J. DIFFERENTIAL GEOMETRY
19 (1984) 483-515

ON THE MODULI SPACES OF SURFACES OF GENERAL TYPE

F. CATANESE

0. Introduction

It is nowadays well known that there exists a coarse moduli space $\mathfrak{M}_g$ for complete smooth curves of genus g, and that $\mathfrak{M}_g$ is a quasiprojective normal irreducible variety of dimension $3g - 3$.

Recently D. Gieseker [11] proved the existence of a quasiprojective coarse moduli space $\mathfrak{M}$ for surfaces of general type modulo birational equivalence.

Whereas, through recent work of J. Harris & D. Mumford [12], much is known about the geometry of $\mathfrak{M}_g$, very little is known about $\mathfrak{M}$, except for particular classes of surfaces.

By the results of E. Bombieri [2] (cf. also [3]), the surfaces of general type with given numerical invariants χ $(= \chi(\mathcal{O}_S))$ and K^2 (= selfintersection of a canonical divisor of the minimal model S) belong to a finite number of families.

Therefore, fixing K^2 and χ, which are invariants (under orientation preserving homeomorphisms) of the oriented topological 4-manifold underlying the minimal model S, isomorphism classes of surfaces S with those invariants are parametrized by a quasiprojective variety $\mathfrak{M}_{K^2,\chi}$ with a finite number of irreducible components $\mathfrak{M}_1,\cdots,\mathfrak{M}_k$ and such that, if you take two points belonging to the same connected component of $\mathfrak{M}_{K^2,\chi}$, they represent isomorphism classes $[S]$, $[S']$ of two minimal models S and S' diffeomorphic to each other.

In analogy with the case of curves, we shall say that $\mathfrak{M}$ is a moduli space for a surface S_0 of general type with given K^2, χ if $\mathfrak{M}$ is the union of all components of $\mathfrak{M}_{K^2,\chi}$ whose points correspond to all the S's which are (orientedly) homeomorphic to S_0.

Received October 18, 1982 and, in revised form, January 5, 1984. Most of this research was done during the academic year 1981–1982 when the author was a visiting member of the Institute for Advanced Study in Princeton, NJ, and was partly supported by National Science Foundation grant MCS 81-03365. The author is a member of G.N.S.A.G.A. of Consiglio Nazionale delle Ricerche, Italy.

The basic questions are:

Is $\mathfrak{M}$ irreducible?

If $\mathfrak{M}$ is reducible, is it pure dimensional?

E. Horikawa [14] showed that surfaces with $K^2 = 5$, $p_g = 4$, $q = 0$ give rise to only one moduli space, connected but with two irreducible components of dimension equal to 40.

The first purpose of this paper is to show that the basic questions have a negative answer "in general", i.e., for surfaces which are quite spread in the geography of surfaces of general type and, as we shall see in the sequel, cannot be considered to be by any means pathological.

To give a more precise statement let us introduce the number $M = M(S)$ which is called the number of moduli of S: by definition, it is the dimension of $\mathfrak{M}$ at the point $[S]$ representing the isomorphism class of S. By what we have said, for given K^2 and χ, M can take only a finite number of values.

Our main result is the following

Theorem A. *For each natural number n there exist positive integers* $M_1 < M_2 < \cdots < M_n$ *and orientedly homeomorphic simply connected minimal models* $S_1, \cdots, S_n$ *of surfaces of general type such that* $M(S_i) = M_i$.

One of the main ideas, which was suggested to us by B. Moishezon, is to use M. Freedman's recent result (cf. [10], [31]) by which if S_1, S_2 are compact oriented differentiable simply connected 4-manifolds with the same intersection form, then they are (orientedly) homeomorphic. The surfaces we construct are "bidouble" covers (i.e., Galois covers with group $(\mathbf{Z}/2)^2$) of $\mathbf{P}^1 \times \mathbf{P}^1$ and their canonical map is a biregular embedding.

The new technique we use is a theory of "natural deformations" of bidouble covers: to keep the paper selfcontained we will pursue elsewhere the general theory of "natural deformations" of abelian covers. This technique allows us to construct irreducible components of the moduli spaces and compute their dimension.

We show then that these components are all different and then, using simple connectivity and Freedman's result, the proof of Theorem A is reduced to a lemma in number theory, which was proved by E. Bombieri and is contained in the appendix.

The second part of the paper (§§5, 6) is devoted to the problem of giving bounds for M in terms of the (topological) invariants K^2, χ.

By the well-known results of Kuranishi [20] and Wavrik [34], a neighborhood of $[S]$ in $\mathfrak{M}$ is a quotient of the Kuranishi family B by the finite group $\mathrm{Aut}(S)$; hence, if T_S is the tangent sheaf to S,

$$\dim H^1(S, T_S) \geqslant M \geqslant \dim H^1(S, T_S)\text{-}\dim H^2(S, T_S) = 10\chi - 2K^2.$$

We prove the following results.

Theorem B. $M \leqslant 10\chi + 3K^2 + 108$.

Theorem C. *If S contains a smooth canonical curve then* $M \leqslant 10\chi + q + 1$.

Finally we consider the case of irregular surfaces.

G. Castelnuovo [4], using incorrect results of F. Severi [30], stated that if S is an irregular surface without irrational pencils, then $M \leqslant p_g + 2q$.

We show that this is false by producing bidouble covers for which, keeping q fixed, M grows asymptotically like $4p_g$.

However, with different techniques we prove a Castelnuovo-like bound:

Theorem D. *Assume* $q \geqslant 3$ *and that there do exist* $\eta_1, \eta_2 \in H^0(S, \Omega^1_S)$ *such that* $\operatorname{div}(\eta_1 \wedge \eta_2)$ *is a reduced irreducible curve (this hypothesis is verified if in particular* $H^0(S, \Omega^1_S)$ *generates* Ω^1_S *outside a finite number of points).*

Then $K^2 \geqslant 6\chi$ *and* $M \leqslant p_g + 3q - 3$.

Of course, two major open problems remain:

(i) Is $\mathfrak{M}$ connected?

(ii) If $\mathfrak{M}^{\text{diff}}(S)$ is the union of the connected components of $\mathfrak{M}(S)$ corresponding to surfaces diffeomorphic to S, is $\mathfrak{M}^{\text{diff}}$ "in general" irreducible, or pure dimensional?

In a sequel to this paper we hope, by studying deformations "in the large" of our surfaces, to be able to prove that question (i) also has a negative answer in general. Question (ii), on the other hand, is intimately related to the problem whether Freedman's result can be made stronger as to give diffeomorphism of the two given differentiable manifolds.

It is a pleasure here to thank E. Bombieri and B. Moishezon for their precious help and warm encouragement.

Conventions. Throughout the paper we shall work over the ground field $\mathbf{C}$ of the complex numbers.

S shall usually denote a minimal model of a surface of general type, T_S will be its tangent sheaf, Ω^1_S will be the sheaf of holomorphic 1-forms (resp. Ω^2_S for the 2-forms).

Divisors will always be Cartier divisors, and $\equiv$ will denote linear equivalence, $\sim$ algebraic equivalence.

K_S will be a canonical divisor, so that $\mathcal{O}_S(K_S) \cong \Omega^2_S$.

For a vector space V, $V^\vee$ will denote its dual.

Given a coherent sheaf $\mathscr{F}$ on a complex space X we shall denote $H^i(\mathscr{F}) = H^i(X, \mathscr{F})$, $h^i(\mathscr{F}) = \dim_{\mathbf{C}} H^i(\mathscr{F})$, $\chi(\mathscr{F}) = \sum_{i=0}^{\dim X}(-1)^i h^i(\mathscr{F})$.

As usual $p_g = h^0(\Omega^2_S)$, $q = h^0(\Omega^1_S)$, and π_1 will denote here the topological fundamental group.

If $a, b, c \in \mathbf{Z}$, $a \equiv b \pmod c$ will mean "a is congruent to b modulo c".

If f, g are functions from $\mathbf{N}$ to $\mathbf{R}$, $f \asymp g$ means

$$\lim_{n \to \infty} f(n)/g(n) = 1.$$

When D is a divisor, we shall say that x is an equation of D if x is a section of $\mathcal{O}_X(D)$ whose associated divisor of zeros is D.

If A and B are divisors on a smooth surface S, $A \cdot B$ denotes their intersection number.

If G, H are two groups $G * H$ is their free product.

The arrow $\twoheadrightarrow$ denotes a surjective homomorphism, and the arrow $\hookrightarrow$ an injective one.

If A, B, C are sets, $A - B - C$ is the set of elements of A not belonging to $B \cup C$.

Remark added in proof. Recently J. Carlsson proved that the image of $\mathfrak{M}$ under the period map ψ, attached to the integrals of the holomorphic 2-forms, has dimension at most $h^{1,1} - 1$. Therefore, if a generic local Torelli theorem would be true for $\mathfrak{M}$ (i.e. if the general fibre of ψ had dimension 0), then one would get the inequality $M \leqslant 10 - K^2 - 1$, which is considerably better than the one given by Theorem C.

1. Smooth abelian covers and their fundamental group

In this section we shall establish a result (Proposition 1.8) which shall be used in the sequel (Proposition 2.7) in order to show that certain surfaces that we shall construct as bidouble covers are simply connected. In the course of doing this we shall consider a more general situation: S and X are smooth (compact) manifolds and π: $S \to X$ is a finite Galois cover with Galois group G abelian, and we shall consider the problem of determining $\pi_1(S)$.

By the theorem of purity of branch locus [36] the critical set of π is a divisor R, the ramification divisor, whose image $\pi(R)$ is called the branch divisor and will here be denoted by D; assuming that X is simply connected and that the components of D are flexible (cf. Definition 1.4) we prove (1.6) that $\pi_1(X - D)$ is abelian and compute (1.8) $\pi_1(S)$ in terms of the cohomology classes of the components of D.

We then conclude the section with a result (Corollary 1.9) of independent interest.

Proposition 1.1. *R and D are divisors with smooth components and with normal crossings. Moreover, if $x \in S$, the stabilizer of x, G_x, is the direct sum of the stabilizers of the components of R passing through x. These last subgroups are cyclic.*

Proof. For $g \in G$, let $\mathrm{Fix}(g)$ be the set $\{x | g(x) = x\}$. If $\Gamma \subset S \times S$ is the graph of g, and Δ is the diagonal, then $\mathrm{Fix}(g)$ is the intersection of Γ with Δ.

Notice that, in general, the stabilizer of a point x, G_x, has, via the differential, a representation ρ_x on $T_{x,S}$.

Now the eigenvalues of $\rho_x(g)$ for $x \in \mathrm{Fix}(g)$ are roots of unity and depend continuously on x, therefore they are constant on the connected components of $\mathrm{Fix}(g)$, and, by the rank theorem, $\mathrm{Fix}(g)$ is a union of smooth subvarieties of dimension equal to the number of $(+1)$ eigenvalues. It follows, in particular, the well-known fact that ρ_x is a faithful representation.

Assume that R_i is a component of R and let G_i be the stabilizer of R_i (i.e. $G_i = \bigcap_{x \in R_i} G_x$).

If $x \in R_i$, then G_i has a faithful representation through ρ_x which is trivial on T_{x,R_i}; since a finite subgroup of $\mathbf{C}^*$ is cyclic, G_i is cyclic.

Assume now that y is a point with nontrivial stabilizer G_y: ρ_y splits as a direct sum of characters $\chi_1, \cdots, \chi_n$ of G_y. Now $\mathrm{Fix}(G_y)$ is the intersection of the components of R passing through y, say $R_1, \cdots, R_k$. By what we have seen if we put $G_i' = \ker(\chi_1, \cdots, \hat{\chi}_i, \cdots, \chi_n)$, G_i' is cyclic and, if $G_i' \neq 0$, then $\mathrm{Fix}(G_i') = R_i'$ is a component of R passing through y. Moreover, all such components are obtained in this way.

Since ρ_y is faithful, the G_i''s give a direct sum $G' = \oplus_{i=1}^n G_i'$.

Consider $Z = S/G'$ and π': $S \to Z$; set $z = \pi'(y)$.

Then z is a smooth point of Z and π'': $Z \to X$ is unramified in codimension 1, hence, by purity of the branch locus, π'' is unramified and $G' = G_y$.

Therefore we can assume $R_i' = R_i$ for $i = 1, \cdots, k$, $R_i' = \varnothing$ for $i > k$, and our assertions are proven if we show that D also has normal crossings, but this follows because, if $x \in D$ and $\pi(y) = x$, a neighborhood of x in X is isomorphic to a neighborhood of z in $Z = S/G_y$. q.e.d.

We remark at this point that π factors canonically as

$$\begin{array}{ccc} S & \xrightarrow{\quad \pi \quad} & X \\ & {\scriptstyle \pi'} \searrow \quad \nearrow {\scriptstyle p'} & \\ & X' & \end{array}$$

where X' is maximal such that p' is unramified. In fact one must take X' as S/G', where G' is the subgroup of G generated by the stabilizers G_x of points of S.

