic, so we develop an appropriate pseudodifferential calculus here. To motivate a description of our methods and results we begin with a sketch of a proof of the Riemannian result along similar lines (see [7] for details). We then point out the differences due to difficulties in carrying the program over to the nonelliptic case.

Let M be a compact oriented Riemannian manifold and let $\Delta = d^*d$ denote the Laplace-Beltrami operator on functions. In local coordinates

$$\Delta = -\sum_{i,j} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} g^{ij} \sqrt{g} \frac{\partial}{\partial x^j}, \tag{0.1}$$

where the metric tensor is given by the matrix (g_{ij}) which has inverse g^{ij} and $g = \det(g_{ij})$. Let $P = \partial/\partial t + \Delta$ operating on functions on $M \times \mathbf{R}$. We seek to construct a parametrix for P, i.e. a pseudodifferential operator Q such that $PQ \equiv QP \equiv I$ modulo smoothing operators. If Q is a parametrix it is given by

Received February 17, 1984 and, in revised form, December 10, 1984. The research of the first author was supported in part by National Science Foundation grant MCS 8104234, the second in part by the National Research Council of Canada and the third in part by National Science Foundation grant MCS 8200442, the Alfred P. Sloan Foundation, and the Max Planck Institut für Mathematik.

integration against a kernel k which differs from the kernel of the heat semigroup $e^{-t\Delta}$ by a kernel which vanishes to infinite order at $t = 0$. Thus k contains all the asymptotic information. The construction of the symbol q of Q can be localized, and k is the partial inverse Fourier transform of q. Now q has an asymptotic expansion

$$q \sim \sum_{j=2}^{\infty} q_{-j}, \qquad q_{-j} = q_{-j}(x, \xi, \tau). \tag{0.2}$$

Here x denotes local coordinates on M with dual coordinates ξ on T^*M and τ is the variable dual to t. The symbol q_{-j} is homogeneous of degree $-j$ with respect to anisotropic dilations in the dual variables:

$$\lambda \cdot (\xi, \tau) = (\lambda\xi, \lambda^2\tau), \qquad \lambda \in \mathbf{R} \setminus 0. \tag{0.3}$$

The heat operator P has symbol $p = p_2 + p_1 + p_0$, where p_j is homogeneous of degree j. The principal symbol p_2 is

$$p_2(x, \xi, \tau) = \sum_{i,j} g^{ij}(x)\xi_i\xi_j + \sqrt{-1}\,\tau. \tag{0.4}$$

The calculus of symbols with this type of expansion is entirely analogous to the Kohn-Nirenberg calculus for symbols with ordinary homogeneity [9]. In particular, the symbol of a composition PQ has asymptotic expansion

$$p \circ q \sim \sum_{\alpha, j, k} \frac{1}{\alpha!} \partial_\xi^\alpha p_j(x, \xi, \tau) D_x^\alpha q_k(x, \xi, \tau). \tag{0.5}$$

Thus to have $PQ \equiv I$ we need $p \circ q \sim 1$. Looking at the term of degree 0 in (0.5) we find from (0.4) that

$$q_{-2}(x, \xi, \tau) = \left[\sum g^{ij}(x)\xi_i\xi_j + \sqrt{-1}\,\tau\right]^{-1}. \tag{0.6}$$

One way to obtain the remaining terms is to take Q_{-2} to have symbol q_{-2} and note that $PQ_{-2} = I - R$, where R has symbol beginning with a term of degree -1. (R is smoothing of order 1 in x and order $1/2$ in t.) Then one should have

$$Q \sim Q_{-2}(I + R + R^2 + R^3 + \cdots), \tag{0.7}$$

and the terms q_{-j} may be computed recursively. Note that $Q_{-2}R^k$ has terms only of degree $\leqslant -2 - k$. It can easily be verified that each term q_{-j} is a finite sum of terms of the form

$$r(x, \xi)\left[\sum g^{ij}(x)\xi_i\xi_j + \sqrt{-1}\,\tau\right]^{-k}, \tag{0.8}$$

where $r(x, \cdot)$ is a polynomial of degree $2k - j$ whose coefficients are polynomials in $g^{-1/2}$ and in the derivatives of the g_{ij}.

The kernel k of Q is related to the symbol by

$$(0.9)\quad k(x,t;y,s)=(2\pi)^{-n-1}\int e^{i(x-y)\cdot\xi+i(t-s)\cdot\tau}q(x,\xi,\tau)\,dy\,ds\,d\xi\,d\tau,$$

and the kernel of $e^{-t\Delta}$ is essentially $k_t(x,y)=k(x,t;y,0)$. Corresponding to the expansion (0.2) is an expansion

$$(0.10)\qquad k\sim\sum_{j\geqslant 2}k_{j-n-2},$$

where k_{j-n-2} is related to q_{-j} by (0.9). Of course (0.9) must be interpreted in the sense of distributions and we need a distribution which agrees with $q_{-j}(x,\cdot)$ for $(\xi,\tau)\neq 0$. Now (0.8) implies that q_{-j} has an extension to a function holomorphic in τ, $\tau\in\mathbf{C}_-$. A consequence is that the corresponding distribution may be taken to be homogeneous with respect to the dilations (0.3). Then the inverse Fourier transform also has a homogeneity property and, in particular,

$$(0.11)\qquad k_m(x,t;x,0)=t^{m/2}k_m(x),\qquad t>0.$$

Taking $\lambda=-1$ we see that $q_{-j}(x,\cdot)$ is an odd function of ξ when j is odd, so $k_m(x)=0$ when $m+n$ is odd. In summary,

$$(0.12)\qquad k_t(x,x)\sim t^{-n/2}\sum_{j=0}^{\infty}t^jk_{2j-n}(x).$$

The functions $k_m(x)$ are independent of our choice of coordinates. We use Riemannian normal coordinates centered at x, so that $g_{ij}(x)=\delta_{ij}$, and take the form (0.6) into account. Then $k_m(x)$ is a polynomial in the derivatives of the metric at x, hence a polynomial in the curvature and its covariant derivatives.

Now let M be a compact oriented CR manifold of dimension $2n+1$ equipped with a Hermitian metric, and suppose the Levi form satisfies condition $Y(q)$; e.g. suppose M is strictly pseudoconvex and $0<q<n$. Let $\square_{b,q}$ denote the $\bar\partial_b$-Laplacian acting on (p,q) forms. There is a corresponding heat semigroup $\exp(-t\square_{b,q})$; (see [15]). We construct a parametrix Q for the corresponding heat operator $\partial/\partial t+\square_{b,q}$ by constructing the asymptotic expansion (0.2) with q_{-j} in an appropriate symbol class. This class is a modification of the class considered in [3] which provides a parametrix construction for $\square_{b,q}$ itself. We take into account the natural weighting of the problem, so that the dual variable in the direction orthogonal to the maximal complex tangent space of M is a symbol of order 2, as is the dual to $\partial/\partial t$.

The principal symbol p_2 of $\partial/\partial t+\square_{b,q}$ is not algebraically invertible and the asymptotic expansion (0.5) is no longer valid in our class of symbols. As in

[3], however, there is an analogous asymptotic expansion in which the pointwise products of (0.5) are replaced by a more complicated composition which amounts to considering the composition of corresponding homogeneous left-invariant operators on a two-step nilpotent group. As in the Riemannian case, one sees that all terms q_{-j} can be computed in principle once one has found q_{-2}. Again the q_{-j} will, because of holomorphy, correspond to homogeneous distributions. Thus the trace of the heat kernel will have an expansion exactly analogous to (0.12).

We develop the necessary pseudodifferential calculus in §3 and show in §4 how to obtain the full asymptotic expansion of a parametrix once one has the principal term. As in the Riemannian case one must consider homogeneous functions as distributions. In the Riemannian case, in effect, one approximates an operator pointwise by a translation-invariant operator, by freezing the coefficients. Here we must approximate by operators which are left-invariant in a nilpotent Lie group structure which varies from point to point. We have to make sense of the convolution of homogeneous functions in order to calculate the symbol of the compositions of homogeneous left-invariant operators. The necessary technical tools are developed in §§1 and 2. In §5 we show how to obtain the principal symbol of the parametrix for second order operators $\partial/\partial t + \Box$, where $\Box$ is an operator on M satisfying a certain criterion of hypoellipticity, and we relate the parametrix to the heat semigroup. We specialize to $\Box_{b,q}$ in §6 and show that the results of §5 apply when M satisfies condition $Y(q)$, so that we have a formula for q_{-2}.

Our symbols q_{-j} no longer have the simple form (0.8) and for an arbitrary Hermitian metric we cannot say anything special about them. If the Levi form is nondegenerate and the metric is a Levi metric, then there is an analogue of (0.8). We introduce the appropriate notion, that of a uniform symbol, at the end of §4. At the end of §5 we derive a sufficient condition for the parametrix of an operator $\partial/\partial t + \Box$ to have a uniform symbol. For $\Box_b$, this condition is that the metric be a Levi metric. Thus in Theorem 6.35 we can give a more complete description of the coefficients of the asymptotic expansion of $\operatorname{tr}\exp(-t\Box_{b,q})$ for a Levi metric. In §7 we show that in the strictly pseudoconvex case with a Levi metric these coefficients have a geometric interpretation: they are integrals of polynomials in the covariant derivatives of the curvature and torsion of the Webster–C. Stanton connection [14], [19]. Thus we have for this case a complete analogue of the Riemannian result. We conclude in §8 by using scaling and $U(n)$-invariance to give a more precise description of the first and second coefficients in the asymptotic expansion.

In the case of a strictly pseudoconvex CR manifold with a Levi metric, Stanton & Tartakoff [17] obtained an exact formula for the kernel of

$\exp(-t\Box_{b,q})$, $0 < q < n$, using successive approximations to solve an integral equation. They used this to give a new proof of the asymptotic expansion in this case. M. Taylor, using a different pseudodifferential calculus, obtained a somewhat less precise expansion in the case of a nondegenerate Levi form (see [18]). In the appendix we show that the formal Neumann series for the parametrix of $\partial/\partial t + \Box_{b,q}$ converges suitably to the exact heat kernel, thereby giving a new proof of the result of Stanton and Tartakoff.

This research was done in part while the third author was a visitor at the Institut des Hautes Études Scientifiques and the Max-Planck-Institut für Mathematik. She wishes to thank these institutions for their hospitality.

1. Homogeneous functions and distributions

We shall consider functions and distributions homogeneous with respect to a certain family of nonisotropic dilations on $\mathbf{R}^{2n+2} = \mathbf{R}^{2n+1} \times \mathbf{R}$. The generic points of $\mathbf{R}^{2n+2}$ and its dual (also denoted $\mathbf{R}^{2n+2}$) will be denoted by

$$z = (x^0, x^1, \cdots, x^{2n}, t) = (x^0, x', t),$$
$$\zeta = (\xi_0, \xi_1, \cdots, \xi_{2n}, \tau) = (\xi_0, \xi', \tau),$$

respectively. The dilations are defined for $\lambda \in \mathbf{R} \setminus 0$ by

$$\lambda \cdot z = (\lambda^2 x^0, \lambda x', \lambda^2 t), \qquad \lambda \cdot \zeta = (\lambda^2 \xi_0, \lambda \xi', \lambda^2 \tau). \tag{1.1}$$

Note that we allow $\lambda < 0$. For a function u on $\mathbf{R}^{2n+2} \setminus 0$, let

$$u_\lambda(z) = u(\lambda \cdot z), \qquad \lambda \in \mathbf{R} \setminus 0. \tag{1.2}$$

This action is extended to distributions by the formula

$$\langle g_\lambda, u \rangle = \lambda^{-2n-4} \langle g, u_{1/\lambda} \rangle, \qquad g \in \mathscr{D}', u \in \mathscr{D}. \tag{1.3}$$

We choose homogeneous norms

$$\|z\| = \left[(x^0)^2 + |x'|^4 + t^2\right]^{1/4}, \qquad \|\zeta\| = \left[\xi_0^2 + |\xi'|^4 + \tau^2\right]^{1/4}, \tag{1.4}$$

where $|\ |$ denotes the euclidean norm in $\mathbf{R}^{2n}$. Then

$$\|\lambda \cdot z\| = |\lambda|\,\|z\|, \quad \|\lambda \cdot \zeta\| = |\lambda|\,\|\zeta\|, \qquad \lambda \in \mathbf{R} \setminus 0. \tag{1.5}$$

A function or distribution f is said to be *homogeneous* of degree $m \in \mathbf{Z}$ (with respect to the dilations (1.1)) if

$$f_\lambda = \lambda^m f, \quad \text{all } \lambda \in \mathbf{R} \setminus 0. \tag{1.6}$$

(1.7) Definition. $\mathscr{F}_m$ is the subspace of $C^\infty(\mathbf{R}^{2n+2} \setminus 0)$ consisting of functions which are homogeneous of degree m.

We have tacitly considered functions either of the variable z or of the dual variable ζ. In the next definition we consider functions of $\zeta = (\xi, \tau)$. Let $\mathbf{C}_-$ be the half-plane $\{\operatorname{Im} \tau < 0\}$ with closure $\overline{\mathbf{C}}_- = \{\operatorname{Im} \tau \leqslant 0\}$.

(1.8) Definition. $\mathscr{F}_{m,h}$ is the subspace of $\mathscr{F}_m$ consisting of functions $f(\xi, \tau)$ which extend to $(\mathbf{R}^{2n+1} \times \overline{\mathbf{C}}_-) \setminus (0)$ in such a way as to be C^∞ in all variables and holomorphic with respect to τ, $\tau \in \mathbf{C}_-$.

The extension is unique and will also be denoted by f; it will continue to be homogeneous with respect to the dilations (1.1), which act on $\mathbf{R}^{2n+1} \times \overline{\mathbf{C}}_-$.

(1.9) Proposition. *If f belongs to $\mathscr{F}_{m,h}$, then there is a distribution g such that g is homogeneous of degree m and g agrees with f on $\mathbf{R}^{2n+2} \setminus 0$.*

Proof. If $m > -2n - 4$, then f is locally integrable and defines a homogeneous distribution. If $m \leqslant -2n - 4$, then $f(\xi, \tau) = O(|\tau|^{-2})$ as $\tau \to \infty$, so the integral $f_1(\xi, \tau) = i\int_\infty^\tau f(\xi, \sigma)\, d\sigma$ exists and is independent of the path of integration in $\mathbf{C}_-$. One sees that f_1 is homogeneous of degree $m + 2$ by choosing the path to be a ray. This process can be repeated until we reach f_j, homogeneous of degree $m + 2j > -2n - 4$. Then f_j defines a homogeneous distribution g_j. We may take $g = D_\tau^j g_j$; it agrees with f on $\mathbf{R}^{2n+2} \setminus 0$ and as a derivative of a homogeneous distribution it is homogeneous.

(1.10) Remark. We extend the last remark. Suppose $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_{2n+1}) \in \mathbf{Z}_+^{2n+2}$. Set

$$\langle \alpha \rangle = 2\alpha_0 + (\alpha_1 + \alpha_2 + \cdots + \alpha_{2n}) + 2\alpha_{2n+1}. \tag{1.11}$$

If f is a smooth function or a distribution which is homogeneous of degree m, then the derivative $D^\alpha f$ is homogeneous of degree $m - \langle \alpha \rangle$.

We normalize the Fourier and inverse Fourier transforms between the Schwartz spaces $\mathscr{S} = \mathscr{S}(\mathbf{R}^{2n+2})$ as follows:

$$\hat{u}(\zeta) = \int e^{-iz\cdot\zeta} u(z)\, dz; \tag{1.12}$$

$$\check{v}(z) = (2\pi)^{-2n-2} \int e^{iz\cdot\zeta} v(\zeta)\, d\zeta. \tag{1.13}$$

As usual, these operations are extended by duality to tempered distributions using the formulas

$$\langle \hat{f}, v \rangle = (2\pi)^{2n+2} \langle f, (\tilde{v})^\vee \rangle, \tag{1.14}$$

$$\langle \check{g}, u \rangle = (2\pi)^{-2n-2} \langle g, (\tilde{u})^\wedge \rangle, \tag{1.15}$$

where $\tilde{v} = v(-\zeta)$ and $\tilde{u}(z) = u(-z)$.

One checks the relation between dilations and the Fourier transform:

$$(g_\lambda)^\vee = \lambda^{-2n-4} (\check{g})_{1/\lambda}, \qquad \lambda \in \mathbf{R} \setminus 0. \tag{1.16}$$

In particular, a tempered distribution is homogeneous of degree m if and only if its Fourier transform is homogeneous of degree $r = r(m) = -m - 2n - 4$.

We now characterize the distributions of Proposition 1.9 in terms of their inverse Fourier transforms.

(1.17) Proposition. *Suppose g is a tempered distribution which is homogeneous of degree m. Then the restriction of g to $\mathbf{R}^{2n+2} \setminus 0$ is smooth if and only if the restriction of $k = \check{g}$ to $\mathbf{R}^{2n+2} \setminus 0$ is smooth. If k also vanishes for $t < 0$, then the restriction of g belongs to $\mathscr{F}_{m,h}$. Conversely, if f belongs to $\mathscr{F}_{m,h}$, then the distribution g of Proposition 1.9 can be chosen so that $k = \check{g}$ vanishes for $t < 0$.*

Proof. Suppose the restriction of g is smooth. Given a multi-order α, homogeneity implies that $D^{\beta}(\zeta^{\alpha} g)$ is integrable at ∞ if $\langle\beta\rangle > \langle\alpha\rangle - r$, where $r = -m - 2n - 4$. Then $D^{\beta}(\zeta^{\alpha} g)$ is the sum of a compactly supported distribution and an integrable function, so $z^{\beta} D^{\alpha} k$ is continuous. It follows that k is smooth away from the origin. The same argument proves the converse.

