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A JORDAN-HÖLDER DECOMPOSITION FOR A CERTAIN CLASS OF INFINITE DIMENSIONAL LIE ALGEBRAS

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Introduction

0.1 The motivation for this paper is a problem in local differential geometry, sometimes called the transitive equivalence problem. This problem will be illustrated with a typical example: Let M be a Riemannian manifold. Take the curvature tensor of M and its covariant derivatives of all orders and combine them by taking their products, contractions, etc. The set of all tensors obtained this way will include certain tensors of type (0, 0), which can be regarded as scalar functions on M. Suppose all of these functions are constant. Question: Is M locally a homogeneous space? The answer due to Singer [13] is that it is.

The problem above makes sense in a much more general setting. Roughly speaking, suppose a differential structure of some sort is given on a manifold and suppose it is impossible to compute differential invariants for it which distinguish one point of the manifold from another. Is the structure, in some sense, a locally homogeneous space? For those who are familiar with Spencer's work on pseudogroup structures, this problem will be given a more precise formulation: Let Γ be a transitive pseudogroup acting on some model space, and let M be a manifold with an almost Γ structure given on it. Problem: When does M admit an underlying Γ structure to which the given almost Γ structure corresponds? To take a simple example: If M is an almost complex manifold, when is there an underlying complex structure, here Γ being the pseudogroup of holomorphic diffeomorphisms of C^n? (Cf. [4] and [15] for definitions.)

It is known that if one is given a transitive analytic pseudogroup Γ and a normal subpseudogroup Γ_0 defined by an invariant foliation, one can define a quotient Γ/Γ_0 with reasonable properties; this is a theorem of Kuranishi and Rodrigues [10]. If we are given a pseudogroup and a descending chain of normal subpseudogroups, the solvability or non-solvability of the problem proposed above seems to depend only on the nature of the quotients which occur in this chain. This is rather analogous to the situation which occurs in Galois theory, where the techniques required to solve an algebraic equation

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depend on the quotients which occur in the Jordan-Hölder decomposition of the Galois group. This fact makes it worthwhile to try to find a kind of "Jordan-Hölder decomposition" for pseudogroups. The main theorem of this paper is a theorem originally conjectured by E. Cartan in the preface to his Infinite Lie Groups paper [2]. It asserts that every transitive pseudogroup admits a finite chain of normal subgroups whose quotients are, in Cartan's language, "simple intransitive groups." Unfortunately, among Cartan's simple intransitive groups are ones which are "simple, improprement dit," that is, not simple in the algebraic sense. These groups are abelian and defined by linear systems of partial differential equations.

The category of pseudogroups is not quite the right setting for Cartan's theorem since the main ideas involved have rather little to do with groups or manifolds. An alternate formulation of the theorem will be given by using a category of Lie algebras introduced by Sternberg and the author in [3]. For the transition from algebra to geometry, this paper or Kuranishi's paper [8] will be referred to.

Remark. Kobayashi and Nagano in [6] have a decomposition of filtered Lie algebras which seems to be rather different from ours.

0.2. A brief outline of the main details of the paper will be given:

Let Δ be an arbitrary field. A category of topological vector spaces over Δ is defined as follows: We give Δ itself the discrete topology so that it becomes a topological field. Let $\mathscr{K}^0_\Delta$ be the category of finite dimensional vector spaces over the field Δ. We make this into a category of linear topological spaces by assigning to each space its discrete topology. We will denote by $\mathscr{K}_\Delta$ the set of all topological vector spaces which can be obtained as projective limits of vector spaces belonging to $\mathscr{K}^0_\Delta$.

Next we define a certain family of Lie algebras, which we will denote by $\mathscr{L}_\Delta$. An object L of $\mathscr{L}_\Delta$ will, by definition, be a Lie algebra whose underlying vector space is a topological vector space belonging to $\mathscr{K}_\Delta$, and whose bracket operation, regarded as a mapping of $L \times L$ into L, is continuous.

Let L belong to $\mathscr{L}_\Delta$. Our first main result is that the following two properties are equivalent.

I. L satisfies the descending chain condition on closed ideals.

II. L admits a neighborhood of the origin containing no ideal except $\{0\}$.

The Lie algebras which are associated with the transitive pseudogroups automatically satisfy condition II, and hence also satisfy condition I.

To get a Jordan-Hölder decomposition one needs an ascending chain condition. We will show by examples that the a.c.c. is too much to hope for, but that there is the following substitute for it. If L belongs to $\mathscr{L}_\Delta$, then L contains a maximal proper closed ideal. Using this result we will prove our main theorem.

Theorem 1. *Let L belong to $\mathscr{L}_\Delta$ and satisfy the d.c.c. on closed ideals. Then L admits a nested sequence of closed ideals*

$$L = I_0 \supset I_1 \supset I_2 \cdots \supset I_k = \{0\} \tag{0.1}$$

such that, for $0 \leq i \leq k$, *one of the following alternatives holds:*

a) I_i/I_{i+1} *is non-abelian, and there are no closed ideals of* L *properly contained between* I_i *and* I_{i+1}.

b) I_i/I_{i+1} *is abelian.*

Moreover, if two decompositions of type (0.1) *are given, the quotients of type* a) *which occur are the same in both decompositions.*

The proof of this theorem depends on a "Schur's Lemma" for the adjoint representation of a simple algebra on itself; this lemma has recently been very much generalized by Quillen [12].

The last sections of this paper deal with the structure of algebras occuring as quotients of type a). Briefly, these results are as follows:

Let L belong to $\mathscr{L}_\Delta$, and I a non-abelian minimal closed ideal of L. Assume, for simplicity, that the base field Δ is of characteristic zero and algebraically closed. One can show that I contains a unique maximal proper closed ideal of itself. Let us denote by R the quotient of I by this maximal ideal. In § 7, we will prove

Theorem 2. *There exists a finite number of indeterminants* $x_1, \cdots, x_n$ *such that* I *is isomorphic as a Lie algebra to the tensor product:*

$$R \mathbin{\hat{\otimes}} \Delta[[x_1, \cdots, x_n]]$$

where $\Delta[[x_1, \cdots, x_n]]$ *is the ring of formal power series in* $x_1, \cdots, x_n$ *defined over the field* Δ, *and the cap over the tensor product symbol indicates that the tensor product is not the usual algebraic tensor product but an appropriate topological completion of it.*

Because of this theorem, we only have to determine R, in order to know the structure of I. However, R is simple; and if Δ is of characteristic zero and algebraically closed, one knows all the simple algebras in $\mathscr{L}_\Delta$ (cf. [5]). Therefore, one gets a complete classification of the type a) quotients occuring in (0.1). The non-algebraically closed case, in particular the case where $\Delta =$ reals is still unsettled; however, a partial picture of what goes on here will be given in § 7.

0.3. This introduction will conclude with a few remarks about what happens when these algebraic results are translated into theorems about geometry. Proofs of most of these remarks will appear in a forthcoming paper on the Kuranishi-Rodrigues theorem.

As was mentioned above, Lie algebras belonging to $\mathscr{L}_\Delta$ and satisfying the d.c.c. correspond to transitive pseudogroups acting on (real or complex) manifolds, if Δ is either the real or complex numbers. It turns out that closed ideals in these algebras correspond to normal subgroups defined by invariant foliations, which are precisely the subgroups to which the Kuranishi-Rodrigues theorem applies. The subgroups corresponding to abelian quotients in the decomposition of Theorem 1 are abelian pseudogroups whose equations of definition are homogeneous *linear* partial differential equations. The prototype of such a pseudogroup is the following: Let $P^\alpha(x, D)$, $\alpha = 1, \cdots, N$ be a collection of linear partial differential equations on R^k with real analytic coefficients. Let Γ consist of all analytic diffeomorphisms of open sets of R^{k+1} onto open sets of R^{k+1} which are of the form

$$(0.2) \qquad \begin{aligned} x &\to x , \\ y &\to y + f(x) , \end{aligned}$$

where $P\ (x, D)f = 0$ for $\alpha = 1, \cdots, N$. In (0.2) x is a k-tuple, y a 1-tuple, and f any real valued analytic function of x satisfying the indicated system of equations. This group leaves fixed an invariant one-dimensional foliation, and on the quotient is just the identity group. Equivalence problems associated with such groups reduce to problems involving overdetermined systems of linear partial differential equations of the form $P^\alpha(x, D)f = g_\alpha$, $\alpha = 1, \cdots, N$.

The pseudogroups corresponding to the non-abelian quotients in the decomposition of Theorem 1 are the groups which Cartan called "simple, proprement dit." They are simple in the sense that they do not contain proper normal subgroups of the original pseudogroup. In the complex analytic case, these groups can be described as follows: Let Γ be a transitive simple pseudogroup acting on a manifold M, N any complex analytic manifold, and π the projection of $M \times N$ on N. Let Γ_N be the pseudogroup consisting of all local holomorphic diffeomorphisms of $M \times N$ which are fiber preserving and project onto the identity with respect to π, and on the fibers, which are just copies of M, induce mappings belonging to Γ. This construction describes (locally) all pseudogroups corresponding to quotient of type a) in Theorem 1.

A final remark: Among the Lie algebras to which Theorem 1 applies there will be some which admit no quotients of type b). These correspond to a family of pseudogroup structures which seem to generalize the multifoliate structures of Kodaira and Spencer in a rather natural way.

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1. Linear compactness

1.1. In this § 1 we list a few general facts about topological vector spaces over discrete fields, which will be needed later on. Most of the results of this § 1 are taken from Lefschetz's book: *Algebraic topology*, Amer. Math. Soc. Coll. Publ., vol. 27, 1942. Also some of this material can be found in Bourbaki [1] or Zariski-Samuel [16].

Throughout this § 1 the symbol Δ will be used to denote a given field which will be topologized by giving it the discrete topology.

Let E be a vector space over Δ. A subset A of E will be called an *affine set* if it is of the form $v + F$, where v is a fixed element of E and F is a subspace. If E is a topological vector space over Δ, we will say that the topology of E is *defined by affine sets* if one can find a basis for the topology consisting only of affine sets.

Proposition 1.1. *Let E be a topological vector space over Δ whose topology is defined by affine sets, and F a subspace of E. The following two statements are equivalent.*

a) *F is open.*

b) *F is closed and E/F is discrete.*

Proof. If E/F is discrete then F must be open. Suppose conversely that F is open. Then for all $v \in E$, $v + F$ is open; therefore, $\bigcup_{v \notin F} v + F$ is open; and hence F, which is the complement of this set, is closed. Since F is open and closed, E/F is discrete.

Definition 1.1. Let E be a topological vector space over Δ whose topology is defined by affine sets, and A an affine subset of E. A will be said to be *linearly compact* if $\bigcap A_\alpha \neq \phi$ for every family $\{A_\alpha\}$ of closed affine subsets of A with the finite intersection property. If E itself is linearly compact we will say that E is a *linearly compact topological vector space.*

Example. Suppose the topology of E is the discrete topology. Then an affine subset is linearly compact if and only if it is finite dimensional.

The following are a list of basic facts about linearly compact spaces.

Proposition 1.2.

a) *If E is linearly compact, and A is a closed affine subset of E, then A is linearly compact.*

b) *If A is a linearly compact affine set in E, then A is closed.*

c) *If d is a continuous linear mapping of E into F and A is a linearly compact affine subset of E, then $d(A)$ is linearly compact.*

d) *If $\{E_\alpha\}$, $\alpha \in I$, is a family of linearly compact spaces, then $\prod_{\alpha \in I} E_\alpha$ is linearly compact.*

e) *If E is the projective limit of linearly compact spaces, then E is linearly compact.*

All these facts are proved exactly as one proves the analogous theorems about compact spaces. For details, see Lefschetz's book.