Definition 1.2. We shall say that π is totally ramified if p' is an isomorphism (i.e., if $G' = G$).

Remark 1.3. If $\pi_1(X) = 0$, then π is totally ramified (actually it suffices that $H_1(X, \mathbf{Z}) = 0$.

In what follows we shall make the (not too restrictive) hypothesis that $\pi_1(X) = 0$; then, with some mild conditions on the components D_i, we can ensure that $\pi_1(X - D)$ is an abelian group by generalizing, with the same method, a result of R. Mandelbaum & B. Moishezon [21, p. 218].

We borrow from them the following definition.

Definition 1.4. A smooth divisor D is said to be flexible if there exists a divisor $D' \equiv D$ such that $D' \cap D \neq \varnothing$, and D' intersects D transversally in codimension 2.

Remark 1.5. Notice that if D is flexible, then it is connected.

In fact, by taking sections of X with a general linear subspace of the $\mathbf{P}^N$ where X is embedded, we can reduce to the case where X is a surface. Then $D^2 > 0$, and if $D = A + B$, with $A \cdot B = 0$, we can assume $A^2 > 0$. By the Index Theorem, $B^2 < 0$ unless $B = 0$, but then B contains a fixed component C of $|D|$. Hence D' contains C, and the intersection is not transversal.

To see that the hypothesis of flexibility is essential in the following theorem, consider the following example:

Let $D = D_1 \cup D_2 \cup D_3$, where D_1, D_2, D_3 are three skew lines in $X = \mathbf{P}^1 \times \mathbf{P}^1$. Then $X - D \cong \mathbf{P}^1 \times (\mathbf{C} - \{0,1\})$ and $\pi_1(X - D) = \mathbf{Z} * \mathbf{Z}$.

Theorem 1.6. *Let X be a simply connected algebraic variety and $D = D_1 \cup \cdots \cup D_k$ a divisor smooth in codimension* 1 *and with normal crossings in codimension* 2. *If the D_i's are flexible, then $\pi_1(X - D)$ is abelian.*

Proof. We can clearly assume, by virtue of Lefschetz's first theorem, and considering again a sufficiently general linear section, that X is a surface.

Let T_i be a tubular neighborhood of D_i, and let ρ_i: $T_i \to D_i$ be a retraction preserving the D_j's for which T_i is a disc bundle over D_i.

Let γ_i, for $i = 1, \cdots, k$, be a simple loop around D_i; i.e., if p_0 is a basepoint in $X - D$, γ_i is conjugate by a path starting from p_0 to a fibre of $\rho_i|_{\partial T_i - D}$.

The proof consists of two steps:

(i) Each element is a product of conjugates of the γ_i's.

(ii) Each γ_i is in the center of the group.

For the first assertion let $\delta \in \pi_1(X - D)$; since X is simply connected, δ can be represented as the boundary of an immersed 2-disc Δ which is transversal to the D_i's, and such that $\Delta \cap T_i$ is a union of fibres of $\rho_i|_{(T_i - D) \cup D_i}$.

Our claim follows then immediately.

To prove (ii), it suffices to show that $\pi_1(T_i - D)$ surjects onto $\pi_1(X - D)$. In fact then one can notice that $T_i - D$ is homotopically equivalent to an S^1-bundle W over $D_i - (\cup_{j \neq i} D_j)$: if $\lambda \in \pi_1(W)$, λ can be represented by a loop which does not project onto D_i, and then, since an S^1-bundle over a 1-complex is trivial, λ commutes with γ_i.

Finally, let us prove e. g. that $\pi_1(T_1 - D)$ surjects onto $\pi_1(X - D)$.

Since D_1 is flexible, we can choose $D_1' \equiv D_1$ transversal to D_1 and contained in T_1: in fact this last property can be ensured by possibly replacing D_1' by a suitable member of the pencil spanned by D_1 and D_1'. This pencil defines a rational map f': $X \dashrightarrow \mathbf{P}^1$; with a blow-up π: $\tilde{X} \to X$ of the basepoints of the pencil, we obtain a morphism f: $\tilde{X} \to \mathbf{P}^1$ which admits as sections the exceptional curves $E_1, \cdots, E_r$ $(r = D^2)$ of the blow-up.

Let $\tilde{D}_j$ be the proper transform of D_j, D'' the proper transform of D_1'. Define $t \in \mathbf{P}^1$ to be a critical value if either $f^{-1}(t)$ is singular, or if $f^{-1}(t) \subset \tilde{D} = \bigcup_{j=1}^{k} \tilde{D}_j$, or if t is a critical value for $f|_{\tilde{D}_j}$. We can also assume D'' to meet transversally the $\tilde{D}_j$'s outside of $E = \bigcup_{h=1}^{r} E_h$; moreover we know that if $\tilde{D}_j \cap E_k$ is not empty, then it is a transversal intersection.

Let $B = \mathbf{P}^1 - \{\text{critical values}\}$, $Y = f^{-1}(B) - \pi^{-1}(D)$. Then $\hat{f} = f|_Y$: $Y \to B$ is a differentiable fibre bundle with fibre $F \cong D'' - \tilde{D} - E$.

Since E_1 is a section of f, we can find a tubular neighborhood T of E_1 and a section Γ of $\hat{f}$: $Y \to B$ such that $\Gamma \subset T$, hence, a fortiori, $\Gamma \subset \pi^{-1}(T_1)$: in fact $T - E_1$ is homotopically equivalent to an S^1-bundle over $\mathbf{P}^1$, therefore its restriction to B is a trivial bundle, moreover, by our choice of B, $\tilde{D}_j$ is transversal to the fibres of $f|_{f^{-1}(B)}$. Therefore $\pi_1(Y)$, by the homotopy exact sequence of a bundle, is a semidirect product of $\pi_1(F)$ and $\pi_1(\Gamma)$.

Hence, if $Y' = \pi(Y)$, $\pi_1(T_1 \cap Y') \twoheadrightarrow \pi_1(Y')$. But clearly, $\pi_1(Y') \twoheadrightarrow \pi_1(X - D)$, $\pi_1(T_1 \cap Y') \twoheadrightarrow \pi_1(T_1 - D)$, hence also $\pi_1(T_1 - D) \twoheadrightarrow \pi_1(X - D)$.

Corollary 1.7. *Let π: $S \to X$ be a finite morphism of smooth varieties of the same dimension, with X simply connected. Then π is an abelian Galois cover if the branching divisor D of π has normal crossing and consists of flexible divisors.*

Proof. By Theorem 1.6, $\pi_1(X - D)$ is abelian; hence $\pi|_{S-R}$ has a group of cover transformations $G \cong \pi_1(X - D)/\pi_1(S - R)$, and by the normality of S, G extends to a group of automorphisms of S such that $S/G \cong X$. q.e.d.

In the hypotheses of Theorem 1.6, we know that $\pi_1(X - D) = H_1(X - D, \mathbf{Z})$. To compute this last group, which we already know to be generated by k elements (the images of $\gamma_1, \cdots, \gamma_k$), we denote by V an open tubular neighborhood of D and by ∂V its boundary.

By Lefschetz's duality

$$H_1(X - D) \cong H_1(X - V) \cong H^{2n-1}(X - V, \partial V) \cong H^{2n-1}(X, \bar{V}),$$

where the last isomorphism is by excision and n is the complex dimension of X.

The exact sequence of the pair, considering that $\bar{V}$ is homotopically equivalent to D, gives us

$$H^{2n-2}(X) \xrightarrow{r} H^{2n-2}(D) \to H^{2n-1}(X, \bar{V}) \to H^{2n-1}(X).$$

But $H^{2n-1}(X)$ is zero by Poincaré duality; hence $\pi_1(X - D) \cong \operatorname{coker}(r)$.

By Poincaré duality, and the formula relating homology to cohomology, $H^{2n-2}(X)$ is a free $\mathbf{Z}$ module and also, since V is orientable, $H^{2n-2}(D) = \oplus \mathbf{Z}[D_i]$, where $[D_i]$ is the fundamental class of D_i. By elementary results on abelian groups, considering the transpose of r, α: $\mathbf{Z}^k \cong H_0(D) \to H_2(X, \mathbf{Z}) \cong H^2(X, \mathbf{Z})$ we obtain the following result.

Proposition 1.8. *In the hypotheses of Theorem* 1.6, *let M be the sublattice of $H^2(X, \mathbf{Z})$ generated by the D_i's, and h its rank. Then $\pi_1(X - D) \cong \mathbf{Z}^{k-h} \oplus H$, where H is the torsion part of $H^2(X, \mathbf{Z})/M$.*

If moreover π: $S \to X$ is an abelian cover with $D = \bigcup D_i$ as branching divisors, R_i is such that $\pi(R_i) = D_i$ and G_i is the cyclic group of order m_i which stabilizes R_i, then $\pi_1(S)$ is the quotient of ker: $(\oplus G_i \twoheadrightarrow G)$ *by* Im $\hat{r}$ *where $\hat{r}$ is the composition of r with the surjection of $\mathbf{Z}^k$ to $\oplus_i G_i$.*

Proof. We are left only to prove the second part of the statement.

By arguing for (S, R) as we did for the pair (X, D) we know that $H^{n-2}(R)$ goes onto the kernel of the surjection $\pi_1(S - R) \twoheadrightarrow \pi_1(S)$. Also $\pi_1(S - R)$ is the kernel of the homomorphism $\mathbf{Z}^k/\operatorname{Im} r \twoheadrightarrow G$.

Under this homomorphism the generators γ_i are mapped to generators of the cyclic group G_i and the relations given by $H^{n-2}(R)$ are that $m_i(\gamma_i)$ is nullhomotopic in S.

Hence $\pi_1(S) \subset \mathbf{Z}^k/\operatorname{Im} r + m\mathbf{Z}^k$ where $m\mathbf{Z}^k$ is the subgroup generated by $m_1 e_1, \cdots, m_k e_k$, $e_1, \cdots, e_k$ being the canonical basis of $\mathbf{Z}^k$. By what we have seen $m\mathbf{Z}^k$ is contained in the kernel of the surjection of $\mathbf{Z}^k \to G$.

Hence $\pi_1(S) = \ker(\oplus G_{i/\operatorname{Im} r} \twoheadrightarrow G)$.

Corollary 1.9. *Given any simply connected algebraic variety X of dimension $n \geqslant 2$ and any abelian group G, there exists an abelian cover S of X with group G^n such that $\pi_1(S) \cong G$.*

Proof. Let $G \cong \oplus_{i=1}^k \mathbf{Z}/m_i Z$ be any decomposition of G into cyclic summands, and let H be a very ample divisor on X. Then we can pick divisors D_{ij} $(i = 1, \cdots, k, j = 1, \cdots, n+1)$ with $m_i H \equiv D_{ij}$, and such that $D = \bigcup_i D_{ij}$ is a divisor with normal crossings. Topologically, the covering of $X - D$ is obtained by considering the following homomorphism of $\pi_1(X - D) \twoheadrightarrow G^n$: let e_{ij} $(j = 1, \cdots, n)$ be the canonical generator of the addendum $\mathbf{Z}/m_i\mathbf{Z}$ in the jth copy of G in G^n; then we map $\gamma_{ij} \to e_{ij}$ for $j \leqslant n$, and $\gamma_{i(n+1)}$ to $\Sigma_{j=1}^n e_{ij}$.

But in this way it is not clear that the cover $S - R \to X - D$ extends to a smooth cover, not just a normal one. But S can be constructed in the following way: let $\tilde{S}$ be the abelian cover of X with group G^{n+1} obtained by taking, $\forall i, j$ in the line bundle H, the m_ith root of D_{ij}. By Proposition 1.8, $\tilde{S}$ is simply connected, and is clearly smooth.

Embed now G diagonally in G^{n+1}: now, since D has normal crossings, it is easy to verify that the resulting action of G is free.

Hence if $S = \tilde{S}/G$, $\pi_1(S) \cong G$ and $S \to X$ is abelian with group $G^{n+1}/G = G^n$.

2. Bidouble covers and their deformations

In this section we shall concentrate on the case where S, X are surfaces and for brevity we shall discuss the case where $G = (\mathbf{Z}/2\mathbf{Z})^2$, which is enough for our present applications.

Our goal is to describe the datum of such a Galois cover π: $S \to X$ in terms of an algebraic setting which allows us to write down explicitly in many cases the local deformation space of S. Anyhow, before proceeding to this special case, let us explain the basic philosophy.

First of all, if G is abelian, $\pi_* \mathcal{O}_S$ is the vector bundle on X which splits as a direct sum $\oplus_{\chi \in G^*} \mathcal{L}_\chi$, where G^* is the group of characters of G and G operates on $\mathcal{L}_\chi$ by the character χ.