If k vanishes for $t < 0$, then the Paley-Wiener-Schwartz theorem implies that $g = \hat{k}$ extends holomorphically to $\operatorname{Im} \tau < 0$, and the preceding argument adapts to show that the extension is smooth. Conversely, suppose f is in $\mathscr{F}_{m,h}$ and let g be constructed as in the proof of Proposition 1.9. It is enough to show that $\check{g}_j$ has support in $\{t \geqslant 0\}$, so we may assume that $m > -2n - 4$ and f is locally integrable. Choose $N > 0$ large enough that

$$f_{\varepsilon}(\xi, \tau) = (1 + i\varepsilon\tau)^{-N} f(\xi, \tau)$$

is integrable for every $\varepsilon > 0$. Then $f_{\varepsilon} \to f = g$ in the topology of $\mathscr{S}'$, so $f_{\varepsilon}^{\vee} \to k$ in S'. Classically, $f_{\varepsilon}^{\vee}$ has support in $\{t \geqslant 0\}$, so k does also.

2. Homogeneous functions and convolutions

Suppose $G = \mathbf{R}^{2n+1} \times \mathbf{R}$ has the composition law

$$(2.1) \qquad (x, s) \cdot (y, t) = (x_0 + y_0 + a(x', y'), x' + y', s + t),$$

where $a\colon \mathbf{R}^{2n} \times \mathbf{R}^{2n} \to \mathbf{R}$ is bilinear. Then G is a Lie group, which is abelian if the form a is symmetric and otherwise is a 2-step nilpotent group. Group translation is an affine map with Jacobian 1, so Lebesgue measure is translation-invariant and convolution is defined by

$$(2.2) \qquad (u * v)(z) = \int u(w^{-1}z) v(w)\, dw = \int u(w) v(zw^{-1})\, dw.$$

The dilations (1.1) are automorphisms of G and

$$(2.3) \qquad (u * v)_{\lambda} = \lambda^{2n+4} u_{\lambda} * v_{\lambda}, \qquad \lambda \in \mathbf{R} \setminus 0.$$

Convolution is associative and satisfies

$$(u * v)(z) = \langle u, (\tilde{v})_z \rangle, \tag{2.4}$$

$$\langle u_1 * u_2, v \rangle = \langle u_2, \tilde{u}_1 * v \rangle = \langle u_1, v * \tilde{u}_2 \rangle, \tag{2.5}$$

where

$$\langle u, v \rangle = \int uv, \qquad \tilde{v}(z) = v(z^{-1}), \qquad v_z(w) = v(wz^{-1}). \tag{2.6}$$

Formulas (2.4) and (2.5) allow one to extend convolution to various pairing of distributions. In particular it is not difficult to check:

(2.7) $\mathscr{E}' + \mathscr{S}$ is an algebra under convolution, and $\mathscr{S}$ is an ideal.

(As usual, $\mathscr{E}'$ is the space of compactly supported distributions.)

Any $g \in \mathscr{S}'$ defines a left-invariant operator $K\colon \mathscr{S}(G) \to \mathscr{E}(G)$ by

$$Ku = \check{g} * u. \tag{2.8}$$

We wish to define a composition of functions $f_j \in \mathscr{F}_{m_j,h}$ which would correspond to the composition of the operators associated, by (2.8), to the homogeneous distributions g_j which extend the f_j. Because of the associativity of convolution, the *formal* prescription is

$$T_0(f_1, f_2) = \text{restriction of } (\check{g}_1 * \check{g}_2)^{\wedge} \text{ to } \mathbf{R}^{2n+2} \setminus 0. \tag{2.9}$$

The difficulty lies in defining the convolution of two distributions which may grow at ∞. We avoid the difficulty by a trick.

(2.10) Definition. A function $g \in \mathscr{C}^\infty(\mathbf{R}^{2n+2})$ is said to be *almost homogeneous* of degree $m \in \mathbf{Z}$ if for each $\lambda \in \mathbf{R} \setminus 0$,

$$\lambda^{-m} g_\lambda - g \text{ belongs to } \mathscr{S}. \tag{2.11}$$

The function g is said to have *homogeneous part* $f \in \mathscr{F}_m$ if for each $N \geqslant 0$ and each α,

$$\lim_{|\zeta| \to \infty} \|\zeta\|^N D^\alpha [g(\zeta) - f(\zeta)] = 0. \tag{2.12}$$

If so, we write

$$f = \hom(g). \tag{2.13}$$

(2.14) Proposition. *If g is almost homogeneous of degree m, then it has a unique homogeneous part.*

Proof. Uniqueness is an immediate consequence of homogeneity and (2.12); indeed we remark for later use that one needs (2.12) only for $\alpha = 0$ and some fixed $N > 0$. To prove existence we set

$$f_r(\zeta) = 2^{-rm} g(2^r \cdot \zeta), \qquad \zeta \neq 0. \tag{2.15}$$

Note that for $\lambda, \mu \neq 0$, $N \geqslant 0$ and $\zeta \neq 0$, (2.11) gives

$$(2.16) \qquad \left|(\lambda\mu)^{-m} g_{\lambda\mu}(\zeta) - \lambda^{-m} g_\lambda(\zeta)\right| \leqslant C(\mu, N)\lambda^{-m-N}\|\zeta\|^{-N}.$$

Taking $\lambda = 2^r$, $\mu = 2$ and $N \geqslant -m + 1$ we obtain

$$(2.17) \qquad |f_{r+1}(\zeta) - f_r(\zeta)| \leqslant C(2, N)2^{-r}\|\zeta\|^{-N}.$$

Therefore the f_r converge uniformly on compact subsets of $\mathbf{R}^{2n+2} \setminus 0$ and the limit f satisfies

$$(2.18) \qquad |f(\zeta) - f_r(\zeta)| \leqslant C(2, N)2^{1-r}\|\zeta\|^{-N}.$$

Taking $\lambda = 2^r$ in (2.16), with μ fixed, we obtain homogeneity of f by letting $r \to \infty$. The estimates (2.12) with $\alpha = 0$ follow from (2.18), and the same argument applies to derivatives.

(2.19) Proposition. *Suppose g_j is almost homogeneous of degree m_j, $j = 1, 2$. Then the inverse Fourier transform k_j belongs to $\mathscr{E}' + \mathscr{S}$. The function $g = (k_1 * k_2)^\wedge$ is almost homogeneous of degree $m_1 + m_2$. The homogeneous part $f = \mathrm{hom}(g)$ is uniquely determined by $f_j = \mathrm{hom}(g_j)$, $j = 1, 2$.*

Proof. Let $r_j = -m_j - 2n - 4$ and $r = -m_1 - m_2 - 2n - 4$. Since g_j is smooth, the growth estimates (2.12) imply that $D^\beta(\zeta^\alpha g_j)$ is integrable when $\langle\beta\rangle - \langle\alpha\rangle > -r_j$, so $z^\beta D^\alpha k_j$ is bounded and continuous. Thus k_j is in $\mathscr{E}' + \mathscr{S}$. It follows that $k = k_1 * k_2$ is in $\mathscr{E}' + \mathscr{S}$ so $g = \hat{k}$ is smooth. By our hypotheses

$$(2.20) \qquad \lambda^{-r_j}(k_j)_\lambda - k_j \text{ belongs to } \mathscr{S}.$$

Now (2.20) and (2.7) imply

$$(2.21) \qquad \lambda^{-r}k_\lambda - k = (\lambda^{-r_1}k_{1,\lambda}) * (\lambda^{-r_2}k_{2,\lambda}) - k \quad \text{belongs to } \mathscr{S}.$$

Therefore $g = \hat{k}$ is almost homogeneous of degree $m = m_1 + m_2$. Finally, suppose g_j' is almost homogeneous with $\mathrm{hom}(g_j') = \mathrm{hom}(g_j)$. Then $g_j' - g_j$ is in $\mathscr{S}$, so (2.7) implies that the corresponding function g' differs from g by an element of $\mathscr{S}$. Thus $\mathrm{hom}(g') = \mathrm{hom}(g)$.

We shall be interested in a slight modification of (2.8) and, consequently, in Proposition 2.19. Set

$$(2.22) \qquad \psi(z) = -(z^{-1}), \qquad g^\# = \check{g} \circ \psi, \qquad k^\flat = (k \circ \psi^{-1})^\wedge.$$

Since ψ commutes with the dilations (1.1), it is easy to check that Proposition 2.19 remains valid if we take

$$(2.23) \qquad k_j = g_j^\#, \qquad g = (k_1 * k_2)^\flat.$$

Note that any $f \in \mathscr{F}_m$ is the homogeneous part of an almost homogeneous g; indeed one may take $g = \chi f$, where $\chi \in C^\infty$ is $\equiv 0$ near 0 and $\equiv 1$ near ∞. Therefore the following construction is well defined.

(2.24) Definition. Suppose f_j belongs to $\mathscr{F}_{m_j}$, $j = 1, 2$. Then $T(f_1, f_2)$ is the element of $\mathscr{F}_m$, $m = m_1 + m_2$, which is defined by

$$T(f_1, f_2) = \hom\left(\left[g_1^{\#} * g_2^{\#}\right]^{\flat}\right), \tag{2.25}$$

where the g_j are almost homogeneous with $\hom(g_j) = f_j$.

The structure of G is implicit in (2.25) through the convolution and also the map ψ of (2.22). We may make G explicit by writing $T(f_1, f_2; G)$.

(2.26) Remark. It is important for the proof which follows to note that the development starting with Definition 2.10 can be carried through with partial smoothness of finitely many derivatives to given order in (2.11) and (2.12). One then obtains a corresponding amount of regularity for the homogeneous part and for $T(f_1, f_2)$.

(2.27) Proposition. *Suppose f_j belongs to $\mathscr{F}_{m_j,h}$, $j = 1, 2$. Then $f = T(f_1, f_2)$ belongs to $\mathscr{F}_{m,h}$, $m = m_1 + m_2$.*

Proof. Choose $\varphi \in \mathscr{D}(\mathbf{R}^{2n+1})$ such that $\varphi \equiv 1$ near the origin. Given $M \in \mathbf{Z}_+$, set

$$\chi_M(\xi, \tau) = 1 - \varphi(\xi)\left[1 - (i\tau)^M(1 + i\tau)^{-M}\right]^M, \qquad \xi \in \mathbf{R}^{2n+1}, \tau \in \overline{\mathbf{C}}_-.$$

Let $g_{M,j} = \chi_M f_j$ and let g_j be an almost homogeneous smooth function with $\hom(g_j) = f_j$. Given any assigned degree of regularity and of agreement at ∞, the $g_{M,j}$ will be that regular and agree with g_j to that degree when M is large. Therefore for M large, $h_M = [g_{M,1}^{\#} * g_{M,2}^{\#}]^{\flat}$ will be well defined and will agree to a prescribed degree at ∞ with $h = [g_1^{\#} * g_2^{\#}]^{\flat}$. Consequently, $\hom(h_M) = T(f_1, f_2)$ for M large. Now convolution preserves the condition of having support in $\{t \geqslant 0\}$, so the Paley-Wiener-Schwartz Theorem implies that h_M has a holomorphic extension in τ which belongs to $C^m(\mathbf{R}^{2n+1} \times \overline{\mathbf{C}}_-)$, where $m = m(M) \to \infty$ as $M \to \infty$. The construction of $\hom(h_M)$ above exhibits $(|\xi|^2 + i\tau)^N \hom(h_M)$ as a uniform limit on compact subsets of $\mathbf{R}^{2n+2}$ of $(|\xi|^2 + i\tau)^N h_{Mr} = H_r$, where $N \geqslant \max(0, 1 - m)$. Because H_r extends holomorphically to $(\mathbf{R}^{2n+1} \times \mathbf{C}_-)\setminus 0$, $[(|\xi|^2 + i\tau)^N \hom(h_M)]^{\vee} = \lim(H_r)^{\vee}$ has support in $\{t \geqslant 0\}$. Thus $(|\xi|^2 + i\tau)^N h_M$ and $h_M = T(f_1, f_2)$ both have holomorphic extensions to $(\mathbf{R}^{2n+1} \times \overline{\mathbf{C}}_-)\setminus 0$. The same argument applies to derivatives, so $T(f_1, f_2)$ is smooth.

3. A class of pseudodifferential operators

Suppose $U \subset \mathbf{R}^{2n+1}$ is open and suppose $\{X_j\colon 0 \leqslant j \leqslant 2n\}$ are vector fields which are a frame for the tangent bundle TU. We adapt a family of group structures on $\mathbf{R}^{2n+1} \times \mathbf{R}$ and a family of pseudodifferential operators on $U \times \mathbf{R}$ to this frame.

Given $y \in U$ there is a unique set of affine coordinates $(x^0, x^1, \cdots, x^{2n})$ on $\mathbf{R}^{2n+1}$ such that y is the origin and

$$\left.\frac{\partial}{\partial x^j}\right|_0 = X_j(0). \tag{3.1}$$

We refer to these, and to the corresponding coordinates $(x^0, \cdots, x^{2n}, t)$ on $\mathbf{R}^{2n+1} \times \mathbf{R}$, as *the y-coordinates*. In the y-coordinates

$$X_j = \frac{\partial}{\partial x^j} + \sum b_{jk}(x)\frac{\partial}{\partial x^k}, \qquad b_{jk}(0) = 0. \tag{3.2}$$

Set

$$c_{jk} = c_{jk}(y) = \frac{\partial}{\partial x^k} b_{j0}(0), \quad 1 \leqslant j, k \leqslant 2n. \tag{3.3}$$

We make $\mathbf{R}^{2n+1}$ a group by defining composition in the y-coordinates as

$$\begin{aligned} (x \cdot z)^0 &= x^0 + z^0 + \sum_{j,k=1}^{2n} c_{jk} x^k z^j, \\ (x \cdot z)^j &= x^j + z^j, \quad 1 \leqslant j \leqslant 2n. \end{aligned} \tag{3.4}$$

Let G_y denote the direct product $\mathbf{R}^{2n+1} \times \mathbf{R}$ where $\mathbf{R}^{2n+1}$ has the composition (3.4). Then G_y has the form considered in §2. We provide it with the dilations (1.1).

By *y-invariant* we mean "invariant with respect to the left translations of G_y". The y-invariant vector fields determined by the y-coordinate directions at the origin are

$$X_0^y = \frac{\partial}{\partial x^0}; \qquad X_j^y = \frac{\partial}{\partial x^j} + \sum_{k=1}^{2n} c_{jk} x^k \frac{\partial}{\partial x^0}, \quad j > 0. \tag{3.5}$$

These provide good approximations to the X_j at y, as we make precise below.

Suppose $q = q(x, \zeta)$ is in $C^\infty(U \times \mathbf{R}^{2n+2})$, and suppose each derivative of q has at most polynomial growth at $\zeta = \infty$. Then q is the *symbol* of a pseudo-differential operator

$$\begin{aligned} Q &= \mathrm{Op}(q) \colon \mathscr{D}(U \times \mathbf{R}) \to \mathscr{E}(U \times \mathbf{R}), \\ Qu(x, t) &= (2\pi)^{-2n-2} \int e^{i(x\cdot\xi + t\tau)} q(x, \zeta) \hat{u}(\zeta)\, d\zeta. \end{aligned} \tag{3.6}$$

When the symbol is independent of τ we shall write $q(x, \xi)$. In particular the imaginary vector fields $-iX_j$ and $-iX_j^y$ have symbols σ_j and σ_j^y in the y-coordinates:

$$\sigma_j(x, \xi) = \xi_j + \sum b_{jk}(x)\xi_k; \tag{3.7}$$

$$\sigma_0^y(x, \xi) = \xi_0; \qquad \sigma_j^y(x, \xi) = \xi_j + \sum_{k=1}^{2n} c_{jk} x^k \xi_0, \quad j > 0. \tag{3.8}$$

Then, because of the choice of the c_{jk}, we have for $|x| \leqslant \delta = \delta(y)$

$$(3.9)\qquad \begin{aligned} &\sigma_j - \sigma_j^y = \sum a_{jk}(x)\sigma_k^y \quad \text{with } a_{jk}(0) = 0 \text{ for all } j, k; \text{ and} \\ &|a_{j0}(x)| \leqslant C(y)\|x\|^2 = C(y)\left((x^0)^2 + |x'|^4\right)^{1/2} \quad \text{for } j > 0. \end{aligned}$$

The group structure has been chosen precisely so that the X_j can be approximated closely at y, in the sense of (3.9), by left-invariant fields.

An affine homeomorphism φ: $\mathbf{R}^{2n+1} \to \mathbf{R}^{2n+1}$ induces a map φ^*: $\mathscr{D}(\varphi(U) \times \mathbf{R}) \to \mathscr{D}(U \times \mathbf{R})$,

$$(3.10)\qquad \varphi^* u(x, t) = u(\varphi(x), t).$$

If $Q = \mathrm{Op}(q)$, set

$$(3.11)\qquad Q_\varphi = (\varphi^*)^{-1} Q \varphi^*.$$

Then $Q_\varphi = \mathrm{Op}(q_\varphi)$, where

$$(3.12)\qquad q_\varphi(x, \xi) = q\left(\varphi^{-1}(x), (d\varphi)'\xi, \tau\right).$$

Since the X_j^y are y-invariant and the symbols $\sigma_j^y(x, \cdot)$ give coordinates on $\mathbf{R}^{2n+1}$, it follows that an operator $Q = \mathrm{Op}(q)$ with symbol defined on $\mathbf{R}^{2n+1} \times \mathbf{R}^{2n+2}$ is y-invariant if and only if q has the form

$$(3.13)\qquad q(x, \xi, \tau) = f(\sigma^y(x, \xi), \tau), \qquad \sigma^y = (\sigma_0^y, \cdots, \sigma_{2n}^y).$$

If so, then Q is a convolution in G_y:

$$(3.14)\qquad Qu = k * u,$$

where (in the y-coordinates) the distribution k is formally

$$(3.15)\qquad k(z^{-1}) = \check{f}(-z).$$

Any symbol $q \in C^\infty(U \times \mathbf{R}^{2n+2})$ can be written in the form

$$(3.16)\qquad q(x, \xi, \tau) = f(x, \sigma(x, \xi), \tau), \qquad f \in C^\infty(U \times \mathbf{R}^{2n+2}).$$

Given $y \in U$ we approximate $Q = \mathrm{Op}(q)$ at y by the y-invariant operator $Q^y = \mathrm{Op}(q^y)$, where

$$(3.17)\qquad q^y(x, \xi, \tau) = f(y, \sigma^y(x, \xi), \tau).$$

This is the nonabelian analogue of "freezing the coefficients". Since $\sigma^y(y, \xi) = \sigma(y, \xi)$, we have

$$(3.18)\qquad Q^y u(y) = Qu(y), \qquad u \in \mathscr{D}(U \times \mathbf{R}).$$

Classical symbols have asymptotic expansions with terms which are homogeneous with respect to the dual variables. We introduce here an analogous class with nonisotropic homogeneity using the form (3.16). Thus we begin with classes of functions f.

