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THE GAUSS MAP OF SURFACES IN $\mathbf{R}^n$

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1. Introduction

Let S be an oriented 2-dimensional surface immersed in a euclidean n-space $\mathbf{R}^n$. We denote by

$$g\colon S \to G_{2,n} \tag{1.1}$$

the generalized Gauss map, where $G_{2,n}$ is the Grassmannian of oriented 2-planes in $\mathbf{R}^n$, and for each point p of S, $g(p)$ is the tangent plane to S at p. Our object in this paper is to study properties of the map g, particularly those related to the geometry of S in $\mathbf{R}^n$ and the conformal structure of S.

The main problems we consider are:

1. Let S_0 be a Riemann surface, and

$$X\colon S_0 \to S \subset \mathbf{R}^n \tag{1.2}$$

a conformal immersion realizing S. What properties does the map

$$G = g \circ X\colon S_0 \to G_{2,n} \tag{1.3}$$

possesses by virtue of being defined via (1.1) as the Gauss map of a surface in $\mathbf{R}^n$?

2. Given a map

$$G\colon S_0 \to G_{2,n} \tag{1.4}$$

defined on a Riemann surface S_0, when does there exist a conformal immersion X of S_0 onto a surface S in $\mathbf{R}^n$ such that G is of the form (1.3), where g is the Gauss map of S?

3. To what extent is a surface S given by (1.2) determined by its Gauss map g?

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To set the background for our results we review some known facts about special classes of Gauss maps.

(a) A surface X: $S_0 \to \mathbf{R}^n$ is a minimal surface if and only if G is anti-holomorphic. The map X is not determined by G. In fact, the same map G: $S_0 \to G_{2,n}$ arises as the Gauss map of infinitely many different minimal surfaces (see Chern and Osserman [4, §1]).

(b) The Gauss map g of a surface in $\mathbf{R}^n$ is holomorphic if and only if the surface lies on a euclidean 2-sphere in some 3-dimensional subspace of $\mathbf{R}^n$ (Hoffman and Osserman [9, Theorem 1.1]).

(c) The Gauss map of a surface in $\mathbf{R}^n$ is harmonic if and only if the surface has parallel mean curvature vector field (Ruh and Vilms [**17**]). By a result of Kenmotsu [14], if S_0 is a simply-connected Riemann surface, and G: $S_0 \to G_{2,3} = S^2$ is a harmonic mapping which is nowhere anti-holomorphic, then G is given by (1.3) for some conformal immersion X: $S_0 \to S \subset \mathbf{R}^3$ where S has constant nonzero mean curvature; the surface S is uniquely determined up to homothety and translation.

(d) The work of Kenmotsu referred to above was motivated by the Weierstrass-Enneper representation theorem for minimal surfaces (see [15, §8]). Kenmotsu considers the following problem: Given a Riemann surface S_0, a real-valued function H, and a map G: $S_0 \to S^2 = G_{2,3}$, when does there exist a conformal immersion X: $S_0 \to \mathbf{R}^3$ with mean curvature H such that G can be written in the form (1.3)? He derives a single second-order equation involving G and H, which is necessarily satisfied by the Gauss map and mean curvature of a conformally immersed surface in $\mathbf{R}^3$. For nowhere anti-conformal maps G this condition turns out to be sufficient to guarantee the existence of a conformal immersion X: $S_0 \to \mathbf{R}^3$ with the prescribed G and H, provided S_0 is simply-connected. (The simple connectivity is required as in the Weierstrass-Enneper formula because the map X is produced by an integral formula involving G and H, where the integrals are taken over paths, and there may be periods over curves not homologous to zero.) The map X is unique up to homothety and translation.

From the preceding facts it is evident that the situation for minimal surfaces is quite different from the general case. In fact we shall see that in many ways minimal surfaces are completely atypical.

Among the results we obtain are the following.