Corollary 1. *If E is linearly compact and F is a closed subspace of E, then E/F is linearly compact.*

Corollary 2. *If E is linearly compact, and F_1 and F_2 are closed subspaces of E, then $F_1 + F_2$ is closed.*

Proof. The image of F_2 in E/F_1 is linearly compact, and hence closed; so the preimage, which is $F_1 + F_2$, is closed.

Corollary 3. *If E is linearly compact and F is open in E, then E/F is finite dimensional.*

Proof. E/F is both discrete and linearly compact, and hence finite dimensional.

1.2. Let E be a topological space over Δ whose topology is defined by affine sets, and E^* the topological dual of E. We topologize E^* by prescribing, for a system of neighborhoods of the origin, the collection of all sets of the form: $F^{\perp}$, where F is a linearly compact subspace of E and $F^{\perp}$ is its annihilator in E^*.

Proposition 1.3. *If E is discrete (linearly compact), then E^* is linearly compact (discrete).*

Proof. If E is linearly compact, then $\{0\} = E^{\perp}$ is a neighborhood of the origin; hence E^* is discrete. Suppose conversely that E is discrete. Let $\{h_\alpha\}$, $\alpha \in I$, be a basis for E, and E_α the one-dimensional subspace of E generated by h_α. It is easy to see that E^* can be identified with the infinite product: $\prod_{\alpha \in I} E_\alpha$, which by part d) of Proposition 1.2 is linearly compact.

Proposition 1.4. *Let E be either discrete or linearly compact. Then there is a canonical isomorphism: $E \cong E^{**}$.*

Proof. We will prove the proposition for the case of linearly compact E. The case of discrete E will be left as an exercise.

An element of E defines a linear functional on E^*, which is continuous, since the topology of E^* is discrete. Therefore, there is a natural linear mapping: $E \to E^{**}$, which will be shown to be an injection. If $e \in E$, $e \neq 0$, there exists an open set $\mathcal{O}$ containing the origin but not E, since E is Hausdorff. By Corollary 3 to Proposition 1.2 we can assume that $\mathcal{O}$ is a subspace of codimension 1; hence, there is a continuous linear functional $E \to \Delta$, which is equal to 1 on e. This proves the injectivity. The mapping: $E \to E^{**}$ is continuous, because the preimages of open subspaces are open. We will show that the image has to be dense. Because of the nature of the topology of E^{**} this amounts to showing that for any set of numbers $c_1, \cdots, c_n \in \Delta$ and any set of linearly independent vectors $f_1, \cdots, f_n \in E^*$ there exists an e in E such that $f_i(e) = c_i$ for $i = 1, \cdots, n$. However, the mapping $\Phi : E \to \overset{n}{\otimes} \Delta$ which sends e onto $\langle f_1(e), \cdots, f_n(e) \rangle$ has to be surjective if $f_1, \cdots, f_n$ are to be linearly independent in E^*; so this condition is always satisfied. By Proposition 1.2 the image of the mapping of E into E^{**} is closed, and since the image is dense, the mapping is surjective. Moreover, by Proposition 1.2, it maps closed subspaces of finite codimension onto closed subspaces of finite codimension; hence by Corollary 3 it maps open sets onto open sets, and therefore is a homeomorphism.

Corollary 1. *A linearly compact space is the topological dual of a discrete space and visa versa.*

Corollary 2. *If a space is linearly compact, then it can be represented as the product of one dimensional spaces.*

Corollary 3. *If a space is linearly compact, then it can be represented as the projective limit of finite dimensional discrete spaces.*

Proof. Let E be linearly compact. By Corollary 2, E can be written as a product of the form $\prod_{\alpha \in I} E_\alpha$ where $\{E_\alpha\}$, $\alpha \in I$, is a family of one dimensional discrete spaces. Let $\mathscr{S}$ be the directed system of finite subsets of I and for each $S \in \mathscr{S}$ let $E_S = \prod_{\alpha \in S} E_\alpha$. Then $E = \lim E_S$.

Corollary 4. *If E is linearly compact, then it is complete in its uniform topology.*

Proof. E is the projective limit of discrete spaces each of which has this property; hence it also has this property. q.e.d.

Let A be a subset of E. We will denote by $A^\perp$ the set of vectors in E^* which annihilate it.

Proposition 1.5. *Let E be either discrete or linearly compact, and F a closed subspace of E. Then $F^{\perp\perp} = F$.*

The proof is an immediate consequence of the following:

Lemma. *Let E be discrete or linearly compact. Let F be a closed subspace of E, and a an element of E not in F. Then there exists an element f in E^* such that $f(a) \neq 0$ and $f | F = 0$.*

Proof. We will give the proof for the case of linearly compact E. The case of discrete E will be left as an exercise.

If E is linearly compact, then E/F is linearly compact, and every continuous linear functional on E/F extends to a continuous linear functional on E; so it is enough to prove the theorem when $F = \{0\}$. Suppose the theorem were false. Then every continuous linear functional on E would have to vanish on a; hence a would be in the kernel of the mapping: $E \longrightarrow E^{**}$ which would contradict Proposition 1.4.

Corollary. *Let E be linearly compact, and F^k, $k = 1, 2, 3, \cdots$ a decreasing chain of closed subspaces of E with $\bigcap_{k=1}^{\infty} F^k = \{0\}$. Let $\mathcal{O}$ be a neighborhood of the origin in E. Then there exists an integer k_0 such that $F^{k_0} \subset \mathcal{O}$.*

Proof. Let $G^k = (F^k)^\perp$ and $G = \bigcup_{k=0}^{\infty} G^k$. Then $G^\perp \subset F^k$, for all k; so $G^\perp = \{0\}$, and $G = \bigcup_{k=0}^{\infty} G^k = E^*$. Since $\mathcal{O}$ is open and contains the origin we can assume it is an open subspace, which means, by Corollary 3 to Proposition 1.2, that $\mathcal{O}$ is closed and of finite codimension. Since $\mathcal{O}^\perp$ must then be finite dimensional, $\mathcal{O}^\perp \subset G^k$ for some large k. Hence, $\mathcal{O} \supset F^k$ for the same k. q.e.d.

The corollary above is a special case of Chevalley's theorem (cf. Zariski-Samuel [16]).

Given two linearly compact spaces E and F their usual algebraic tensor product would not, in general, be linearly compact. However we can define a topological tensor product which is linearly compact as follows. We form the ordinary tensor product $E^* \otimes F^*$ and give it the discrete topology. Then we define the topological tensor product of E and F to be the space $(E^* \otimes F^*)^*$, which will be denoted by $E \hat{\otimes} F$.

2. Some Lie algebras with topological properties

2.1. We introduce a special family of Lie algebras which throughout this sub-section will be designated by the symbol $\mathscr{L}_\Delta$.

Definition 2.1. A Lie algebra L belongs to $\mathscr{L}_\Delta$ if and only if its underlying vector space is a linearly compact topological vector space over Δ, and the bracket operation, regarded as a mapping of $L \times L$ into L, is continuous.

Proposition 2.1. *Let L belong to $\mathscr{L}_\Delta$, and I a closed ideal of L. Then L/I belongs to $\mathscr{L}_\Delta$.*

Proof. LTR.

Proposition 2.2. *Let L belong to $\mathscr{L}_\Delta$, and $\mathcal{O}$ a neighborhood of the origin in L. Then there exists an open subspace $\mathcal{O}'$ contained in $\mathcal{O}$ such that $[L, \mathcal{O}'] \subset \mathcal{O}$.*

Proof. We can assume that $\mathcal{O}$ is a subspace. Because of the continuity of the bracket mapping, there exists an open subspace $\mathcal{O}''$ contained in $\mathcal{O}$ such that $[\mathcal{O}'', \mathcal{O}''] \subset \mathcal{O}$. $\mathcal{O}''$ is of finite codimension; therefore we can find a finite number of elements $x_1, \cdots, x_r$ of L spanning a complement of $\mathcal{O}''$ in L. Moreover for each x_i we can find an open subspace $\mathcal{O}_i$ such that $(ad\, x_i)(\mathcal{O}_i) \subset \mathcal{O}$. Let $\mathcal{O}' = \mathcal{O}'' \cap (\bigcap_{i=1}^{r} \mathcal{O}_i)$.

Corollary. *If $\mathcal{O}$ is an open subspace of L, then there exists an open subalgebra of L contained in $\mathcal{O}$.*

Proof. Let $D_L\mathcal{O} = \{X \in \mathcal{O} \mid [L, X] \subset \mathcal{O}\}$. $D_L\mathcal{O}$ is an open subspace of $\mathcal{O}$, by Proposition 2.2. We will prove that it is a subalgebra. If X_1 and X_2 are in $D_L\mathcal{O}$, then $[X_1, X_2] \in \mathcal{O}$. Moreover, if $Y \in L$, then $[Y, [X_1, X_2]] = [[Y, X_1], X_2] + [X_1, [Y, X_2]]$, and hence is in $\mathcal{O}$. This proves that $[X_1, X_2]$ is in $D_L\mathcal{O}$. q.e.d.

The technique used to prove the corollary to Proposition 2.2 will be useful later on; therefore we will incorporate it in a definition.

Definition 2.2. Let L be a Lie algebra, defined over the field Δ, and A a subspace of L. We will denote by D_LA the set $\{X \in A \mid [L, X] \in A\}$.

The following facts will be left for the reader to varify.

Proposition 2.3.

a) *If A and B are subspaces of L, then $D_L(A \cap B) = D_LA \cap D_LB$.*

b) *D_LA is a subalgebra of L.*

c) *If A is itself a subalgebra of L, then D_LA is an ideal of A (NOT necessarily an ideal of L, however).*

d) *If A is open (closed), D_LA is open (closed).*

We can interate the operation: $A \to D_L A$. If A is a subspace of L, we will define inductively: $D_L^1 A = D_L A$, $D_L^i A = D_L(D_L^{i-1} A)$, $i = 1, 2, \cdots$. We will let $D_L^\infty A = \bigcap_{i=1}^{\infty} D_L^i A$.

Proposition 2.4. *$D_L^\infty A$ is an ideal of L. Moreover, every ideal of L which is contained in A is contained in $D_L^\infty A$.*

Proof. If X is an element of L, then $[X, D_L^i A] \subset D_L^{i-1} A$ by definition. Hence $[X, D_L^\infty A] \subset D_L^\infty A$. This proves the first part of the proposition. If I is an ideal of L contained in A, then I is contained in $D_L A$ since $[L, I] \subset A$. Repeating this argument, we see that $I \subset D_L(D_L A)$, and by induction, $I \subset D_L^i A$ for all i.

2.2. This sub-section is mostly devoted to studying the class of Lie algebras given by the following definition.

Definition 2.3. A Lie algebra L belonging to $\mathscr{L}_\Delta$, *has no small ideals*, if there is a neighborhood of the origin in L containing no ideals of L except the trivial ideal.

Before discussing this definition (in § 3), we will list a few examples:

Example 1. Let L be a Lie algebra belonging to $\mathscr{L}_\Delta$, and A an open subspace of L. Let $L' = L/D_L^\infty A$. By Proposition 2.4, L' has no small ideals.