(3.19) Definition. For $m \in \mathbf{Z}$, let $\mathscr{F}_{m,h}(U)$ be the subspace of $C^\infty(U \times [(\mathbf{R}^{2n+1} \times \overline{\mathbf{C}}_-) \setminus 0])$ consisting of functions which are holomorphic on $\mathbf{C}_-$ with respect to the last variable and which are homogeneous of degree m with respect to the dilations (1.1) in the last $2n + 2$ variables.

(3.20) Definition. $\mathscr{F}_h^m(U)$ is the subspace of $C^\infty(U \times \mathbf{R}^{2n+2})$ consisting of functions f which have an asymptotic expansion

$$f \sim \sum_{j=0}^{\infty} f_{m-j}, \qquad f_{m-j} \in \mathscr{F}_{m-j,h}. \tag{3.21}$$

Here (3.21) means that for any multi-orders α, β and any $N \geqslant 0$,

$$\begin{gathered} \left| D_x^\alpha D_\zeta^\beta \left[f - \sum_{j < m+N} f_{m-j} \right] \right| \leqslant C_{\alpha\beta N} \|\zeta\|^{-N-\langle\beta\rangle}, \\ \text{if } 1 \leqslant \|\zeta\| = \|(\sigma, \tau)\| = \left(\sigma_0^2 + |\sigma'|^4 + \tau^2\right)^{1/4}, \end{gathered} \tag{3.22}$$

where $C_{\alpha\beta N}$ is a locally bounded function on U. Here again $\langle\beta\rangle = 2\beta_0 + (\beta_1 + \cdots + \beta_{2n}) + 2\beta_{2n+1}$.

(3.23) Definition. $S_{m,h}(U \times \mathbf{R})$ is the subspace of

$$C^\infty\left(U \times \left[\left(\mathbf{R}^{2n+1} \times \overline{\mathbf{C}}_-\right) \setminus 0\right]\right)$$

consisting of functions of the form (3.16), where f belongs to $\mathscr{F}_{m,h}(U)$.

(3.24) Definition. $S_h^m(U \times \mathbf{R})$ is the subspace of $C^\infty(U \times \mathbf{R}^{2n+2})$ consisting of functions of the form (3.16), where f belongs to $\mathscr{F}_h^m(U)$.

(We write $S_h^m(U \times \mathbf{R})$ rather than $S^m(U)$ because the corresponding operators are taken to act on functions on $U \times \mathbf{R}$.)

There is an asymptotic expansion for symbols in $S_h^m(U \times \mathbf{R})$ corresponding to (3.21), which we write as

$$q \sim \sum_{j=0}^{\infty} q_{m-j}, \qquad q_{m-j} \in S_{m-j,h}(U \times \mathbf{R}). \tag{3.25}$$

Note that the terms in (3.21) and (3.25) are unique.

(3.26) Remarks. Similar symbol classes were introduced in [3], though the variable τ was missing and homogeneity was assumed only for $\lambda > 0$. As in [3] the corresponding class of *operators* depends only on the sub-bundle of the tangent bundle TU which is generated by the X_j for $j > 0$. In particular, the operator class is independent of the affine structure of U and of the choice of frame for the sub-bundle. This and various assertions to follow can be proved with small modifications in the arguments of [3].

Given a sequence $q_{m-j} \in S_{m-j,h}(U \times \mathbf{R})$, $j \in \mathbf{Z}_+$, there is a symbol $q \in S_h^m(U \times \mathbf{R})$ with asymptotic expansion (3.25). This symbol is unique modulo the usual equivalence relation

$$q \sim q' \quad \text{if for each } \alpha, \beta, N,$$
$$\lim_{|\zeta| \to \infty} \|\zeta\|^N D_x^\alpha D_\zeta^\beta (q - q') = 0 \quad \text{uniformly on compact subsets of } U. \tag{3.27}$$

Easy estimates on derivatives establish inclusion relations between $S_h^m(U \times \mathbf{R})$ and certain Hörmander classes [8]:

$$S_h^m(U \times \mathbf{R}) \subset S_{1/2,1/2}^p(U \times \mathbf{R})$$
$$\text{with } p = m \text{ if } m \geqslant 0 \text{ and } p = \tfrac{1}{2}m \text{ if } m < 0. \tag{3.28}$$

Given any $q \in S_h^m(U \times \mathbf{R})$, there is a symbol $q' \in S_h^m(U \times \mathbf{R})$ such that

(3.29) $q \sim q'$ and $Q' = \mathrm{Op}(q')$ is properly supported, i.e. Q' maps $\mathscr{D}(U \times \mathbf{R})$ to itself and extends to map $\mathscr{E}(U \times \mathbf{R})$ to itself.

Our goal is to show that the properly supported operators with symbols in $\bigcup_m S_h^m(U \times \mathbf{R})$ are an algebra, and to describe the terms in the asymptotic expansion of a composition. To begin, we compose top-order terms by adapting the composition (2.25) to the present situation. Suppose

$$q_j(x, \xi, \tau) = f_j(x, \sigma(x, \xi), \tau), \qquad j = 1, 2, \tag{3.30}$$

where f_j is in $\mathscr{F}_{m_j,h}(U)$. Choose $\chi \in C^\infty(\mathbf{R}^{2n+2})$ with $\chi \equiv 0$ near 0 and $\chi \equiv 1$ near ∞. Set

$$q_j^y(x, \xi, \tau) = f_j(y, \sigma^y(x, \xi), \tau)\chi(\sigma^y(x, \xi), \tau), \tag{3.31}$$

and let Q_j^y be the y-invariant operator $\mathrm{Op}(q_j^y)$. In the y-coordinates the functions $q_j^y(y, \cdot)$ are almost homogeneous. In view of (3.15) we find that the homogeneous part of the symbol of $Q_1^y Q_2^y$ at y is obtained from

$$f(y, \cdot) = T(f_1(y, \cdot), f_2(y, \cdot); G_y). \tag{3.32}$$

Therefore we define

$$q_1 \# q_2 = q, \quad \text{where } q(y, \xi, \tau) = f(y, \sigma(y, \xi), \tau). \tag{3.33}$$

Now f depends smoothly on y, so f is in $\mathscr{F}_{m,h}(U)$, $m = m_1 + m_2$. We have proved

$$\# : S_{m_1,h}(U \times \mathbf{R}) \times S_{m_2,h}(U \times \mathbf{R}) \to S_{m_1+m_2,h}(U \times \mathbf{R}). \tag{3.34}$$

(3.35) Definition. If, in a given coordinate system, $q(x, \xi, \tau) = f(x, \sigma(x, \xi), \tau)$, then

$$q^{(\delta)}(x, \xi, \tau) = [\partial_\sigma^\delta f](x, \sigma(x, \xi), \tau), \tag{3.36}$$

$$\begin{aligned} q^{(\alpha\beta\gamma)}(x, \xi, \tau) &= [\partial_x^\alpha \partial_\sigma^\beta f](x, \sigma(x, \xi), \tau)\sigma(x, \xi)^\gamma \\ &= f^{(\alpha\beta\gamma)}(x, \sigma(x, \xi), \tau). \end{aligned} \tag{3.37}$$

(3.38) Theorem. *Suppose $Q_j = \mathrm{Op}(q_j)$, $j = 1, 2$, where q_j belongs to $S_h^{m_j}(U \times \mathbf{R})$. Suppose one or both of Q_j is properly supported, so that $Q = Q_1 Q_2$ is well defined. Then $Q = \mathrm{Op}(q)$ where q belongs to $S_h^m(U \times \mathbf{R})$, $m = m_1 + m_2$.*

If $q_{j,s} \in S_{s,h}(U \times \mathbf{R})$ are the terms in the asymptotic expansion of the q_j, then the term of order r in the asymptotic expansion of q has the form

$$\sum_{|\beta|=|\gamma|} h_{\alpha\beta\gamma\delta} q_{1,s}^{(\delta)} \# q_{2,t}^{(\alpha\beta\gamma)}; \tag{3.39}$$

$$r = s + t - \langle\delta\rangle - \langle\beta\rangle + \langle\gamma\rangle, \qquad \langle\delta\rangle \geqslant |\alpha| + |\beta| + \langle\gamma\rangle - \langle\beta\rangle. \tag{3.40}$$

The functions $h_{\alpha\beta\gamma\delta}$ are polynomials in the derivatives of the coefficients of the vector fields X_j.

Proof. A slightly less precise version of this theorem is proved for the analogous class of operators in [3]. The present version follows the same lines, so we sketch it briefly. It is enough to consider the case when each of q_1, q_2 has a single term in its asymptotic expansion. Suppose

$$q_2(x, \xi, \tau) = g(x, \sigma(x, \xi), \tau). \tag{3.41}$$

Fix a point $y \in U$. For convenience we work in the y-coordinates, so we identify y with $0 \in \mathbf{R}^{2n+1}$. Consider the Taylor expansion of (3.41) around the point $(0, \sigma^0(x, \xi), \tau)$:

$$\begin{aligned} q_2(x, \xi, \tau) &\sim \sum \frac{1}{\alpha!\beta!} g^{(\alpha,\beta)}(0, \sigma^0(x, \xi), \tau) x^\alpha (\sigma - \sigma^0)^\beta \\ &\sim \sum h_{\alpha\beta\gamma}(x) g^{(\alpha\beta\gamma)}(0, \sigma^0(x, \xi), \tau). \end{aligned} \tag{3.42}$$

Here $\alpha!\beta! h_{\alpha\beta\gamma}(x) = x^\alpha c_{\beta\gamma}(x)$, where

$$[\sigma(x, \xi) - \sigma^0(x, \xi)]^\beta = \sum_{|\gamma|=|\beta|} c_{\beta\gamma}(x) \sigma^0(x, \xi)^\gamma. \tag{3.43}$$

We want to compose Q_1 with the operator which corresponds to a single term in the second sum in (3.42), and to consider the symbol of the composition at $x = 0$. For this purpose we may replace Q_1 by Q_1^y, $y = 0$, and we compose Q_1^y first with the operation of multiplication by $h_{\alpha\beta\gamma}$. This last composition involves an operator of type $(1/2, 1/2)$ with a classical operator of type $(1, 0)$, so the standard asymptotic expansion is valid for the symbol:

$$\sum \frac{1}{\delta!} \partial_\xi^\delta q_1^y(0, \xi, \tau) D_x^\delta h_{\alpha\beta\gamma}(0). \tag{3.44}$$

We take $\delta! h_{\alpha\beta\gamma\delta} = D_x^\delta h_{\alpha\beta\gamma}$ and obtain the formal expansion (3.39). To show that only finitely many terms of given degree occur we want the limitation in (3.40),

$$\langle\delta\rangle \geqslant |\alpha| + |\beta| + \langle\gamma\rangle - \langle\beta\rangle. \tag{3.45}$$

Now (3.9) implies

$$(3.46)\qquad |c_{\beta\gamma}(x)| \leqslant c_{\beta\gamma}\|x\|^{|\beta|+\langle\gamma\rangle-\langle\beta\rangle} \quad \text{when } |\gamma| = |\beta| = 1.$$

Taking products gives (3.46) for all $|\gamma| = |\beta|$, so

$$(3.47)\qquad |h_{\alpha\beta\gamma}(x)| \leqslant C_{\alpha\beta\gamma}\|x\|^{|\alpha|+|\beta|+\langle\gamma\rangle-\langle\beta\rangle}$$

It follows that the derivatives of $h_{\alpha\beta\gamma}$ in (3.44) vanish at $x = 0$ except for δ satisfying (3.45).

To complete the proof we must estimate the error term which arises from truncating the Taylor expansion at $|\alpha + \beta| = N - 1$ in (3.42). The remainder term is the sum over $|\alpha + \beta| = N$ of

$$(3.48)\qquad \frac{1}{(N-1)!}\int_0^1 (1-a)^{N-1} g^{(\alpha,\beta)}(ax, \sigma^a, \tau)x^\alpha(\sigma - \sigma^0)^\beta\, da,$$
$$\sigma^a = (1-a)\sigma^0 + a\sigma.$$

We estimate for fixed $a \in [0, 1]$, then integrate. Write

$$(3.49)\qquad x^\alpha(\sigma - \sigma^0)^\beta = \sum_{|\gamma|=|\beta|} h^a_{\alpha\beta\gamma}(\sigma^a)^\gamma,$$

$$(3.50)\qquad q_{2,a}^{(\alpha\beta\gamma)} = g^{(\alpha,\beta)}(ax, \sigma^a(x,\xi), \tau)[\sigma^a(x,\xi)]^\gamma.$$

We introduce *weight functions*

$$(3.51)\qquad \Phi_a(x,\xi) = 1 + \|(\sigma^a(x,\xi),\tau)\|, \qquad \varphi_a = \left(1 + |(\sigma^a,\tau)|\right)^{-1}\Phi_a.$$

Dropping the subscript a, we work in the weighted pseudodifferential calculus [2]. Then

$$(3.52)\qquad q_{2,a}^{(\alpha\beta\gamma)} \in S^{s,0}_{\Phi,\varphi}, \qquad s = m_2 - \langle\beta\rangle + \langle\gamma\rangle.$$

If $q_1(x,\xi,\tau) = f(x, \sigma(x,\xi), \tau)$, set

$$(3.53)\qquad q_{1,a}^{(\delta)}(x,\xi,\tau) = \partial_\sigma^\delta f(0, \sigma^a(x,\xi), \tau).$$

Then

$$(3.54)\qquad q_{1,a}^{(\delta)} \in S^{r,0}_{\Phi,\varphi}, \qquad r = m_1 - \langle\delta\rangle.$$

It follows that when δ satisfies (3.45), the symbol of the composition of the operators with symbols (3.50) and (3.53) belongs to $S^{m-N,0}_{\Phi,\varphi}$. Now

$$(3.55)\qquad S^{t,0}_{\Phi,\varphi} \subset S^{t/2}_{1/2,1/2} \quad \text{if } t \leqslant 0.$$

Therefore if we argue as before and note that one obtains estimates uniform with respect to $a \in [0, 1]$, we find that the error term has symbol in $S^{(m-N)/2}_{1/2,1/2}$ when $m - N \leqslant 0$. Choosing N large, we control the error as desired.

4. Parametrices and kernels

In this section we complete the construction of the general machinery. Recall that an operator R: $\mathscr{D}(U \times \mathbf{R}) \to \mathscr{E}(U \times \mathbf{R})$ is said to be *smoothing* if it extends to a continuous map from $\mathscr{E}'(U \times \mathbf{R})$ to $\mathscr{E}(U \times \mathbf{R})$. This is true if and only if R has symbol which is rapidly decreasing in ζ, together with all derivatives, as $\zeta \to \infty$ [8]. Equivalently, R has a smooth kernel:

$$Ru(x,t) = \int r(x,t,y,s)u(y,s)\,dy\,ds,$$

$r \in C^\infty(U \times \mathbf{R} \times U \times \mathbf{R})$.

A *parametrix* for an operator P from $\mathscr{D}(U \times \mathbf{R})$ to $\mathscr{E}(U \times \mathbf{R})$ is a properly supported operator Q such that $QP - I$ and $PQ - I$ are smoothing.

(4.1) Theorem. *An operator $P = \mathrm{Op}(p)$ with $p \in S_h^m(U \times \mathbf{R})$ has a parametrix $Q = \mathrm{Op}(q)$ with $q \in S_h^{-m}(U \times \mathbf{R})$ if and only if there is a symbol q_{-m} in $S_{-m,h}(U \times \mathbf{R})$ such that*

$$q_{-m} \# p_m = 1 = p_m \# q_{-m}, \tag{4.2}$$

where p_m is the first term in the asymptotic expansion of p.

Proof. Suppose P has such a parametrix Q with leading symbol $q_{-m} \in S_{-m,h}(U \times \mathbf{R})$. The first term in the asymptotic expansion of $QP - I$ is $q_{-m} \# p_m - 1$ and the remaining terms vanish at $\zeta = \infty$, so since $QP - I$ is smoothing we must have $q_{-m} \# p_m = 1$ by homogeneity. Similarly, $p_m \# q_{-m} = 1$.

Conversely, suppose (4.2) holds. Let Q_{-m} be a properly supported operator with symbol in $S_h^{-m}(U \times \mathbf{R})$ having q_{-m} as the unique term in its asymptotic expansion. Then Theorem 3.38 implies that $Q_{-m}P = I - R$, where $R \in \mathrm{Op}\, S_h^{-1}(U \times \mathbf{R})$. We may assume that P is properly supported, so R is also. Then

$$(I + R + R^2 + \cdots + R^k)Q_{-m}P = I - R^{k+1}, \quad R^{k+1} \in \mathrm{Op}\, S_h^{-k-1}(U \times \mathbf{R}).$$

The term q_{-m-j} in the asymptotic expansion of $(I + R + \cdots + R^k)Q_{-m}$ is independent of k when $k \geqslant j$. Therefore we may choose $q \sim \sum q_{-m-j}$ and conclude that $Q = \mathrm{Op}(q)$, chosen to be properly supported, has the property

$$QP - I \in \bigcap_k \mathrm{Op}\, S_h^{-k}(U \times \mathbf{R}).$$

Thus $QP - I$ is smoothing. In the same way we construct Q' so that $PQ' - I$ is smoothing. As usual this implies that $Q - Q' = (QP - I)Q' - Q(PQ' - I)$ is smoothing, so we may replace Q' by Q. This completes the proof.