S' is determined by the $\mathcal{O}_X$-algebra structure of the integral algebra $\pi_* \mathcal{O}_S$, in turn determined by nonzero homomorphisms of $\mathcal{L}_\chi \oplus \mathcal{L}_{\chi'} \to \mathcal{L}_{\chi+\chi'}$ satisfying compatibility conditions. These homomorphisms are determined by the equations x_i of the D_i's. Deforming the D_i's one gets new abelian covers of X: to obtain a bigger natural way of deforming S, one factors $\forall i = 1, \cdots, k$ ($R = R_1 \cup \cdots \cup R_k$) π as π_i: $S \to Y_i$, $Y_i \overset{p_i}{\to} X$ where $Y_i = S/G_i$, G_i being the stabilizer of R_i. S is an m_ith cyclic cover of Y_i, and there is a natural way of deforming a cyclic cover since S is given as an hypersurface on a line bundle with base Y_i; the bulk of the problem is to put together these independent deformations in a unique smooth family.

Assume now $G = (\mathbf{Z}/2\mathbf{Z})^2$ and let $\{\sigma_1, \sigma_2, \sigma_3\} = G - \{0\}$.

Let $Y_i = S/\sigma_i$ be as above (notice that it has some ordinary quadratic singularities at the isolated fixpoints of σ_i, given by $R_j \cap R_k$, where (i, j, k) is a permutation of $(1, 2, 3)$).

Let χ_i be the character orthogonal to σ_i, and nonzero.

It is clear that $(p_i)_* \mathcal{O}_{Y_i} = \mathcal{O}_X \oplus \mathcal{L}_{\chi_i}$.

The double cover Y_i is ramified on $D_j + D_k$, therefore one can choose an equation of D_i, x_i ($i = 1, 2, 3$), such that $x_j x_k$ is a section giving the homomorphism of $\mathcal{L}_{\chi_i}^{\otimes 2} \to \mathcal{O}_X$ which gives the algebra structure to $(p_i)_* \mathcal{O}_{Y_i}$. We write therefore $\mathcal{L}_{\chi_i} = \mathcal{O}_X(-L_i)$ and we have

$$2L_i \equiv D_j + D_k. \tag{2.1}$$

We observe also that one has a homomorphism of

$$\mathcal{O}_X(-L_i) \otimes \mathcal{O}_X(-L_j) \to \mathcal{O}_X(-L_k),$$

which is given by x_k (notice that $2(L_i + L_j - L_k) \equiv 2D_k$ by (2.1) and that if X has no 2-torsion then (2.1) implies the following linear equivalence (2.2)).

We have therefore also

$$D_k + L_k \equiv L_i + L_j. \tag{2.2}$$

Proposition 2.3. *A smooth bidouble cover $\pi\colon S \to X$ is uniquely determined by the data of effective divisors D_1, D_2, D_3, and divisors L_1, L_2, L_3 such that* (2.1) *and* (2.2) *hold and $D = \bigcup D_i$ has normal crossings.*

Proof. Define S as $\operatorname{Spec}(\mathcal{O}_X \oplus (\oplus_{i=1}^3 \mathcal{O}_X(-L_i)))$ where the algebra structure is given by the equations x_i as described above and you get how to associate to the given data the bidouble cover.

It is more satisfactory, for later use, to embed S in a vector bundle V over X, and to write equations for S. Define V to be $\oplus_{i=1}^3 \mathcal{O}_X(L_i)$ and denote by w_1, w_2, w_3 fibre coordinates relative to the three summands (i.e. the w_i's are given by linear functionals on the fibres and are expressed by functions $w_{i\alpha}$ when one fixes a trivializing cover (U_α) for V: anyhow we shall avoid the use of the index α, to simplify our notations).

Consider now in V the subvariety S given by the following equations:

$$\begin{aligned} w_i^2 &= x_j x_k, \quad (i, j, k) \text{ being a permutation,} \\ x_k w_k &= w_i w_j \quad \text{of } (1,2,3) \text{ as usual.} \end{aligned} \tag{2.4}$$

One notices that the surface $\hat{S}$ given by the first set of three equations, i.e. $w_i^2 = x_j x_k$, is a Galois $(\mathbf{Z}/2\mathbf{Z})^3$ covering of X.

$\hat{S}$ is a complete intersection in V, and splits into two components, $S = S^+$ and S^-, where S^- is determined by the equations $x_k w_k = -w_i w_j$.

Though $\hat{S}$ is singular, S^+ and S^- are smooth.

Now $S^+ \cap S^-$ consists of $R = R_1 \cup R_2 \cup R_3$ and $R_i - R_j - R_k$ is a nodal curve for S, while at the points of $R_i \cap R_j$ the singularity of $\hat{S}$ is the one given by two quadrics in 3-space tangent at one point and with the same pair of asymptotic lines.

In fact $x_k w_k = -w_i w_j = w_i w_j$ $(k = 1,2,3)$ implies that at least 2 of the coordinates w_i are zero.

Then $S^+ \cap S^-$ is the union of the three components defined by the equations $w_i = w_j = x_k = 0$ and $w_k^2 = x_i x_j$, i.e., the R_k's.

Clearly $\hat{S}$ can be singular only over D; at the points of $R_k - R_i - R_j$, x_i, x_j are not zero and $\hat{S}$ is contained in the smooth threefold $w_k^2 = x_i x_j$, $x_k = w_i^2/x_j$ with local coordinates w_i, y_k, w_j (if x_k, y_k are local coordinates on X at the point of projection) and there $\hat{S}$ is defined by the equation $w_j^2/x_i = w_i^2/x_j$.

On the other hand, at the points of $R_i \cap R_j$, x_i, x_j and the w_i's are local coordinates for V and $\hat{S}$ is contained in the smooth threefold $w_j^2/x_k = x_i$, $w_i^2/x_k = x_j$ with local coordinates w_i, w_j, w_k and there $\hat{S}$ is defined by the single equation

$$w_k^2 = \left(w_j w_i/x_k\right)^2.$$

Hence we have shown that S is smooth; moreover, on S, are defined sections z_i of $\mathcal{O}_S(R_i)$ such that

$$z_i^2 = x_i, \quad z_i z_j = w_k. \tag{2.5}$$

We notice that the only ambiguity is that, if (z_1, z_2, z_3) is a choice, $(-z_1, -z_2, -z_3)$ is the only other allowable one. We remark also that the formulas (2.5) do not define S (up to the above involution, cf. Corollary 1.9) unless there exist divisors L_i' on S such that

$$2L_i' \equiv D_i, \quad L_j' + L_k' \equiv L_i. \tag{2.6}$$

With these sections, we notice that, to have local coordinates on S, one can take (z_i, z_j) at the points of $R_i \cap R_j$, while, if (x_i, y_i) are local coordinates around $D_i - D_j - D_k$ (notice that x_i is the equation of D_i), (z_i, y_i) are local coordinates around $R_i - R_j - R_k$.

In the remaining points of S one can lift local coordinates on X to obtain local coordinates on S.

Now the action of σ_i on S is induced by the following action on V:

$$w_i \to w_i, \quad w_j \to -w_j, \quad w_k \to -w_k,$$

which in the local coordinates around $R_i - R_j - R_k$ is expressed by $\sigma_i(z_i, y_i) = (-z_i, y_i)$.

At the points of $R_i \cap R_j$, fixed by the group G, the action is given as

$$\sigma_k(z_i, z_j) = (-z_i, -z_j),$$
$$\sigma_i(z_i, z_j) = (-z_i, z_j),$$
$$\sigma_j(z_i, z_j) = (z_i, -z_j).$$

To finish the proof, we need only to notice that multiplying x_i by a nonzero constant, say c^2, we obtain an isomorphic surface S (multiply w_j, w_k by c). q.e.d.

We can rephrase Proposition 1.8 as follows.

Proposition 2.7. *Let π: $S \to X$ be a smooth bidouble cover with X simply connected and the D_i's flexible. Then $\pi_1(S) = \mathbf{Z}/2\mathbf{Z}$ if and only if each D_i is not empty and is 2-divisible in* Pic(X).

In the remaining cases S is simply connected.

Proof. Since, as we tacitly assumed until now, S is connected, at most one of the D_i's is empty.

In fact, we have, by Proposition 1.8, a surjection $(\mathbf{Z}/2)^t \twoheadrightarrow (\mathbf{Z}/2)^2$, where t = number of nonempty divisors D_i, and by Proposition 1.8, $\pi_1(S)$ is trivial if $t = 2$.

If $t = 3$, by Proposition 1.8 again, $\pi_1(S) = \mathbf{Z}/2\mathbf{Z}$ if and only if $\operatorname{Im} r \subset (2\mathbf{Z})^3$, i.e., every D_i is 2-divisible in homology, hence in Pic(X).

Definition 2.8. Given a bidouble cover $S \xrightarrow{\pi} X$ as in Proposition 2.3 and expressed as a subvariety of $V = \oplus_{i=1}^{3} \mathcal{O}_X(L_i)$ by the equations (2.4), $S' \subset V$ is called a natural deformation of S if it is given by equations

$$(2.9) \qquad \begin{aligned} w_i^2 &= (\gamma_j w_j + x_j')(\gamma_k w_k + x_k'), \\ w_j w_k &= x_i' w_i + \gamma_i w_i^2 \end{aligned}$$

for each permutation (i, j, k) of $(1,2,3)$, and where $x_j' \in H^0(\mathcal{O}_X(D_j))$, $\gamma_j \in H^0(\mathcal{O}_X(D_j - L_j))$.

Hence natural deformations are parametrized by a product of projective spaces over an open set of which one has a smooth family of deformations of S. We shall see now under which conditions every small deformation of S is a natural one.

Definition 2.10. Let $D_1, \cdots, D_k$ be divisors in a smooth manifold X and $x_1, \cdots, x_k$ equations for them.

Define $\Omega_X^1(\log D_1, \cdots, \log D_k)$ to be the subsheaf (as $\mathcal{O}_X$ module) of $\Omega_X^1(D_1 + \cdots + D_k)$ generated by Ω_X^1 and by dx_i/x_i, for $i = 1, \cdots, k$.

Proposition 2.11. *Assume that the D_i's of Definition* 2.10 *are reduced divisors which form locally a regular sequence. Then one has the following exact sequence*:

$$(2.12) \qquad 0 \to \Omega_X^1 \to \Omega_X^1(\log D_1, \cdots, \log D_k) \xrightarrow{R} \bigoplus_{i=1}^{k} \mathcal{O}_{D_i} \to 0.$$

Proof. Clearly, $\Omega_X^1 \subset \Omega_X^1(\log D_1, \cdots, \log D_k)$ and if we have a form which is locally written as $\sum_{i=1}^{k} a_i\, dx_i/x_i + \eta$, where $\eta \in \Omega_X^1$, we define its image by the residue map R to be the collection of the residue classes $a_i \pmod{x_i}$. Clearly the sequence is exact provided the definition is well posed. That is, we must show that, if $\sum a_i\, dx_i/x_i$ is regular (i.e., in Ω_X^1), then $x_i | a_i$. We shall prove this result by induction on k. For $k = 1$ we have a form of type $\eta = (a/x)\, dx$, and, if $y_1, \cdots, y_n$ are local coordinates on X, then $\eta = (a/x)\sum_{j=1}^{n} (\partial x/\partial y_j)\, dy_j$ hence, for each j, $(a/x)\, \partial x/\partial y_j$ is regular. By the Weierstrass preparation theorem we can assume that x and $\partial x/\partial y_1$ have no common factors, hence x divides a.

Finally, assume that $(x_1,\cdots,x_k)$ is a regular sequence: then x_k is not a zero divisor in $\mathcal{O}/(x_i)$ for $i < k$ (here we exploit the fact that $\mathcal{O}$ is a regular local ring, hence the condition of being a regular sequence is independent of the ordering, cf. [23]). We can assume, by the inductive assumption, since $\sum_{i=1}^{k-1} x_k a_i\, dx_i/x_i$ is a regular form, that x_i divides $x_k a_i$ $(i = 1,\cdots,k-1)$; by the above remark $x_i|a_i$, hence $a_k\, d$ is a regular form and $x_k|a_k$. q.e.d.

We remark now that, for any sheaf $\mathcal{F}$ on S, $\pi_*\mathcal{F}$ splits as a direct sum according to the characters of G.

In particular, we have seen that

$$\pi_*(\mathcal{O}_S) = \mathcal{O}_X \oplus \left(\bigoplus_{i=1}^{3} \mathcal{O}_X(-L_i) \right). \tag{2.13}$$

Now, it is well known that

$$K_S \equiv \pi^*(K_X) + R; \tag{2.14}$$

hence K_S is not in general a pullback if S is simply connected, but we have in any case the following

Lemma 2.15.

$$\pi_*(\mathcal{O}_S(K_S)) \cong \mathcal{O}_X(K_X) \oplus \left(\bigoplus_{i=1}^{3} \mathcal{O}_X(K_X + L_i) \right).$$

Proof. Consider again the factorization of π given by

$$\pi_i\colon S \to Y_i, \quad \rho_i\colon Y_i \to X.$$

Now $\pi_* = (\rho_i)_*(\pi_i)_*$, and ω_{Y_i} is the σ_i-invariant part of $(\pi_i)_*\omega_S$. Hence, since $(\rho_i)_*\omega_{Y_i} = \omega_X \oplus \omega_X(L_i)$, the invariant summand of $\pi_*(\omega_S)$ is ω_X, and the one corresponding to the character χ_i is $\omega_X(L_i)$. q.e.d.