Example 2. Let $x_1, \cdots, x_d$ be indeterminants, and F the ring of formal power series in $x_i, \cdots, x_d$ with coefficients in the field Δ. Let L be the Lie algebra of derivations of F. To define a topology in L we first note that F has a canonical filtration, $F = F^0 \supset F^1 \supset \cdots$, where F^k is the set of formal power series whose leading terms are of degree k. Let L^k be the subspace of L consisting of derivations which map F into F^{k+1}. It is clear that L^k is of finite codimension in L and that $\bigcap_{k=0}^{\infty} L^k = \{0\}$. We define a topology on L by letting $\{L^k\}$ be a system of neighborhoods of the origin. We will lèt the reader verify that $L \in \mathscr{L}_\Delta$, the main point being to establish the inclusion:

$$(*) \qquad [L^s, L^t] \subset L^{s+t},$$

which shows that the bracket operation is continuous in this topology. The inclusion $(*)$ also shows that $L^k = D_L^k L^0$; therefore by Proposition 2.4 there are no non-trivial ideals of L contained in L_0. Since L_0 is open, L has no small ideals.

Example 3. Let L be the algebra discussed in example 2, and M a closed subalgebra of L. M will be called *transitive* if the mapping $M \to L/L_0$ is surjective. Let $M^k = M \cap L^k$. It is not hard to show that if M is transitive, then $M^k = D_M^k M^0$. Therefore, by Proposition 2.4, there are no ideals of M contained in M^0, and, since M^0 is open in M, the condition of Definition 2.3 is satisfied.

Algebras of the type discussed in Example 3 have been studied in detail in [3]. Using the imbedding theorem proved in [3], together with results proved below, one can show that if a Lie algebra has no small ideals, then it is isomorphic to a Lie algebra of the type discussed in Example 3.

3. Filtrations

3.1. Definition 3.1. Let L be a Lie algebra belonging to $\mathscr{L}_\Delta$. Let $\mathscr{F} = \{F^i\}$, $-\infty < i < +\infty$, be a collection of closed subspaces of L. We will say that $\mathscr{F}$ is a *filtration* of L if:

a) $F^i \supset F^{i+1}$ for all i,

b) $\bigcup_{i=-\infty}^{\infty} F^i = L$ and $\bigcap_{i=-\infty}^{\infty} F^i = \{0\}$,

c) $[F^i, F^j] \subset F^{i+j}$ for all i and j.

The filtration will be called *open* if all the F^i are open.

If $\{F^i\}$, $-\infty < i < \infty$, is an open filtration of L, then $F^i = L$ for some i, since all the F^i are of finite codimension and their union is equal to L. Let i_0 be the largest integer for which $F^i = L$. If L is infinite dimensional over Δ, and has no small ideals, then i_0 must be ≤ -1; for, if not, we would have $[L, F^i] = [F^0, F^i] \subset F^i$; hence all the F^i's would be ideals and, by Chevalley's theorem, every neighborhood of the origin would contain some F^i. If $i_0 = -k$, where $k > 1$, then the filtration of L obtained by setting $\bar{F}^i = F^{ki}$, $-\infty < i < \infty$, has the property that $\bar{F}^{-1} = L$.

The remarks above justify our making the following assumption: Given an open filtration of L we will always assume, unless otherwise stated, that all the terms in the filtration of degree ≤ -1 are equal to L and that the term of degree 0 is unequal to L.

We will show now that a Lie algebra admits an open filtration provided it has no small ideals. First we need the following notion.

Definition 3.2. Let L be a Lie algebra belonging to $\mathscr{L}_\Delta$, and A an open subalgebra of L. We will say that A is *fundamental* if it contains no ideals of L except $\{0\}$.

If L possesses a fundamental subalgebra, then L has no small ideals. Conversely if L has no small ideals, then there is an open neighborhood of the origin containing no non-trivial ideals of L, and, therefore, by Proposition 2.2, L possesses a fundamental subalgebra.

Let A be a fundamental subalgebra of L. We will define an open filtration of L as follows: $A^k = L$ for $k \leq -1$, $A^0 = A$, and $A^k = D^k_L A$ for $k > 0$. By proposition 2.3, A^k is open for all k, and Proposition 2.4, coupled with the fact that A is fundamental, implies that $\bigcap_{i=0}^{\infty} A^i = \{0\}$. We let the reader verify that $[A^i, A^j] \subset A^{i+j}$ for all i and j (proof by induction on Proposition 2.3). Therefore, $\{A^k\}$, $-\infty < k < \infty$, is an open filtration.

3.2. Let $\mathscr{F} = \{F^i\}$, $-\infty < i < \infty$, be any filtration of L. The graded vector space

$$\sum_{i=-\infty}^{\infty} F^i/F^{i+1}$$

has a natural Lie algebra structure. We will denote this graded Lie algebra by $gr(L, \mathscr{F})$, and also sometimes write $gr(L, \mathscr{F})^i$ for F^i/F^{i+1}.

Suppose now that $\mathscr{F}$ is an open filtration of L. Because of our convention about degrees, $gr(L, \mathscr{F})^i = 0$ if $i < -1$. Let $V = gr(L, \mathscr{F})^{-1}$. If we bracket V with itself we land in $gr(L, \mathscr{F})^{-2}$, which is 0. Therefore V is an abelian subalgebra of $gr(L, \mathscr{F})$.

Let $gr(L, \mathscr{F})^*$ be the graded dual space of the vector space $gr(L, \mathscr{F})$. As a vector space it is the direct sum:

$$\sum_{i=-\infty}^{\infty} (F^i/F^{i+1})^* .$$

There is a natural linear representation of $gr(L, \mathscr{F})$ on $gr(L, \mathscr{F})^*$, namely the transpose of the adjoint representation. If we restrict this representation to the subalgebra V of $gr(L, \mathscr{F})$ we get a representation of V on $gr(L, \mathscr{F})^*$.

Given a representation of a Lie algebra, it has a canonical extension to a representation of its universal enveloping algebra. However, the universal enveloping algebra of the abelian algebra V is just the polynomial ring of V, which we will denote by $S(V)$. Hence we have established the following result.

Proposition 3.1. *The dual $gr(L, \mathscr{F})^*$ of the graded Lie algebra associated with the filtration $\mathscr{F}$ has the structure of a graded $S(V)$ module.*

Remark. It is easy to check that if $x \in S(V)$ is of degree i and $a \in gr(L, \mathscr{F})^*$ is of degree j, then xa is of degree $i + j$.

It will be important to know when the module $gr(L, \mathscr{F})^*$ is finitely generated. The proposition below is a criterion for this.

Proposition 3.2. *The following two statements are equivalent:*

a) *There exists an integer n_0 such that for all $n > n_0$, $F^n = D_L F^{n-1}$.*

b) *The module $gr(L, \mathscr{F})^*$ is finitely generated as an $S(V)$ module.*

Proof. Assume the first statement is true. We will show that $gr(L, \mathscr{F})^*$ is generated by its terms of degree $\leq n_0$. To show this we have to show that the mapping

$$V \otimes (F^n/F^{n+1})^* \to (F^{n+1}/F^{n+2})^* \tag{3.1}$$

defined by multiplication by V is surjective if $n > n_0$. This is equivalent to proving that the dual mapping:

$$F^{n+1}/F^{n+2} \to F^n/F^{n+1} \otimes V^* \tag{3.2}$$

is injective for $n > n_0$. Suppose a is in the kernel of (3.2). This means that $[V, a] = 0$, or, if α is a representative of a in F^{n+1}, that $[L, \alpha] \subset F^{n+1}$, which is equivalent to saying that α is in $D_L F^{n+1}$. By assumption, $D_L F^{n+1} = F^{n+2}$ for

$n > n_0$; so α is in F^{n+2}, and $a = 0$. This proves the injectivity of (3.2). The argument we have just given can be reversed to show that b) implies the injectivity of (3.2) providing n_0 is chosen to be larger than the degree of any of the generators of $gr(L, \mathscr{F})^*$, and from the injectivity of (3.2) we can conclude, as above, that $D_L F^{n+1} = F^{n+2}$.

Definition 3.3. If the filtration $\mathscr{F}$ possesses either, and hence both, of the properties of Proposition 3.2, we will call it *admissible*.

We have shown above that if L has no small ideals, it possesses an admissible filtration. We will use this fact to prove

Theorem 3.1. *Let L belong to $\mathscr{L}_\Delta$. Then the following two statements are equivalent:*

a) *L has no small ideals.*

b) *L satisfies the descending chain condition on closed ideals.*

We will first prove

Lemma. *Let $\{F^i\}$, $-\infty < i < \infty$, be an open filtration of L. Let I_1 and I_2 be closed ideals of L with I_1 contained in I_2, and*

$$gr(I_k, \mathscr{F}) = \sum_{s=-\infty}^{\infty} (I_k \cap F^s + F^{s+1})/F^s\,, \qquad k = 1, 2\,,$$

the graded ideal in $gr(L, \mathscr{F})$ corresponding to I_k. If $gr(I_1, \mathscr{F}) = gr(I_2, \mathscr{F})$, then $I_1 = I_2$.

Proof. Suppose, by induction, we have shown that $I_2 \subset I_1 + F^r$. We will show that $I_2 \subset I_1 + F^{r+1}$. By assumption $gr(I_1)^r = gr(I_2)^r$. Therefore $I_1 \cap F^r + F^{r+1} = I_2 \cap F^r + F^{r+1}$, and $I_2 \subset I_1 + I_2 \cap F^r + F^{r+1} \subset I_1 + F^{r+1}$. By Chevalley's theorem F^r eventually gets inside of every open neighborhood of the origin. Since I_1 is closed, $I_1 = \bigcap_{r=0}^{\infty} I_1 + F^r$, and hence $I_2 \subset I_1$.

Proof of Theorem 1.1. That b) implies a) is trivial. To prove that a) implies b), let $I_1 \supset I_2 \supset I_3 \cdots$ be a descending chain of closed ideals. We want to show that this chain stabilizes after a finite number of terms. Since each term in the sequence is closed it is enough, by the lemma, to show that the sequence $gr(I_k, \mathscr{F})$, $k = 0, 1, 2, \cdots$, stabilizes after a finite number of terms, where $\mathscr{F}$ is an admissible filtration of L.

Let $gr(I_k, \mathscr{F})^\perp$ be the annihilator of $gr(I_k, \mathscr{F})$ in $gr(L, \mathscr{F})^*$. This subspace is stable with respect to the transpose of the adjoint representation; so, in particular, it is an $S(V)$ submodule of $gr(L, \mathscr{F})^*$. Since $(gr(I_k, \mathscr{F})^\perp)^\perp = gr(I_k, \mathscr{F})$, it is enogh to show that the increasing sequence of submodules $gr(I_k, \mathscr{F})^\perp$ eventually stabilizes. However, $gr(L, \mathscr{F})^*$ is finitely generated over a Noetherian ring; hence every increasing sequence of submodules stabilizes.

4. Primitive and simple Lie algebras, Schur's lemma

4.1. In § 6 we will need the following notion:

Definition 4.1. Let L be a Lie algebra belonging to $\mathscr{L}_\Delta$. L is called primitive if it contains a maximal proper subalgebra which is fundamental.

Proposition 4.1. *If L is primitive, then every closed ideal of L except* $\{0\}$ *is of finite codimension.*

Proof. Let M be a maximal proper subalgebra which is fundamental, i.e. contains no ideals of L except $\{0\}$. Let I be an ideal of L. Since there are no subalgebras strictly contained between L and M, $I + M = L$. Hence the homomorphism: $M \to L/I$ is surjective. Since $D_L^k M$ is an ideal of M, its image is an ideal of L/I. But L/I satisfies the descending chain condition; therefore the sequence $I + D_L^k M$ stabilizes after a finite number of terms. Since I is closed, $I = \bigcap_{k=0}^{\infty} I + D_L^k M = I + D_L^k M$ for large k. Thus, I is of finite codimension.

4.2. In this sub-section we prove some elementary results about simple algebras.

Lemma 4.1. *Let L belong to $\mathscr{L}_\Delta$ and satisfy the descending chain condition, and $\{F^k\}$ be an admissible filtration of L. Then there exists an integer k_0 such that for all $k > k_0$ the bracket mapping*

$$(F^{-1}/F^0) \times (F^k/F^{k-1}) \to (F^{k-1}/F^k)$$

is surjective.

