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TWO TOPOLOGICAL EXAMPLES IN MINIMAL SURFACE THEORY

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This paper gives an account of examples which the author has constructed to answer two questions posed by W. H. Meeks III.

Question 1. Given a set Γ of disjoint smooth Jordan curves on the standard 2-sphere S^2, such that Γ bounds two homeomorphic embedded compact connected minimal surfaces F and G in B^3, is there an isotopy of B^3 fixing Γ and taking F to G ([2, Problem 1], [1, conjecture 5])? Meeks has shown that such surfaces always split B^3 into two handlebodies; it then follows that such an isotopy exists if Γ consists of a single curve or if F and G are annuli [2]. We give two counterexamples: one where F and G are planar domains with three boundary components and one where F' and G' have genus one and two boundary components.

Question 2. Can a Jordan curve on the boundary of a convex set in $\mathbf{R}^3$ bound a minimal disc that is not embedded [1, conjecture 2]? We give an example of a smooth Jordan curve on S^2 that bounds an immersed stable minimal disc that is not embedded.

Our examples depend upon the "bridge principle" for minimal surfaces, which, roughly stated, is to this effect: if we have two stable minimal surfaces X and Y in $\mathbf{R}^3$ and an arc c joining their boundaries, there exists a new stable minimal surface Z which consists of surfaces close to X and Y together with a thin "bridge" running along c. We also require that the boundaries of X, Y and Z should lie on S^2. A version of the bridge principle has been proved by Meeks and Yau [4], although without this further requirement; we are able to justify our examples by a slight modification of their technique.

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Example for Question 1. We shall first describe the construction informally in two steps, then prove that the surfaces are not isotopic in B^3 and then give a proof that such surfaces exist.

Step 1. In any system of polar coordinates on S^2, consider a pair of latitudes l_1 and l_2. If they are close enough to the equator they bound a stable embedded minimal annulus in B^3 (Fig. 1(a)). They also bound a pair of stable embedded minimal discs (Fig. 1(b)). Take a great circle g orthogonal to l_1 and l_2; this contains two arcs p and q which join l_1 to l_2. Form a new pair C of disjoint smooth Jordan curves by erasing a short arc from l_1 or l_2 at each intersection with g and adding four arcs nearly parallel to p and q. If the bridge principle holds for C, C bounds a stable minimal annulus consisting of surfaces near the discs spanning l_1 and l_2 connected by bridges along p and q (Fig. 2(b)). By—for example—the minimal Dehn lemma of Meeks and Yau [3], each component of C also bounds a stable embedded minimal disc inside its convex hull; the convex hulls are disjoint and hence so are these discs (Fig. 2(a)).

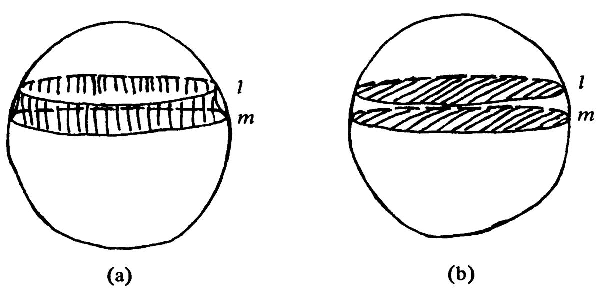

FIG. 1

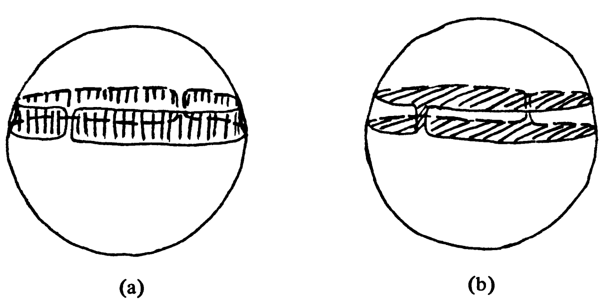

FIG. 2

Step 2. Apply the construction of Step 1 to three pairs of latitudes, so as to obtain six Jordan curves on S^2 each of which bounds a disc in S^2 disjoint from the others. Denote these curves by a_i, b_i $(i = 1, 2, 3)$ so that each pair (a_i, b_i) bounds an annulus E_i from Step 1. Topologically but not metrically, the annuli are arranged as in Fig. 3. To obtain a counterexample where the surfaces F and G are planar domains with three boundary components, we connect these annuli by three further bridges α, β and γ as shown in Fig. 4. The surface F consists of annuli close to E_1 and E_3 and discs spanning a_2 and b_2, connected by bridges. The surface G consists of annuli close to E_2 and E_3 and discs spanning a_1 and b_1, connected by bridges.

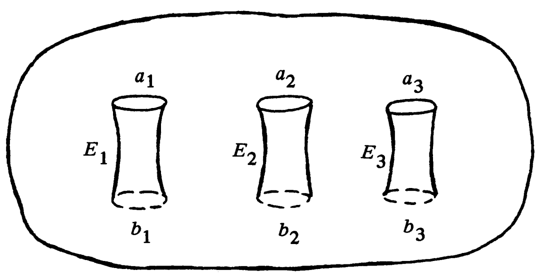

FIG. 3

To obtain a counterexample where the surfaces F' and G' have genus one and two boundary components, we modify the previous example by adding a fourth bridge δ connecting b_2 and b_3. F' and G' are the surfaces obtained by adding the additional bridge δ to the construction of F and G. We omit the proofs that F' and G' exist and are not isotopic, which are similar to those for F and G.