An operator Q with symbol $q \in S_h^m(U \times \mathbf{R})$ may be expressed symbolically as an integral operator. Indeed the symbol $q(x, \cdot)$ has an inverse Fourier transform $k_x \in \mathcal{S}'$ and

$$Qu(x,t) = \int_{U\times\mathbf{R}} k_x(x-y, t-s)u(y,s)\,dy\,ds \tag{4.3}$$

provided (4.3) is taken in the sense of distributions. The argument used to prove Proposition 1.17 shows that $k_x(\cdot)$ is smooth on $\mathbf{R}^{2n+2}\setminus 0$; indeed the function

$$K(x,y,t) = k_x(x-y,t) \tag{4.4}$$

is smooth except on $\{x = y, t = 0\}$. Moreover, if $m < -2n - 4$, then $q(x, \cdot)$ is integrable and K is continuous on $U \times U \times \mathbf{R}$.

(4.5) Theorem. *Suppose $Q = \mathrm{Op}(q)$, $q \in S_h^{-m}(U \times \mathbf{R})$. Modulo the addition of a smoothing operator, the kernel* (4.4) *associated to Q vanishes for $t \leqslant 0$ and has an asymptotic expansion for $t > 0$:*

$$K(x,x,t) \sim t^{-r}\sum_{j=0}^{\infty} t^j K_j(x), \qquad r = \text{greatest integer} \leqslant n + 2 - \frac{m}{2}. \tag{4.6}$$

Here (4.6) *means that for any $N \in \mathbf{Z}_+$,*

$$K(x,x,t) - t^{-r}\sum_{j=0}^{M} t^j K_j(x) \in C^N(U\times[0,\infty)) \quad \text{if } M \geqslant M(N). \tag{4.7}$$

Proof. From the asymptotic expansion for q we may write

$$k_x \sim \sum_{j=0}^{\infty} k_{-m-j,x} \tag{4.8}$$

with $k_{-m-j,x}$ the inverse Fourier transform of $q_{-m-j}(x, \cdot)$. When expressed in the x-coordinates, the distribution $k_{-m-j,x}$ is homogeneous of degree $m + j - 2n - 4$ and it vanishes for $t \leqslant 0$. For any $N \in \mathbf{Z}_+$ the map

$$x \mapsto k_x - \sum_{j\leqslant M} k_{-m-j,x} \tag{4.9}$$

is a C^N map from U to $C^N(\mathbf{R}^{2n+2})$ for M large. Therefore, modulo a smooth kernel, K vanishes for $t \leqslant 0$. Since $q_{-m-j}(x, \cdot)$ is homogeneous, it is an odd function of the variables $(\xi_1, \cdots, \xi_{2n})$. Therefore $k_{-m-j,x}(0,t) = 0$ when $m + j$ is odd. When $m + j$ is even, $k_{-m-j,x}(0,t)$ is homogeneous of degree $\frac{1}{2}(m + j - 2n - 4)$ in t. This completes the proof.

We shall want to examine the terms of the asymptotic expansion (4.6) in greater detail under some special assumptions, when Q is a parametrix for a differential operator P. If P has symbol in $S_h^m(U \times \mathbf{R})$, then P has the form

$$P = \sum_{\langle \alpha \rangle \leqslant m} a_\alpha X^\alpha, \qquad a_\alpha \in C^\infty(U), \tag{4.10}$$

where we take

$$X^\alpha = (X_0)^{\alpha_0}(X_1)^{\alpha_n} \cdots (X_{2n+1})^{\alpha_{2n+1}}, \qquad X_{2n+1} = \frac{\partial}{\partial t}. \tag{4.11}$$

Conversely, if P has the form (4.10) then its symbol is in $S_h^m(U \times \mathbf{R})$.

Our special assumption concerns the principal term of a parametrix for P. To phrase it precisely, we take an enumeration $\{d_k\}_{k=1}^\infty$ of all formal derivatives of the coefficients of the X_j and of P. Given a coordinate chart on U, these specialize to functions which we also denote by $\{d_k\}$.

(4.12) Definition. A symbol q is *uniform* if for each $y \in U$ there is a coordinate chart on U sending y to $0 \in \mathbf{R}^{2n+1}$ such that in these coordinates each x-derivative has the form

$$D_x^\alpha q(0, \xi, \tau) = \sum_{k=1}^\infty f_{\alpha k}(d_1(0), \cdots, d_k(0)) g_{\alpha k}(\xi, \tau). \tag{4.13}$$

Here the $f_{\alpha k}$ are polynomials (of which all but finitely many vanish, for fixed α) and the $g_{\alpha k}$ are functions, and the $f_{\alpha k}$ and $g_{\alpha k}$ do not depend on y.

(4.14) Theorem. *Suppose P has the form* (4.10) *and suppose P has a parametrix Q. Suppose the principal term q_{-m} of the symbol of Q is uniform. Then each term $K_j(y)$ in the asymptotic expansion* (4.6) *may be obtained by evaluating at y a universal polynomial in derivatives of the coefficients of the vector fields X_j and the coefficients a_α in the special coordinate system of* (4.12). *The polynomial may be chosen to depend only on n, m, j and the functions $f_{\alpha k}$, $g_{\alpha k}$ of* (4.13).

Proof. The properly supported pseudodifferential operators with symbols belonging to $\bigcup S_h^r(U \times \mathbf{R})$ are an algebra. Consider the subset consisting of operators with the property that each term in the asymptotic expansion of the symbol is uniform, with the same choice of coordinates at each point $y \in U$. The proof of Theorem 3.38 shows that this subset is a subalgebra. The operator P belongs to this subalgebra and we have assumed that Q_{-m} does also. Therefore the parametrix Q constructed in Theorem 4.1 also belongs to this subalgebra. To complete the proof we need only note that the inverse Fourier transform of a summand in (4.13), taken at $x = 0$ and $t > 0$, has the form

$$c_{\alpha k} f_{\alpha k}(d_1(0), \cdots, d_k(0)) t^a, \tag{4.15}$$

where $c_{\alpha k}$ depends on $g_{\alpha k}$ and a is an integer depending on n and the degree of homogeneity of the symbol.

5. The heat equation on a manifold

Suppose M is a compact smooth manifold of dimension $2n + 1$ and suppose $\mathscr{V} \subset TM$ is a sub-bundle of rank $2n$. On any sufficiently small coordinate neighborhood $U \subset M$ there is a frame $X_0, \cdots, X_{2n}$ for TU such that $X_1, \cdots, X_{2n}$ is a frame for $\mathscr{V}$.

(5.1) Definition. $\operatorname{Op} S^m(M \times \mathbf{R}, \mathscr{V})$ is the space of pseudodifferential operators from $\mathscr{D}(M \times \mathbf{R})$ to $\mathscr{E}(M \times \mathbf{R})$ which have, in each small coordinate neighborhood U, a symbol belonging to the class $S_h^m(U \times \mathbf{R})$ determined by the frame $\{X_j\}$ on U.

As noted earlier, the class of operators just defined is independent of the local choices of the X_j and the local coordinate charts. (It is also the case that in the applications we make to the asymptotic expansion of the heat kernel, this independence is not needed: everything may be done in local charts of one's choice.)

Consider now a second order differential operator $\square$: $\mathscr{E}(M) \to \mathscr{E}(M)$. Then $\square$ can be considered as mapping $\mathscr{D}(M \times \mathbf{R})$ to itself, and we assume

$$\square \text{ belongs to } \operatorname{Op} S^2(M \times \mathbf{R}, \mathscr{V}). \tag{5.2}$$

Then $P = \partial/\partial t + \square$ also belongs to $\operatorname{Op} S^2(M \times \mathbf{R}, \mathscr{V})$. We assume

$$P = \frac{\partial}{\partial t} + \square \text{ has a parametrix } Q \in \operatorname{Op} S^{-2}(M \times \mathbf{R}, \mathscr{V}). \tag{5.3}$$

Finally, we assume that M has a smooth density dx, and a corresponding Hermitian inner product

$$(u, v) = \int_M u(x)\overline{v(x)}\,dx, \qquad u, v \in \mathscr{E}(M). \tag{5.4}$$

We assume $\square$ is formally positive:

$$(\square u, u) \geqslant 0, \qquad u \in \mathscr{E}(M). \tag{5.5}$$

(5.6) Theorem. *Suppose* (5.2), (5.3), *and* (5.5) *are satisfied by* $\square$. *Then the following hold.*

(a) $\square$ *has a unique self-adjoint extension in* $L^2(M)$ *and the extension is nonnegative.*

(b) $\square$ *has eigenvalues* $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ *(counting multiplicities) with* $\lambda_j \to +\infty$. *The corresponding eigenfunctions* $\{u_j\}$ *belong to* $\mathscr{E}(M)$ *and their closed span is* $L^2(M)$.

(c) *For each* $t > 0$, *the operator* $e^{-t\square}$ *is a smoothing operator with trace*

$$\operatorname{tr}(e^{-t\square}) = \sum e^{-t\lambda_j}. \tag{5.7}$$

(d) *The trace has an asymptotic expansion*

$$\mathrm{tr}(e^{-t\Box}) \sim t^{-n-1} \sum_{j=0}^{\infty} k_j t^j \quad \text{as } t \to 0+. \tag{5.8}$$

(e) *The constants k_j in the expansion* (5.8) *have the form*

$$k_j = \int_M K_j(x)\, dx. \tag{5.9}$$

$K_j(x)$ can be computed from the term of degree $-2-2j$ in the asymptotic expansion of the symbol of the parametrix Q in local coordinates.

Proof. $\Box$ has at least one nonnegative self-adjoint extension, the Friedrichs extension [20]; we denote it also by $\Box$. Uniqueness will follow from (b), which implies that the closure of $\Box$ in $L^2(M)$ is self-adjoint.

Let $C_+(\mathbf{R}; L^2(M))$ denote the space of continuous functions $u\colon \mathbf{R} \to L^2(M)$ with the property that $u(t) = 0$ for $t \leqslant t_u$. Let us define

$$\begin{gathered} \tilde{Q}\colon C_+(\mathbf{R}; L^2(M)) \to C_+(\mathbf{R}; L^2(M)), \\ \tilde{Q}u(t) = \int_{-\infty}^{t} e^{(s-t)\Box} u(s)\, ds. \end{gathered} \tag{5.10}$$

Now $e^{-t\Box}$ is strongly differentiable and maps to the domain of $\Box$ for $t > 0$, and

$$\frac{d}{dt}(e^{-t\Box}) = -\Box e^{-t\Box}; \qquad e^{-t\Box}u \to u \quad \text{as } t \to 0+. \tag{5.11}$$

Therefore

$$\begin{gathered} \left(\frac{d}{dt} + \Box\right)\tilde{Q}u = u = \tilde{Q}\left(\frac{d}{dt} + \Box\right)u, \\ u \in \mathscr{D}(\mathbf{R}; \mathscr{E}(M)) = \mathscr{D}(M \times \mathbf{R}). \end{gathered} \tag{5.12}$$

By assumption

$$\left(\frac{\partial}{\partial t} + \Box\right)Q = I - R_1, \qquad Q\left(\frac{\partial}{\partial t} + \Box\right) = I - R_2, \tag{5.13}$$

where R_1, R_2 are smoothing. We assume that Q is properly supported, so R_1 and R_2 are also. Thus

$$\tilde{Q} = \tilde{Q}\left[\left(\frac{\partial}{\partial t} + \Box\right)Q + R_1\right] = Q + \tilde{Q}R_1, \tag{5.14}$$

$$\tilde{Q} = \left[Q\left(\frac{\partial}{\partial t} + \Box\right) + R_2\right]\tilde{Q} = Q + R_2\tilde{Q}. \tag{5.15}$$

Now (5.15) implies that $\tilde{Q}$ maps $\mathscr{D}(M \times \mathbf{R})$ to $\mathscr{E}(M \times \mathbf{R})$, which implies that $\tilde{Q}R_1$ is smoothing. Thus, from (5.14), $Q - \tilde{Q}$ is smoothing. In particular, $\tilde{Q}$

belongs to Op $S^{-2}(M \times \mathbf{R}, \mathscr{V})$ and has a distribution kernel

$$[\tilde{Q}u](x, t) = \int_{M \times \mathbf{R}} K(x, y, t - s) u(y, s)\, dy\, ds. \tag{5.16}$$

Comparing (5.16) and (5.10) we see that $K(x, y, t) = 0$ for $t < 0$ and $K(x, y, t)$ is the distribution kernel of $e^{-t\Box}$ for $t > 0$. But we know from §4 that K is smooth on $M \times M$ for $t > 0$, and this implies that $e^{-t\Box}$ is smoothing.

Parts (d) and (e) of Theorem 5.6 now follow from Theorem 4.5. To derive (b) and (c) we note that $e^{-\Box}$ is positive and self-adjoint, since $\Box$ is. Also, since $e^{-\Box}$ is smoothing, it is compact. Therefore $L^2(M)$ has an orthonormal basis consisting of eigenfunctions $\{u_j\}$ of $e^{-\Box}$ with eigenvalues $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant 0$, $\mu_j \to 0$. The eigenspace corresponding to a given eigenvalue μ is finite-dimensional, contained in the domain of $\Box$ and invariant for $\Box$ since $e^{-\Box}\Box = \Box e^{-\Box}$ on dom $\Box$. Therefore $\Box$ diagonalizes on this eigenspace and clearly each eigenvalue is $-\log \mu$. Thus the u_j are eigenvectors of $\Box$ with eigenvalues $\lambda_j = -\log \mu_j \to +\infty$.

Finally, the function $v_j(x, t) = e^{-\lambda_j t} u_j(x)$ is annihilated by $\partial/\partial t + \Box$, so (5.13) gives $v_j = R_2 v_j \in \mathscr{E}(M \times \mathbf{R})$, which implies $u_j \in \mathscr{E}(M)$.

(5.17) Remark. Note that the entire preceding development applies equally to systems. We may assume that the second order differential operator $\Box$ acts on smooth sections of a vector bundle over M. In local coordinates the kernels will be matrix-valued and the functions K_j of (5.9) will be matrix traces.

We turn now to the question of the existence of a parametrix for $P = \partial/\partial t + \Box$. In view of Theorem 4.1, one seeks the principal term q_{-2}. Let U, $\{X_j\}$ and $\mathscr{V}$ be as above. Let θ be the 1-form on U which is characterized by the conditions

$$\theta(X_0) \equiv 1; \qquad \theta(X_j) \equiv 0, \quad 1 \leqslant j \leqslant 2n. \tag{5.18}$$

We assume that in U, $\Box$ has the form

$$\Box = -\sum_{j=1}^{2n} X_j^2 - i\lambda X_0 + i \sum_{j=1}^{2n} \mu_j X_j + \nu, \tag{5.19}$$

where λ, μ_j and ν are smooth functions. Note that formal self-adjointness implies that these functions are real.

The matrix of smooth functions

$$a_{jk} = i\theta([X_j, X_k]), \qquad 1 \leqslant j, k \leqslant 2n, \tag{5.20}$$

is Hermitian and purely imaginary, so it has eigenvalues $\{a_j(y)\}$ which we number so that

$$a_j(y) \geqslant 0, \quad a_{n+j}(y) = -a_j(y) \qquad 1 \leqslant j \leqslant n. \tag{5.21}$$

(5.22) Theorem. *Suppose that for each $y \in M$ there is a coordinate neighborhood U and a frame $\{X_j\}$ for TU such that $\Box$ has the form* (5.19). *Suppose also that*

$$|\lambda(y)| < \sum_{j=1}^{n} a_j(y), \tag{5.23}$$

where the a_j are the nonnegative eigenvalues of the matrix (5.20) *at y. Then $P = \partial/\partial t + \Box$ has a parametrix Q which belongs to* $\mathrm{Op}\, S_h^{-2}(M \times \mathbf{R}, \mathscr{V})$.

Proof. Fix $y \in M$. In the associated y-coordinates (3.1) we know that

$$X_0^y = \frac{\partial}{\partial x^0}; \qquad X_j^y = \frac{\partial}{\partial x^j} + \sum_{k>0} c_{jk} x^k \frac{\partial}{\partial x^0}, \quad j > 0. \tag{5.24}$$

From (3.2), (3.3), and (5.20) one obtains

$$i a_{jk}(y) = c_{jk} - c_{kj}. \tag{5.25}$$

The matrix (c_{jk}) may be brought to anti-symmetric form by a quadratic coordinate transformation, in which the coordinate function x^0 is replaced by

$$(x^0)' = x^0 - \tfrac{1}{4} \sum_{j,k>0} (c_{kj} + c_{jk}) x^j x^k, \tag{5.26}$$

while the other coordinate functions are unchanged. Then in the new coordinates, (5.24) holds with c_{jk} replaced by the anti-symmetric part

$$c'_{jk} = \tfrac{1}{2} i a_{jk}(y). \tag{5.27}$$

Let us refer to these as the *anti-symmetric y-coordinates*. (They are uniquely determined by the original coordinates on U and the choice of a frame.)

There is an orthogonal transformation which brings the matrix (c'_{jk}) to normal form. This transformation allows us to replace $\{X_j\}_{j>0}$ by linear combinations and to make a corresponding orthogonal transformation of coordinates so that now

$$\begin{aligned} X_0^y &= \frac{\partial}{\partial x^0}; \quad X_j^y = \frac{\partial}{\partial x^j} - \frac{1}{2} a_j x^{n+j} \frac{\partial}{\partial x^0}, \\ X_{n+j}^y &= \frac{\partial}{\partial x^{n+j}} - \frac{1}{2} a_{n+j} x^j \frac{\partial}{\partial x^0}, \qquad 1 \leqslant j \leqslant n, \end{aligned} \tag{5.28}$$

where $a_j = a_j(y)$. We refer to these as *normal y-coordinates*. The form (5.19) is preserved, though the coefficients μ_j and ν may change.