Proof. Let $W = F^{-1}/F^0$. The dual of the bracket mapping described above is the mapping of the dual spaces

$$W \times (F^{k-1}/F^k)^* \to (F^k/F^{k+1})^* .$$

The dual from of the statement we want to prove is the following: If $a \in (F^{k-1}/F^k)$ and $(w, a) \to 0$ for all $w \in W$, then $a = 0$. By Proposition 3.2, $gr(L, \mathscr{F})^*$ is a finitely generated $S(W)$ module. Hence the submodule of elements which get annihilated when multiplied by elements of W is finitely generated over $S(W)$, and since it is in effect a "Δ" module, it is finitely generated over Δ, i.e. a finite dimensional vector space over Δ. Thus, if k is large enough the condition $(w, a) \to 0$ for all $w \in W$ implies that $a = 0$.

Lemma 4.2. *Let L belong to $\mathscr{L}_\Delta$, $\{F^i\}$ be an admissible filtration of L, and H be a subspace of L complementary to the subspace F^0. Then there exists an integer k_0 such that $[H, F^k] = F^{k-1}$ for $k > k_0$.*

Proof. Let k_0 be as in Lemma 4.1, $x_1, x_2, \cdots, x_n$ a basis of H, and a an element of F^k. By Lemma 4.1 we can construct a sequence of elements $a_{k+u}, b_{r,k+u} \in F^{k+u}$, $u = 1, 2, \cdots, r = 1, 2, \cdots, n$, such that $a_k = a$ and

$$a_{k+u-1} = \sum_{r=1}^{n} [x_r, b_{r,k+u}] + a_{k+u} .$$

Let $b_r = \sum_{u=1}^{\infty} b_{r,k+u}$, which converges, since the subspaces F^k form a neighborhood base for the topology of L at the origin, and L is complete in its topology. By construction,

$$a = \sum_{r=1}^{n} [x_r, b_r] .$$

Proposition 4.2. *Let L belong to $\mathscr{L}_\Delta$ and satisfy the descending chain condition on closed ideals, and I be a dense ideal of L. Then $I = L$.*

Proof. Let $\{F^k\}$ be an admissible filtration of L. Since I is dense in L, it contains a subspace H complementary to F^0, and therefore contains F^k for k sufficiently large by Lemma 4.2. This means that I is open, and hence closed and equal to L.

Definition 4.2. Let L be a Lie algebra belonging to $\mathscr{L}_\Delta$. L will be called algebraically simple if it contains no non-trivial ideals, and topologically simple if it contains no non-trivial closed ideals.

Proposition 4.3. *The above definition is redundant. L is algebraically simple if and only if it is topologically simple.*

Proof. If L is topologically simple, then every non-trivial ideal of L is dense in L; hence by Proposition 4.2 there are no non-trivial ideals in L.

4.3. Definition 4.3. Let L be a Lie algebra over a field Δ. We will denote by Δ_L the set of Δ-linear mappings of L into L which commute with all of the mappings $ad(x): L \to L$, $x \in L$. Δ_L will be called the commutator set of L.

If we take the sum of two mappings belonging to Δ_L or take their composition, then the resulting mapping belongs to Δ_L; therefore Δ_L has the structure of an associative ring. If a is an element of the base field Δ, then the mapping of L onto L defined by $x \to ax$ belongs to Δ_L. Hence there is an injection $\Delta \to \Delta_L$.

There is a very simple criterion for Δ_L to be a commutative ring, namely the following:

Lemma 4.3. *If $[L, L] = L$, then Δ_L is a commutative ring.*

Proof. Let ρ_1 and ρ_2 be elements of Δ_L. Applying $\rho_1\rho_2$ to an element in L of the form $[x, y]$ we get

$$\rho_1\rho_2[x, y] = [\rho_1 x, \rho_2 y] = \rho_2\rho_1[x, y] .$$

Hence ρ_1 and ρ_2 commute on $[L, L]$, and if $[L, L] = L$, then Δ_L is a commutative ring. q.e.d.

In the next two sections we will make extensive use of the following proposition which is an infinite dimensional version of a special case of Schur's lemma.

Proposition 4.4. *Let L belong to $\mathscr{L}_\Delta$, and be non-abelian and simple.*

Then Δ_L is a commutative field, which can be obtained from Δ by a finite algebraic extension.

Proof. If L is finite dimensional over Δ, then Proposition 4.4 follows from Lemma 4.3 and the usual form of Schur's lemma. To prove the Proposition when L is infinite dimensional over Δ we imitate the standard proof of Schur's lemma. Let ρ be an element of Δ_L. Both the kernel and image of the mapping ρ are ideals of L. Hence either ρ is the zero mapping or ρ is bijective and has a two-sided inverse which also belongs to Δ. Therefore Δ_L is a division ring. Since L is simple and non-abelian, $[L, L] = L$. Thus, by Lemma 4.3, Δ_L is a field. Let $\{F^i\}$ be an admissible filtration of L. By Lemma 4.2, there exists an integer k_0 such that $F^k = [L, F^{k+1}]$ for $k > k_0$. If ρ is an element of Δ_L, then $\rho F^k = [\rho L, F^{k+1}] = [L, F^{k+1}] = F^k$. Therefore, Δ_L has a representation as a ring of endomorphisms of L/F^k. This representation is faithful, since Δ_L is a division ring and is faithful on Δ. Since L/F^k is finite dimensional over Δ, standard theorems in linear algebra imply that Δ_L is a finite algebraic extension of Δ. q.e.d.

We conclude this sub-section with a few remarks about simple Lie algebras belonging to $\mathscr{L}_\Delta$, when Δ is of characteristic zero and algebraically closed. It was conjectured by Sophus Lie that these algebras consist of the finite dimensional simple Lie algebras plus four families of infinite dimensional algebras, which correspond roughly to the set of all diffeomorphisms of n space, the set of all volume preserving diffeomorphisms of n space, the set of all simplectic diffeomorphisms of $2n$ space, and the set of all conctact transformations on $2n + 1$ space. The validity of Lie's conjecture was proved by Eli Cartan [2], but there were certain gaps in Cartan's proof which were recently filled in by Quillen, Sternberg and the author in [5].

If Δ is not algebraically closed or of characteristic zero, not much is known about the simple algebras belonging to $\mathscr{L}_\Delta$. In the case where $\Delta =$ reals, a list of simple Lie algebras containing all known examples can be found in [14].

5. Some applications of Schur's lemma

5.1. Let L be a simple non-abelian Lie algebra belonging to $\mathscr{L}_\Delta$, and Δ_L the commutator set of L. By Proposition 4.4, Δ_L is a commutative field which can be obtained from Δ by a finite algebraic extension.

Let W be a finite dimensional vector space over Δ, and consider the tensor product[1]

$$L \underset{\Delta}{\otimes} W .$$

[1] For reasons which will be clear below we will start indicating tensor products with respect to Δ by putting a subscript Δ below the tensor product sign.

There is a natural representation of L on this space, namely the tensor product of the adjoint representation with the trivial representation of L on W. We will denote this representation by γ. If a is an element of L, and $b \otimes w$ is an element of $L \otimes W$, then $\gamma(a)(b \otimes w) = [a, b] \otimes w$. For certain applications we have in mind in §§ 6–7, we will need to determine all invariant subspaces of $L \underset{\Delta}{\otimes} W$ with respect to this representation.

We first point out that there is a standard way to manufacture out of W a vector space over the field Δ_L: we just take the tensor product $\Delta_L \underset{\Delta}{\otimes} W$, and Δ_L obviously acts on this on the right. When we speak below of subspaces of $\Delta_L \underset{\Delta}{\otimes} W$ we will mean "Δ_L" or "Δ_L-invariant" subspaces.

Since L is a module over Δ_L there is a natural Δ-linear mapping

$$L \underset{\Delta}{\otimes} \Delta_L \to L . \tag{5.1}$$

If we tensor the right hand side of (5.1) by W, then we get a mapping

$$\wedge : L \underset{\Delta}{\otimes} (\Delta_L \otimes W) \to L \underset{\Delta}{\otimes} W .$$

Let D be an invariant subspace of $L \underset{\Delta}{\otimes} W$ with respect to the representation γ. We associate with D a subspace U_D of $\Delta_L \otimes W$ as follows:

$$u \in U_D \Longleftrightarrow \widehat{a \otimes u} \in D , \qquad \forall a \in L .$$

Our main result is the following:

Proposition 5.1. $D = L \underset{\Delta_L}{\otimes} U_D$.

Proof. Let us introduce the following terminology. We will say that a subspace V of W is minimal, if $D \cap (L \underset{\Delta}{\otimes} V) \neq \{0\}$ and no proper subspace of V has this property. To prove Proposition 5.1. we will first prove

Lemma 5.1. *If V is a minimal subspace of W, then $D \cap (L \underset{\Delta}{\otimes} V) \subset L \underset{\Delta_L}{\otimes} U_D$.*

Proof. Let $\omega_1, \cdots, \omega_k$ be a basis of V. Then every element of $D \cap (L \underset{\Delta}{\otimes} V)$ can be written in the form

$$a_1 \otimes \omega_1 + \cdots + a_k \otimes \omega_k , \tag{5.2}$$

where $a_i \in L$, $i = 1, \cdots, k$. The space $D \cap (L \underset{\Delta}{\otimes} V)$ is invariant with respect to γ; hence for all $a \in L$

$$[a, a_1] \otimes \omega_1 + \cdots + [a, a_k] \otimes \omega_k$$

is in this space providing (5.2) is in this space. This shows that the set of all coefficients of ω_1 occuring in (5.2) forms an ideal of L. Since L is simple and this ideal is non-zero, every element of L must occur as a coefficient of ω_1 in (5.2). Moreover, by the minimality of V, every coefficient can occur just once because if there were duplications we could get a non-zero element of D involving fewer of the ω's by subtraction. In other words, the intersection $D \cap (L \underset{\Delta}{\otimes} W)$ consists of

$$(5.3) \qquad a \otimes \omega_1 + \rho_2(a) \otimes \omega_2 + \cdots + \rho_k(a) \otimes \omega_k\,, \quad V_a \in L\,,$$

where $\rho_2, \cdots, \rho_k$ are certain linear mappings of L into L. Applying $\gamma(b)$ to a typical expression of the above form we get

$$[b, \rho_i(a)] = \rho_i[a, b]$$

for all a, b in L. This proves that $\rho_i \in \Delta_L$ for all i. Thus, if we set $u = \omega_1 + \rho_2 \otimes \omega_2 + \cdots + \rho_k \otimes \omega_k$, then we get

$$D \cap (L \underset{\Delta}{\otimes} V) = \{a \mathbin{\hat{\otimes}} u,\, a \in L\}\,,$$

which proves the lemma.

We will now conclude the proof of Proposition 5.1. We will say that an element α of D is a minimal element if it is contained in an intersection of the form

$$D \cap (L \underset{\Delta}{\otimes} V)\,,$$

where V is a minimal space. We will prove the proposition by showing that every element of D can be written as the sum of minimal elements. We assume this statement is false, and argue by contradiction. Let α be an element of D which is not the sume of minimal elements. Then α will be of the form

$$(5.4) \qquad \alpha = a_1 \otimes \omega_1 + \cdots + a_r \otimes \omega_r\,, \quad a_i \in L\,, \quad \omega_i \in W\,.$$

We can always assume α chosen to involve as few of the ω's as possible. Since α is non-zero the subspace of W spanned by $\omega_1, \cdots, \omega_r$ contains a minimal subspace V. Moreover in this subspace we can always find an element of the form

$$a_1 \otimes \omega_1 + a_2' \otimes \omega_2' + \cdots + a_r' \otimes \omega_r'$$

(with a_1 as above and the other a''s possibly different). Subtracting this expression from (5.4) we get an element which is not the sum of minimal elements and involves fewer of the ω's. This contradiction proves the proposition.