For use later on in the study of the infinitesimal deformations of S, we compute $\pi_*(\Omega^1_S \otimes \Omega^2_S)$.

Theorem 2.16.

$$\pi_*\left(\Omega^1_S \otimes \Omega^2_S\right) = \Omega^1_X(\log D_1, \log D_2, \log D_3) \otimes \Omega^2_X$$
$$\oplus \left(\bigoplus_{i=1}^{3} \Omega^1_X(\log D_i) \otimes \Omega^2_X \otimes \mathcal{O}_X(L_i) \right).$$

Moreover, the first summand is the G-invariant one, the others correspond to the three nontrivial characters χ_i.

Proof. Let us compute first the G-invariant summand. Locally around $R_j \cap R_k$, $\Omega_S^1 \otimes \Omega_S^2$ is spanned over $\mathcal{O}_X$ by

$$\left(dz_j \wedge dz_k\right) dz_l \otimes \begin{cases} 1, \\ z_j, \\ z_k, \\ z_j z_k, \end{cases}$$

where $l = j, k$.

Now all these eight generators are eigenvectors for the action of G.

If we impose invariance by the action of σ_i we remain only with

$$\left(dz_j \wedge dz_k\right) dz_l \otimes z_h \qquad (l, h = j, k).$$

Invariance also by σ_j, σ_k implies that we are left only with

$$\left(dz_j \wedge dz_k\right) dz_j \otimes z_k, \quad \left(dz_j \wedge dz_k\right) dz_k \otimes z_j,$$

i.e., up to constants, $(dx_j \wedge dx_k) dx_j / x_j$, $(dx_j \wedge dx_k) dx_k / x_k$, the two local generators of $\Omega_X^1(\log D_1, \log D_2, \log D_3) \otimes \Omega_X^2$.

Around $D_i - D_j - D_k$ we get a sheaf locally isomorphic to $\Omega_X^2 \otimes \Omega_X^1(\log D_i)$, on $X - D$ a sheaf isomorphic to $\Omega_X^1 \otimes \Omega_X^2$ and the proof follows since there is a canonical inclusion of $\Omega_X^1(\log D_1, \log D_2, \log D_3) \otimes \Omega_X^2$ in $\pi_*(\Omega_S^1 \otimes \Omega_S^2)^G$.

The summand corresponding to the character χ_i is spanned by σ_i-invariants, σ_j, σ_k-anti-invariants.

Around $R_j \cap R_k$ we get

$$\left(dz_j \wedge dz_k\right)\left(dz_j\right) z_j, \quad \left(dz_j \wedge dz_k\right) dz_k \left(z_k\right),$$

i.e.,

$$\frac{dx_j \wedge dx_k}{z_j z_k} dx_j, \quad \frac{dx_j \wedge dx_k}{z_j z_k} dx_k, \quad \text{or} \quad \frac{dx_j \wedge dx_k}{w_i} dx_l \qquad (l = j, k).$$

Whereas around $R_i \cap R_j$ anti-invariance by σ_k leaves us with

$$\left(dz_i \wedge dz_j\right) dz_l \otimes \begin{cases} 1, \\ z_i z_j, \end{cases} \qquad (l = i, j).$$

Invariance by σ_i implies that we are left with

$$\left(dz_i \wedge dz_j\right) dz_i, \quad \left(dz_i \wedge dz_j\right) dz_j \left(z_i z_j\right),$$

which can be replaced, since z_k is a unit around $R_i \cap R_j$ (and G-invariant locally), by

$$\left(dz_i \wedge dz_j\right) \frac{dz_i}{z_k}, \quad \left(dz_i \wedge dz_j\right) \frac{dz_j}{z_k} \left(z_i z_j\right),$$

i.e.,

$$\frac{(dx_i \wedge dx_j)}{w_i}\frac{dx_i}{x_i}, \quad \frac{(dx_i \wedge dx_j)}{w_i}dx_j.$$

By the same argument as above, the desired isomorphism of $\pi_*(\Omega^1_S \otimes \Omega^2_S)_{\chi_i}$ with $\Omega^1_X(\log D_i) \otimes \Omega^2_X(L_i)$ follows. q.e.d.

It follows immediately from 2.16 and (2.12) that there is an exact sequence

$$\begin{aligned}
&0 \to \left(\Omega^1_X \otimes \Omega^2_X\right) \oplus \left(\bigoplus_{i=1}^{3} \Omega^1_X \otimes \Omega^2_X(L_i)\right)\\
(2.17)\qquad &\qquad \| \\
&\qquad \Omega^1_X \otimes \left(\pi_*\Omega^2_S\right)\\
&\to \pi_*\left(\Omega^1_S \otimes \Omega^2_S\right) \to \left(\bigoplus_i \mathcal{O}_{D_i}(K_X)\right) \oplus \left(\bigoplus_i \mathcal{O}_{D_i}(K_X + L_i)\right) \to 0.
\end{aligned}$$

Its exact cohomology sequence is dual, via Serre duality and since $\Omega^1_X \otimes (\pi_*\Omega^2_S) = \pi_*(\pi^*\Omega^1_X \otimes \Omega^2_S)$, to the exact sequence

$$\begin{aligned}
&0 \to H^0(T_S) \to H^0(\pi^*T_X)\\
(2.18)\qquad &\to \bigoplus_{i=1}^{3} H^0\left(\mathcal{O}_{D_i}(D_i) \oplus \mathcal{O}_{D_i}(D_i - L_i)\right) \xrightarrow{\partial} H^1(T_S) \to H^1(\pi^*T_X)\\
&\to H^1(\cdots).
\end{aligned}$$

As explained in the following theorem, the meaning of (2.18) is: $\operatorname{Im}\partial$ are the infinitesimal deformations obtained by means of natural deformations, which are equivalent modulo the action of Aut(X), and not all the deformations of X lift to deformations of S (e.g. because the cohomology classes of the D_i's do not remain of type (1,1) under the deformation of X).

More precisely, let

$$\rho\colon \bigoplus_{i=1}^{3} H^0\left(\mathcal{O}_X(D_i) \oplus H^0(\mathcal{O}_X(D_i - L_i))\right) \to H^1(T_S)$$

be the composition of the direct sum of the restriction maps of the ith term to $H^0(\mathcal{O}_{D_i}(D_i)) \oplus H^0(\mathcal{O}_{D_i}(D_i - L_i))$ with ∂. Then we have

Theorem 2.19. $\rho((\delta_i, \gamma_i))$, *for* $\delta_i \in H^0(\mathcal{O}_X(D_i))$, $\gamma_i \in H^0(\mathcal{O}_X(D_i - L_i))$, *is the Kodaira-Spencer class of the* 1-*parameter family given by the natural deformations corresponding to* $x_i' = x_i + t\delta_i$, $t\gamma_i$ (*cf. Definition* 2.8), *where* $t \in \mathbf{C}$.

Proof. By the linearity of the Kodaira-Spencer map associated to natural deformations it is enough to prove the result when $\delta_2 = \delta_3 = \gamma_2 = \gamma_3 = 0$ and either $\delta_1 \neq 0$, $\gamma_1 = 0$, or $\delta_1 = 0$, $\gamma_1 \neq 0$.

In both cases, consider the factorization

$$S \overset{\pi_1}{\rightarrow} Y \overset{P_1}{\rightarrow} X.$$

Observe that if $(S_t)_{t\in\mathbf{C}}$ is our 1-parameter family, we still have a factorization

$$S_t \overset{\pi_1}{\rightarrow} Y_1 \overset{P_1}{\rightarrow} X;$$

moreover, if $D_1' = P_1^*(D_1)$,

$$\mathcal{O}_{D_1}(D_1) \oplus \mathcal{O}_{D_1}(D_1 - L_1) \cong (P_1)_*\big(\mathcal{O}_{D_1'}(D_1')\big).$$

Now, though Y_1 has singular points, nodes, the map π_1: $S_t \rightarrow Y_1$ has as branch locus the nodes of Y_1 plus a movable divisor $D_1'(t)$, hence our result follows from known facts about double covers (cf. [14], [27]).

An alternate proof can be given as follows: since we have a deformation of S as a submanifold of V, if $N_{S|V}$ is the normal bundle of S in V, the infinitesimal deformations determines a section of $N_{S|V}$ and the Kodaira-Spencer class is given through the coboundary ∂': $H^0(N_{S|V}) \rightarrow H^1(T_S)$ of the long cohomology sequence associated to the exact sequence

$$0 \rightarrow T_S \rightarrow T_V \otimes \mathcal{O}_S \rightarrow N_{S|V} \rightarrow 0$$

(cf. [15]).

On the other hand we have a projection $T_V \otimes \mathcal{O}_S \twoheadrightarrow \pi^*(T_X)$ by which the following diagram is commutative (observe that $\pi_*(\mathcal{O}_{R_i}(\pi^*D_i)) \cong \mathcal{O}_{D_i}(D_i) \oplus \mathcal{O}_{D_i}(D_i - L_i)$):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_S & \longrightarrow & T_V \otimes \mathcal{O}_S & \longrightarrow & N_{S|V} & \longrightarrow & 0 \\
 & & \wr\| & & \downarrow & & \beta\downarrow & & \\
0 & \longrightarrow & T_S & \longrightarrow & \pi^*(T_X) & \longrightarrow & \bigoplus_i (\mathcal{O}_R(\pi^*D_i)) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Therefore ∂' factors as $\partial \circ H^0(\beta)$, and we omit the rest of the verification.

Corollary 2.20. *If* π: $S \rightarrow X$ *is, as usual, a smooth bidouble cover and* $H^1(\pi^*T_X) = 0$ (*i.e.* $H^1(T_X) = H^1(T_X(-L_i)) = 0$), *then every small deformation of S is a natural deformation. In particular, the Kuranishi family B of S is smooth* (*and its dimension can be computed by* (2.18)).

Proof. Natural deformations give a family of deformations with surjective Kodaira-Spencer map: since natural deformations are parametrized by a smooth variety, their image in B contains a neighborhood of 0 in $H^1(T_X)$. q.e.d.

We conclude this section by computing the numerical invariants of bidouble covers.

Since $K_S = \pi^* K_X + R$, and $2R = \pi^* D$,

$$K_S^2 = (2K_X + D)^2. \tag{2.21}$$

On the other hand,

$$\begin{aligned}\chi(\mathcal{O}_S) &= \chi(\pi_*(\mathcal{O}_S))\\ &= 4\chi(\mathcal{O}_X) + \frac{1}{2}\sum_{i=1}^{3}(-L_i)(-L_i - K_X) \text{ (by the Riemann-Roch theorem)}\\ &= 4\chi(\mathcal{O}_X) + \frac{1}{2}K_X \cdot \Big(\sum_i L_i\Big) + \frac{1}{2}\sum_i L_i^2,\end{aligned}$$

an equality which can also be expressed as

$$\chi(\mathcal{O}_S) = 4\chi(\mathcal{O}_X) + \frac{1}{2}K_X \cdot D + \frac{1}{8}\Big(D^2 + \sum_i D_i^2\Big). \tag{2.22}$$

We also derive a useful formula in case $H^1(\pi^* T_X)$ is zero; then the Kuranishi family B of S is smooth of dimension equal to

$$\sum_{i=1}^{3}\Big(h^0(\mathcal{O}_X(D_i)) - 1 + h^0\big(\mathcal{O}_X(D_i - L_i)\big)\Big) - h^0(\pi^* T_X) - h^0(T_S). \tag{2.23}$$

We just remark that if the L_i's are sufficiently ample, then S is of general type, hence $h^0(T_S) = 0$, and moreover $h^0(\pi^* T_X) = h^0(T_X) = \dim_{\mathbf{C}} \operatorname{Aut}(X)$.

3. Double and bidouble covers of $\mathbf{P}^1 \times \mathbf{P}^1$

In this section we shall consider covers of $Q = \mathbf{P}^1 \times \mathbf{P}^1$ and we shall denote by $\mathcal{O}_Q(a, b)$ the line bundle $\mathrm{pr}_1^*(\mathcal{O}_{\mathbf{P}^1}(a)) \otimes \mathrm{pr}_2^*(\mathcal{O}_{\mathbf{P}^1}(b))$, pr_1, pr_2 being the two projections on the two factors.

Also, if $m \in \mathbf{Z}$ we shall denote by $[m]^+ = \max(m, 0)$. For later use, we write down a table for the dimensions of the cohomology groups of line bundles on Q.