In normal y-coordinates the principal term of the symbol of P^y, the y-invariant approximation to P at y, is

$$p_2^y = i\tau + \sum_{j>0} (\sigma_j^y)^2 + \lambda \sigma_0^y, \qquad \lambda = \lambda(y). \tag{5.29}$$

Here

$$\begin{aligned}\sigma_0^y(x,\xi) &= \xi_0; \quad \sigma_j^y(x,\xi) = \xi_j - \tfrac{1}{2}a_j x^{n+j}\xi_0, \\ \sigma_{n+j}^y(x,\xi) &= \xi_{n+j} - \tfrac{1}{2}a_{n+j}x^j\xi_0, \qquad 1 \leqslant j \leqslant n.\end{aligned} \tag{5.30}$$

We seek to determine the symbol q_{-2}^y so that the composition of the corresponding operators satisfies

$$P_2^y Q_{-2}^y = I = Q_{-2}^y P_2^y. \tag{5.31}$$

Now the symbol p_2^y is a polynomial in all variables, so the exact symbol of the composition can be calculated; (5.31) becomes

$$\sum_{|\alpha|\leqslant 2} \frac{1}{\alpha!}(\partial_\xi^\alpha p_2^y)(D_x^\alpha q_{-2}^y) \equiv 1 \equiv \sum_{|\alpha|\leqslant 2} \frac{1}{\alpha!}(\partial_\xi^\alpha q_{-2}^y)(D_x^\alpha p_2^y). \tag{5.32}$$

Because of y-invariance, q_{-2}^y has the form

$$q_{-2}^y(x,\xi,\tau) = f_y(\sigma^y(x,\xi),\tau). \tag{5.33}$$

Moreover, it is enough to have (5.32) hold at $x = 0$.

The normal form (5.29), (5.30) implies that P_2^y is invariant with respect to rotations in the (x^j, x^{n+j}) plane, $1 \leqslant j \leqslant n$. The same should be true of Q_{-2}^y, which implies finally

$$\sigma_j \frac{\partial}{\partial\sigma^{n+j}} f_y - \sigma_{n+j}\frac{\partial}{\partial\sigma^j} f_y \equiv 0 \quad \text{for } 1 \leqslant j \leqslant n. \tag{5.34}$$

Once (5.33) and (5.34) are taken into account, equation (5.32) at $x = 0$ becomes the single equation

$$\sum_{j=1}^{2n}\left[\sigma_j^2 - \tfrac{1}{4}a_j^2\sigma_0^2\left(\frac{\partial}{\partial\sigma_j}\right)^2\right] f + (i\tau + \lambda\sigma_0) f \equiv 1. \tag{5.35}$$

This equation has a solution

$$f_y(\sigma,\tau) = \int_0^\infty e^{-i\tau s - \lambda\sigma_0 s} G(\sigma,s)\,ds, \tag{5.36}$$

$$G(\sigma,s) = \prod_{j=1}^{2n} \cosh(a_j\sigma_0 s)^{-1/2} \exp\left(-\sigma_j^2 \frac{\tanh(a_j\sigma_0 s)}{a_j\sigma_0}\right). \tag{5.37}$$

(We take $b^{-1}\tanh(bs) = s$ if $b = 0$.) The assumption (5.23) gives absolute convergence of the integral (5.36) for $\operatorname{Im}\tau \leqslant 0$ and $(\sigma,\tau) \neq 0$, and one checks that f_y belongs to $\mathscr{F}_{-2,h}$.

It remains to be shown that these pointwise symbols fit together smoothly. Three changes must be considered: (i) the passage from normal y-coordinates to anti-symmetric y-coordinates; (ii) the passage to the original y-coordinates; (iii) the passage to the original fixed coordinates in U.

For the transition (i) let $A = A(y)$ be the linear transformation in T_y^*U which has matrix $(a_{jk}(y))$ in the anti-symmetric y-coordinates. Thus in normal coordinates A is diagonal and (5.37) takes the form

$$G(\sigma, s) = \left[\det\cosh(s\sigma_0 A)\right]^{-1/2}\exp\left(-\sigma_0^{-1}A^{-1}\tanh(s\sigma_0 A)\sigma'\cdot\sigma'\right),\quad \sigma' = (\sigma_1,\cdots,\sigma_{2n}). \tag{5.38}$$

The transition (i) is linear and we obtain (5.36), (5.38) as the expression in anti-symmetric y-coordinates.

For the transition (ii), suppose φ is a diffeomorphism of $\mathbf{R}^N$ with Jacobian $\equiv 1$ and with $\varphi(0) = 0$. Let us find at $x = 0$ the symbol q_φ in terms of the symbol q, where $Q_\varphi u = [Q(u\circ\varphi^{-1})]\circ\varphi$. (This notation is opposite to the convention adopted in (3.11), but it is more convenient here.) Then

$$Q_\varphi u(0) = c_N\int q(0,\eta)\int e^{-iz\cdot\eta}u\left(\varphi^{-1}(z)\right)dz\,d\eta,\qquad c_N = (2\pi)^{-N}. \tag{5.39}$$

Let $y = \varphi^{-1}(z)$ and express $u(y)$ by the Fourier inversion formula to obtain

$$q_\varphi(0,\xi) = c_N\int e^{-i\varphi(y)\cdot\eta + iy\cdot\xi}q(0,\eta)\,dy\,d\eta. \tag{5.40}$$

We apply this in $\mathbf{R}^{2n+2}$ with variables (x, t), dual variables (ξ, τ), and a quadratic transformation

$$\varphi(x,t) = (x^0 + Bx'\cdot x', x', t),\qquad x' = (x^1,\cdots,x^{2n}), \tag{5.41}$$

where B is a symmetric linear transformation. Then one may integrate in the first and last variables and dual variables in (5.40) to obtain

$$\begin{aligned} q_\varphi&(0,\xi,\tau)\\ &= c_{2n}\int \exp i\left[(\xi'-\eta')\cdot y' - \xi_0 By'\cdot y'\right]q(0,\xi_0,\xi',\tau)\,d\eta'\,dy'. \end{aligned} \tag{5.42}$$

In our case above

$$q\left(0,\xi_0,\eta',\tau\right) = \int_0^\infty F(\xi_0,\tau,s)\exp\left(-T_s\eta'\cdot\eta'\right)ds, \tag{5.43}$$

$$F(\xi_0,\tau,s) = e^{-i\tau s - \lambda\xi_0 s}\left[\det\cosh(s\xi_0 A)\right]^{-1/2}, \tag{5.44}$$

$$T_s = T_{s,\xi_0} = \xi_0^{-1}A^{-1}\tanh(s\xi_0 A). \tag{5.45}$$

Now

$$\begin{aligned} c_{2n}&\int\exp\left(-i\eta'\cdot y' - T_s\eta'\cdot\eta'\right)d\eta'\\ &= (4\pi)^{-n}(\det T_s)^{-1/2}\exp\left[-\tfrac{1}{4}T_s^{-1}y'\cdot y'\right]. \end{aligned} \tag{5.46}$$

Set $z = \frac{1}{2}T_s^{-1/2}y'$ and $\zeta = 2T_s^{1/2}\xi'$. Then (5.42), (5.43) and (5.45) give

$$(5.47)\qquad q_\varphi(0,\xi,\tau) = \pi^{-n}\int F(\xi_0,\tau,s)\exp(iz\cdot\zeta - C_s z\cdot z)\,ds\,dz,$$

where

$$(5.48)\qquad C_s = C_{s,\xi_0} = I + 4i\xi_0 T_s^{1/2}BT_s^{1/2}.$$

The transformation C_s can be diagonalized by an orthogonal transformation, so analytic continuation from the case $C_s > 0$ gives

$$(5.49)\qquad \pi^{-n}\int \exp(iz\cdot\zeta - C_s z\cdot z)\,dz = (\det C_s)^{-1/2}\exp(-\tfrac{1}{4}C_s^{-1}\zeta\cdot\zeta).$$

But

$$(5.50)\qquad C_s^{-1}\zeta\cdot\zeta = 4S_s\xi'\cdot\xi', \qquad S_s = \left(T_s^{-1} + 4i\xi_0 B\right)^{-1}.$$

Thus our expression in the y-coordinates is

$$(5.51)\qquad q_\varphi(0,\xi,\tau) = \int_0^\infty F(\xi_0,\tau,s)(\det S_s)^{1/2}(\det T_s)^{-1/2}e^{-S_s\xi'\cdot\xi'}\,ds.$$

The matrices of B and of iA in the y-coordinates are the symmetric and anti-symmetric parts of the matrix (c_{jk}), which itself consists of first partial derivatives of coefficients of the $\{X_j\}$. Moreover, T_s is a function of A and ξ_0. Thus in the original y-coordinates

$$(5.52)\qquad q_{-2}^y(0,\xi,\tau) = f(y,\xi,\tau), \qquad f\in\mathcal{F}_{-2,h}(U).$$

Transforming back to the original coordinates in U by the appropriate affine map one obtains

$$(5.53)\qquad q_{-2}(y,\xi,\tau) = f(y,\sigma(y,\xi),\tau)$$

with σ_j the symbol of X_j in the original coordinates. Thus q_{-2} does belong to $S_{-2,h}(M\times\mathbf{R},\mathscr{V})$.

To complete our general discussion, we turn to the question of the applicability of Theorem 4.14 in the present case. Thus we need to consider when the symbol q_{-2} is uniform (Definition 4.12).

Take as the special coordinate chart of Definition 4.12 the anti-symmetric y-coordinates. The symbol q_{-2} itself will have the form (5.52) with f independent of y, provided that $\lambda(y)$ and the $a_j(y)$ are independent of y and all the $a_j(y)$ are equal, $1\leqslant j\leqslant n$. Indeed, in this case

$$(5.54)\qquad \begin{aligned} &q_{-2}(0,\xi,\tau)\\ &\quad= \int_0^\infty e^{-i\tau s-\lambda\xi_0 s}\cosh(as\xi_0)^{-n}\ \exp\left(-\frac{\tanh(as\xi_0)}{a\xi_0}|\xi'|^2\right)ds\end{aligned}$$

in the anti-symmetric y-coordinates, where a is the common value of $a_j(y)$, $1 \leqslant j \leqslant n$, $\lambda = \lambda(y)$ and $\xi' = (\xi_1, \cdots, \xi_{2n})$. To see that derivatives also have the requisite form, note that in the anti-symmetric y-coordinates

$$q_{-2}(x, \xi, \tau) = h(d_1(x), \cdots, d_N(x), \xi, \tau), \tag{5.55}$$

where $d_1, \cdots, d_N$ are the coefficients of the X_j and their first derivatives, and where we have shown that h and $h(d_1(0), \cdots, d_N(0), \xi, \tau)$ do not depend on y.

6. The heat equation on a CR manifold

In this section we specialize to the Kohn Laplacian $\square_b$ on a compact connected oriented CR manifold M with a Hermitian metric. The CR structure is defined by a complex rank n sub-bundle $T_{1,0}$ of the complexified tangent bundle $\mathbf{C} \otimes TM$, having the properties

$$T_{1,0} \cap T_{0,1} = \{0\}, \quad \text{where } T_{0,1} = \overline{T}_{1,0}; \tag{6.1}$$

$$\text{if } Z \text{ and } W \text{ are sections of } T_{0,1}, \text{ so is } [Z, W]. \tag{6.2}$$

M is equipped with a Riemannian metric which extends to a Hermitian metric compatible with the CR structure:

(6.3) $T_{1,0} \perp T_{0,1}$ and complex conjugation is an isometry in $\mathbf{C} \otimes T_xM$.

Then there is a unique (real) line bundle $\mathcal{N} \subset TM$ such that

$$\mathbf{C} \otimes TM = T_{1,0} \oplus T_{0,1} \oplus \mathbf{C}\mathcal{N}. \tag{6.4}$$

The $\bar\partial_b$ complex of Kohn and Rossi [10] may be realized as follows. The bundle of covectors of type $(1,0)$ is

$$\Lambda^{1,0} = \{\text{annihilator of } T_{0,1} \oplus \mathcal{N}\} \subset \mathbf{C} \otimes T^*M. \tag{6.5}$$

Similarly,

$$\Lambda^{0,1} = \{\text{annihilator of } T_{1,0} \oplus \mathcal{N}\} \subset \mathbf{C} \otimes T^*M. \tag{6.6}$$

The bundle of covectors of type (p, q) is

$$\Lambda^{p,q} = (\Lambda^{1,0})^p \wedge (\Lambda^{0,1})^q \subset \Lambda^{p+q} = \Lambda^{p+q}(\mathbf{C} \otimes T^*M), \tag{6.7}$$

where the exponents refer to the iterated wedge product. The Hermitian form induces by duality an inner product on each fiber of $\mathbf{C} \otimes T^*M$ and therefore an inner product $\langle\ ,\ \rangle$ on each fiber of Λ^{p+q}. Let $\pi_{p,q}$ denote the orthogonal projection

$$\pi_{p,q}\colon \Lambda^{p+q} \to \Lambda^{p,q}. \tag{6.8}$$

A (p, q) form is a section of $\Lambda^{p,q}$; the space of (p, q) forms is denoted $\mathscr{E}^{p,q}$. From now on we fix p and set

$$(6.9) \qquad \bar\partial_{b,q}: \mathscr{E}^{p,q} \to \mathscr{E}^{p,q+1}, \qquad \bar\partial_{b,q} = \pi_{p,q+1} \circ d,$$

where d is the exterior derivative. Then

$$(6.10) \qquad \bar\partial_b: \mathscr{E}^p = \bigoplus_q \mathscr{E}^{p,q} \to \mathscr{E}^p$$

is a chain complex.

The metric induces an inner product in $\mathscr{E}^{p,q}$,

$$(6.11) \qquad (f, g) = \int_M \langle f(x), g(x) \rangle dV(x),$$

where dV is the volume form. Then $\bar\partial_{b,q}$ has a formal adjoint $\vartheta_{b,q}$. The associated Laplacian is

$$(6.12) \qquad \square_{b,q} = \vartheta_{b,q}\bar\partial_{b,q} + \bar\partial_{b,q-1}\vartheta_{b,q-1},$$

and it is this operator which plays the role of the operator $\square$ of §5.

Locally we may choose an orthonormal frame $\theta^1, \theta^2, \cdots, \theta^n$ for $\Lambda^{1,0}$; then $\bar\theta^1, \bar\theta^2, \cdots, \bar\theta^n$ is an orthonormal frame for $\Lambda^{0,1}$. The $2n$-form

$$(6.13) \qquad \omega = i^n \theta^1 \wedge \bar\theta^1 \wedge \cdots \wedge \theta^n \wedge \bar\theta^n$$

is real and is independent of the choice of the frame; thus ω can be considered as a globally defined element of $\mathscr{E}^{n,n}$. Locally there is a real 1-form θ of length 1 which is orthogonal to $\Lambda^{1,0} \oplus \Lambda^{0,1}$. Note that θ is unique up to sign and can be specified uniquely by requiring that the map

$$(6.14) \qquad f \to \int_M f\theta \wedge \omega, \qquad f \in C_c(U),$$

define a positive measure on the domain U of θ. Therefore θ, so chosen, is a uniquely determined global one-form and

$$(6.15) \qquad dV = \theta \wedge \omega$$

is the volume form on M. Note that θ annihilates $T_{1,0} \oplus T_{0,1}$.

(6.16) Definition. The *Levi form* is the Hermitian function-valued form defined on sections of $T_{1,0}$,

$$(6.17) \qquad L(Z, W) = i\theta([Z, \overline{W}]),$$

where θ is the annihilator of $T_{1,0} \oplus T_{0,1}$ chosen above.

Note that

$$(6.18) \qquad L(Z, W) = -\mathrm{i}d\theta(Z, \overline{W}) \quad \text{if } Z \text{ and } W \text{ are sections of } T_{1,0}.$$

Therefore L induces (or is induced) pointwise by the form $-\mathrm{i}d\theta_x$ on the Hermitian vector space $(T_{1,0})_x$.

(6.19) Definition. The *eigenvalues* of the Levi form at $x \in M$ are the eigenvalues of the Hermitian form $L_x = -i d\theta_x$ with respect to the inner product $\langle\ ,\ \rangle$ on $(T_{1,0})_x$.

(6.20) Definition. Given $0 \leqslant q \leqslant n$, the Levi form L is said to satisfy *condition* $Y(q)$ at $x \in M$ if the set of eigenvalues of L at x cannot be converted to a set of nonnegative reals by changing exactly q signs.

(6.21) Theorem. *If the Levi form satisfies condition* $Y(q)$ *at each point of* M, *then* $\partial/\partial t + \square_{b,q}$ *has a parametrix belonging to* $\operatorname{Op} S^{-2}(M \times \mathbf{R}, \mathscr{V})$, *where* $\mathscr{V}$ *is the bundle* $\operatorname{Re}(T_{1,0}) + \operatorname{Im}(T_{1,0})$.