Corollary. *Let D be an invariant subspace of $L \underset{\Delta}{\otimes} W$ with respect to the representation γ. Then $\gamma(L)D = D$.*

Proof. $[L, L] = L$; therefore $\gamma(L)D = \gamma(L)(L \underset{\Delta_L}{\otimes} U_D) = [L, L] \underset{\Delta_L}{\otimes} U_D = L \underset{\Delta_L}{\otimes} U_\Delta = D$.

5.2. In this sub-section we will prove a generalization of Schur's lemma which will be needed in § 7.

Let Δ be a field, and F the ring of formal power series $\Delta[[x_1, \cdots, x_n]]$ in n indeterminants over Δ. Let L be a simple Lie algebra belonging to $\mathscr{L}_\Delta$, whose commutator field Δ_L is equal to Δ. The ring F with its Krull topology is linearly compact; hence the tensor product $L \hat{\otimes} F$ is defined. The ordinary tensor product $L \otimes F$ has the structure of a Lie algebra, and we leave it as an exercise to verify that there is a unique Lie algebra structure on $L \hat{\otimes} F$ such that the bracket operation is continuous and such that the mapping

$$L \otimes F \to L \hat{\otimes} F$$

is a Lie algebra homomorphism. A corollary of this remark is that $L \hat{\otimes} F$ belongs to $\mathscr{L}_\Delta$.

Proposition 5.2. *The ring of commutators of $L \hat{\otimes} F$, i.e. $\Delta_{L \hat{\otimes} F}$, is isomorphic to F.*

Proof. For every multi-index $\alpha_1, \cdots, \alpha_n$, α_i a non-negative integer, we will denote by x^α the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in the indeterminates $x_1, \cdots, x_n$. Let ρ be an element of $\Delta_{L \hat{\otimes} F}$. We will regard L as imbedded in $L \hat{\otimes} F$ and look at $\rho \mid L$. If $A \in L$, then we can write

$$\rho(A) = \sum_{\alpha=0}^{\infty} \rho_\alpha(A) X^\alpha , \qquad \rho_\alpha(A) \in L ,$$

where ρ_α depends linearly on A. If we bracket this expression by an element B of L, then we get $[\rho_\alpha(A), B] = \rho_\alpha[A, B]$ for all A, B in L (i.e. $\rho_\alpha \in \Delta$), since $\rho[A, B] = [\rho(A), B]$. Hence we can write

$$\rho(A) = A \otimes \sum_{\alpha=0}^{\infty} \rho_\alpha X^\alpha \qquad \text{for all } A \in L .$$

Thus $\rho \mid L$ is just multiplication by the formal power series $\sum \rho_\alpha X^\alpha$. By Lemma 6.2, $[L, L \hat{\otimes} F] = L \hat{\otimes} F$. Hence, if $\rho \in \Delta_{L \otimes F}$, it is determined completely by its restriction to L. This completes the proof.

5.3. Let M be any Lie algebra, and $\mathrm{Der}(M)$ the Lie algebra of its derivations. There is a natural representation of $\mathrm{Der}(M)$ on the commutator ring Δ_M of M defined as follows. If $X \in \mathrm{Der}(M)$ and $\rho \in \Delta_M$, then

$$X \circ \rho - \rho \circ X$$

is a mapping of M into M, which commutes with Lie brackets and hence belongs to Δ_M. We will denote this element of Δ_M by $\mathscr{L}_X\rho$. One easily verifies the following identities:

$$\mathscr{L}_X\rho_1\rho_2 = (\mathscr{L}_X\rho_1)\rho_2 + \rho_1\mathscr{L}_X\rho_2\,,$$
$$\mathscr{L}_X\mathscr{L}_Y\rho - \mathscr{L}_Y\mathscr{L}_X\rho = \mathscr{L}_{[X,Y]}\rho\,,$$

which show that the mapping $X \rightarrow \mathscr{L}_X$ is a homomorphism of the ring $\mathrm{Der}(M)$ into the ring $\mathrm{Der}(\Delta_M)$. The kernel of this homomorphisms consists of all "Δ_M-linear" derivations of M.

We will apply these general remarks to the above situation. Since F is the commutator ring of $L \hat{\otimes} F$ there is a natural homomorphism:

$$\pi : \mathrm{Der}(L \hat{\otimes} F) \rightarrow \mathrm{Der}(F)\,. \tag{5.5}$$

Every derivation of F, however, induces a derivation on $L \hat{\otimes} F$; so there is a homomorphism the other way

$$l : \mathrm{Der}(F) \rightarrow \mathrm{Der}(L \hat{\otimes} F)\,.$$

It is easy to verify that $\pi \circ l =$ identity, which implies that (5.5) is surjective. The kernel of (5.5) consists of all F-linear derivations of $L \hat{\otimes} F$. Such a derivation is determined completely by its restriction to L; hence we can describe the kernel of (5.5) as being the set of all mappings

$$\alpha : L \rightarrow L \hat{\otimes} F$$

satisfying the identity

$$\alpha[X, Y] = [X, \alpha(Y)] + [\alpha(X), Y]$$

for all $X, Y \in L$. We will denote this set by $\mathrm{Der}(L) \hat{\otimes} F$. (We *define* $\mathrm{Der}(L) \hat{\otimes} F$ in this way since we have not defined a topology on $\mathrm{Der}(L)$.)

We summarize the above remarks:

Proposition 5.3. *There is a split exact sequence of Lie algebras:*

$$0 \rightarrow \mathrm{Der}(L) \hat{\otimes} F \rightarrow \mathrm{Der}(L \hat{\otimes} F) \underset{l}{\overset{\pi}{\rightleftarrows}} \mathrm{Der}(F) \rightarrow 0\,. \tag{5.6}$$

6. A Jordan-Hölder decomposition

6.1. The following example shows that it is impossible to get a strict Jordan-Hölder decomposition for the type of Lie algebras we have been considering.

Let $X_1, \cdots, X_n$ be indeterminants, and F the ring of formal power series in $X_1, \cdots, X_n$ with coefficients in Δ. Let F^k be the subspace of F consisting of power series whose leading terms are of degree $\geq k$. Let L be the set of derivations of the form: $c_1 \frac{\partial}{\partial X_1} + \cdots + c_n \frac{\partial}{\partial X_n}$, $c_i \in \Delta$, $i = 1, \cdots, n$. L is an abelian Lie algebra and F is an L module; so we can form the semidirect abelian extension $L \sharp F$ in the category of Lie algebras. As a vector space over Δ, this is just the product of L and F, which can be given the structure of a linearly compact topological space in an obvious way, using the given filtration on F. The bracket operation will be continuous in this topology; so $L \sharp F$ belongs to the family $\mathscr{L}_\Delta$. We claim there are no ideals of L contained in F^1; in fact, every formal power series, if differentiated sufficiently often, will have a leading term of degree zero. Therefore, $L \sharp F$ has no small ideals.

Let F_k denote the set of all polynomials in $X_1, \cdots, X_n$ of degree $\leq k$. This set is stable with respect to L; so $I_k = L \sharp F_k$ is an ideal in $L \sharp F$. Since I_k is finite dimensional, it is closed, and I_k is properly contained in I_{k+1}; therefore the ascending chain condition is violated. To say that $L \sharp F$ has a Jordan-Hölder decomposition in the usual sense of the term means it is possible to find a finite chain of closed ideals, each member of the chain containing the preceding member, so that no closed ideal is strictly contained between any two members of the chain. However, this would imply that $L \sharp F$ satisfy both chain conditions on closed ideals, and, as we have seen, the a.c.c. is violated.

We note that $L \sharp F$ does have a finite chain of closed ideals such that every quotient is abelian. In fact, the following sequence of Lie algebras

$$0 \to F \to L \sharp F \to L \to 0$$

is exact, and both end terms are abelian. We will see that we can get a decomposition, analogous to the classical Jordan-Hölder decomposition, for Lie algebras with no small ideals, provided we allow abelian quotients of the type above which are not, strictly speaking, simple.

6.2. The following proposition is a kind of weak substitute for the a.c.c. on closed ideals.

Proposition 6.1. *Let L be a Lie algebra belonging to $\mathscr{L}_\Delta$. Then there exists a proper closed ideal I of L which is strictly maximal in the sense that there are no ideals of L lying properly between L and I.*

Proof. Among the open subalgebras of L not identical with L itself pick a maximal one, say M. Let I_M be the largest ideal of L contained in M. Since M is closed, I_M is closed. Let $L' = L/I_M$, and let M' be the image of M in L'. M' is a fundamental algebra of L' by construction, and there are no subalgebras of L' lying strictly between L' and M'; hence L' is primitive. If L' is simple we are done. Otherwise L' admits a closed ideal of finite codimension, which is proper by Proposition 5.1. Let I' be a maximal proper closed ideal of finite codimension in L'. Then L'/I' is simple; hence if I is the preimage of I' in L, L/I is simple. q.e.d.

For applications of Proposition 6.1 we will need some topological lemmas:

Lemma 6.1. *Let L be a Lie algebra belonging to $\mathscr{L}_\Delta$, I a closed ideal of L, and S an open subspace of I with respect to the relative topology. Then the normalizer of S is open in L.*

Proof. By the normalizer we mean the set of $x \in L$ such that $[x, S] \subset S$. This set is clearly closed; therefore, to show that it is open we have to show that it is of finite codimension. Since S is open in I, S is of the form $I \cap \mathcal{O}$ where $\mathcal{O}$ is an open subspace of L. Let A be an open subalgebra of L contained in $\mathcal{O}$. It is clear that $[A, I] \subset I$ and $[A, I \cap A] \subset I \cap A$; hence there is a morphism of algebras:

$$A \to \operatorname{End}_\Delta(I/I \cap A) .$$

The kernel of this mapping is of finite codimension in A and is obviously contained in the normalizer of S.

Lemma 6.2. *Let L be a Lie algebra, A be a subalgebra of L, $D_L A = \{X \in A, [L, X] \in A\}$, and N be the normalizer of A. Then N is contained in the normalizer of $D_L A$.*

Proof. Let $X \in N$, $Y \in D_L A$ and $Z \in L$. Then

$$[Z, [X, Y]] = [[Z, X], Y] + [X, [Z, Y]]$$

by Jacobi's identity. The first term on the right is in A since Y is in $D_L A$, and the second term is in A since $[Z, Y]$ is in A and X is in the normalizer of A. Hence the sum is in A. Since Z was arbitrary this proves that $[X, Y]$ is in $D_L A$.

Proposition 6.2. *Let L be a Lie algebra belonging to $\mathscr{L}_\Delta$, I a closed ideal of L, and J a closed maximal ideal of I. Then the normalizer of J in L is open.*

Proof. Let S' be a proper open subalgebra of I/J, S its preimage in I, and I_S the union of all ideals of I contained in S. It is clear that I_S itself is a closed ideal of I, and equals J since it contains J and is unequal to I. Therefore we can represent J as the infinite intersection $\bigcap_{k=0}^{\infty} D_I^k S$. Applying Lemma 6.2 inductively we see that the normalizer of S is contained in the normalizer of J. Thus, by Lemma 6.1 the normalizer of J is open in L.

6.3. We will consider the following situation: L belongs to $\mathscr{L}_\Delta$. I is a closed ideal of L. J is a closed maximal ideal of I, and I^∞ is the largest ideal of L contained in J.