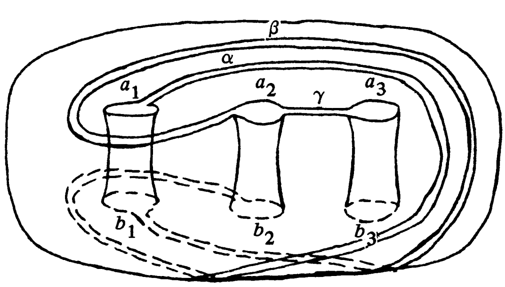

FIG. 4

Proof that the surfaces F and G are not isotopic. Each of the three components of $\Gamma = \partial F = \partial G$ bounds a disc in S^2 which is disjoint from the others; call this the disc *inside* that component. Regard B^3 as embedded in $\mathbf{R}^3$, and attach an unknotted 1-handle h to B^3 along the discs inside the components of Γ which meet b_2 and b_3. Each of F and G splits B^3 into a ball and a ball with two handles. For G, the union of this ball with h is a solid torus standardly embedded in $\mathbf{R}^3$. For F it is embedded as a regular neighborhood of a trefoil knot. Therefore F and G are not isotopic in B^3.

Proof that the surfaces F and G exist. Following Meeks and Yau [4] we shall construct for each surface a 3-manifold M with piecewise-smooth boundary embedded in $\mathbf{R}^3$, which may be thought of as a regular neighborhood of the surface, and show that a minimal surface with the required properties exists inside M. The boundary of M is required to satisy the following condition (C):

(C1) M is contained in the interior of a compact submanifold N of $\mathbf{R}^3$ and there is a smooth stratification of N such that ∂M is a union of strata.

(C2) Each 2-dimensional stratum of ∂M has nonnegative mean curvature with respect to the inward normal.

(C3) Each 2-dimensional stratum H of ∂M extends to a properly-embedded smooth surface K in N such that $H = K \cap M$.

We shall prove the existence of F, the argument for G being similar.

The pairs of curves (a_1, b_1) and (a_3, b_3) are to be spanned by annuli. For each pair we start from the pair of latitudes l_1 and l_2 from which it was constructed in Step 1. For $i = 1, 2$ we take another two latitudes λ_i and μ_i, a short distance from l_i on either side. λ_i and μ_i bound an annular region on S^2, which is to be part of the boundary of a ball B_i; ∂B_i will be piecewise-smooth and satisfy condition (C). The remainder of ∂B_i is formed by the smaller caps of spheres of radius $r_i > 1$ passing through λ_i and μ_i.

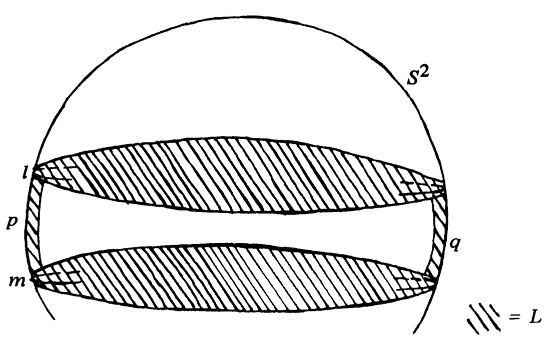

FIG. 5

The bridges connecting the latitudes l_1 and l_2 of Step 1 run along arcs p and q of a great circle g. The $\frac{1}{4}$-neighborhood of g has positive mean curvature with respect to the inward normal. We define the three-manifold L to be the union of B_1 and B_2 with the portions of the $\frac{1}{4}$-neighborhood of g which lie inside S^2 and along p and q. (L is shaded in Fig. 5). We note that in the construction we have just described, we are free to take all the latitudes l_i, λ_i and μ_i to be as close as we please to the equator, and r_i to be as large as we please.

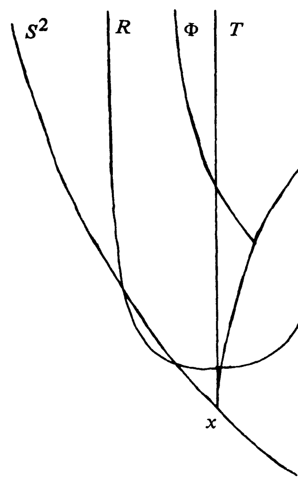

FIG. 6(A)