	$h^0(\mathcal{O}_Q(a,b))$	$h^1(\mathcal{O}_Q(a,b))$	$h^2(\mathcal{O}_Q(a,b))$
$a < 0, b < 0$	0	0	$(a+1)(b+1)$
$a \geqslant -1, b \geqslant -1$	$(a+1)(b+1)$	0	0
$b \geqslant 0, a \leqslant -2$, or $a \geqslant 0, b \leqslant -2$	0	$-(a+1)(b+1)$	0

TABLE I

Also bear in mind that $T_Q \cong \mathcal{O}_Q(2,0) \oplus \mathcal{O}_Q(0,2)$.

The situation for double coverings is quite simple: every small deformation is still a double covering, if S is of general type.

In fact we have (with almost the same proof of 2.16 and that which follows).

Proposition 3.1. *Let $X \xrightarrow{\pi} Y$ be a smooth double cover with $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y \oplus \mathcal{O}_Y(-L)$. Then, if D is the branch divisor,*

$$\pi_*\left(\Omega_X^1 \otimes \Omega_X^2\right) \cong \left(\Omega_Y^1(\log D) \otimes \Omega_Y^2\right) \oplus \left(\Omega_Y^1 \otimes \Omega_Y^2(L)\right).$$

Moreover we have the exact sequence

$$(3.2)\qquad \begin{aligned} 0 &\to H^0(T_X) \to H^0(\pi^*T_Y) \to H^0(\mathcal{O}_D(D)) \xrightarrow{\partial} H^1(T_X) \\ &\to H^1(\pi^*T_Y) \to \cdots . \end{aligned}$$

Assume now $Y = Q$, $L = \mathcal{O}_Q(n, m)$ $(n \geqslant 0, m \geqslant 0)$. Then

$$\begin{aligned} H^1(\pi^*T_Q) &= H^1\left(T_Q \oplus T_Q(-n,-m)\right) \\ &= H^1\left(\mathcal{O}_Q(2,0) \oplus \mathcal{O}_Q(0,2) \oplus \mathcal{O}_Q(2-n,-m) \oplus \mathcal{O}_Q(-n,2-m)\right) \end{aligned}$$

which, by Table I, can be nonzero, if we assume $n \geqslant m$, only if $m \leqslant 2$, $n \geqslant 2$.

Let us pass now to consider bidouble covers.

Let $\mathcal{O}_Q(D_i)$ be the line bundle $\mathcal{O}_Q(n_i, m_i)$. Then $n_i \equiv n_j \pmod 2$, $m_i \equiv m_j \pmod 2$, and

$$\mathcal{O}_Q(L_i) \equiv \mathcal{O}_Q\left(\frac{n_j + n_k}{2}, \frac{m_j + m_k}{2}\right) = \mathcal{O}_Q(a_i, b_i).$$

Since $H^1(T_Q) = 0$, we want $H^1(T_Q(-L_i)) = 0$; i.e., by the remarks above, it is sufficient to have $a_i, b_i \geqslant 3$, in order that every small deformation of S be a natural deformation. Also, in this case, by Table I, $H^0(T_Q(-L_i)) = 0$; moreover, in this case $\mathcal{O}_Q(D) = \mathcal{O}_Q(\Sigma n_i, \Sigma m_j)$, hence $K_S^2 > 0$ and $|K_S| \neq \varnothing$, so that S is of general type, hence $H^0(T_S) = 0$, and the dimension of the Kuranishi family B of S is equal to $h^1(T_S)$, and, by (2.18),

$$(3.3)\qquad \begin{aligned} h^1(T_S) &= \sum_{i=1}^{3} (n_i + 1)(m_i + 1) \\ &\quad + \frac{1}{4}\left[(2n_i - n_j - n_k)(2m_i - m_j - m_k)\right]^+ - 9. \end{aligned}$$

Lemma 3.4. *S is a minimal model of a surface of general type if $\Sigma a_i \geqslant 5$, $\Sigma b_i \geqslant 5$.*

Proof. $2K_S \equiv \pi^*(D + K_Q)$, therefore in the above hypotheses $|2K_S|$ has no fixed part and maps to a surface.

Example 3.5. This is a case when not all the deformations are natural (cf. [7]).

Consider the case where the three divisors are of type (1,3), (3,1), (1,1) respectively; Lemma 3.4 applies and we obtain, if the D_i's are transversal, a simply connected surface of general type with $K^2 = 2$, $p_g = 1$, by formulas (2.21) and (2.22).

L_1, L_2, L_3 are divisors of type (2,1), (1,2), (2,2), respectively. It is then clear that $\Omega^1_Q \otimes \Omega^2_Q$, $\Omega^1_Q \otimes \Omega^2_Q(L_i)$ have no nonzero sections. The same is true for the sheaves $\mathcal{O}_{D_i} \otimes \Omega^2_Q$, $\mathcal{O}_{D_i}(K_Q + L_i)$, $i = 1, 2$, while $\mathcal{O}_{D_3}(K_Q + L_3) \cong \mathcal{O}_{D_3}$.

Therefore the cohomology long exact sequence associated to (2.17) reduces to

$$(3.6) \qquad 0 \to H^0\big(\Omega^1_Q(\log D_3)\big) \to H^0\big(\mathcal{O}_{D_3}\big) \xrightarrow{\partial} H^1\big(\Omega^1_Q\big) \to \cdots$$

and, if ∂ is injective, we conclude that $H^0(\Omega^1_S \otimes \Omega^2_S) = 0$ (i.e., $H^2(T_S) = 0$) and the Kuranishi family B of S is smooth of dimension 16, while in this case all the natural deformations are Galois covers, and give a 11-dimensional subvariety of B.

The injectivity of ∂ follows from the following lemma.

Lemma 3.7. *In the cohomology exact sequence associated to the exact sequence*

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \to \mathcal{O}_D \to 0,$$

where $D = \bigcup D_i$ is a disjoint union of connected reduced divisors D_i, ∂: $H^0(\mathcal{O}_D) = \oplus_i H^0(\mathcal{O}_{D_i}) \to H^1(\Omega^1_X)$ has as image the subspace generated by the first Chern classes of the D_i's.

Proof. More precisely we shall show that, if 1_{D_i} is the function which is $\equiv 1$ on D_i and 0 elsewhere, then $\partial(1_{D_i}) = c_1(D_i)$. Let then $\{f_\alpha = 0\}$ be a set of local equations for D_i on a covering $(U_\alpha) = \mathfrak{U}$. 1_{D_i} lifts to the 0-cochain $d(\log f_\alpha)$, with values in $\Omega^1_X(\log D)$. Let $f_\alpha = f_{\alpha\beta} f_\beta$ on $U_\alpha \cap U_\beta$. Clearly $\partial(1_{D_i})$ is then represented in $H^1(\mathfrak{U}, \Omega^1_X)$ by the 1-cocycle given by $d(\log f_\alpha) - d(\log f_\beta) = d(\log f_{\alpha\beta})$, which is nothing else than $c_1(D_i)$ (cf. [18, p. 64]). q.e.d.

We are now going to study the behavior of the canonical map of a bidouble cover of $\mathbf{P}^1 \times \mathbf{P}^1$; as a biproduct of this investigation we shall obtain the fact that corresponding to different choices of the (n_i, m_i)'s (up to permutations of $\{1, 2, 3\}$ and the symmetry of $\mathbf{P}^1 \times \mathbf{P}^1$ exchanging the factors) one gets different families of surfaces.

Theorem 3.8. *Let S be a smooth bidouble covering of $\mathbf{P}^1 \times \mathbf{P}^1$ corresponding to three divisors of types (n_i, m_i). Assume that $m_i + m_j$, $n_i + n_j \geqslant 8$ $(\forall i, j)$. Then the natural deformations of S give the unique component of the moduli space of S containing the point $[S]$, of dimension given by* (3.3). *S is simply connected unless all the n_i's, m_i's are even and (for every $i = 1, 2, 3$) $m_i + n_i > 0$.*

Moreover, the canonical system $|K_S|$ *is without basepoints and gives a biregular embedding of* S. *The singular points of* R *are characterized by their inflectionary behavior under the canonical map (hence they are invariant under any automorphism of* S*), and the* n_i*'s,* m_i*'s are uniquely determined by* S *(and not by the datum* $\pi\colon S \to Q$*).*

Proof. The base space of the natural deformations of S is a Zariski open set $\mathfrak{U}$ of a projective space. We have a morphism φ of $\mathfrak{U}$ into the moduli space $\mathcal{M}$, and we know that $\varphi(\mathfrak{U})$ contains a neighborhood of $[S]$ in $\mathfrak{M}$; therefore φ is dominant. Again since a neighborhood of $[S]$ in $\mathcal{M}$ is a quotient of the Kuranishi family B of S by $\mathrm{Aut}(S)$, $\overline{\varphi(\mathfrak{U})}$ is the only component of $\mathcal{M}$ passing through $[S]$.

Remark that this component is unirational.

For the assertions regarding the canonical system, recall that

$$H^0(K_S) \cong z_i z_j z_k \pi^*\Big(\underset{\substack{\| \\ 0}}{H(K_Q)}\Big) \oplus \Big(\bigoplus_{i=1}^{3} z_i \pi^* H^0\big(\mathcal{O}_Q(L_i + K_Q)\big)\Big),$$

cf. Lemma 2.15. By assumption L_i is of type (a_i, b_i) with $a_i, b_i \geqslant 4$, hence $\big|\mathcal{O}_Q(L_i + K_Q)\big|$ gives an embedding of Q.

Since the sections z_i have no common zeros, $|K_S|$ has no basepoints and gives a local embedding at every point. It suffices now to show that the canonical map separates points. This is obvious now if $\pi(x) \neq \pi(y)$, while, if $y = g(x)$, choose three sections $s_i \in H^0(\mathcal{O}_Q(L_i + K_Q))$ not vanishing at $\pi(x) = \pi(y)$. Then the linear functionals given by evaluation at x, y cannot be proportional, since $z_i\pi^*(s_i)(gx) = \chi_i(g)(z_i\pi^*(s_i))(x)$ and either

(i) $z_i(x) = z_j(x) = 0$ and then $g(x) = x$, or

(ii) $z_i(x) \neq 0$ for each i, but then must be $\chi_i(g) = 1$ for each i, so that $g =$ identity, or

(iii) $z_i(x) = 0$, $z_j(x)$, $z_k(x) \neq 0$; then $\chi_j(g) = \chi_k(g) \Rightarrow g = \sigma_i$, and hence $\sigma_i(x) = x$.

To check which are the points of inflection of the image, it suffices to see for which $p \in S$ the following homomorphism r_p is not surjective:

$$0 \to H^0\big(\mathcal{M}_p^3 \mathcal{O}_S(K_S)\big) \to H^0(\mathcal{O}_S(K_S)) \xrightarrow{r_p} \mathcal{O}_P/\mathcal{M}_p^3. \tag{3.9}$$

Since any very ample linear system on Q of type (α_1, α_2) with $\alpha_j \geqslant 2$ has no inflectionary behavior, clearly r_p is onto if $p \notin R$.

Assume now $p \in R_i - R_k$, and choose local coordinates z_i, y_i, where $z_i^2 = x_i$, as usual, and x_i, y_i are local coordinates at $\pi(p)$. It suffices to remark that any jet of second order at p can be written as $f(z_i^2, y_i) + g(z_i^2, y_i)z_i$.

If, instead, $p \in R_i \cap R_j$, z_i, z_j are local coordinates, and we only get jets of the type $f(z_i^2, z_j^2) + z_i g(z_i^2, z_j^2) + z_j h(z_i^2, z_j^2)$; therefore r_p has a cokernel of dimension 1. Moreover, $r_p(H^0(\mathscr{M}_p^2 \mathscr{O}_S(K_S)))$ consists of jets of the form $\alpha z_i^2 + \beta z_j^2$, hence there are uniquely determined two tangent directions at the point p.

Observe now that if S is general we can assume that D_i and D_j are not tangent to the lines of Q passing through $\pi(p)$; therefore, if we set

(3.10)
$$\begin{aligned} \lambda_i = \max\{\lambda \mid & \lambda \text{ is the order of zeros at } p \text{ of a section } s \in H^0(K_S) \\ & \text{such that the image of } s \text{ in } \mathscr{M}_p^{\lambda}/\mathscr{M}_p^{\lambda+1} \text{ has the tangent} \\ & \text{line corresponding to } z_i = 0 \text{ as a simple root}\}, \end{aligned}$$

then $\lambda_i = -1 + 2a_i b_i$.

Setting further $\nu_i = \max\{\nu |$for a general tangent direction L, $2\nu + 1$ is the order of zero at p of a section $s \in H^0(K_S)$ such that the image of s in $\mathscr{M}_p^{2\nu+1}/\mathscr{M}_p^{2\nu+2}$ has the tangent direction of $z_i = 0$ as a simple root, but the tangent direction to $z_j = 0$ is not a root, and L is a root of multiplicity $\nu\}$ it follows that $\nu_i = \min(a_i, b_i)$.

Hence λ_i, ν_i determine the (unordered) pair (a_i, b_i).