The closure of I^∞ is a closed ideal of L and is contained in J; since J is closed, so I^∞ must be closed. We will show that if I/J is non-abelian, there are no closed ideals of L contained between I and I^∞. The proof of this fact will involve several steps. We will first define a filtration on I and prove a comparable theorem for the graded algebra associated with this filtration, and then use this result to prove the original theorem.

We define a filtration on I as follows: we set $I^0 = I$, $I^1 = J$ and, by induction, $I^k = D_L I^{k-1}$ for $k > 1$. The intersection $\bigcap_{k=0}^{\infty} I^k$ is the largest ideal of L contained in J, by Proposition 2.4; so this intersection is equal to I^∞. It is easy to see, by repeated application of Proposition 2.3, that $[I^k, I^l] \subset I^{k+l}$; therefore, $\{I^k\}$, $k = 0, 1, 2, \cdots$, is a filtration of I in the sense of Definition 3.1 (except that condition b) is not satisfied). Let $\mathscr{I}$ be the graded algebra associated with this filtration:

$$\mathscr{I} = \sum_{k=0}^{\infty} I^k / I^{k+1} .$$

We will show that $\mathscr{I}$, in addition to being an algebra, is a module over a ring of polynomials. To show this, let N be the normalizer of J in L, and W the finite dimensional abelian Lie algebra over Δ, whose underlying vector space is L/N. Since I^k is equal to $D_L^{k-1}J$, $[N, I^k] \subset I^k$ by Lemma 6.2. Hence the mapping $L \times I \to I$ factors through N to give a mapping:

$$L/N \times I_{k+1}/I_{k+2} \to I_k/I_{k+1} \qquad \text{for all } k , \tag{6.1}$$

or, summing on k, a mapping:

$$W \times \mathscr{I} \to \mathscr{I} . \tag{6.1$'$}$$

If w_1 and w_2 are elements of W, and x and y are representatives for w_1 and w_2 in L, then, for all elements $\alpha \in I^k$, the brackets $[x, [y, \alpha]]$ and $[y, [x, \alpha]]$ are in I^{k-2}, and, by Jacobi's identity, their difference is in I^{k-1}. So if we pass to quotients, the mappings that are induced on $\mathscr{I}$ by w_1 and w_2, using (6.1)$'$, cummute with each other. This shows that (6.1)$'$ is a representation of the Lie algebra W on $\mathscr{I}$. This representation extends to a representation of the universal enveloping algebra of W, which is just the symmetric algebra $S(W)$. So we have demonstrated: $\mathscr{I}$ is an $S(W)$ module in a canonical way.

Let $\mathscr{I}^k$ be the k-th graded term in the module $\mathscr{I}$, and S^l the l-th graded term in the ring $S(W)$. If $a \in \mathscr{I}^k$ and $p \in S^l$, then $pa \in \mathscr{I}^{k-l}$. Since $\mathscr{I}^0 = I/J$, the pairing of $\mathscr{I}^k$ with S^k gives a mapping:

$$\mathscr{I}^k \to \mathrm{Hom}_\Delta(S^k, I/J) .$$

By summing this mapping over k we get a morphism of $S(W)$ modules:

$$\mathscr{I} \to \mathrm{Hom}_\Delta(S(W), I/J) . \tag{6.2}$$

We will show that (6.2) is injective. Suppose by induction that the injectivity has been proved up to degree $k - 1$. If the mapping fails to be injective in degree k, then there exists an $a \in \mathscr{I}^k$ such that for all $w \in W$, $wa = 0$ in $\mathscr{I}^{k-1}$. If α is a representative for a in I^k, then $[L, \alpha] \subset I^k$ or $\alpha \in D_L I^k$. By definition, $I^{k+1} = D_L I^k$; therefore $a = 0$ and (6.2) is injective in degree k. Thus we have proved

Lemma 6.3. *There is an injective morphism of $S(W)$ modules:*

$$\mathscr{I} \to \operatorname{Hom}_d(S(W), I/J) . \tag{6.2$'$}$$

Let W^* be the vector space dual of W. Using the fact that $S(W)^* = S(W^*)$, we will write the second term in the sequence (6.2) as $I/J \otimes S(W^*)$. Since I/J is a Lie algebra, and $S(W^*)$ is a graded associative commutative algebra, their tensor product is a Lie algebra.

Proposition 6.3. *The mapping*

$$\mathscr{I} \to I/J \otimes S(W^*) \tag{6.3}$$

obtained from (6.2) *is a morphism of graded Lie algebras.*

Proof. The module structure of $\mathscr{I}$ is compatible with its structure as a Lie algebra; so to prove that (6.3) is a morphism of Lie algebras we only have to show this in degree zero. But, in degree zero, (6.3) is just the identity mapping.

6.4. We will now assume that I/J is non-abelian. We will prove

Proposition 6.4. *If A is a proper closed ideal of I containing I^∞, then A is contained in J.*

Proof. The Lie algebra $I/J = \mathscr{I}^0$ is simple and non-abelian by hypothesis; therefore, $[\mathscr{I}^0, \mathscr{I}^k] = \mathscr{I}^k$ by Proposition 6.3 and the corollary to Proposition 5.1. This implies the following identity in the filtered algebra:

$$I^{k-1} = [I, I^{k-1}] + I^k . \tag{6.4}$$

Suppose now that A is not contained in J. Then we get $A + J = I$, since J is maximal. We will prove by induction that $A + I^k = I$. We have just proved this for the case where $k = 1$. Assume it has been proved for the case $k - 1$, i.e. assume that $A + I^{k-1} = I$. Combining this with (6.4) we get

$$I = A + I^{k-1} = A + [I, I^{k-1}] + I^k .$$

Replacing the term I which occurs in brackets on the right by $A + I^1$, we get $I = A + I^k$ as claimed. This establishes the induction.

The intersection $\bigcap_{k=0}^{\infty} A + I^k$ is equal to A, since A is closed and contains I^∞; so, by what we have just proved, $A = I$, which contradicts the assumption that A is proper.

Corollary. *There are no closed ideals of L strictly contained between I and I^∞.*

Proof. Let A be an ideal of L strictly contained in I and containing I^∞. Since A is an ideal of L it is, a fortiori, an ideal of I; hence it is contained in J. However, the largest ideal of L contained in J is I^∞; so $A = I^\infty$. q.e.d.

If we combine the corollary to Proposition 6.4 with the descending chain condition on closed ideals we get the following Jordan-Hölder decomposition:

Theorem 6.1. *Suppose the Lie algebra L belongs to $\mathscr{L}_\Delta$ and satisfies the descending chain condition on closed ideals. Then there exists a sequence*

$$L = I_0 \supset I_1 \supset I_2 \supset \cdots \supset I_k = \{0\}$$

of closed ideals of L such that for each $0 \leq i \leq k$ one of the following two alternatives holds:

a) *I_i/I_{i+1} is non-abelian, and there are no closed ideals of L properly contained between I_i and I_{i+1}.*

b) *I_i/I_{i+1} is abelian.*

Moreover, suppose two chain decompositions of L are given, for which the above conditions are satisfied. Let $Q_1, \cdots, Q_r$ be the set of non-abelian quotients occuring in the first decomposition, counted with multiplicities, and let $Q'_1, \cdots, Q'_s$ be the set of non-abelian quotients occuring in the second decomposition. Then $r = s$, and the Q_i's and Q'_j's are pairwise isomorphic as modules over L.

Proof. We will first prove that such a decomposition exists. Let I_1 be a maximal closed ideal of L, and J a maximal closed ideal of I_1. If I/J is abelian, let I_2 be the closure of $[I_1, I_1]$. This is contained in J; therefore, it is properly contained in I_1. If I/J is non-abelian, we let I_2 be the largest ideal of L contained in J. By the corollary to Proposition 6.4, there are no closed ideals of L contained between I_1 and I_2. Now we repeat the argument we have just given with I_1 replaced by I_2. We can construct by induction a sequence of closed ideals each ideal in the sequence being properly contained in the preceding one, so that the conclusions of the theorem are satisfied. By the d.c.c. this sequence must terminate at $\{0\}$ after a finite number of terms.

Suppose now that $I_0 \supset I_1 \supset \cdots \supset I_m$ and $I'_0 \supset I'_1 \supset \cdots \supset I'_n$ are two chain decompositions satisfying the conditions of Theorem 6.2. Let $Q_1, Q_2, \cdots, Q_r$, $r \leq m$ and $Q'_1, \cdots, Q'_s$, $s \leq n$, be the non-abelian quotients which occur, counted with multiplicity. For every i between 1 and r there exists a t between 0 and m such that $Q_i = I_t/I_{t+1}$. Consider the expression: $I_t \cap I'_u + I_{t+1}$, for each u between 0 and n. This is an ideal lying between I_t and I_{t+1}, and it is closed by Proposition 1.2, Corollary 2. Hence it must be equal to either I_t or I_{t+1}. If $u = 0$, then $I'_u = L$ and it is equal to I_t, while if $u = n$, then $I'_u = \{0\}$ and it is equal to I_{t+1}. Therefore, there exists a unique integer ν, $0 \leq \nu < n$, such that $I_t \cap I'_\nu + I_{t+1} = I_t$ and $I_t \cap I'_{\nu+1} + I_{t+1} = I_{t+1}$. This implies that the following mapping of L modules:

$$(*) \qquad \frac{I_t \cap I'_\nu}{I_t \cap I'_{\nu+1}} \to \frac{I_t}{I_{t+1}},$$

is surjective. On the other hand there is an injective mapping of L modules

$$(**) \qquad \frac{I_t \cap I'_\nu}{I_t \cap I'_{\nu+1}} \to \frac{I'_\nu}{I'_{\nu+1}}.$$

The quotient, $I'_\nu/I'_{\nu+1}$, can not be abelian, because this would force I_t/I_{t+1} to be abelian; so this quotient is simple, and (**) is bijective. From (*) it follows that I_t/I_{t+1} and $I'_\nu/I'_{\nu+1}$ are isomorphic as L modules.

By the above procedure we get an explicit mapping of the set of Q's into the set of Q''s such that each Q gets mapped onto an isomorphic Q'. However the same construction can be applied to the Q''s to get a mapping in the opposite direction. We let the reader verify that these mappings are inverses of each other.

7. Structure of the non-abelian minimal closed ideals

7.1. We want to find out what kinds of algebras can occur as quotients of type a) in the Jordan-Hölder decomposition of § 6. We will actually look at a more general question: What kinds of algebras can occur as non-abelian minimal closed ideals of algebras L which belong to $\mathscr{L}_\Delta$? We will assume throughout this § 7 that the base field Δ is of characteristic zero.

Let I be a non-abelian minimal closed ideal of L, and J a maximal closed ideal of I. We will show J is unique.

Proposition 7.1. *Every closed ideal of I except I itself is contained in J.*

Proof. This proposition follows from Proposition 6.4 provided I/J is non-abelian. If I/J were abelian, then $[I, I]$ would be contained in J, and its closure would also be contained in J. Since I is minimal, we would have to have $[I, I] = 0$, which would contradict the hypothesis of non-abelian I.

Let R denote the quotient I/J. We will study the relation between I and R. Our starting point is the filtration $I^k = D_L^{k-1}J$, $k = 0, 1, 2, \cdots$, which we discussed in § 6. We will continue to use the notations which we used before; in particular, we will denote by $\mathscr{I}$ the graded Lie algebra

$$\sum_{k=0}^{\infty} I^k/I^{k+1},$$

and by W the quotient of L by the normalizer of J. We will let Δ_R be the commutator ring of the algebra R (cf. § 4.3). Since R is simple, Δ_R is a field, in fact, a finite algebraic extension of the base field Δ.