(A) Suppose G: $S_0 \to G_{2,n}$ arises in the form (1.3) from the Gauss map g of a surface in $\mathbf{R}^n$. Let E be the set of points in S_0 where $d\overline{G}$ is complex-linear, and let $E' = S_0 \setminus E$. On E, G satisfies a system of first-order differential equations (the Cauchy-Riemann equations), while on E' a more complicated pair of integrability conditions must be satisfied:

(i) From the fact that the mean curvature vector is nonzero on E' comes a requirement that a certain complex n-vector field V, constructed from G, be a real n-vector field times a complex function, say f.

(ii) The argument of f, say α, must satisfy an equation of the form

$$\Delta\alpha = \beta, \tag{1.5}$$

where β is a first-order differential expression derived from G. Since α can be explicitly written in terms of the first derivatives of G, (1.5) is a third-order differential equation satisfied by G (Theorem 2.3).

(B) **Existence.** Let G be a smooth map of S_0 into $G_{2,n}$. We may then define the subsets E and E' of S_0 as in (A) above, since the definition depends only on the map G. If D is a simply-connected subdomain of S_0 lying in E, then $\overline{G}$ is holomorphic on D, and we may define a minimal surface S in $\mathbf{R}^n$ by a map X: $D \to \mathbf{R}^n$ such that $G = g \circ X$. If $D \subset E'$, then there exists a surface defined by a conformal immersion X: $D \to \mathbf{R}^n$ such that $G = g \circ X$, provided that both the necessary conditions (i) and (ii) described in (A) above are satisfied in D (Theorem 2.6).

Thus the conditions (i) and (ii) above may be considered as the necessary and sufficient conditions in the generic case for a given map G to arise as a Gauss map in the manner described. If the domain D contains points of both E and E', then it is more difficult to give explicit conditions to guarantee the existence of a regular surface with Gauss map G. We can formulate sufficient conditions in that case, but they are not of the same directly verifiable nature as those given in (A).

(C) **Uniqueness.** Suppose X: $S_0 \to \mathbf{R}^n$ is a conformally immersed surface with Gauss map G, and suppose the mean curvature vector of X does not vanish identically. Then X is determined uniquely by G, up to translation and homothety (Theorem 2.5).

Corollary. *Let S_1 and S_2 be two surfaces in $\mathbf{R}^n$, such that at least one of them has a point where the mean curvature vector is not zero. Let f: $S_1 \to S_2$ be a conformal map. If S_1 and S_2 have the same tangent planes at each pair of points p and $f(p)$, then f is the restriction of a translation and homothety of $\mathbf{R}^n$. If at some point p of S_1, either df_p is an isometry, or the mean curvature of S_1 at p and that of S_2 at $f(p)$ are equal and nonzero, then f is a translation.*

Remarks. 1. This corollary generalizes a result of Abe and Erbacher [1, Theorem on p. 197, part (D)]. They prove that in the case $n = 3$, if f is a global isometry and the mean curvature does not vanish, then f is a translation.

2. The assumption that the mean curvature be not identically zero is necessary; not only is the conclusion false for minimal surfaces in $\mathbf{R}^n$, but in $\mathbf{R}^3$ essentially the exact reverse is true. (See the discussion following the proof of

Theorem 2.5 below.) Also the well-known example of the catenoid and helicoid (see, for example, [6, p. 213]) shows that there can exist an isometric map between minimal surfaces, preserving the tangent planes, which is not a congruence.

(D) A surface in $\mathbf{R}^n$ with nonvanishing mean curvature may be represented by its Gauss map (Theorem 2.7). This is essentially a generalization of the Kenmotsu representation in $\mathbf{R}^3$ which was mentioned above. In fact specializing our results to $n = 3$, we can produce new proofs of the theorems of Kenmotsu (see [11]). We mention here that our point of view is somewhat different from his in that we ask what can be determined from the Gauss map alone, whereas he prescribes both the Gauss map and the mean curvature function H. However, as we have seen, the Gauss map determines the whole surface up to similarilty whenever $H \not\equiv 0$, and it therefore determines the function H up to a constant factor.