Proof. We simply want to calculate $\square_{b,q}$ in local coordinates and apply a matrix version of Theorem 5.22. Given $y \in M$ choose a coordinate neighborhood U in which we can choose an orthonormal frame $\theta, \theta_1, \cdots, \theta_n, \bar{\theta}_1, \cdots, \bar{\theta}_n$ as above. Let $X_0, Z_1, \cdots, Z_n, \bar{Z}_1, \cdots, \bar{Z}_n$ be the dual frame for $\mathbf{C} \otimes TU$. For functions $f \in \mathscr{D}(U)$,

$$df = \sum (Z_j f)\theta^j + \sum (\bar{Z}_j f)\bar{\theta}^j + (X_0 f)\theta.$$

Therefore

$$\bar{\partial}_{b,0} f = \sum (\bar{Z}_j f)\bar{\theta}^j. \tag{6.22}$$

Given multi-indices $J = (j_1, \cdots, j_p)$ and $K = (k_1, \cdots, k_q)$, with $|J| = p$ and $|K| = q$, set

$$\theta^{J,K} = \theta^{j_1} \wedge \cdots \wedge \theta^{j_p} \wedge \bar{\theta}^{k_1} \wedge \cdots \wedge \bar{\theta}^{k_q}. \tag{6.23}$$

Restricting to J, K with entries in strictly increasing order, we obtain an orthonormal frame for $\mathscr{E}^{p,q}(U)$. Now

$$\bar{\partial}_{b,q}(f\theta^{J,K}) = \sum (\bar{Z}_j f)\bar{\theta}^j \wedge \theta^{J,K} + fr, \tag{6.24}$$

where r is a $(p, q+1)$-form whose coefficients with respect to our orthonormal frame are derivatives of the coefficients of $\{\theta^k\}$ and $\{\bar{\theta}^k\}$. It follows after some calculation [5] that

$$\begin{aligned} \square_{b,q}\left(\sum f_{J,K}\theta^{J,K}\right) &= \sum (\square_K f_{J,K})\theta^{J,K} \\ &\quad + \sum_{i \notin K, j \in K} \left[Z_i, \bar{Z}_j\right] f_{J,K} \bar{\theta}^j \lrcorner (\bar{\theta}^i \wedge \theta^{J,K}) \\ &\quad + \sum (Z_j f) r_j + \sum (\bar{Z}_j f)\bar{r}_j + (X_0 f) r_0 + fs, \end{aligned} \tag{6.25}$$

where $(Z_j f) r_j = \sum (Z_j f_{J,K}) r_{j,J,K}$, etc., and

$$\square_K = -\tfrac{1}{2}\sum\left(\bar{Z}_j Z_j + Z_j \bar{Z}_j\right) + \tfrac{1}{2}\sum_{j \in K}\left[Z_j, \bar{Z}_j\right] - \tfrac{1}{2}\sum_{j \notin K}\left[Z_j, \bar{Z}_j\right]. \tag{6.26}$$

The r_j, $\bar{r}_j$ and s in (6.25) are (p, q) forms whose coefficients are polynomials in the derivatives of the coefficients of X_0, $\{X_j\}$, $\{\bar{X}_j\}$, θ, $\{\theta^k\}$, $\{\bar{\theta}^k\}$. (Derivatives from the volume form occur, but in view of (6.13) and (6.15) these are already accounted for.)

It is important to note that

$$(6.27) \qquad [Z_j, \bar{Z}_k] = -iL(Z_j, Z_k)X_0 \qquad \operatorname{mod}(T_{1,0} \oplus T_{0,1}).$$

Suppose that our frames have been chosen so that the Z_j diagonalize L at y:

$$(6.28) \qquad L(Z_j, Z_k)_y = \lambda_j \delta_{jk}, \qquad 1 \leqslant j, k \leqslant n.$$

Let X_j and X_{n+j} be the real and imaginary parts of $\bar{Z}_j$:

$$(6.29) \qquad Z_j = X_j - iX_{n+j}, \quad \bar{Z}_j = X_j + iX_{n+j}, \qquad 1 \leqslant j \leqslant n.$$

Then since $\theta([Z_j, Z_k]) \equiv 0$, (6.28) implies

$$(6.30) \qquad \theta([X_j, X_k])_y = \theta([X_{n+j}, X_{n+k}])_y = 0, \qquad 1 \leqslant j, k \leqslant n;$$

$$(6.31) \qquad \theta([X_j, X_{n+k}])_y = -\tfrac{1}{2}\lambda_j \delta_{jk}, \qquad 1 \leqslant j, k \leqslant n.$$

Thus the principal part of $\Box_K$ at y is

$$(6.32) \qquad -\sum_{j=1}^{2n} X_j^2 - i\left[\sum_{j \in K} \tfrac{1}{2}\lambda_j - \sum_{j \notin K} \tfrac{1}{2}\lambda_j\right] X_0,$$

and the principal part of $\Box_{b,q}$ at y is diagonal, with the terms (6.32) on the diagonal. The condition (5.23) of Theorem 5.22 becomes

$$(6.33) \qquad \left| \sum_{j \in K} \tfrac{1}{2}\lambda_j - \sum_{j \notin K} \tfrac{1}{2}\lambda_j \right| < \tfrac{1}{2} \sum_{j=1}^{n} |\lambda_j|,$$

since (6.30) and (6.31) imply that the set of eigenvalues $\{a_j(y)\}$ is precisely $\{\pm \tfrac{1}{2}\lambda_j\}$. Clearly $Y(q)$ is equivalent to the validity of (6.33) for every multi-index K with $|K| = q$. Thus $Y(q)$ implies the existence of pointwise inverses for the localized principal terms $(\partial/\partial t + \Box_{b,q})_2^y$. As in the proof of Theorem 5.22 this provides a principal symbol q_{-2}^y and thus a parametrix.

We now specialize further.

(6.34) Definition. The Hermitian metric on M is a *Levi metric* if $L_x^2 = I$ at each point $x \in M$.

Thus the eigenvalues of the Levi metric are all ± 1. We have assumed M connected, so there are integers n_+ and $n_- = n - n_+$ such that the multiplicity of ± 1 as an eigenvalue is $n_\pm = n_\pm(L)$ at every point. In this case, condition $Y(q)$ reads simply $q \neq n_\pm$.

(6.35) Theorem. *Suppose the metric on the CR manifold M is a Levi metric. If $q \neq n_{\pm}(L)$ and $0 \leqslant q \leqslant n$, then $\partial/\partial t + \Box_{b,q}$ has a parametrix $Q \in S_h^{-2}(M \times \mathbf{R}, \mathscr{V})$. Each term in the asymptotic expansion* (5.8) *for the trace of* $\exp(-t\Box_{b,q})$ *has the form* (5.9), *where each term $K_j(x)$ may be obtained by evaluating at x a universal polynomial in the derivatives of the coefficients of a local frame θ, $\{\theta^k\}$, $\{\bar{\theta}^k\}$ in suitable coordinates at x.*

Proof. Take vector fields X_j as above and set $\omega^0 = \theta$, $\omega^j = 2\operatorname{Re}\theta^j$ and $\omega^{n+j} = 2\operatorname{Im}\theta^j$. The frames $\{X_j\}$, $\{\omega^j\}$ are dual, so in any coordinate system if

$$X_j = \sum \alpha_j^k \frac{\partial}{\partial x^k}, \qquad \omega^j = \sum \beta_k^j \, dx^j$$

it follows that (α_j^k) is the inverse of (β_k^j). In y-coordinates or in anti-symmetric y-coordinates, these matrices are the identity at y. Therefore the derivatives of the α_j^k at y are universal polynomials in the derivatives of the β_k^j. For a Levi metric we are in the situation discussed at the end of §5, in which Theorem 4.14 is applicable. Thus as special coordinates we may take the anti-symmetric coordinates for a frame which diagonalizes the Levi form, and the conclusion follows.

To obtain more information about the possible polynomials which can occur in representing the terms of the asymptotic expansion (5.8), (5.9), it is useful to consider the effect of a *change of scale* in the Levi metric. Note that a Levi metric $\langle\ ,\ \rangle$ is determined by its restriction to $T_{1,0}$, together with the choice of an orthocomplement $\mathscr{N}$ for $T_{1,0} \oplus T_{0,1}$. In fact this restriction determines the metric on $T_{1,0} \oplus T_{0,1}$ and the requirement that the eigenvalues of the Levi form be ± 1 determines the 1-form θ up to sign. Then a section X_0 of $\mathscr{N}$ is uniquely determined by the requirement $\theta(X_0) \equiv 1$. Since one wants $\langle X_0, X_0\rangle \equiv 1$, the metric is then determined on $\mathscr{N}$.

Given $\lambda > 0$ we change the scale of a given Levi metric $\langle\ ,\ \rangle$ by setting

$$\langle Z, W\rangle_\lambda = \lambda^2 \langle Z, W\rangle \quad \text{if } Z, W \text{ are sections of } T_{1,0} \oplus T_{0,1} \tag{6.36}$$

and using the same orthocomplement. If θ, $\{\theta^k\}$, $\{\bar{\theta}^k\}$ is an orthonormal frame as above with dual frame X_0, $\{Z_j\}$, $\{\bar{Z}_j\}$, then we may take

$$\theta_\lambda = \lambda^2\theta, \quad \theta_\lambda^k = \lambda\theta^k, \quad X_{0,\lambda} = \lambda^{-2}X_0, \quad Z_{j,k} = \lambda^{-1}Z_j \tag{6.37}$$

to get an orthonormal frame for $\langle\ ,\ \rangle_\lambda$. The corresponding volume form is then

$$dV_\lambda = \lambda^{2n+2}\, dV. \tag{6.38}$$

The relation between the inner products in $\Lambda^{p,q}$ is given by

$$\langle\ ,\ \rangle_\lambda = \lambda^{-2p-2q}\langle\ ,\ \rangle \quad \text{in } \Lambda^{p,q}. \tag{6.39}$$

(6.40) Proposition. $\Box_{b,\lambda} = \lambda^{-2}\Box_b$.

Proof. The projectors $\pi_{p,q}$ are unchanged, so

$$\bar{\partial}_{b,\lambda} = \bar{\partial}_b. \tag{6.41}$$

We need to prove

$$\vartheta_{b,\lambda} = \lambda^{-2}\vartheta_b. \tag{6.42}$$

Suppose u is in $\mathscr{E}^{p,q+1}$ and v is in $\mathscr{E}^{p,q}$, and let $d = 2n + 2 - 2p - 2q$. Then

$$\begin{aligned}(\vartheta_{b,\lambda}u, v)_\lambda &= (u, \bar{\partial}_{b,\lambda}v)_\lambda = (u, \bar{\partial}_b v)_\lambda \\ &= \int \langle u, \bar{\partial}_b v\rangle_\lambda \, dV_\lambda = \lambda^{d-2}\int \langle u, \bar{\partial}_b v\rangle dV \\ &= \lambda^{d-2}(u, \bar{\partial}_b v) = \lambda^{d-2}(\vartheta_b u, v) = \cdots = \lambda^{-2}(\vartheta_b u, v)_\lambda.\end{aligned} \tag{6.43}$$

We take the *kernel* of $\exp(-t\Box_{b,\lambda})$ on $\mathscr{E}^{p,q}$ to be the unique map on $M \times M$,

$$(x, y) \mapsto G_{t,\lambda}(x, y) \in \operatorname{Hom}(\Lambda_y^{p,q}, \Lambda_x^{p,q}) \tag{6.44}$$

such that

$$\exp(-t\Box_{b,\lambda})u(x) = \int_M G_{t,\lambda}(x, y)u(y)\, dV_\lambda(y), \qquad x \in M,\ u \in \mathscr{E}^{p,q},\ t > 0. \tag{6.45}$$

Because of (6.40) and (6.38) one has

$$G_{t,\lambda}(x, y) = \lambda^{-2n-2}G_{\lambda^{-2}t}(x, y), \tag{6.46}$$

where $G_t = G_{t,1}$. As in §5 there is an asymptotic expansion

$$G_{t,\lambda}(x, x) \sim t^{-n-1}\sum_{j=0}^{\infty} t^j K_{j,\lambda}(x). \tag{6.47}$$

The terms in this expansion are unique, so (6.46) and (6.47) imply

$$K_{j,\lambda}(x) = \lambda^{-2j}K_j(x). \tag{6.48}$$

Thus for the expansion of the trace on $\mathscr{E}^{p,q}$ we have

$$\begin{aligned}\operatorname{tr}[\exp(-t\Box_{b,\lambda})] &= \int \operatorname{tr}[G_{t,\lambda}(x, x)]\, dV_\lambda(x) \\ &\sim t^{-n-1}\sum_{j=0}^{\infty}\int k_{j,\lambda}(x)\, dV_\lambda(x)\end{aligned} \tag{6.49}$$

with

$$k_{j,\lambda}(x) = \operatorname{tr} K_{j,\lambda}(x) = \lambda^{-2j}k_j(x). \tag{6.50}$$

7. Geometry of Levi metrics

In this section we specialize to the case of compact strictly pseudoconvex CR manifolds. Suppose M is a $(2n+1)$-dimensional compact CR manifold equipped with a Levi metric and the corresponding real 1-form θ. Suppose further that M is strictly pseudoconvex, i.e. that the Levi form is positive definite.

As before, locally there is an orthonormal frame $Z_1, \cdots, Z_n$ for $T_{1,0}$ such that

$$L(Z_\alpha, Z_\beta) = \delta_{\alpha\beta}, \qquad 1 \leqslant \alpha, \beta \leqslant n. \tag{7.1}$$

there is also a unique real vector field X_0 orthogonal to $T_{1,0}$ such that

$$\langle \theta, X_0 \rangle \equiv 1. \tag{7.2}$$

Let $\theta_1, \cdots, \theta_n$ be the dual frame of $(1,0)$-forms. Then

$$d\theta = i\theta^\alpha \wedge \bar{\theta}^\alpha + \theta \wedge \tau. \tag{7.3}$$

(Here and below, repeated Greek indices are summed from 1 to n.) The 1-form τ is unique if we require

$$\tau \equiv 0 \mod\{\theta^\alpha, \bar{\theta}^\alpha\}. \tag{7.4}$$

Also,

$$d\theta^\alpha = \theta^\beta \wedge \theta_\beta{}^\alpha + \theta \wedge \tau^\alpha + \tfrac{1}{2}\theta^\alpha \wedge \tau. \tag{7.5}$$

The 1-forms $\theta_\beta{}^\alpha$ and τ^α can be chosen uniquely, subject to the conditions

$$\theta_\alpha{}^\beta + \bar{\theta}_\beta{}^\alpha = 0; \tag{7.6}$$

$$d\tau = 2\,\mathrm{Im}\,\bar{\tau}^\alpha \wedge \theta^\alpha; \tag{7.7}$$

$$\text{if } \tau^\alpha \equiv m_\beta{}^\alpha \theta^\beta \mod\{\theta, \bar{\theta}^\beta\}, \text{ then } m_\beta{}^\alpha = \overline{m}_\alpha{}^\beta. \tag{7.8}$$

This was proved for special Levi metrics by Webster [19] and for general Levi metrics by C. Stanton [14]. They use the forms $\theta_\alpha{}^\beta$ to introduce a connection D on M, defined for the given local frame by

$$DX_0 = 0, \qquad DZ_\alpha = \theta_\alpha{}^\beta Z_\beta, \qquad D\bar{Z}_\alpha = \overline{DZ}_\alpha. \tag{7.9}$$

This is a metric connection.

The forms τ, τ^α, $\bar{\tau}^\alpha$ play the role of the *torsion forms* of D. Indeed the classical torsion forms for our local frame are

$$\begin{aligned} T^0 &= d\theta = i\theta^\alpha \wedge \bar{\theta}^\alpha + \theta \wedge \tau, \\ T^\alpha &= d\theta^\alpha + \theta_\beta{}^\alpha \wedge \theta_\beta = \theta \wedge \tau^\alpha + \tfrac{1}{2}\theta^\alpha \wedge \tau, \\ \bar{T}^\alpha &= \theta \wedge \bar{\tau}^\alpha + \tfrac{1}{2}\bar{\theta}^\alpha \wedge \tau. \end{aligned} \tag{7.10}$$

The *curvature forms* are

$$\Pi_\alpha{}^\beta = d\theta_\alpha{}^\beta + \theta_\gamma{}^\beta \wedge \theta_\alpha{}^\gamma, \qquad \overline{\Pi}_\alpha{}^\beta. \tag{7.11}$$

As usual, *normal coordinates* at a point $x_0 \in M$ are obtained from a frame at x_0 by exponentiating along the geodesics of the connection. Given the frame above, we take

$$\{X_0, \operatorname{Re} Z_\alpha, -\operatorname{Im} Z_\alpha\} \tag{7.12}$$

as a frame for $T_{x_0}M$ and exponentiate to obtain coordinates in a neighborhood of x_0. The frames $\{Z_\alpha\}$, $\{\theta^\alpha\}$ can be extended by parallel transport along geodesics from x_0. We call the resulting frame a *normal frame* at x_0. Because the connection is a metric connection, by (7.9) the normal frame gives us orthonormal frames for $T_{1,0}$ and $\Lambda^{1,0}$.

(7.13) Remark. Webster and C. Stanton also construct a canonical connection in the case of a Levi metric and a nondegenerate indefinite Levi form, but this connection preserves the Levi form rather than the metric. Thus a normal frame is no longer orthonormal. For this reason we have specialized to the strictly pseudoconvex case.

Suppose we have chosen normal coordinates at x_0 and the associated normal frame $\{Z_\alpha\}$, $\{\theta^\alpha\}$, and let $X_\alpha = \operatorname{Re} Z_\alpha$, $X_{n+\alpha} = -\operatorname{Im} Z_\alpha$, $1 \leqslant \alpha \leqslant n$. In these coordinates let

$$r = \left(\sum_{j=0}^{2n} x_j^2\right)^{1/2} \tag{7.14}$$

be the radial function and let

$$\sum_{j=0}^{2n} \frac{x^j}{r}\frac{\partial}{\partial x^j} = \frac{1}{r}\mathscr{R} \tag{7.15}$$

be the corresponding radial vector field. Since rays are geodesics and since the normal frame is obtained by parallel transport, we have

$$\mathscr{R} = \sum_{j=0}^{2n} x^j X_j. \tag{7.16}$$

Therefore

$$\mathscr{R} \lrcorner \theta = x^0, \quad \mathscr{R} \lrcorner \theta^\alpha = \tfrac{1}{2} z^\alpha, \qquad 1 \leqslant \alpha \leqslant n, \tag{7.17}$$

where $z^\alpha = x^\alpha + i x^{n+\alpha}$. Since $D_{\mathscr{R}}\theta^\alpha = 0$, one has

$$\mathscr{R} \lrcorner \theta_\alpha{}^\beta = 0, \qquad 1 \leqslant \alpha, \beta \leqslant n. \tag{7.18}$$

Given normal coordinates and a normal frame, by the *components* of the frame, or of the curvature or torsion forms, we mean the components with respect to the coordinate frames $\{\partial/\partial x^j\}$ and $\{dx^j\}$, $\{dx^j \wedge dx^k;\ j < k\}$. The following is the analogue of a classical Riemannian result (see the appendix of [1]).