By Proposition 6.3 there is an injective morphism of graded Lie algebras:

$$\mathscr{I} \to R \otimes S(W^*) . \tag{7.1}$$

This mapping sends $\mathscr{I}^1$ onto a subspace of $R \otimes W^*$ which is invariant with respect to the adjoint action of R. Therefore, by Proposition 5.1, there exists a Δ_R invariant subspace U of $W^* \underset{\Delta}{\otimes} \Delta_R$ such that the image of $\mathscr{I}^1$ is equal to $R \underset{\Delta_R}{\otimes} U$. Let $S(U)$ be the ring of polynomials over U regarded as an algebra over the field Δ_R.

Proposition 7.2. *The mapping* (7.1) *maps* $\mathscr{I}$ *isomorphically onto* $R \underset{\Delta_R}{\otimes} S(U)$.

Proof. By Proposition 5.1, the image of $\mathscr{I}$ is of the form $R \underset{\Delta_R}{\otimes} S_0$, where S_0 is a graded subspace of $S(W^*) \underset{\Delta}{\otimes} \Delta_R$. Since $[R, R] = R$, this graded subspace is a graded subalgebra; therefore, $S_0^k \supset S^k(U)$. We will prove by induction that these two spaces are equal.

The ring $S(W^*) \otimes \Delta_R$ is a module over the ring $S(W)$; in fact it is isomorphic to the $S(W)$ module $\mathrm{Hom}_\Delta(S(W), \Delta_R)$.

Proposition 6 implies that S_0 is a submodule. Let $u_1, \cdots, u_n$ be a basis for U over Δ_R, and $v_1, \cdots, v_n$ the dual basis for U^*. Since $U^* \subset W \otimes \Delta_R$, each v_i induces a mapping D_{v_i} of $S(W^*) \otimes \Delta_R$ into itself. S_0 is stable with respect to this mapping, and is also so with respect to multiplication by u_i since it is an algebra and contains U.

We will prove by induction that $S_0^k = S^k(U)$. This is true for $k = 1$ by hypothesis. Suppose it has been established for $k - 1$. If p is in S_0^k we can write

$$p = \frac{1}{k} \sum_{i=1}^{n} u_i(D_{v_i} p)$$

by Euler's identity. Since $D_{v_i} p$ is in $S^{k-1}(U)$, p is in $S^k(U)$.

7.2. A rough statement of the theorem we want to prove is that I is isomorphic to its graded algebra. We will make this statement a little more precise:

Let F be the ring of formal power series over the vector space U, and F^i the set of formal power series whose leading terms are of degree $\geq i$. The filtration $\{F^i\}$ defines a topology on F with respect to which F is linearly compact. R is also linearly compact; therefore, the tensor product $R \underset{\Delta_R}{\hat{\otimes}} F$ is defined. Since R is a Lie algebra and F is an associative commutative algebra, their ordinary tensor product is a Lie algebra. One can easily see that there is a unique Lie algebra structure on $R \underset{\Delta_R}{\hat{\otimes}} F$ such that the natural mapping

$R \underset{\Delta_R}{\otimes} F \to R \underset{\Delta_R}{\hat{\otimes}} F$ is a homomorphism and $R \underset{\Delta_R}{\hat{\otimes}} F$ belongs to $\mathscr{L}_\Delta$. Our main result is

Theorem 7.1. *I and $R \underset{\Delta_R}{\hat{\otimes}} F$ are isomorphic as Lie algebras over Δ.*

For the sake of brevity we will only prove a special case of Theorem 7.1, namely the case where $\Delta = \Delta_R$. If we assume this, we can drop the unwieldy notation involved in distinguishing between Δ linearity and Δ_R linearity; in particular we can drop the subscripts when writing tensor products. The proof of Theorem 7.1 is basically the same without this assumption except for a few small details which the reader will have no trouble filling in. For the rest of this sub-section we will assume $\Delta = \Delta_R$.

It will be convenient to prove Theorem 7.1 as a corollary of a slightly stronger result. Before we state this result, we will make a few preliminary remarks: Let $\mathrm{Der}(R \hat{\otimes} F)$ be the Lie algebra of derivations of $R \hat{\otimes} F$. There is a natural homomorphism $R \hat{\otimes} F \to \mathrm{Der}(R \hat{\otimes} F)$, which is injective since the center of $R \hat{\otimes} F$ is trivial; therefore, we can think of $R \hat{\otimes} F$ as a subalgebra of $\mathrm{Der}(R \hat{\otimes} F)$. If I were isomorphic to $R \hat{\otimes} F$, this isomorphism would induce a homomorphism $L \to \mathrm{Der}(R \hat{\otimes} F)$, since L acts as an algebra of derivations on I. The technique of our proof will be to construct an isomorphism of I onto $R \hat{\otimes} F$ and a homomorphism of L into $\mathrm{Der}(R \hat{\otimes} F)$ simultaneously.

The algebra $R \hat{\otimes} F$ has a canonical filtration, given by its subalgebras $R \hat{\otimes} F^i$, $i = 0, 1, 2, \cdots$. The corresponding graded algebra is $R \otimes S(U)$, which by Proposition 7.2 is isomorphic to the graded algebra $\mathscr{I}$. We will prove

Theorem 7.2. *There is a homomorphism ϕ mapping the pair of Lie algebras L, I into $\mathrm{Der}(R \hat{\otimes} F)$, $R \hat{\otimes} F$ and satisfying the following conditions:*

a) *The restriction of ϕ to I maps I into $R \hat{\otimes} F$ so as to preserve the filtrations.*

b) *The map $gr(\phi): \mathscr{I} \to R \otimes S(U)$ induced by ϕ is the isomorphism of Proposition* 7.2.

Remark. Theorem 7.2 implies Theorem 7.1 since $\phi | I$ is bijective if condition b) is satisfied.

To understand what is involved in proving Theorem 7.2 we will examine a few of its implications. We recall that in § 5.3 we constructed the following split exact sequence of Lie algebras:

$$0 \to \mathrm{Der}(R) \hat{\otimes} F \to \mathrm{Der}(R \hat{\otimes} F) \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathrm{Der}(F) \to 0 .$$

Suppose that $\phi: L \to \mathrm{Der}(R \hat{\otimes} F)$ is a mapping which satisfies the conditions of Theorem 7.2. Then $\lambda = \pi \circ \phi$ is a homomorphism of L into $\mathrm{Der}(F)$. Let τ be the mapping $\phi - l \circ \pi \circ \phi$. Since $\pi \circ \tau = 0$, τ is a mapping of L into $\mathrm{Der}(R) \hat{\otimes} F$.

Given $X \in L$ let us denote by $\mathscr{L}_X$ the derivation induced on $\mathrm{Der}(R) \hat{\otimes} F$ by the derivation $\lambda(X)$ of the ring F. Then τ has to satisfy the following identity:

$$\mathscr{L}_X \tau(Y) - \mathscr{L}_Y \tau(X) = \tau([X, Y]) - [\tau(X), \tau(Y)], \tag{7.2}$$

for all X and Y in L.

Conversely given a homomorphism $\lambda: L \to \mathrm{Der}(F)$ and a mapping $\tau: L \to \mathrm{Der}(R) \hat{\otimes} F$ satisfying the identity (7.2) then the mapping $\phi = \tau + l \circ \lambda$ is a homomorphism of L into $\mathrm{Der}(R \hat{\otimes} F)$. Our strategy for proving Theorem 7.2 is to construct first the mapping λ, and then to solve equation (7.2) for τ.

7.3. To construct a mapping τ satisfying the identity (7.2) we will need some facts about the Spencer-Kozul complex of the polynomial ring $S(U)$ (cf. [3]). We recall briefly how this complex is defined.

An element w of U^* is by definition a linear mapping of U into Δ. It extends in a natural way to a derivation

$$D_w: S(U) \to S(U)$$

of degree -1. Let $u_1, \cdots, u_n$ be a basis of U, and $w_1, \cdots, w_n$ a basis of U^*. We define a mapping

$$\varepsilon: S^k(U) \to S^{k-1}(U) \otimes U$$

by the formula $\varepsilon(r) = \sum_{i=1}^{n} D_{w_i} r \otimes u_i$.

Let $\Lambda^l = \Lambda^l(U)$ be the space of alternating l-forms over U. There is a canonical "wedge" mapping $U \otimes \Lambda^l \to \Lambda^{l+1}$. We will denote by δ the mapping obtained by composing the two mappings in the sequence:

$$S(U) \otimes \Lambda^l \xrightarrow{\varepsilon \otimes id} S(U) \otimes U \otimes \Lambda^l \xrightarrow{\text{wedge}} S(U) \otimes \Lambda^{l+1}.$$

It is easy to verify that $\delta \circ \delta = 0$. So the vector space $S(U) \otimes \Lambda$ equipped with δ is a complex, which is bigraded since both S and Λ are graded; δ is a boundary operator of bidegree $(-1, 1)$. We will show that it is an acyclic complex by constructing an explicit homotopy operator.

If $\alpha \in S^k \otimes \Lambda^l$ is of the form $a \otimes u_{i_0} \wedge \cdots \wedge u_{i_r}$, then we will set

$$D\alpha = \frac{1}{k + r + 1} \sum_{s=0}^{r} (-1)^s u_{i_s} a \otimes u_{i_0} \wedge \cdots \wedge \hat{u}_{i_s} \wedge \cdots \wedge u_{i_r}.$$

Since every element of $S \otimes \Lambda$ can be uniquely expressed as a linear combination of elements of this form, D is unambiguously defined for all of $S \otimes \Lambda$, except in bidegree (0, 0) when the term in the denominator is zero. One easily verifies the identity $D\delta + D\delta = I$, from which one can conclude

Lemma 7.1. *The cohomology of the complex* $\{S \otimes \Lambda, \delta\}$ *is zero except in bidegree* (0, 0).

In the examples considered below we will be considering complexes of the form $\{V \otimes S \otimes \Lambda, \delta_v\}$, where V is a vector space and δ_v is $i_v \otimes \delta$, i_v being just the identity map on V.

The acyclicity of these complexes can be proved in the same way as above.

7.4. Before we return to the proof of Theorem 7.2, we will discuss the structure of Der(F) in a little more detail. We pointed out in § 7.3 that an element w of U^* induces a derivation D_w on $S(U)$; it also induces a derivation D_w on F, since F is the ring of formal power series over U. Let $w_1, \cdots, w_n$ be a basis of U^*. It is not hard to show that every derivation of F is of the form

$$\sum_{i=1}^{n} f_i D_{w_i}, \quad f_i \in F, \quad i = 1, \cdots, n.$$

The set of all derivations of the above form with $f_i \in F^1$, $i = 1, \cdots, n$, forms a subalgebra of $\operatorname{Der} F$, called the isotropy algebra to be denoted by $\operatorname{Der}^0 F$. The quotient of $\operatorname{Der} F$ by $\operatorname{Der}^0 F$ can be identified with U^* (as a vector space).

The first step in the proof of Theorem 7.2 is the following lemma which was proved by Sternberg and the author in [3].

Lemma 7.2. *Let L be a Lie algebra belonging to $\mathscr{L}_\Delta$, N be an open subalgebra of L, $U = (L/N)^*$, and F be the ring of formal power series over U. Then there exists a homomorphism*

$$\lambda : L, N \to \operatorname{Der} F, \operatorname{Der}^0 F$$

such that the mapping

$$L/N \to \operatorname{Der}/\operatorname{Der}^0 = U^* = L/N,$$

induced by λ, is the identity mapping. Moreover, given two homomorphisms λ_1 and λ_2 satisfying the above conditions, there exists an automorphism ρ of F such that $\lambda_2 = \rho_* \lambda_1$, where ρ_* is the automorphism of $\operatorname{Der} F$ induced by ρ.