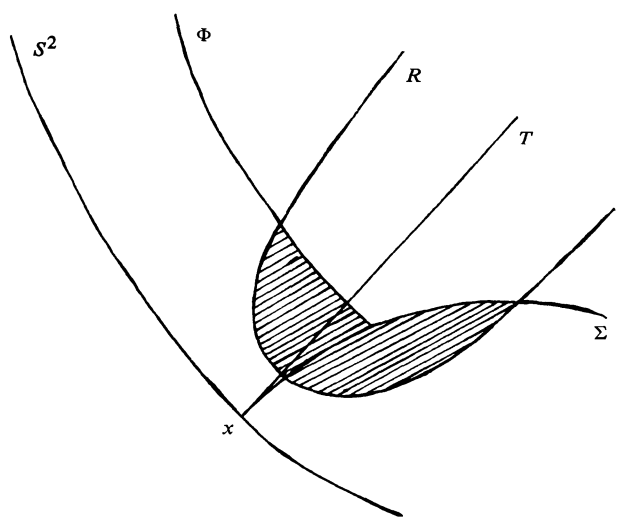

FIG. 6(B)

We shall modify L to make its boundary satisfy condition (C). Consider the point x in which p or q intersects ∂B_i ($i = 1$ or 2). In a neighborhood of x, L consists of part of B_i and part of the $\frac{1}{4}$-neighborhood of g, and ∂L consists of a surface Σ which is part of a sphere of radius r_i, a surface Φ which is part of the $\frac{1}{4}$-neighborhood of g and part of S^2 (Fig. 6). Consider a catenoid R whose axis of symmetry is the normal to Σ at x and which is also symmetrical about T, the plane tangent to Σ at x; there is a one-parameter family of such catenoids which differ by a homothety centered at x. Fig. 6 shows the intersection of R with the plane P through the geodesic g. If the angle θ between T and the plane tangent to S^2 at x is small, then, as we scale down R, $R \cap P$ begins to intersect g near x, and continues to do so for arbitrarily small values of the scale factor (Fig. 6(a)). If θ is close to $\pi/2$, there is a neighborhood of x in which $R \cap P$ does not intersect g for any scale factor (Fig. 6(b)). It is the latter condition that we require, and we choose l_i, λ_i, μ_i and r_i so that it is satisfied. When R has been scaled down sufficiently, the component of $R \cap L$ which lies closest to x is a disc D. D, Σ and Φ cut off three balls from L, and we modify L by removing these three balls. (The portions to be removed are shaded in Fig. 6(b).) With this modification at each such point ∂L satisfies condition (C).

We make one further modification to ∂L. Take a stable catenoid spanning two circles close to g, chosen so as to cut off small "channels" from L along the arcs p and q. By removing these "channels" from L we allow space for the bridges of Step 2.

The curves a_2 and b_2 are to be spanned by discs. Here we take two new latitudes a little outside the pair of latitudes from which a_2 and b_2 were constructed, and form a single ball satisfying condition (C) in the same way as before. From this ball we remove the portion that lies between two planes parallel and close to the plane of g.

It remains to complete the construction of M by adding neighborhoods of three curves on S^2 corresponding to the bridges α, β and γ of Step 2. Where these meet the rest of M we intersect with a catenoid as before.

M has been constructed in such a manner that the set of curves $\Gamma = \partial F = \partial G$ may be taken to lie on that part of S^2 which is in the boundary of M. A theorem of Meeks and Yau [4, Theorem 5], together with their technique for handling a piecewise smooth boundary [4, Proof of Theorem 1], yields the existence of an embedded minimal surface with the properties required of F. (I am grateful to S. T. Yau for drawing my attention to this theorem. An earlier version of the proof used the geometric Dehn lemma and thus was restricted to genus 0.)

Example for Question 2. We take two disjoint Jordan curves on S^2 that bound stable minimal discs with interiors that intersect, and connect them by a bridge with boundary in S^2.

More precisely, let l_i $(i = 1, 2, 3)$ be latitudes close to the equator, l_2 lying between l_1 and l_3. Form a Jordan curve $\Gamma_1 \subset S^2$ by connecting l_1 and l_2 with two arcs nearly parallel to a longitude. By the bridge principle, Γ_1 bounds a minimal disc D_1 consisting of discs close to the flat discs spanned by l_1 and l_2 connected by a thin bridge. D_1 does not minimize area, so Γ_1 bounds another minimal disc D_2 which lies close to S^2. Form a Jordan curve Γ_2 by connecting Γ_1 and l_3 with two arcs nearly parallel to a longitude. Let D_3 be the flat disc bounded by l_3. Then Γ_2 bounds a stable minimal disc D_4 consisting of discs close to D_1 and D_3 joined by a bridge, and Γ_2 also bounds a stable minimal disc D_5 consisting of discs close to D_2 and D_3 joined by a bridge. Let Γ_2' be a Jordan curve close to but not intersecting Γ_2. Then Γ_2 and Γ_2' bound stable minimal discs that intersect, and these may be connected by a bridge. As with the example for Question 1, we may prove that such a disc exists by constructing a manifold satisfying condition (C), which in this case is immersed in $\mathbf{R}^3$ rather than embedded. (The reason for the disc D_3 is to allow us to use this method to attach the final bridge.)

Remark. The Jordan curve we have constructed appears to bound at least five stable minimal discs, four obtained by the bridge principle together with the disc of least area. It seems that only one of these fails to be embedded. Note that Meeks has shown that an extremal curve that bounds a minimal disc that is not embedded must bound at least two embedded stable minimal discs [4].

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