Considering instead sections of $H^0(K_S)$ vanishing of even order at p one can determine in an analogous way the pair (a_k, b_k).

In fact a section vanishing at p of even maximal order $2a_k b_k$ is such that 4 tangent directions are determined, the pullback of the two rulings on $\mathbf{P}^1 \times \mathbf{P}^1$; 2 of these directions, L_1, L_1', appear with multiplicity a_k, 2 with multiplicity b_k, L_2, L_2'.

Choosing a section of maximal odd order vanishing at p and with the tangent direction of $z_i = 0$ as a simple root one determines the ordered pair (a_i, b_i), a_i = multiplicity of L_1, L_l', b_i = multiplicity of L_2, L_2'.

4. Nonpure dimensionality of the moduli spaces

In this section we shall finally prove Theorem A, tying up together many results proved so far. We start by recalling some known facts on the topology of compact complex surfaces.

Let S be a complex compact surface. S is in a natural way an oriented compact 4-manifold.

Let b_i $(i = 1, \cdots, 4)$ be the Betti numbers of S, i.e., $b_i = \dim_{\mathbf{C}} H^i(S, \mathbf{C})$, and let $e = \sum_{i=0}^4 (-1)^i$ be the topological Euler-Poincaré characteristic of S.

Assume in the sequel that S is simply connected. Then $H^2(S, \mathbf{Z})$ is a lattice of rank equal to b_2, and Poincaré duality gives a unimodular integral quadratic form q: $H^2(S, \mathbf{Z}) \to \mathbf{Z}$.

Let b^+ (b^-) be the number of $(+1)$ $((-1))$ eigenvalues of q and let $\tau = b^+ - b^-$ be the signature of q (since q is nondegenerate, $b_2 = b^+ + b^-$).

It is well known that the two holomorphic invariants χ, K^2 are uniquely determined by the topological invariants e, τ.

In fact one has $K^2 = 3\tau + 2e$ and $12\chi = (3\tau + 3e)$.

The quadratic form q is said to be even if $q(x) \equiv 0 \pmod 2$ $\forall x \in H^2(S, \mathbf{Z})$.

Remark 4.1. If $q(x) \neq -1$ $\forall x$, then S is necessarily a minimal surface. In fact an exceptional curve of the first kind has a Chern class x with $q(x) = -1$. In particular, if q is even, then S is minimal.

We also recall that if S is a minimal model of surface of general type, the following inequalities hold:

$K^2 \geqslant 1, \quad \chi \geqslant 1 \quad$ (G. Castelnuovo),

$K^2 \geqslant 2p_g - 4 \geqslant 2\chi - 6 \quad$ (Noether),

$K^2 \leqslant 9\chi \quad$ (Bogomolov-Miyaoka-Yau).

S. T. Yau [35] also proved that if $K^2 = 9\chi$ and K is ample, then the universal cover of S is the unit ball in $\mathbf{C}^2$.

From the above results also follows (cf. [25], [32]) the corollary:

(4.2) $\mathbf{P}^2$ is the only simply connected complex surface for which the quadratic form q is positive definite.

In fact if $b_2 = b^+ = \tau$, then S is minimal. Now

$$K^2 \leqslant 9\chi \Leftrightarrow 4K^2 \leqslant 3(12\chi)$$
$$\Leftrightarrow 3\tau \leqslant e = 2 + b_2 - 2b_1.$$

Therefore, if $b_2 = \tau$, $b_2 = 1$ and $b_1 = 0$, then $K^2 = 9$, $\chi = 1$.

Now, if S is minimal and $K^2 = 9$, then either $S = \mathbf{P}^2$ or S is of general type, and since $b_2 = 1$, K is ample; but then $\pi_1(S)$ is an infinite group by the theorem of Yau.

Remark 4.3. The above arguments show that $b_2 = \tau$ imply that S is minimal and $K^2 = 9$, $\chi = 1$, $b_1 = 0$.

Moreover, we recall that if S is a minimal surface, and $K^2 \geqslant 1$, then either S is of general type, or S is a minimal rational surface, i.e. either $\mathbf{P}^2$ or a ruled surface $\mathbf{F}_n$ ($n = 0, 2, 3, \cdots$), for which $K^2 = 8$, $\chi = 1$.

For the reader's convenience, we also recall the recent result of M. Freedman ([10], [31]) which, together with earlier results of Milnor [24], Novikov [26] and Wall [33], gives the following theorem.

Theorem. *Let S_1, S_2 be oriented differentiable compact 4-manifolds. Assume that S_1, S_2 are simply connected and that they have quadratic forms q_1, q_2, isometric to each other. Then S_1, S_2 are homeomorphic (by an orientation preserving homeomorphism).*

We immediately deduce the following proposition.

Proposition 4.4. *Let S_1, S_2 be complex surfaces with $K^2 \neq 9$. If S_1, S_2 are simply connected and have the same invariants K^2, χ, then they are (orientedly) homeomorphic if and only if either*

(a) $K_{S_i} \in 2\,\mathrm{Pic}(S_i)$ $(i = 1, 2)$, *or*

(b) $K_{S_i} \notin 2\,\mathrm{Pic}(S_i)$ $(i = 1, 2)$.

Proof. Since, by [13, p. 43], for each element D in $H^2(S_i, \mathbf{Z})$, $DK \equiv D^2 \pmod 2$, in case (a) the quadratic forms q_i are both even and in case (b) they are both odd. In particular, in case (a) the S_i's are minimal by Remark 4.1.

It is classical, (see e.g. [29, pp. 92–93]) that two integral unimodular indefinite quadratic forms are completely determined by their rank, their signature and their parity. In our case K^2, χ determine the rank $= e - 2$ and the signature τ; we only have to prove that q_i is indefinite. q_i cannot be positive definite by 4.3, since K^2 is $\neq 9$, and q_i can never be negative definite by the following lemma, hence we can apply Freedman's theorem.

Lemma 4.5. *For no complex surface with $b_1 \neq 1$ the quadratic form q is negative definite.*

Proof. Kodaira proved that $b^+ = 2p_g + 1$ if b_1 is even and $b^+ = 2p_g$ if b_1 is odd, $b_1 = 2q - 1$.

Now, if $\chi < 0$, then S is ruled, hence algebraic; therefore $b^+ > 0$ and our assumption implies $q = 1$, $b_1 = 1$, $p_g = 0$.

In this last case $\chi = 0$, hence $\tau = -e = -(2 + b_2 - 2) = -b_2$.

Theorem 4.6. *Let S be a smooth bidouble cover of $\mathbf{P}^1 \times \mathbf{P}^1$ branched on two divisors of types $(2a, 2b)$, $(2n, 2m)$, respectively, with $a, b, n, m \geqslant 3$. If $a \equiv n \pmod 2$, and $b \equiv m \pmod 2$, then S satisfies the hypotheses of Proposition 4.4.*

Moreover

$$K_S^2 = 8(n + a - 2)(m + b - 2),$$

$$\chi(\mathcal{O}_S) = 2(n + a - 1)(m + b - 1) + 2 - \tfrac{1}{2}(m + b)(n + a) + \tfrac{1}{2}(a - n)(b - m),$$

$$M(S) = h^1(T_S) = (2n + 1)(2m + 1) + (2a + 1)(2b + 1) - 8 + [(2n - a)(2m - b)]^+ + [(2a - n)(2b - m)]^+.$$

If $a > 2n$, $m > 2b$, then the bidouble covers of $\mathbf{P}^1 \times \mathbf{P}^1$ of the same numerical type as S form a Zariski open set in the moduli space $\mathcal{M}$ of S, and $M(S) = (2n + 1)(2m + 1) + (2a + 1)(2b + 1) - 8$.

Proof. $\pi_1(S) = 0$ by Proposition 2.7. Moreover, since $\pi_1(S) = 0$, $\mathrm{Pic}(S)$ has no torsion, therefore $K_S \equiv \pi^*(K_{\mathbf{P}^1 \times \mathbf{P}^1}) + R \equiv \pi^*(\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(n + a - 2, m + b - 2))$ and is therefore divisible by 2 if $n + a$, $m + b$ are even numbers.

The numerical assertions are restatements of the more general formulas (2.21), (2.22), (3.3), and the last assertion follows from (2.18) since then $H^0(\mathcal{O}_{D_i}(D_i - L_i)) = H^0(\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(D_i - L_i)) = 0$ for each i.

Theorem A. *For each natural number k, there exist positive integers $M_1 < M_2 < \cdots < M_k$ and (orientedly) homeomorphic simply connected minimal surfaces of general type $S_1, \cdots, S_k$ such that $M(S_i) = M_i$.*

Proof. Let $S_1, \cdots, S_k$ be smooth bidouble covers of $\mathbf{P}^1 \times \mathbf{P}^1$ of types $(a_i, b_i)(n_i, m_i)$, as in Theorem 4.6. By Proposition 4.4 they are homeomorphic if and only if the numerical invariants K^2, χ are the same for them, and the theorem is proved if the numbers $M_i = M(S_i)$ are all distinct.

So the problem is reduced to finding, for each k, k 4-tuples of numbers, satisfying certain inequalities, for which the two quadratic polynomials K^2, χ take the same values, but the polynomial M takes all different values.

To simplify the numerical problem, we set

$$u = (n + a - 2), \quad v = (m + b - 2), \quad w = (a - n), \quad z = (m - b),$$

where, of course, we want u, v, w, z to be even numbers, since $a \equiv n \pmod 2$, $b \equiv m \pmod 2$.

The inequalities $a > 2n$, $m > 2b$, $n \geqslant 3$, $b \geqslant 3$ read out now as

$$(*) \qquad u - 4 \geqslant w > u/3 + 2/3, \quad v - 4 \geqslant z > v/3 + 2/3.$$

In terms of the new variables we have now

$$K^2 = 8uv, \quad \chi = (3/2)uv + (u + v) + 2 - \tfrac{1}{2}wz,$$
$$M = 2uv + 6(u + v) + 10 - 2wz.$$

We remark now, simply, that

$$M = 2(uv) + 10 + 4\left[(u + v) - \tfrac{1}{2}wz\right] + 2(u + v)$$
$$= 4\chi + 2 - \tfrac{1}{2}K^2 + 2(u + v).$$

Therefore, we want to solve the equations

$$uv = \text{constant}, \quad (u + v) - \tfrac{1}{2}wz = \text{constant}$$

with u, v, w, z even and satisfying the inequalities $(*)$, and moreover have at least k distinct values for $(u + v)$. Upon dividing all the four numbers by 2, the proof is reduced to the following lemma.

Lemma 4.7 (*E. Bombieri*). *For any given number k, one can solve the equations*

$$(*) \qquad uv = M, \quad wz - (u + v) = N$$

together with

$$(**)\qquad \frac{u+1}{3} < w < u - 2, \quad \frac{v+1}{3} < z < v - 2$$

in integer u, v, w, z, *and have at least* k *distinct values for* $(u + v)$, *for suitable* $M, N \to \infty$.

Proof. The proof given in the letter of E. Bombieri, reproduced in the appendix, applies mutatis mutandis in this case.

Remark 4.8. By Freedman's result it follows that all homotopy $K3$ surfaces (see [17]) are homeomorphic to a smooth quartic in $\mathbf{P}^3$; therefore we get thus a countable number of homeomorphic surfaces which are not deformations of each other.

5. Bounds for the number of moduli in terms of topological invariants

As mentioned in the Introduction, by the Kodaira-Spencer-Kuranishi theory of deformations, and the result of Wavrik ([18]–[20]) a neighborhood of $[S]$ in $\mathcal{M}$ is isomorphic to $B/\mathrm{Aut}(S)$, where B is the base of the Kuranishi family of deformations.

Hence $M = \dim B \geqslant 10\chi - 2K^2$, and $M \leqslant h^1(T_S)$.

In fact, since e.g. $\mathrm{Aut}(S)$ is a finite group [22], S being of general type, $h^0(T_S) = 0$; moreover, by Serre duality, $h^2(T_S) = h^0(\Omega_S^1 \otimes \Omega_S^2)$, therefore also

$$(5.1)\qquad M \leqslant 10\chi - 2K^2 + h^0\big(\Omega_S^1 \otimes \Omega_S^2\big),$$

and one wants to give an upper bound for $h^0(\Omega_S^1 \otimes \Omega_S^2)$.

Theorem C. *If* S *contains a smooth canonical curve, then* $M \leqslant 10\chi + q + 1$.

Proof. Let C be a smooth canonical curve. C is connected by Lemma 1 of [2] and therefore $h^0(\mathcal{O}_C) = 1$.