For the proof of Lemma 7.2 see [3] and [14].

Remark. If λ satisfies the hypotheses of Lemma 7.2, then the kernel of λ is the largest ideal of L contained in N (since $\operatorname{Der}^0 F$ is a fundamental subalgebra of $\operatorname{Der} F$). In particular if I is an ideal of L contained in N, then λ maps it into zero.

7.5. Let J be the unique maximal closed ideal of I, and N its normalizer in L. We claim that $(L/N)^*$ is isomorphic to U as a vector space.[2] By definition, U is a subspace of $(L/N)^*$. Suppose that $X \in L$ is annihilated by U. Then by Proposition 6.3 and Proposition 7.2, $[X, J] \subset J$; therefore X is in N, which proves our assertion.

We can apply Lemma 7.2 of § 7.4 with L, N and U taken to be as above. Let λ be a homomorphism of L into Der F satisfying the conditions of Lemma 7.2. From λ we will construct a mapping

$$\tau : L \to \mathrm{Der}(R) \hat{\otimes} F$$

satisfying the identity (7.2); this will be done by an inductive procedure. Given a mapping

$$\tau_k : L, I \to \mathrm{Der}(R) \hat{\otimes} F , \quad R \hat{\otimes} F ,$$

we will say that it satisfies condition $(7.2)_k$ if

$$\mathscr{L}_X \tau_k(Y) - \mathscr{L}_Y \tau_k(X) - \tau_k([X, Y]) - [\tau_k(X), \tau_k(Y)]$$

is in $\mathrm{Der}(R) \hat{\otimes} F^k$ for all X, Y in L.

The main step in the induction is the following:

Proposition 7.3. *Let τ_{k-1} be a mapping of L, I into $\mathrm{Der}(R \hat{\otimes} F)$, $R \hat{\otimes} F$ satisfying condition $(7.2)_{k-1}$, $k \geq 1$. Then there exists a mapping τ_k of L, I into $\mathrm{Der}(R \hat{\otimes} F)$, $R \hat{\otimes} F$ such that:*

a) *τ_k satisfies condition $(7.2)_k$.*

b) *$\tau_k(X) - \tau_{k-1}(X)$ is in $\mathrm{Der}(R) \hat{\otimes} F^k$ for all X in L.*

Proof. We assume the mapping τ_{k-1} is given. We will construct τ_k by first constructing certain chains in the Spencer-Kozul complex:

$$\mathrm{Der}(R) \otimes S(U) \otimes \Lambda(U) .$$

We will show that these chains are cycles, and use the elements which they bound to construct τ_k.

For the rest of the proof we will fix an injective linear mapping $i: U^* \to L$ such that the composition of i with the projection of L on $L/N = U^*$ is the identity.

For every pair of elements u, v in U^* we will define an element $\gamma_{u,v}$ in $\mathrm{Der}(R) \otimes S^{k-1}(U)$ by the following construction. Let $X = i(u)$ and $Y = i(v)$. Consider the expression:

$$\mathscr{L}_X \tau_{k-1}(Y) - \mathscr{L}_Y \tau_{k-1}(X) - \tau_{k-1}([X, Y]) - [\tau_{k-1}(X), \tau_{k-1}(Y)] ,$$

[2] This is assuming that $\Delta = \Delta_R$; see the remark in § 7.2.

which is an element of $\mathrm{Der}(R) \mathbin{\hat{\otimes}} F^{k-1}$ by hypothesis. We will define the element $\gamma_{u,v}$ to be the "leading term" of this element; that is, the projection of this element on $\mathrm{Der}\, R \otimes (F^{k-1}/F^k)$.

If we let u and v vary, then $\gamma_{u,v}$ is a linear expression in u and v, and $\gamma_{u,v} = -\gamma_{v,u}$. Therefore, the mapping $u, v \to \gamma_{u,v}$ can be regarded as a linear mapping of $\Lambda^2 U^*$ into $\mathrm{Der}\, R \otimes S^{k-1}$, or, alternatively, it can be regarded as an element of $\mathrm{Der}\, R \otimes S^{k-1} \otimes \Lambda^2(U)$; call this element c^2.

Next let A be a fixed element of N. For each $u \in U^*$ we will construct an element $\gamma_{A,u}$ of $\mathrm{Der}(R) \otimes S^{k-1}$ by the same procedure as above. Let $X = i(u)$. We will define $\gamma_{A,u}$ to be the "leading term" of the expression

$$\mathscr{L}_X \tau_{k-1}(A) - \mathscr{L}_A \tau_{k-1}(X) - \tau_{k-1}([X, A]) - [\tau_{k-1}(X), \tau_{k-1}(A)] ,$$

which is an element of $\mathrm{Der}\, R \otimes F^{k-1}$ by hypothesis. $\gamma_{A,u}$ depends linearly on u, and hence defines an element of $\mathrm{Der}\, R \otimes S^{k-1} \otimes U$ which we will denote by c^1_A.

Finally, given any pair of elements A, B in N we will associate with them an element $c^0_{A,B}$ of $\mathrm{Der}(R) \otimes S^{k-1}$. By definition $c^0_{A,B}$ will be the "leading term" of the expression

$$\mathscr{L}_A \tau_{k-1}(B) - \mathscr{L}_B \tau_{k-1}(A) - \tau_{k-1}([A, B]) - [\tau_{k-1}(A), \tau_{k-1}(B)] .$$

Lemma 7.3. *The elements c^2, c^1_A, and $c^0_{A,B}$ of the Spencer-Kozul complex $\mathrm{Der}(R) \otimes S \otimes \Lambda$ are cycles ($\delta c^2 = \delta c^1_A = \delta c^0_{A,B} = 0$).*

Proof. We will just prove that $\delta c^2 = 0$, and leave the other two cases as exercises for the reader. Let u, v, and w be elements of U^*. If we think of c^2 as a mapping of $\Lambda^2 U^*$ into $S^{k-1}(U)$, then its boundary is the expression

$$\delta c^2(u, v, w) = D_u c_2(v, w) + D_v c_2(w, u) + D_w c_2(u, v) .$$

We can compute this expression as follows: Let $X = i(u)$, $Y = i(v)$ and $Z = i(w)$. Then $\delta c^2(u, v, w)$ is the $k-2$ component of the expression

$$\begin{aligned} \mathscr{L}_X \{ \mathscr{L}_Y \tau_{k-1}(Z) - \mathscr{L}_Z \tau_{k-1}(Y) - \tau_{k-1}([X, Y]) \\ + [\tau_{k-1}(X), \tau_{k-1}(Y)] \} + \cdots , \end{aligned} \tag{7.3}$$

where the dots indicate cyclic permutations of the first term. We will prove that the $k-2$ component of (7.3) is zero. Expanding the first two terms of (7.3) and their cyclic permutations, we get

$$\begin{aligned} &\mathscr{L}_X \mathscr{L}_Y \tau_{k-1}(Z) - \mathscr{L}_Y \mathscr{L}_X \tau_{k-1}(Z) + \cdots \\ &\quad = \mathscr{L}_{[X,Y]} \tau_{k-1}(Z) + \mathscr{L}_{[Y,Z]} \tau_{k-1}(X) + \mathscr{L}_{[Z,X]} \tau_{k-1}(Y) . \end{aligned}$$

Substituting this back into (7.3) we get

$$\mathscr{L}_{[X,Y]}\tau_{k-1}(Z) - \mathscr{L}_Z\tau_{k-1}([X, Y]) \\ + [\mathscr{L}_X\tau_{k-1}(Y) - \mathscr{L}_Y\tau_{k-1}(X), \tau_{k-1}(Z)] + \cdots$$

By the inductive hypothesis $(7.2)_{k-1}$, this expression is equal to

$$\tau_{k-1}([[X, Y], Z]) - [\tau_{k-1}[X, Y], \tau_{k-1}(Z)] + \cdots \\ + [\tau_{k-1}[X, Y], \tau_{k-1}(Z)] - [[\tau_{k-1}(X), \tau_{k-1}(Y)], \tau_{k-1}(Z)] + \cdots ,$$

where the dots now indicate cyclic permutations of the first term plus terms of order $k - 1$. In this expression the middle terms cancel and the first and last terms, if permuted cyclically, are zero by Jacobi's identity. This proves that (7.3) is of degree $k - 1$, and hence that $\delta c^2 = 0$. q.e.d.

If k is greater than 1, the condition $\delta c^0_{A,B} = 0$ implies that $c^0_{A,B} = 0$ since $c^0_{A,B}$ is of bidegree $(k - 1, 0)$ and hence can not be the boundary of anything. Since $\delta c^1_A = \delta c^2 = 0$ we can find an element b^0_A in $\mathrm{Der}(R) \otimes S^k(U)$ and an element b^1 in $\mathrm{Der}(R) \otimes S^k(U) \otimes \Lambda^1(U)$ snch that $\delta b^0_A = c^1_A$ and $\delta b^1 = c^2$. We now define τ_k as follows. We regard $\mathrm{Der}(R) \otimes S^k$ as a subspace of $\mathrm{Der}(R) \otimes F^k$. (This identification can obviously be made in a canonical way.) We then define τ_k by the formulas:

a) if $A \in N$, we set $\tau_k(A) = \tau_{k-1}(A) + b^0_A$,

b) if $X \in L$ is of the form $i(u)$ where $u \in U^*$, we set

$$\tau_k(X) = \tau_{k-1}(X) + b^1(u) .$$

Since the mapping $i: U^* \to L$ was chosen to map U^* onto a complement of N in L, the formulas a) and b) define τ_k unambiguously for all elements of L. The reader can check that the condition $(7.3)_k$ is satisfied automatically because of the way we defined c^1_A and c^2. Condition b) is satisfied because b^1 and b^0_A are both of degree k. This concludes the proof of Proposition 7.3. q.e.d.

To set up the induction we note that since N is the normalizer of the maximal ideal J of I there is a canonical homomorphism

$$\nu: N \to \mathrm{Der}(R) .$$

We define a mapping $\tau_0: L \to \mathrm{Der}(R) \mathbin{\hat{\otimes}} F$ in the following way:

a) If $A \in N$ we set $\tau_0(A) = \nu(A)$.

b) If X is of the form $i(u)$ where u is in U^*, we set $\tau_0(X) = 0$.

The mapping τ_0 is defined unambiguously for all elements of L by a) and b). Given τ_0 we now define $\tau_1, \tau_2, \cdots$ inductively using Proposition 7.3. We let τ be the limit of the τ_k. (This limit exists because of condition b) in the hypotheses of Proposition 7.3.) It is clear that τ satisfies condition (7.2), and also that τ maps I homomorphically into $R \mathbin{\hat{\otimes}} F$. In fact if X and Y are in $\mathscr{I}$, then $\mathscr{L}_X$ and $\mathscr{L}_Y$ are zero (cf. remark at the end of § 7.4); hence, equation (7.2) becomes

$$\tau([X, Y]) = [\tau(X), \tau(Y)] .$$

To complete the proof of Theorem 7.2 we must show that τ behaves as prescribed on the graded structure. We note that if X is in L and Y is in I, then $\mathscr{L}_Y = 0$, and, therefore, equation (7.2) becomes

$$\mathscr{L}_X \tau(Y) = \tau([X, Y]) - [\tau(X), \tau(Y)] . \tag{7.5}$$

Let $X = i(u)$ where $u \in U^*$. The identity (7.5) implies that the mapping

$$gr(\tau) : \mathscr{I} \to R \otimes S(U)$$

commutes with the derivation D_u for all $u \in U^* = L/N$. However, we defined τ_0 so that this mapping would be the identity mapping in degree zero, and if this mapping is to commute with the derivations D_u it must be the identity mapping in all degrees. Hence the proof of Theorem 7.2 is finished.

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