(E) It is in the case $n = 4$ that our results provide the most new information. In that case the Grassmannian $G_{2,4}$ is isometrically and biholomorphically equivalent to the product $S_1 \times S_2$ of two 2-spheres, and the map $G: S_0 \to G_{2,4}$ may be represented by a pair of maps g_1, g_2 of S_0 into the factors S_1, S_2. The two necessary conditions of (A) take a particularly simple form ((3.7) and (3.8) below) and may be directly applied to a number of geometric problems:

(i) Chern [3] proved that S is minimal if and only if both g_1 and g_2 are anti-holomorphic. It follows from (3.7) that for an arbitrary surface in $\mathbf{R}^4$ it is impossible to have one of the g_k anti-holomorphic and the other not; i.e., if *either* g_1 or g_2 is anti-holomorphic, then they both are, and the corresponding surface is minimal.

(ii) The map $G = (g_1, g_2)$ is harmonic if and only if each g_i is harmonic as a map into S^2. Hence by the theorem of Ruh-Vilms mentioned above, a surface S has parallel mean curvature if and only if g_1 and g_2 are harmonic. Moreover, either g_1 and g_2 are both everywhere anti-conformal (corresponding to S minimal) or else each g_i is nowhere anti-conformal. Conversely, given two harmonic maps g_1, g_2' from S_0 into S^2 which are either everywhere or nowhere anti-conformal, the necessary and sufficient conditions for $G = (g_1, g_2)$ to be a Gauss map reduce to one equation, (3.7). This equation is not always satisfied by a pair of harmonic maps and therefore the situation is radically different from that in $\mathbf{R}^3$ where, by Kenotsu's theorem (see (C) above), every nowhere anti-conformal harmonic map is realizable as the Gauss map of a surface.

(iii) The condition (3.7) on the $\bar{z}$-spherical derivatives of g_1 and g_2 mentioned above is a key ingredient in proving the following theorem (Hoffman, Osserman and Schoen [12]): *Let S be a complete oriented surface in $\mathbf{R}^4$ with parallel nonzero mean curvature vector field. Then neither g_1 nor g_2 can take values in an* *open hemisphere. If either $g_1(S)$ or $g_2(s)$ lies in a closed hemisphere, then S must be either a right circular cylinder in some $\mathbf{R}^3 \subset \mathbf{R}^4$, or else a product of circles.*

The following related results will be discussed in a subsequent paper [11].

(a) If we assume that $G = (g_1, g_2)$ is both harmonic and conformal and that neither g_1 nor g_2 is anti-holomorphic, then the existence theorem produces a minimal surface in some $S^3 \subset \mathbf{R}^4$. Thus we obtain a representation theorem for minimal surfaces in S^3 analogous to the classical Weierstrass-Enneper integral representation theorem for minimal surfaces in $\mathbf{R}^3$. Walter Seaman [19, §1.4] has shown further that to each harmonic map g_1 corresponds a unique harmonic g_2 such that the pair (g_1, g_2) is the Gauss map of a minimal surface in S^3, provided that g_1 is nowhere anti-conformal.

(b) Certain subsets of the Grassmannian can never occur as the image under the Gauss map of any surface in $\mathbf{R}^4$. An example is any set of the form $p \times C$ where p is a point of S_1 and C is a curve in S_2 (or, more generally, C is a subset of S_2 which is neither a point nor an open set).

For related results, see Weiner [21, §5].

(c) New proofs of classical theorems of Blaschke [2] and Chern and Spanier [5] (see also Weiner [21]), concerning the area of the image of the projections g_1, g_2 of the Gauss map, are given. Namely, if J_1 and J_2 are the Jacobians of g_1 and g_2 respectively, then

$$K = J_1 + J_2, \qquad K^N = J_1 - J_2,$$

where K and K^N are the Gauss and normal curvatures, respectively, of the surface in $\mathbf{R}^4$. From these formulas it follows that for a compact surface S in $\mathbf{R}^4$

$$\int_S K\, dA = 2\pi(d_1 + d_2), \quad \int_S K^N dA = 2\pi(d_1 - d_2), \qquad d_j = \deg(g_j).$$

Using the Gauss-Bonnet formula, it follows that

$$\chi(S) = \frac{1}{2\pi}\int_S K\, dA = d_1 + d_2.$$

Furthermore, $\chi(S) = 2d_1 = 2d_2$ if and only if $\int_S K^N dA = 0$. This last equation is implied by a variety of geometrically interesting conditions among which are: S is embedded; S lies on a three-sphere in $\mathbf{R}^4$.