(7.19) Proposition. *Let $\{x^j\}$ be normal coordinates at x_0. Then for any component of the normal frame X_0, $\{Z_\alpha\}$, θ, $\{\theta^\alpha\}$, each Taylor coefficient at the origin is a universal polynomial in the components of the covariant derivatives of the curvature and torsion tensors of the Webster-Stanton connection.*

Proof. Let $\mathscr{R}$ denote the radial vector field (7.15) and also the Lie derivative with respect to this vector field. If u is a form, let $\hat{u}(k)$ denote the term homogeneous of degree k in the (component-wise) Taylor expansion of u at the origin. Now (7.3), (7.17), and (7.18) give

$$(7.20)\qquad \begin{aligned} \mathscr{R}\theta &= \mathscr{R}\lrcorner d\theta + d(\mathscr{R}\lrcorner\theta) = \mathscr{R}\lrcorner(i\theta^\alpha \wedge \bar{\theta}^\alpha + \theta \wedge \tau) + dx^0 \\ &= \frac{i}{2}(z^\alpha\bar{\theta}^\alpha - \bar{z}^\alpha\theta^\alpha) + x^0\tau - (\mathscr{R}\lrcorner\tau)\theta + dx^0. \end{aligned}$$

Next, (7.5), (7.17), and (7.18) give

$$(7.21)\qquad \begin{aligned} \mathscr{R}\theta^\alpha &= \mathscr{R}\lrcorner d\theta^\alpha + d(\mathscr{R}\lrcorner\theta^\alpha) \\ &= \mathscr{R}\lrcorner\left(\theta^\beta \wedge \theta_\beta{}^\alpha + \theta \wedge \tau^\alpha + \tfrac{1}{2}\theta^\alpha \wedge \tau\right) + \tfrac{1}{2}dz^\alpha \\ &= \tfrac{1}{2}z^\beta\theta_\beta{}^\alpha + x^0\tau^\alpha - (\mathscr{R}\lrcorner\tau^\alpha)\theta + \tfrac{1}{4}z^\alpha\tau - \tfrac{1}{2}(\mathscr{R}\lrcorner\tau)\theta^\alpha + \tfrac{1}{2}dz^\alpha. \end{aligned}$$

Note that for a function a, $(\mathscr{R}a)^\wedge(k) = k\hat{a}(k)$. Therefore

$$(7.22)\qquad \begin{aligned} \left[\mathscr{R}\left(a_j dz^j\right)\right]^\wedge(k) &= \left[\left(\mathscr{R}a_j\right)dx^j + a_j dx^j\right]^\wedge(k) \\ &= (k+1)\left(a_j dx^j\right)^\wedge(k). \end{aligned}$$

Thus (7.21) and (7.22) give

$$(7.23)\qquad \hat{\theta}^\alpha(0) = \tfrac{1}{2}dz^\alpha,$$

while (7.20) and (7.22) give

$$(7.24)\qquad \hat{\theta}(0) = dx^0,$$

$$(7.25)\qquad \hat{\theta}(1) = \tfrac{i}{8}(z^\alpha d\bar{z}^\alpha - \bar{z}^\alpha dz^\alpha) + \tfrac{1}{2}x^0\hat{\tau}(0) - \tfrac{1}{2}(\mathscr{R}\lrcorner\tau)^\wedge(1)\,dx^0.$$

Inductively, (7.20), (7.21), and (7.22) show that $\hat{\theta}(k)$ and $\hat{\theta}^\alpha(k)$ are polynomials in $\{\hat{\tau}(l), \hat{\tau}^\alpha(l), \hat{\theta}_\beta{}^\alpha(l), l < k\}$. Next,

$$(7.26)\qquad \begin{aligned} \mathscr{R}\theta_\beta{}^\alpha &= \mathscr{R}\lrcorner d\theta_\beta{}^\alpha + d\left(\mathscr{R}\lrcorner\theta_\beta{}^\alpha\right) = \mathscr{R}\lrcorner d\theta_\beta{}^\alpha \\ &= \mathscr{R}\lrcorner\left(\Pi_\beta{}^\alpha - \theta_\gamma{}^\alpha \wedge \theta_\beta{}^\gamma\right) = \mathscr{R}\lrcorner\Pi_\beta{}^\alpha, \end{aligned}$$

so $\hat{\theta}_\beta{}^\alpha(l)$ is a polynomial in $\hat{\Pi}_\beta{}^\alpha(l-1)$ for $l > 0$, and

(7.27) $$\hat{\theta}_\beta{}^\alpha(0) = 0.$$

We have now proved the result for the 1-forms θ, $\{\theta^\alpha\}$, with ordinary derivatives in place of covariant derivatives. To replace the ordinary derivatives by covariant derivatives, we let A denote the matrix of the coframe $\{\theta, \theta^\alpha, \bar{\theta}^\alpha\}$ in terms of the coframe $\{dx^0, \cdots, dx^{2n}\}$. Let $\theta' = \{\theta_i'^j\}$ denote the connection matrix with respect to the coframe $\{dx^0, \cdots, dx^{2n}\}$, and let $\underline{\theta}$ denote the connection matrix with respect to $\{\theta, \theta^\alpha, \bar{\theta}^\alpha\}$. Then

(7.28) $$\theta' = A^{-1}\underline{\theta}A - A^{-1}dA.$$

By (7.20), (7.21), (7.22), (7.26) and (7.27), we see that the components of $\check{\theta}'(k)$ are polynomials in the real and imaginary parts of $\{\hat{\Pi}_\alpha{}^\beta(l-1), \hat{\tau}(l), \hat{\tau}^\alpha(l), l \leqslant k\}$. Thus, inductively, we may replace derivatives of components of the torsion and curvature tensors by components of their covariant derivatives.

Finally, the result for the components of the vector fields follows, since these components can be obtained by inverting the matrix A, and the determinant of A at the origin is a constant depending only on n.

(7.29) Remark. As a consequence of (7.20), (7.21) and (7.22) we see that

$$X_j = \frac{\partial}{\partial x^j} + \sum b_{jk}(x)\frac{\partial}{\partial x^k}, \qquad b_{jk}(0) = 0,$$

$$c_{jk} = \frac{\partial b_{j0}}{\partial x^k}(0) = \begin{cases} \frac{1}{4}, & k = j + n, \\ -\frac{1}{4}, & j = k + n, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, normal coordinates at x_0 are anti-symmetric x_0-coordinates in the sense of §5.

Combining Proposition 7.19 and Remark 7.29 with Theorem 6.35, we obtain a geometric interpretation of the terms in the trace of the heat kernel for $\square_b$ for a strictly pseudoconvex CR manifold with a Levi metric. We summarize our main results from Theorems 5.6, 6.21, and 6.35, together with the remarks just made.

(7.30) Theorem. *Let M be a compact CR manifold of dimension $2n+1$, and suppose the Levi form satisfies condition $Y(q)$ at each point of M. Then*

(7.31) $$\operatorname{tr}(e^{-t\square_{b,q}}) \sim t^{-n-1}\sum_{j=0}^{\infty} k_j t^j \quad \text{as } t \to 0+,$$

where

(7.32) $$k_j = \int_M K_j(x)\, dV(x);$$

K_j is a locally computable function on M and $\Box_{b,q}$ operates on $\Lambda^{p,q}$ for some fixed p, $0 \leqslant p \leqslant n$. In the special case that M is strictly pseudoconvex (so $Y(q)$ is satisfied for $0 < q < n$) and that the metric is a Levi metric, the function K_j may be calculated at $x \in M$ by evaluating a polynomial in the components of the curvature and torsion of the Webster–C. Stanton connection, and their covariant derivatives, computed in normal coordinates at x; this polynomial depends only on n, p, q, and j.

8. The terms $K_0(x)$ and $K_1(x)$ in the expansion

As in §7, we assume that M is a compact oriented CR manifold of dimension $2n+1$ with a positive definite Levi form and a Levi metric. We seek more information about the terms in the asymptotic expansion of the trace of the heat kernel for $\Box_{b,q}$, $0 < q < n$. For this purpose we use scale changes, $U(n)$ invariance and conjugation. The method gives additional information about each term $K_j(x)$, but the answer is already complicated for $K_1(x)$ and we consider here only $K_0(x)$ and $K_1(x)$. The main result, Theorem 8.31, follows from three lemmas.

Recall from the end of §6 that the scale change $\langle\ ,\ \rangle_\lambda = \lambda^2\langle\ ,\ \rangle$ on $T_{1,0}$ leads to the changes in orthonormal frames:

$$X_{0,\lambda} = \lambda^{-2}X_0,\quad Z_{\alpha,\lambda} = \lambda^{-1}Z_\alpha,\quad \theta_\lambda = \lambda^2\theta,\quad (\theta^\alpha)_\lambda = \lambda\theta^\alpha. \tag{8.1}$$

It follows immediately from (7.3), (7.5), and (8.1) that

$$\tau_\lambda = \tau,\quad (\tau^\alpha)_\lambda = \lambda^{-1}\tau^\alpha,\quad (\theta_\beta{}^\alpha)_\lambda = \theta_\beta{}^\alpha. \tag{8.2}$$

In particular, the connection is unchanged, and (7.11) implies that the curvature form is also unchanged:

$$(\Pi_\alpha{}^\beta)_\lambda = \Pi_\alpha{}^\beta. \tag{8.3}$$

Since the connection is not changed, while the initial conditions for normal coordinates scale by (8.1), it follows that normal coordinates $\{x^j\}$ scale by

$$\{x_\lambda^j\} = \{\lambda^2 x^0, \lambda x^1, \cdots, \lambda x^{2n}\}. \tag{8.4}$$

We shall denote components with respect to $\{dx^j\}$ and $\{dx^j \wedge dx^k\}$ by (further) subscripts, so for example $\tau = \Sigma(\tau)_j dx^j$ and $\Pi_\beta{}^\alpha = \Sigma_{j<k}(\Pi_\beta{}^\alpha)_{jk}dx^j \wedge dx^k$. Using the same convention with respect to $\{dx_\lambda^j\}$ and $\{dx_\lambda^j \wedge dx_\lambda^k\}$ for τ_λ and so on, we have

$$\begin{aligned} (\tau_\lambda)_j &= \lambda^{-\langle j\rangle}(\tau)_j, & (\tau^\alpha{}_\lambda)_j &= \lambda^{-1-\langle j\rangle}(\tau^\alpha)_j \\ \left[(\theta_\beta{}^\alpha)_\lambda\right]_j &= \lambda^{-\langle j\rangle}(\theta_\beta{}^\alpha)_j, & \left[(\Pi_\beta{}^\alpha)_\lambda\right]_{jk} &= \lambda^{-\langle j\rangle-\langle k\rangle}(\Pi_\beta{}^\alpha)_{jk}, \end{aligned} \tag{8.5}$$

where

$$\langle 0\rangle = 2, \qquad \langle j\rangle = 1 \quad \text{if } 1 \leqslant j \leqslant 2n. \tag{8.6}$$

Note that, by (7.4), τ_0 vanishes at $x = 0$; also

$$\frac{\partial}{\partial x^j_\lambda} = \lambda^{-\langle j\rangle}\frac{\partial}{\partial x^j}. \tag{8.7}$$

Let D_j denote covariant differentiation in the $\partial/\partial x^j$ direction at $x = 0$. Then

$$D_j\theta = 0 = D_j\theta^\alpha. \tag{8.8}$$

Thus, at $x = 0$ we have

$$D_{j,\lambda} = \lambda^{-\langle j\rangle}D_j. \tag{8.9}$$

Consider now the first two terms in the asymptotic expansion (7.31).

(8.10) Lemma. *The term K_0 in the expansion* (7.31) *is a constant depending only on n, p, and q. The term K_1 is a linear combination of*

$$(\tau)_j(\tau)_k, \quad D_j\tau_k, \quad (\tau^\alpha)_j, \quad (\Pi_\beta{}^\alpha)_{jk}, \qquad 1 \leqslant j, k \leqslant 2n, \tag{8.11}$$

and their complex conjugates.

Proof. The statement about K_0 follows from the parametrix construction. It may also be deduced from the form of the K_j as stated in Theorem 7.30, together with the scaling results (6.48) and (8.5). Similarly, by (6.48) K_1 scales by λ^{-2} and (8.5) shows that the only monomials in components of covariant derivatives of curvature and torsion which scale by λ^{-2} are those of (8.11) and their conjugates. (Note that the $(\tau)_j$ are real, so there are no mixed quadratic terms to consider).

Next we use $U(n)$ invariance.

(8.12) Lemma. *The term K_1 in the asymptotic expansion* (7.31) *has the form*

$$K_1 = aR_\alpha{}^\alpha{}_\beta{}^\beta + ba_\alpha\bar{a}_\alpha + cZ_\alpha\bar{a}_\alpha + d\bar{Z}_\alpha a_\alpha + em_\alpha{}^\alpha, \tag{8.13}$$

where a, b, c, d, e are constants depending only on n, p, and q, $m_\beta{}^\alpha$ is given by (7.8), *while a_α and $R_\alpha{}^\beta{}_\gamma{}^\delta$ are defined by*

$$\begin{gathered}\tau = a_\alpha\theta^\alpha + \bar{a}_\alpha\bar{\theta}^\alpha,\\ \Pi_\alpha{}^\beta \equiv R_\alpha{}^\beta{}_\gamma{}^\delta\theta^\gamma \wedge \bar{\theta}^\delta \quad \operatorname{mod}\{\theta \wedge (\Lambda^{1,0} \oplus \Lambda^{0,1}), \Lambda^{2,0}, \Lambda^{0,2}\}\end{gathered} \tag{8.14}$$

and repeated Greek indices are summed from 1 *to n.*

Proof. Fix $x \in M$. The unitary group $U(n)$ acts on normal frames at x. If $\{X_0, Z_\alpha, \theta, \theta^\alpha\}$ is a normal frame and $U = (U^\alpha_\beta)$ belongs to $U(n)$, we take the action to be

$$\theta \mapsto \theta, \quad \theta^\alpha \mapsto U^\alpha_\beta\theta^\beta, \quad X_0 \to X_0, \quad Z_\alpha \mapsto (U^{-1})^\beta_\alpha Z_\beta. \tag{8.15}$$

This corresponds to an action on normal coordinates and induces an action on all tensors. Because $K_1(x)$ is independent of the choice of normal frame, the linear combination of Lemma 8.10 is invariant under the $U(n)$ action. To describe the action on the terms in (8.11) and their conjugates, we find it convenient to use the metric to raise and lower indices and thereby eliminate conjugates. Let $Z^\alpha = \overline{Z}_\alpha$ and $\theta_\alpha = \bar{\theta}^\alpha$. Then, e.g.

$$(8.16) \qquad \tau = a_\alpha \theta^\alpha + a^\alpha \theta_\alpha, \quad a^\alpha = \bar{a}_\alpha, \qquad \Pi_\alpha{}^\beta = R_\alpha{}^\beta{}_\gamma{}^\delta \theta^\gamma \wedge \theta_\delta.$$

After we eliminate conjugates in this way, the $U(n)$ action can be described by saying that upper Greek indices transform by U and lower Greek indices transform by (U^{-1}) as in (8.15).

Let $V = \mathbf{C}^n$, with the standard basis $\{e^1, \cdots, e^n\}$, considered as a $U(n)$ module in the usual way. Let V^* be the dual space, with dual basis $\{e_1, \cdots, e_n\}$, considered as a $U(n)$ module with respect to the inverse transpose action. Let

$$V_s^r = V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*,$$

where there are r copies of V and s copies of V^*, with the induced $U(n)$ action. We can imbed the data of (8.11) and the conjugates in a direct sum E of copies of the V_s^r. For example

$$(8.17) \qquad \begin{aligned} \{(\tau)_j(\tau)_k\}_{j,k=1}^{2n} &\mapsto a_\alpha a_\beta e^\alpha \otimes e^\beta + a^\alpha a_\beta e_\alpha \otimes e^\beta + a^\alpha a^\beta e_\alpha \otimes e_\beta \\ &\in V_2^0 \oplus V_1^1 \oplus V_0^2. \end{aligned}$$

This imbedding is compatible with the $U(n)$ actions. Let v_0 be the image of the data (8.11) and the conjugates, for our chosen frame. Then Lemma 8.10 says that there is a linear functional

$$(8.18) \qquad f \colon E \mapsto \mathbf{C}$$

such that

$$(8.19) \qquad f(Uv_0) = K_1(x); \qquad U \in U(n).$$

We may replace f by its average over $U(n)$ and assume that f is $U(n)$-invariant. The restriction to each summand is also $U(n)$-invariant. By classical invariant theory [13], a $U(n)$-invariant linear functional on V_s^r is zero unless $r = s$. Checking the transformation laws of the various components in (8.11), one sees that the only ones which imbed in some V_r^r are

$$(8.20) \qquad a^\alpha a_\beta, \quad m_\alpha{}^\beta, \quad Z^\alpha a_\beta, \quad Z_\alpha a^\beta, \quad R_\alpha{}^\beta{}_\gamma{}^\delta, \quad R^\alpha{}_\beta{}^\gamma{}_\delta.$$

Moreover, any $U(n)$-invariant linear functional on V_r^r is a multiple of a complete contraction [13]. Thus our functional f is a linear combination of

$$(8.21) \quad a^\alpha a_\alpha, \quad m_\alpha{}^\alpha, \quad Z^\alpha a_\alpha, \quad Z_\alpha a^\alpha, \quad R_\alpha{}^\alpha{}_\beta{}^\beta, \quad R_\alpha{}^\beta{}_\beta{}^\alpha, \quad R^\alpha{}_\alpha{}^\beta{}_\beta, \quad R^\beta{}_\alpha{}^\alpha{}_\beta.$$

By (7.6) and (7.11), $(\Pi_\alpha{}^\beta)$ is skew-symmetric, so

$$R_\alpha{}^\alpha{}_\beta{}^\beta = \overline{R}_\alpha{}^\alpha{}_\beta{}^\beta = R^\alpha{}_\alpha{}^\beta{}_\beta, \qquad R_\alpha{}^\beta{}_\beta{}^\alpha = R^\beta{}_\alpha{}^\alpha{}_\beta. \tag{8.22}$$

Finally, by the curvature identities in [14], $R_\alpha{}^\alpha{}_\beta{}^\beta = R_\alpha{}^\beta{}_\beta{}^\alpha$. Thus $f|U(n)v_0$ is a linear combination of the terms

$$a^\alpha a_\alpha, \quad m_\alpha{}^\alpha, \quad Z^\alpha a_\alpha, \quad Z_\alpha a^\alpha, \quad R_\alpha{}^\alpha{}_\beta{}^\beta. \tag{8.23}$$

The lemma follows from (8.19).