Consider now the exact sequences

$$(5.2)\qquad 0 \to \Omega_S^1 \to \Omega_S^1 \otimes \Omega_S^2 \to \Omega_S^1 \otimes \Omega_S^2 \otimes \mathcal{O}_C \to 0,$$

$$(5.3)\qquad 0 \to \mathcal{O}_C \to \Omega_S^1 \otimes \Omega_S^2 \otimes \mathcal{O}_C \to \mathcal{O}_C(3C) \to 0.$$

By the associated cohomology sequences we get

$$h^0\big(\Omega_S^1 \otimes \Omega_S^2\big) \leqslant q + 1 + h^0\big(\mathcal{O}_C(3C)\big) = q + 1 + 2K^2. \quad \text{q.e.d.}$$

In the general case, we are going to apply the following theorem of Bombieri [2, Theorem 2, p. 184].

Theorem. *The linear system* $|(m+1)K|$ *is free from basepoints if*:

(i) $m \geqslant 3$,

(ii) $m = 2, K^2 \geqslant 2, p_g \geqslant 1$ *or* $K^2 \geqslant 3$,
(iii) $m = 1, K^2 \geqslant 5, p_g \geqslant 3$ *or* $p_g \geqslant 3, q = 0$.

Theorem B. $M \leqslant 10\chi + 3K^2 + 108$.

Proof. We follow the same method used for Theorem C.

Let Γ be a smooth curve in $|(m+1)K|$. Again Γ is connected and we can consider the two exact sequences

$$(5.4) \qquad 0 \to \Omega^1_S(-mK) \to \Omega^1_S(K) \to \Omega^1_S(K) \otimes \mathcal{O}_\Gamma \to 0,$$

$$(5.5) \qquad 0 \to \mathcal{O}_\Gamma(-mK) \to \Omega^1_S(K) \otimes \mathcal{O}_\Gamma \to \mathcal{O}_\Gamma((m+3)k) \to 0.$$

Clearly,

$$h^0(\mathcal{O}_\Gamma((m+3)K)) = (m+3)(m+1)K^2 - \frac{(m+1)(m+2)}{2}K^2$$
$$= \frac{1}{2}(m+1)(m+4)K^2;$$

moreover, if $m \geqslant 1$, $H^0(\Omega^1_S(-mK)) = 0$ by Bogomolov's lemma (see e.g. [1], [25], [32]).

Therefore $h^0(\Omega^1_S \otimes \Omega^2_S) \leqslant (K^2/2)(m+1)(m+4)$ if $|(m+1)K|$ is free from basepoints, by Bertini's theorem.

Hence, in case (iii), $M \leqslant 10\chi + 3K^2$.

Now, if $p_g \leqslant 2$, then $\chi \leqslant 3$; hence $K^2 \leqslant 27$.

So, if (iii) does not apply, but (ii) does, $K^2 \leqslant 27$ and $M \leqslant 10\chi + 7K^2 \leqslant 10\chi + 3K^2 + 108$.

Finally, if (ii) does not apply, then either $K^2 \leqslant 2, p_g = 0$, or $K^2 = p_g = 1$. In the second case ([5], [6]) the number of moduli M equals $10\chi - 2K^2$, in the first we apply (i) with $m = 3$ to get $M \leqslant 10\chi + 12K^2 \leqslant 10\chi + 3K^2 + 18$.

It is natural to ask how sharp are these upper bounds, at least asymptotically as $K^2, \chi \to \infty$.

In this regard the bound given by Theorem C should be considered the more natural one although surfaces which do not possess a smooth canonical curve do not belong to a finite number of families. We show now that at least 4χ is needed.

Proposition 5.6. *Given any algebraic surface X, there exist bidouble covers S_j of X with the following properties*:

(i) $q(S_j) = q(X)$, *and, if X is simply connected, S_j is simply connected too.*

(ii) $M(S_j)/\chi(S_j) \to 4$.

Proof. Let H be a sufficiently very ample divisor on X such that $H^1(\mathcal{O}_X(-rH)) = 0$ for any $r \neq 0$ and also $h^i(T_X(-rH)) = 0$ for any $r > 0$, $i = 0, 1$.

Let D_1, D_2, D_3 be smooth transversal divisors with $D_1 \in |(2r+1)H|$, D_2, $D_3 \in |H|$.

We may further assume that H is not divisible by 2 in $\mathrm{Pic}(X)$; let then $L_1 \equiv H$, $L_2 = L_3 \equiv (r+1)H$, and consider the bidouble cover S of X associated to the L_i's, D_i's.

By (2.13), $h^1(\mathcal{O}_S) = h^1(\mathcal{O}_X)$, and, if $\pi_1(X) = 0$, S is simply connected by Proposition 2.7.

By (2.22),

$$\chi(\mathcal{O}_S) = 4\chi(\mathcal{O}_X) + \tfrac{1}{2}(2r+3)H \cdot K_X + \tfrac{1}{8}H^2\left[(2r+3)^2 + (2r+1)^2 + 2\right].$$

Moreover, since S is of general type, by (2.18) the natural deformations of S give a smooth subvariety of the base B of the Kuranishi family of S of dimension

$$h^0(\mathcal{O}_X(D_1)) + h^0(\mathcal{O}_X(D_1 - L_1)) - 1 - h^0(T_X)$$
$$= 2\chi(\mathcal{O}_X) - 1 + h^0(T_X) + \tfrac{1}{2}\left((2r+1)^2H^2 + (2r)^2H^2 - (4r+1)H \cdot K_X\right).$$

Therefore, $M(S) \geqslant 4r^2H^2 +$ (terms of lower order in r), while $\chi(\mathcal{O}_S) = H^2r^2 +$ (lower order terms).

6. Irregular surfaces without irrational pencils

In this last section, before proving Theorem D we want, on the one hand, to discuss the good properties shared by surfaces without irrational pencils, on the other hand we want to analyze the reasons why Castelnuovo's claim does not hold true.

Definition 6.1. An irrational pencil on S is a surjective morphism $f\colon S \to B$ where B is a curve of genus $g \geqslant 1$.

Remark 6.2. If B is a curve of genus $\geqslant 1$, any dominant rational map $f\colon S \to B$ is necessarily a morphism.

Let $\mathbf{P}(T_S) = W$ be the projective bundle of lines in the tangent bundle of S, so that $H^0(\Omega^1_S) \cong H^0(\mathcal{O}_W(1))$.

Let moreover $\alpha\colon S \to A$ be the Albanese map of S. We assume, in this paragraph, that $q = \dim A > 0$.

Remark 6.3. If S is irregular and has no irrational pencils, then $q(S) \geqslant 2$ and $\dim(\alpha(S)) = 2$.

If $A = \mathrm{Alb}(S)$ is simple, then S has no irrational pencils unless $\alpha(S)$ is a curve. In fact, by the universal property of the Albanese map, if $f\colon S \to B$ is an irrational pencil, and J is the Jacobian variety of B, f factors through α; hence $J \cong A$ and $\alpha(S) = B$.

We recall some known facts.

(6.4) (Castelnuovo-De Franchis, cf. [2], [0], [25]). Let η_1, η_2 be two independent sections of Ω^1_S, and assume that $\eta_1 \wedge \eta_2 \equiv 0$. Then there exists an irrational pencil $f\colon S \to B$ of genus $\geqslant 2$ and two holomorphic 1-forms φ_1, φ_2 on B such that $\eta_i = f^*(\varphi_i)$.

Definition 6.5. Let D be an effective divisor and η a section of $\Omega^1_S(D)$. A curve C is said to be an integral curve of η if η is in the kernel of the restriction map $r_C\colon H^0(\Omega^1_S(D)) \to H^0(\Omega^1_C(D))$.

(6.6) (Severi-Bogomolov, cf. [30], [0], [9]). If S has no irrational pencils and $\eta \in H^0(\Omega^1_S) - \{0\}$, η has only finitely many integral curves. If $\eta' \in H^0(\Omega^1_S(D))$ has infinitely many integrals, there exists a rational map $f\colon S \to B$ such that η' is a pullback of a rational 1-form on B.

(6.7) (Kodaira-Ramanujam-Bombieri). Let $a(C) = \dim\ker(r_C\colon H^0(\Omega^1_S) \to \Omega^1_C)$. Then [28, Remark, p. 48] $a(C) = h^1(\mathcal{O}_S(-C)) - h^0(\mathcal{O}_C) + 1$, and ([28], [2]) $a(C) = a(C_{\text{red}})$. If $a(C) > 0$, then $C^2 \leqslant 0$ ([2, Corollary p. 178] implies that if $C^2 > 0$, $|nC|$ for $n \gg 0$ is a movable linear system noncomposed of an irrational pencil, hence by Theorem 2 of [2] $h^1(\mathcal{O}_S(-C)) = 0$ and $a(C) = 0$). If S has no irrational pencils, $a(C) > 0 \Rightarrow \dim|nC| = 0$ for each $n > 0$.

Remark 6.8. Ramanujam [28] observes that $a(D) = \dim(\ker(r'_D\colon \mathrm{Pic}(S) \to \mathrm{Pic}(D)))$. If $a(D) = q$, then the Albanese map of S contracts the connected components of D to points, therefore, if S does not have an irrational pencil of genus q, and $D_1, \cdots, D_k$ are the irreducible (reduced) components of D, the intersection matrix $(D_i D_j)$ is negative definite. If, on the other hand, $0 < a(D) < q$, then $\mathrm{Alb}(S)$ is not simple and, if B is the Abelian variety dual to the connected component of the identity in $\ker(r'_D)$, B has positive dimension and there exists a surjective homomorphism $f\colon A \to B$, such that $f \circ \alpha(D)$ is a point. Clearly, $Y = f \circ \alpha(S)$ is not a point, and if S has no irrational pencils, then Y is a surface and again the intersection matrix $(D_i D_j)$ is negative definite.

In particular, if S has no irrational pencils and $a(C) \neq 0$, then $C^2 < 0$.

The same arguments of Remark 6.8. can be used to prove

Proposition 6.9. *Let η be a nonzero holomorphic 1-form, and assume that the surface S has no irrational pencils. Then the irreducible integral curves of η are numerically independent, hence their number is strictly less than the Picard number ρ of S ($\rho = \operatorname{rank}\mathrm{Num}(S)$).*

So, if S is an irregular surface without irrational pencils, there are few curves which are integrals of some holomorphic 1-form, and, a fortiori, few curves over which a nonzero η can vanish.

In the paper [30], Severi claimed that the 1-forms on S had no common zeros if S had no irrational pencil of genus equal to q. This is however almost always false, e.g. in the situation described in Proposition 5.6. In fact, if X is an irregular surface there without irrational pencils, the Albanese map of S_j factors through π: $S_j \to X$ and the Albanese map of X, hence also S_j has no irrational pencils; but every 1-form on S_j vanishes at the singular points of the ramification divisor R of π.

To get now a surface S as above where every 1-form vanishes on a curve, take an X where every 1-form vanishes at a point p, and a double cover ψ': $S' \to X$ such that $\Psi'^{-1}(p)$ is a rational double point of S': then let S be a minimal desingularization of S'.

We can again easily obtain that S has no irrational pencils, and that every 1-form on S is pullback of a 1-form on X; hence, if Ψ: $S \to X$ is the double cover and $\Psi(C) = p$, every 1-form vanishes on C.

Using these incorrect results of Severi, Castelnuovo in [4] claimed that if S has no irrational pencils, then $M(S) \leqslant p_g + 2q$. Proposition 5.6. shows that this is indeed false, anyhow we are going to state same results close in spirit to the one claimed by Castelnuovo.

Theorem 6.10. *Let $\mathcal{G}$ be the subsheaf of Ω^1_S generated by global sections, and assume that* $\operatorname{supp}(\Omega^1_S/\mathcal{G})$ *has dimension zero. Then $K^2 \geqslant 6\chi$, equality holding if and only if $\mathcal{G} = \Omega^1_S$ (in which case $q = 3$); moreover, $M \leqslant p_g + 3q - 3$.*

Proof. Let p: $W \to S$ be the natural projection of the projectivized tangent bundle. By our assumption the linear system $|\mathcal{O}_W(1)|$ has a base locus contained in a set of the form $p^{-1}(N)$, where N is a finite set of points in S.

By Bertini's theorem we can pick up three divisors D_1, D_2, D_3 in $|\mathcal{O}_W(1)|$ with the following properties:

(a) D_i contains only a finite number of vertical fibres (we can in fact assume D_i to be irreducible, so that p: $D_i \to S$ is birational).

(b) D_i is smooth outside $p^{-1}(N)$ and $D_i \cap D_j - p^{-1}(N)$ is a smooth and irreducible curve which projects to a curve via p.

(c) $D_1 \cap D_2 \cap D_3 \subset p^{-1}(N)$, in particular, if one can take $N = \varnothing$, then $D_1 \cap D_2 \cap D_3 = \varnothing$.

In terms of the three corresponding holomorphic 1-forms η_1, η_2, η_3 the above conditions mean:

(a) η_i has only isolated zeros.

(b) $\eta_i \wedge \eta_i = 0$ is an irreducible reduced canonical curve C.

(c) η_1, η_2, η_3 generate Ω^1_S outside N, and if $\mathcal{G} = \Omega^1_S$, η_1, η_2, η_3 generate Ω^1_S.

