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GROWTH OF FINITELY GENERATED SOLVABLE GROUPS

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This note is intended as an addendum to the preceeding paper [1] by J. A. Wolf. We will prove the following

Theorem. *Let Γ be a solvable group which is not polycyclic, and S a finite set of generators for Γ. Then there exists an exponential lower bound*

$$g_S(m) \geq (\text{constant})^m > 1$$

for the growth function g_S of Γ.

Briefly, Γ has "exponential growth." For definitions and explanations the reader is referred to [1]. Note that the results of [1] provide a partial answer to a problem which was posed by the author in Amer. Math. Monthly **75** (1968) 685-686.

The proof will be based on the study of a group extension

$$1 \longrightarrow A \longrightarrow B \xrightarrow{\varphi} C \longrightarrow 1 ,$$

where we will always assume that A is abelian and that B is finitely generated. Let Z denote the ring of integers.

Lemma 1. *If B does not have exponential growth, then for each $\alpha \in A$ and $\beta \in B$ the set of all conjugates $\beta^k \alpha \beta^{-k}$, with $k \in Z$, spans a finitely generated subgroup of A.*

Proof. For each sequence $i_1, i_2, \cdots, i_m$ of 0's and 1's consider the expression

$$\beta\alpha^{i_1}\beta\alpha^{i_2} \cdots \beta\alpha^{i_m} \in B .$$

If these 2^m expressions all represented distinct elements of B, then the growth function g_S of B, computed using any set S of generators for B which contains both β and $\beta\alpha$, would satisfy

$$g_S(m) \geq 2^m .$$

But this would contradict the hypothesis. Hence there must exist a nontrivial relation of the form

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$$\beta\alpha^{i_1}\cdots\beta\alpha^{i_m} = \beta\alpha^{j_1}\cdots\beta\alpha^{j_m}$$

for some integer m.

It will be convenient to temporarily introduce the abbreviation

$$\alpha_k = \beta^k\alpha\beta^{-k} ,$$

and to note that

$$\beta\alpha^{i_1}\beta\alpha^{i_2}\cdots\beta\alpha^{i_m} = \alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_m^{i_m}\beta^m .$$

Thus our relation becomes

$$\alpha_1^{i_1}\cdots\alpha_m^{i_m} = \alpha_1^{j_1}\cdots\alpha_m^{j_m} ,$$

or briefly

$$\alpha_1^{i_1-j_1}\alpha_2^{i_2-j_2}\cdots\alpha_m^{i_m-j_m} = 1 ,$$

where the exponents $i_k - j_k$ take the values 0 or ± 1, and are not all zero. In fact, choosing m as small as possible, we may assume that $i_1 \neq j_1$ and $i_m \neq j_m$.

It follows that α_m can be expressed as a word in $\alpha_1, \cdots, \alpha_{m-1}$. Conjugating by β it follows that α_{m+1} can be expressed as a word in $\alpha_2, \cdots, \alpha_m$ and hence also as a word in $\alpha_1, \cdots, \alpha_{m-1}$. Continuing inductively we see that every α_k with $k \geq m$ can be expressed as a word in $\alpha_1, \cdots, \alpha_{m-1}$. Similarly every α_k with $k \leq 0$ can be expressed in terms of $\alpha_1, \cdots, \alpha_{m-1}$. This completes the proof.

Lemma 2. *If the quotient group $C = B/A$ has a finite presentation, then there exist finitely many elements $\alpha_1, \cdots, \alpha_l \in A$ so that every element of A can be expressed as a product of conjugates of the α_j.*

Proof. Choose generators $\beta_1, \cdots, \beta_k$ for B and note that the images $\varphi(\beta_1), \cdots, \varphi(\beta_k)$ generate C. Since C is finitely presentable, it has a presentation with these given elements $\varphi(\beta_1), \cdots, \varphi(\beta_k)$ as generators, subject only to a finite number of relations

$$r_1(\varphi(\beta_1), \cdots, \varphi(\beta_k)) = \cdots = r_l(\varphi(\beta_1), \cdots, \varphi(\beta_k)) = 1 ,$$

(compare Kurosh [2, p. 73]).

Setting $\alpha_j = r_j(\beta_1, \cdots, \beta_k)$, it follows easily that every element of A can be expressed as a product of conjugates of the α_j.

Lemma 3. *If C is polycyclic, and B does not have exponential growth, then B must be polycyclic also.*

Proof. Choose generators $\gamma_1, \cdots, \gamma_p$ for C so that every element of C can be expressed as a product

$$\gamma_1^{i_1}\gamma_2^{i_2}\cdots\gamma_p^{i_p}$$

with $i_1, \cdots, i_p \in \boldsymbol{Z}$. Choose elements $\beta_1, \cdots, \beta_p \in B$ so that

$$\varphi(\beta_1) = \gamma_1, \cdots, \varphi(\beta_p) = \gamma_p .$$

According to Lemma 2 there exist elements $\alpha_1, \cdots, \alpha_l \in A$ so that every element of A can be expressed as a product of conjugates of the α_j. Clearly each conjugate of α_j can be written as

$$(\beta_1^{i_1} \cdots \beta_p^{i_p})^{-1}\alpha_j(\beta_1^{i_1} \cdots \beta_p^{i_p}) .$$

Let A_0 denote the subgroup of A spanned by $\alpha_1, \cdots, \alpha_l$. Applying Lemma 1 to the elements α_j and β_1 we see that there exists a finitely generated group A_1 which is spanned by all conjugates of the form

$$\beta_1^{-i_1}\alpha_j\beta_1^{i_1}, \quad \text{with} \quad 1 \leq j \leq l, \quad i_1 \in Z .$$

Similarly applying Lemma 1 to each generator of A_1 and to β_2 we see that all of the

$$\beta_2^{-i_2}(\beta_1^{-i_1}\alpha_j\beta_1^{i_1})\beta_2^{i_2}$$

span a finitely generated group A_2. Continuing inductively we construct $A_1 \subset A_2 \subset \cdots \subset A_p$, and it follows that $A = A_p$ is also a finitely generated abelian group.

Thus A is polycyclic. Since C is polycyclic, it follows that B is polycyclic also.

We are now ready to prove the Theorem. Let

$$\Gamma = \Gamma^0 \supset \Gamma^1 \supset \cdots \supset \Gamma^{S+1} = 1$$

be the commutator series of the finitely generated solvable group Γ. If Γ did not have exponential growth, then applying Lemma 3 to the group extension

$$1 \to \Gamma^S \to \Gamma \to \Gamma/\Gamma^S \to 1 ,$$

an easy induction on s would show that Γ must be polycyclic.

Thus, if we assume that Γ is not polycyclic, it follows that Γ must have exponential growth. Hence the proof of the Theorem is complete.

References

[1] J. A. Wolf, *Growth of finitely generated solvable groups and curvature of Riemannian manifolds*, J. Differential Geometry **2** (1968) 421–446.

[2] A. G. Kurosh, *Theory of groups*, Vol. II, translated by K. A. Hirsch, Chelsea, New York, 1956.

[3] J. Milnor, *A note on curvature and fundamental group*, J. Differential Geometry **2** (1968) 1-7.

Bibliographic addendum. The author would like to call attention to the following paper by Švarc, which contains many of the ideas utilized in [3]:

A. S. Švarc, *A volume invariant of coverings*, Dokl. Akad. Nauk SSSR **105** (1955) 32–34 (Russian).

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