To obtain more information, we use conjugation.

(8.24) Lemma. *There are real constants a, b and c depending only on n, p and q, such that*

$$K_1(x) = aR_\alpha{}^\alpha{}_\beta{}^\beta + ba_\alpha\bar{a}_\alpha + c\left(Z_\alpha\bar{a}_\alpha + \overline{Z}_\alpha a_\alpha\right). \tag{8.25}$$

Proof. We define a new CR structure $T'_{1,0}$ on M by $T'_{1,0} = T_{0,1}$. Let $\theta' = -\theta$. The original metric is a Levi metric for the new structure and complex conjugation is an isometry of the structures. The new sublaplacian $\square'_b$ is given by

$$\square'_b\mu = \left(\square_b\bar{\mu}\right)^-, \tag{8.26}$$

so the new eigenforms of $\square'_b$ on $\Lambda'^{p,q}$ are the conjugates of the eigenforms of $\square_b$ on $\Lambda^{p,q}$, and the eigenvalues are the same. Therefore $\operatorname{tr} K_t(x, x)$ is (real and) unchanged, so

$$K'_j(x) = K_j(x), \quad \text{all } j \text{ and } x, \tag{8.27}$$

where $\{K'_j\}$ are the terms in the asymptotic expansion for $\square'_b$.

If $\{\theta^\alpha\}$ is an orthonormal frame for $\Lambda^{1,0}$, the conjugates form an orthonormal frame $\{\theta'^\alpha\}$ for $(\Lambda')^{1,0}$. Set

$$\theta'^\beta_\alpha = \bar{\theta}_\alpha{}^\beta, \quad \tau' = \tau, \quad \tau'^\alpha = -\bar{\tau}^\alpha.$$

These satisfy (7.3)–(7.8) for the new frame, and therefore give the connection and torsion forms. The curvature satisfies $\Pi'^\beta_\alpha = \overline{\Pi}_\alpha{}^\beta$. By (8.27) and (8.13).

$$\begin{aligned} K_1 &= \tfrac{1}{2}\left(K_1 + K'_1\right) \\ &= \tfrac{1}{2}a\left(R_\alpha{}^\alpha{}_\beta{}^\beta + \overline{R}_\alpha{}^\alpha{}_\beta{}^\beta\right)_1 + ba_\alpha\bar{a}_\alpha + \tfrac{1}{2}(c+d)\left(Z_\alpha\bar{a}_\alpha + \overline{Z}_\alpha a_\alpha\right) \\ &= aR_\alpha{}^\alpha{}_\beta{}^\beta + ba_\alpha\bar{a}_\alpha + \tfrac{1}{2}(c+d)\left(Z_\alpha\bar{a}_\alpha + \overline{Z}_\alpha a_\alpha\right). \end{aligned} \tag{8.28}$$

Here we have used (8.22) and the fact that

$$m'_\alpha{}^\alpha = -\overline{m}_\alpha{}^\alpha = -m_\alpha{}^\alpha \tag{8.29}$$

by (7.8). Since K_1 is real, as are $R_\alpha{}^\alpha{}_\beta{}^\beta$, $a_\alpha\bar{a}_\alpha$ and $Z_\alpha\bar{a}_\alpha + \overline{Z}_\alpha a_\alpha$, we may replace the constants a, b, and $\frac{1}{2}(c+d)$ in (8.28) by their real parts to complete the proof.

(8.30) Definition. The *scalar curvature* is $\kappa = R_{\alpha}{}^{\alpha}{}_{\beta}{}^{\beta}$. Thus κ is a function on M, independent of the choice of frame.

The next theorem is largely a summary of Lemmas 8.10, 8.12 and 8.24.

(8.31) Theorem. (i) *The term* K_0 *in the asymptotic expansion* (7.31) *is the constant function*

$$K_0 \equiv \binom{n}{p}\binom{n}{q}\frac{1}{2^n\pi^{n+1}}\int_{-\infty}^{\infty} e^{-(n-2q)\mu}\left(\frac{\mu}{\sinh\mu}\right)^n d\mu.$$

(ii) *The term* K_1 *in the asymptotic expansion* (7.31) *has the form*

$$K_1 = a\kappa + b\langle\tau,\tau\rangle + c\operatorname{Re}\bar\partial_b^*\pi_{0,1}\tau,$$

where a, b *and* c *are real constants depending only on* n, p *and* q.

Proof. To prove (i) we examine the parametrix, using the forms $\theta^{J,K}$ of (6.23) to trivialize $\Lambda^{p,q}$ in a neighborhood of x. By (5.54) and (6.32), since each $\lambda_j = 1$ we have

$$\begin{aligned}k_{-2n-2,x}(0,t) = \left(\frac{1}{2\pi}\right)^{2n+2}\Bigg[\int e^{it\tau}\Bigg\{\int_0^\infty & e^{-i\tau s-(n-2q)\xi_0 s/2} \\ &\times\left(\cosh\tfrac12 s\xi_0\right)^{-n}\exp\left(-2|\xi'|^2\xi_0^{-1}\tanh\tfrac12\xi_0 s\right)ds\Bigg\}\,d\xi\,d\tau\Bigg]I,\end{aligned}\tag{8.32}$$

where I is the $\binom{n}{p}\binom{n}{q}$ identity matrix. We are working in normal coordinates centered at x, so $dV(x) = 2^{-n}dx$ by (7.23) and (7.24). Hence (i) follows from

$$t^{-n-1}K_0(x)\,dV(x) = \operatorname{tr}k_{-2n-2,x}(0,t)\,dx \tag{8.33}$$

and evaluation of (8.32). To evaluate (8.32) we integrate first with respect to ξ', then use the Fourier inversion formula to integrate with respect to s and τ, and finally set $\mu = \frac12\xi_0 t$.

To prove (ii) we use (8.25). Because $\{\theta^\alpha\}$ is an orthonormal frame for $\Lambda^{1,0}$, $a_\alpha\bar a_\alpha = \frac12\langle\tau,\tau\rangle$. Furthermore, $\pi_{0,1}\tau = \bar a_\alpha\bar\theta^\alpha$, so

$$\bar\partial_b^*\pi_{0,1}\tau = -Z_\alpha\bar a_\alpha - \left(a^{-1}Z_\alpha a\right)\bar a_\alpha, \tag{8.34}$$

where the volume element is $dV = adx$. To calculate $Z_\alpha(a)$ evaluated at x, we write

$$\begin{aligned}\theta = \textstyle\sum a_j^0\,dx^j, \qquad \theta^\alpha &= a_0^\alpha dx^0 + a_\beta^\alpha dz^\beta + a_{n+\beta}^\alpha d\bar z^\beta, \\ \bar\theta^\alpha &= a_0^{n+\alpha}dx^0 + a_\beta^{n+\alpha}dz^\beta + a_{n+\beta}^{n+\alpha}d\bar z^\beta,\end{aligned}\tag{8.35}$$

so $a = d\det(a_k^j)$ for some constant d. By (7.23) and (7.24), $2a_k^j(0) = \delta_k^j$ for j, $k \neq 0$, and $a_k^0(0) = \delta_k^0$. Thus since x is the center of the normal coordinates, $(a^{-1}Z_\alpha a)(x) = (Z_\alpha\sum a_j'^j)(0)$, where $a_j'^j$ is the first order part of the Taylor expansion of a_j^j about 0. By (7.23),

$$Z_\alpha(0) = 2\frac{\partial}{\partial z^\alpha}. \tag{8.36}$$

By (7.25) and (7.17),

$$a_0'^0 = -\tfrac{1}{2}(\mathscr{R}\lrcorner\tau)\hat{\ }(1) = -\tfrac{1}{4}\left(a_\beta z^\beta + \bar{a}_\beta \bar{z}^\beta\right). \tag{8.37}$$

By (7.21)–(7.24),

$$a_\beta'^\alpha = \tfrac{1}{2}\left\{x^0\hat{\tau}^\alpha(0) + \tfrac{1}{4}z^\alpha a_\alpha - \tfrac{1}{2}\left(a_\beta z^\beta + \bar{a}_\beta \bar{z}^\beta\right)\right\}. \tag{8.38}$$

In (8.37)–(8.38) the functions a_β, $\bar{a}_\beta$ are evaluated at 0. Combining (8.36)–(8.38), we obtain

$$\left(Z_\alpha \sum a_j'^j\right)(0) = \frac{\partial}{\partial z^\alpha}\left\{-\left(n+\tfrac{1}{4}\right)\left(a_\beta z^\beta + \bar{a}_\beta \bar{z}^\beta\right)\right\} = -\left(n+\tfrac{1}{4}\right)a_\alpha. \tag{8.39}$$

Substituting (8.39) in (8.34), we see that

$$\bar{\partial}_b^*\pi_{0,1}\tau = -\left\{Z_\alpha \bar{a}_\alpha - \left(n+\tfrac{1}{4}\right)a_\alpha \bar{a}_\alpha\right\} = \tfrac{1}{2}\left(n+\tfrac{1}{4}\right)\langle\tau,\tau\rangle - Z_\alpha \bar{a}_\alpha. \tag{8.40}$$

In view of (8.40) we have

$$K_1 = a\cdot\kappa + b_1\langle\tau,\tau\rangle + c_1\,\mathrm{Re}\left(\bar{\partial}_b^*\pi_{0,1}\tau\right), \tag{8.41}$$

where a, b_1 and c_1 are real. This completes the proof.

(8.42) Remark. The constant K_0 was first calculated in [17]. The present formula is slightly different because of different normalizations in the definition of Levi metric.

(8.43) Corollary. *In the special case that* $\tau \equiv 0$, *there is a real constant* a *depending only on* p, q *and* n *such that* $K_1 = a\kappa$.

(8.44) Remark. If M is a compact strictly pseudoconvex CR manifold, it is always possible to give M a Levi metrics satisfying the hypothesis of the corollary. In fact, each choice of θ determines a unique such Levi metric; these are the metrics studied by Webster [19].

Appendix: The exact heat kernel

The machinery developed above makes possible a quick proof of the main result of [17], that the *exact* heat kernel of $\Box_b$ can be obtained by iteration. We sketch the proof in the more general context of §5 for the scalar case.

As in §5, assume that $P = \partial/\partial t + \Box$ has a parametrix Q with symbol in $S_h^{-2}(M \times \mathbf{R}, \mathscr{V})$. Let Q_0 correspond to the leading term, so that

$$PQ_0 - I = R \quad \text{has symbol in } S_h^{-1}(M \times \mathbf{R}, \mathscr{V}). \tag{A1}$$

Then Q has a formal expansion

$$Q \sim \sum_{k=0}^{\infty} Q_0 R^k. \tag{A2}$$

We shall show that this series converges in a certain sense and gives the exact heat kernel.

Let a_k and b_k be the distribution kernels of $Q_0 R^k$ and of R^k, respectively. Then

$$a_k(x, t; y, s) = a_{k,t-s}(x, y), \tag{A3}$$

where $a_{k,t} = 0$ for $t < 0$ and $a_{k,t} \in \mathscr{E}(M \times M)$ for $t > 0$. The same is true for $b_{k,t}$.

(A4) Theorem. *The series $\sum_{k=0}^{\infty} a_{k,t}$ converges in $C^\infty(M \times M)$ to the kernel K_t of $e^{-t\Box}$ for $t > 0$; moreover the convergence is uniform with respect to t in bounded subsets of $\mathbf{R}_+$.*

Proof. Let $A_k(t)$ and $B_k(t)$ be the operators in $L^2(M)$ with kernels $a_{k,t}$ and $b_{k,t}$, $t > 0$. Then formally at least

$$A_{j+k}(t) = A_j * B_k(t) = \int_0^t A_j(t-s) B_k(s)\, ds. \tag{A5}$$

Given any integer $m \geqslant 0$, identify $C^m(M \times M)$ with a subalgebra of $\mathscr{L}(L^2(M))$ by identifying kernels and operators. Then any admissible norm in $C^m(M \times M)$ gives a norm in this subalgebra, such that

$$|AB| \leqslant C_m |A|\,|B|. \tag{A6}$$

The continuous functions on $[0, \infty)$ with values in $C^m(M \times M)$ form an algebra with respect to the convolution composition (A5). For such functions set

$$|B|_T = \sup\{|B(t)| : 0 \leqslant t \leqslant T\}, \tag{A7}$$

$$B^{(1)} = B, \qquad B^{(k+1)} = B * B^{(k)}. \tag{A8}$$

Inductively one sees that

$$k!\,|B^{(k+1)}|_T \leqslant (C_m T)^k (|B|_T)^{k+1}. \tag{A9}$$

Now choose $\nu > 2n + 4 + m$. Any operator with symbol in $S_h^{-\nu}(M \times \mathbf{R}, \mathscr{V})$ has kernel in $C^m(M \times \mathbf{R}, M \times \mathbf{R})$. In particular, for any integer $N \geqslant 2\nu$ the function $A_N(\cdot)$ satisfies

$$A_N = A_{\nu+j} * B_{k\nu} = A_{\nu+j} * B_\nu^{(k)}, \tag{A10}$$

where $N = (k+1)\nu + j$ with $0 \leqslant j < \nu$. The estimates (A9) imply the convergence of the series $\sum a_{j,t}$ in $C^m(M \times M)$, uniformly for $0 < t \leqslant T$. It remains to show that the sum is K_t. Let

$$Q_N = Q_0 \sum_{j \leqslant N} R^j. \tag{A11}$$

Then for $u \in C_+(\mathbf{R}; L^2(M))$,

$$Q_N u(t) = \sum_{j \leqslant N} \int_{-\infty}^{t} A_j(t-s)u(s)\,ds. \tag{A12}$$

The estimates (A9) imply that $Q_N u$ converges in $C_+(\mathbf{R}; L^2(M))$ as $N \to \infty$. On the other hand, for $u \in \mathscr{D}(M \times \mathbf{R}) = \mathscr{D}(\mathbf{R}; \mathscr{E}(M))$,

$$PQ_N u = u - R^N u \tag{A13}$$

and the estimates (A9) imply that $R^N u \to 0$ in $C_+(\mathbf{R}; L^2(M))$ as $N \to \infty$. Thus for such functions u,

$$\lim_{N \to \infty} Q_N u = \tilde{Q} u, \tag{A14}$$

where $\tilde{Q}$ is the inverse for P given by (5.10). It follows that the kernel K_t associated to $\tilde{Q}$ is indeed the sum $\sum a_{j,t}$.

References

[1] M. Atiyah, R. Bott & V. K. Patodi, *On the heat equation and the index theorem*, Invent. Math. **19** (1973) 279–330; corrections, ibid. **28** (1975) 277–280.

[2] R. Beals & C. Fefferman, *Spatially inhomogeneous pseudodifferential operators*, Comm. Pure Appl. Math. **27** (1974) 1–24.

[3] R. Beals & P. C. Greiner, *Pseudodifferential operators associated to hyperplane bundles*, Rend. Sem. Mat. Torino, (1983) 7–40.

[4] M. Berger, *Geometry of the spectrum*. I, Proc. Sympos. Pure Math., Vol. 27, Part 2, Amer. Math. Soc., Providence, RI, 1975, 129–152.

[5] G. Folland & E. M. Stein, *Estimates for the $\bar\partial_b$-complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. **27** (1974) 429–522.

[6] P. Gilkey, *Curvature and the eigenvalues of the Laplacian for elliptic complexes*, Advances in Math. **10** (1973) 344–382.

[7] P. C. Greiner, *An asymptotic expansion for the heat equation*, Arch. Rational Mech. Anal. **41** (1971) 163–218.

[8] L. Hörmander, *Pseudodifferential operators and hypoelliptic equations*, Proc. Sympos. Pure Math., Vol. 10, Amer. Math. Soc., Providence, RI, 1967, 138–183.

[9] J. J. Kohn & L. Nirenberg, *An algebra of pseudodifferential operators*, Comm. Pure Appl. Math. **18** (1965) 269–305.

[10] J. J. Kohn & H. Rossi, *On the extension of holomorphic functions from the boundary of a complex manifold*, Ann. of Math. (2) **81** (1965) 451–472.

[11] H. P. McKean, Jr., & I. M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Differential Geometry **1** (1967) 43–69.

[12] I. M. Singer, *Eigenvalues of the Laplacian and invariants on manifolds*, Proc. Internat. Congr. Math. I, Vancouver 1974, 187–200.

[13] M. Spivak, *A comprehensive introduction to differential geometry*, Vol. 5, Publish or Perish, Boston, 1975.

[14] C. M. Stanton, *Intrinsic connections for Levi metrics*, in preparation.

[15] N. K. Stanton, *The fundamental solution for the heat equation associated to the $\bar\partial$-Neumann problem*, J. Analyse Math. **34** (1978) 265–274.

[16] ______, *The heat equation in several complex variables*, Bull. Amer. Math. Soc. **11** (1984) 65–84.

[17] N. K. Stanton & D. S. Tartakoff, *The heat equation for the* $\bar{\partial}_b$*-Laplacian*, Comm. Partial Differential Equations **7** (1985) 597–686.

[18] M. Taylor, *Noncommutative microlocal analysis*. I, Mem. Amer. Math. Soc. No. 313, 1984.

[19] S. M. Webster, *Pseudo-hermitian structures on a real hypersurface*, J. Differential Geometry **13** (1978) 25–41.

[20] K. Yosida, *Functional analysis*, Springer, Berlin, 1971.

YALE UNIVERSITY
UNIVERSITY OF TORONTO
UNIVERSITY OF NOTRE DAME