Consider now the two exact sequences

$$0 \to \mathcal{O}_S^2 \xrightarrow{\eta} \Omega^1_S \to \mathcal{F} \to 0, \tag{6.11}$$

$$0 \to \mathcal{O}_C \xrightarrow{\eta_3} \mathcal{F} \to \Delta \to 0. \tag{6.12}$$

Here η is given by (η_1, η_2) and the cokernel, $\mathscr{F}$, is supported on $C = \operatorname{div}(\eta_1 \wedge \eta_2)$.

$\mathscr{F}$ is a rank 1 torsion-free $\mathcal{O}_C$ sheaf, and the section η_3: $\mathcal{O}_S \to \Omega_S^1$ induces on the quotient the homomorphism (6.12), whose cokernel Δ has support contained in N ($\Delta = 0$ if $N = \varnothing$).

To prove the first assertion, which is equivalent to $c_1^2(\Omega_S^1) = c_1^2 \geqslant c_2 = c_2(\Omega_S^1)$, we consider the total Chern classes of the sheaves in question, which are multiplicative for exact sequences.

Since Δ has 0-dimensional support, $c(\Delta) = 1 - \text{length}(\Delta)$, but $c(\Delta)$ can also be expressed as

$$c(\mathscr{F}) \cdot c(\mathcal{O}_C)^{-1} = c(\Omega_S^1) \cdot c(\mathcal{O}_S(-C))$$
$$= (1 + K + c_2)(1 - K) = 1 - (K^2 - c_2).$$

Therefore $c_1^2 - c_2 = K^2 - c_2 = \text{length}(\Delta) \geqslant 0$, and, if equality holds, $\Delta = 0$. But then $h^0(\mathscr{F}) = 1$ by (6.12) and $h^0(\mathscr{F}) \geqslant q - 2$ by (6.11).

To prove the second assertion, it suffices to tensor the sequences (6.11), (6.12) by the invertible sheaf $\Omega_S^2 = \mathcal{O}_S(K)$.

Therefore

$$h^0(\Omega_S^1 \otimes \Omega_S^2) \leqslant 2p_g + h^0(\mathscr{F}(K)) \leqslant 2p_g + \text{length}(\Delta) + h^0(\mathcal{O}_C(K)).$$

By the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(K) \to \mathcal{O}_C(K) \to 0,$$

$h^0(\mathcal{O}_C(K)) \leqslant p_g - 1$ and, by the above expression for length(Δ) $(= 2K^2 - 12\chi)$, we obtain

$$M \leqslant h^1(T_S) \leqslant 10\chi - 2K^2 + 2p_g + 2K^2 - 12\chi + p_g + 1 - 1 = p_g + 3q - 3.$$

q.e.d.

We can sharpen the previous result: if $q \geqslant 3$ and one has a sequence like (6.11), then $h^0(\mathscr{F}) \geqslant 1$ and $\mathscr{F}$ is a rank-1 torsion free $\mathcal{O}_C$ sheaf if $C = \operatorname{div}(\eta_1 \wedge \eta_2)$ is a reduced curve. Therefore, if η_3 is a nonzero section of $\mathscr{F}$, one has also a sequence like (6.12) and supp(Δ) has dimension zero if furthermore C is irreducible.

The proof then goes on in exactly the same way, and, if equality holds $(K^2 = 6\chi)$, then $q = 3$ and the Albanese map α: $S \to A$ is unramified.

Thus we have

Theorem D. *If* $q \geqslant 3$ *and there exist two holomorphic* 1*-forms* η_1, η_2 *such that* $\operatorname{div}(\eta_1 \wedge \eta_2)$ *is a reduced irreducible curve, then* $M \leqslant p_g + 3q - 3$; *moreover,* $K^2 \geqslant 6\chi$, *equality holding if and only if the Albanese map is an unramified map into an Abelian* 3*-fold.*

In particular, the same conclusions hold if Ω^1_S *is generated by global sections outside a finite number of points.*

Question. If $q = 3$ and $\alpha: S \to A$ is unramified, then is it also an embedding?

Appendix: Letter of E. Bombieri written to the author on March 2, 1982

Here is a solution to your problem. You want to show that one can solve the equations

$$(*) \qquad uv = M, \quad wz - 2(u + v) = N$$

together with

$$(**) \qquad \frac{u+2}{3} < w < u - 4, \quad \frac{v+2}{3} < z < v - 4$$

in integers u, v, w, z and have at least k distinct values for $u + v$, for suitable $M, N \to \infty$.

Let ε be a small positive number, $0 < \varepsilon < 1 - 3^{-1/3}$. The first step consists in constructing an integer M_0 such that we have k distinct factorizations

$$u_i v_i = M_0, \qquad i = 1, \cdots, k,$$

with

$$(1 - \varepsilon) M_0^{1/2} < u_i < v_i < (1 - \varepsilon)^{-1} M_0^{1/2}.$$

This is easy to do and we can choose for example $M_0 = 6^m$ for sufficiently large m.

In the next step we choose a sequence $w_1 < w_2 < \cdots < w_k$ of large prime numbers, with $w_k/w_1 < (1 - \varepsilon)^{-1}$; by the Prime Number Theorem, we can do this with k and w_1 arbitrarily large. Now we can use the Chinese Remainder Theorem and construct N_0 such that

$$N_0 \equiv -2(u_i + v_i) \pmod{w_i}$$

for $i = 1, \cdots, k$. Let also $N_\nu = N_0 + w_1 \cdots w_k \nu$ be a solution to the above system of congruences and let us write $N_\nu = w_i z_i - 2(u_i + v_i)$ for $i = 1, \cdots, k$.

The set $(M_0, N_\nu; u_i, v_i, w_i, z_i)$ satisfies $(*)$ for $i = 1,\cdots,k$ but it does not usually satisfy the inequalities $(**)$. This is done as follows:

Since

$$(1-\varepsilon)M_0^{1/2} < u_i, \quad v_i < (1-\varepsilon)^{-1}M_0^{1/2}, \quad 1 < w_i/w_1 < (1-\varepsilon)^{-1},$$

we have

$$\frac{z_i}{v_i} > \frac{N_\nu}{w_i v_i} > (1-\varepsilon)\frac{N_\nu}{w_1 v_i} > (1-\varepsilon)^2\frac{N_\nu}{w_1 M_0^{1/2}},$$

and similarly we have

$$\frac{z_i}{v_i} < (1-\varepsilon)^{-1}\frac{N_\nu + 8M_0^{1/2}}{w_1 M_0^{1/2}}, \quad (1-\varepsilon)\frac{w_1}{M_0^{1/2}} < \frac{w_i}{u_i} < (1-\varepsilon)^{-2}\frac{w_1}{M_0^{1/2}}.$$

We deduce that if $3(1-\varepsilon)^3 > 1$ then we can find two integers R_ν, S_ν such that

$$\frac{z_i}{v_i} < R_\nu < 3\frac{z_i}{v_i}, \quad \frac{w_i}{u_i} < S_\nu < 3\frac{w_i}{u_i},$$

for $i = 1,\cdots,k$ and all large ν.

Let $T_\nu = R_\nu S_\nu$. Then the data

$$M(\nu) = T_\nu^2 M_0, \quad N(\nu) = T_\nu N_\nu, \quad u_{i,\nu} = T_\nu u_i,$$
$$v_{i,\nu} = T_\nu v_i, \quad w_{i,\nu} = R_\nu w_i, \quad z_{i,\nu} = S_\nu z_i$$

satisfy both $(*)$ and $(**)$ for $i = 1,\cdots,k$ and all large ν, as we wanted.

References

[0] F. A. Bogomolov, *Families of curves on a surface of general type*, Soviet Math. Dokl. **18** (1977) 1294–1297.

[1] ______, *Holomorphic tensors and vector bundles on projective varieties*, Math. U.S.S.R. Izv. **13** (1979) 499–555.

[2] E. Bombieri, *Canonical models of surfaces of general type*, Inst. Hautes Études Sci. Publ. Sci. **42** (1973) 171–219.

[3] E. Bombieri & D. Husemoller, *Classification and embeddings of surfaces* (Algebraic Geometry Arcata 1974), Proc. Sympos. Pure Math., Vol. 29, Amer. Math. Soc., Providence, RI, 1975, 329–420.

[4] G. Castelnuovo, *Sul numero dei moduli di una superficie irregolare*. I, II, Rend. Accad. Lincei **7** (1949) 3–7, 8–11.

[5] F. Catanese, *Surfaces with $K^2 = p_g = 1$ and their period mapping* (Algebraic Geometry Copenhagen 1978), Lecture Notes in Math., Vol. 732, Springer, Berlin, 1979, 1–26.

[6] ______, *The moduli and the global period mapping of surfaces with $K^2 = p_g = 1$: a counter-example to the global Torelli problem*, Compositio Math. **41** (1980) 401–414.

[7] F. Catanese & O. Debarre, *Surfaces with $K^2 = 2, p_g = 1, q = 0$*, to appear.

[8] O. Debarre, *Inegalités numeriques pour les surfaces de type general*, Bull. Soc. Math. France **110** (1982) 319–346.

[9] M. Deschamps, *Courbes de genre geometrique borné sur une surface de type general (d'après F. A. Bogomolov)*, Sem. Bourbaki 500, Feb. 1977, 1–12.

[10] M. H. Freedman, *The topology of four dimensional manifolds*, J. Differential Geometry **17** (1982) 357–453.

[11] D. Gieseker, *Global moduli for surfaces of general type*, Invent. Math. **43** (1977) 233–282.

[12] J. Harris & D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982) 23–86.

[13] F. Hirzebruch, W. D. Neumann & S. S. Koh, *Differentiable manifolds and quadratic forms*, Marcel Dekker, New York, 1971.

[14] E. Horikawa, *On deformations of quintic surfaces*, Invent. Math. **31** (1975) 43–85.

[15] K. Kodaira, *A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds*, Ann. of Math. **75** (1962) 146–162.

[16] ______, *On the structure of compact complex analytic surfaces*. I, Amer. J. Math. **86** (1964) 781–798.

[17] ______, *On homotopy K3 surfaces*, Essays on Topology and Related Topics, Springer, New York, 1970, 58–69.

[18] K. Kodaira & J. Morrow, *Complex manifolds*, Holt-Rinehart-Winston, New York, 1971.

[19] K. Kodaira & D. C. Spencer, *A theorem of completeness for complex analytic fibre spaces*, Acta Math. **100** (1958) 281–294.

[20] M. Kuranishi, *New proof for the existence of locally complete families of complex structures* (Proc. Conf. Complex Analysis, Minneapolis) Springer, Berlin, 1965, 142–154.

[21] R. Mandelbaum & B. Moishezon, *On the topology of algebraic surfaces*, Trans. Amer. Math. Soc. **260** (1980) 195–222.

[22] H. Matsumura, *On algebraic groups of birational transformations*, Rend. Accad. Lincei Ser. 8, **34** (1963) 151–155.

[23] ______, *Commutative algebra*, Benjamin, New York, 1970.

[24] J. Milnor, *On simply connected 4-manifolds* (Sympos. Internac. Topologia Alg., Mexico 1956), 1958, 122–128.

[25] Y. Miyaoka, *On the Chern numbers of surfaces of general type*, Invent. Math. **42** (1977) 225–237.

[26] S. P. Novikov, *Homotopically equivalent smooth manifolds*. I, Amer. Math. Soc. Transl. (2) **48** (1965) 271–396.

[27] U. Persson, *Double coverings and surfaces of general type* (Proc. Conf. Alg. Geom. Trömso), Lecture Notes in Math. Vol. 687, Springer, Berlin, 1978, 168–195.

[28] C. P. Ramanujam, *Remarks on the Kodaira vanishing theorem*, J. Indian Math. Soc. **36** (1972) 41–51; supplement in J. Indian Math. Soc. **38** (1974) 121–124.

[29] J. P. Serre, *Cours d'arithmetique*, Presse Universitaire de France, Paris, 1970.

[30] F. Severi, *La serie canonica e la teoria delle serie principali di gruppi di punti sopra una superficie algebrica*, Comment. Math. Helv. **4** (1932) 268–326.

[31] L. Siebenmann, *La conjecture de Poincaré topologique en dimension* 4 (*d'après M. H. Freedman*), Sem. Bourbaki 588, Feb. 1982, 1–30.

[32] A. Van de Ven, *Some recent results on surfaces of general type*, Sem. Bourbaki 500, Feb. 1977, 1–12.

[33] C. T. C. Wall, *On simply connected 4-manifolds*, Proc. London Math. Soc. **39** (1964) 141–149.

[34] J. J. Wavrik, *Obstructions to the existence of a space of moduli*, Global Analysis, Princeton Math. Series **29** (1969) 403–414.

[35] S. T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977) 1798–99.

[36] O. Zariski, *On the purity of the branch locus of algebraic functions*, Proc. Nat. Acad. Sci. U.S.A. **44** (1958) 791–796.

UNIVERSITÀ DI PISA, ITALIA