We note as one final consequence of our investigation, the emergence of a certain third-order operator defined on complex functions of a complex variable, whose properties seem to be of considerable interest. The operator, denoted here by $T(f)$, is defined at points where $f_{\bar z} \neq 0$ by the formula

$$T(f) = \frac{\partial}{\partial \bar z}\left(\frac{f_{z\bar z}}{f_{\bar z}} - \frac{2\bar f f_z}{1 + |f|^2}\right).$$

The vanishing of the imaginary part of $T(f)$ is essentially equivalent to f being the stereographic projection of the Gauss map of a conformally immersed surface in $\mathbf{R}^3$, [11], while the vanishing of the real part is the precise condition for the validity of a certain formula proved earlier for the special case of harmonic maps by Schoen and Yau [18]. For other results on the operator $T(f)$, see Theorem 3.1 and [11].

2. Surfaces in $\mathbf{R}^n$

We take as model for the Grassmannian of oriented two-planes in $\mathbf{R}^n$ the quadric Q_{n-2} in $\mathbf{C}P^{n-1}$ defined by the equation

$$\sum_{k=1}^{n} z_k^2 = 0. \tag{2.1}$$

For any point $P = (z_1,\cdots,z_n)$ on Q_{n-2}, if $z_k = a_k + ib_k$, we obtain a pair of real vectors $A = (a_1,\cdots,a_n)$ and $B = (b_1,\cdots,b_n)$ which satisfy

$$|A| = |B|, \qquad A \cdot B = 0, \tag{2.2}$$

these equations being equivalent to (2.1). Furthermore, A and B cannot be zero since homogeneous coordinates $(z_1,\cdots,z_n)$ of a point in $\mathbf{C}P^{n-1}$ are not all zero. Hence the pair A, B form an orthogonal basis of an oriented two-plane Π. Different homogeneous coordinates for P determine the same plane. Conversely, for any two-plane Π we may choose an orthogonal basis A, B of Π and set $z_k = a_k + ib_k$, obtaining a point P of Q_{n-2}. (For more details on this, see Hoffman and Osserman [9].)

We shall use the notation $P = [z]$ to denote the point in $\mathbf{C}P^{n-1}$ corresponding to the point $z = (z_1,\cdots,z_n)$ in $\mathbf{C}^n \backslash \{0\}$.

By a *surface S in* $\mathbf{R}^n$, we shall mean a pair (S_0, X), where S_0 is a Riemann surface and $X\colon S_0 \to \mathbf{R}^n$ is a locally conformal map. Thus all our surfaces are oriented. Of course, that is no restriction for local results or for simply-connected surfaces, and using the standard two-sheeted covering of a non-orientable surface one can generally translate global results from oriented surfaces to nonorientable ones.

Note also that our surfaces are all immersed, and therefore have a well-defined tangent plane at each point. If $z = \xi + i\eta$ is a local parameter on the Riemann surface S_0, and $(x_1,\cdots,x_n)$ are coordinates in $\mathbf{R}^n$, then the map defining S is given locally in the form

$$X(z), \qquad X = (x_1,\cdots,x_n). \tag{2.3}$$

The conformality of the map is expressed by

$$\left|\frac{\partial X}{\partial \xi}\right| = \left|\frac{\partial X}{\partial \eta}\right| \neq 0, \qquad \frac{\partial X}{\partial \xi}\cdot\frac{\partial X}{\partial \eta} = 0. \tag{2.4}$$

The tangent plane to S, spanned by $\partial X/\partial\xi$, $\partial X/\partial\eta$, corresponds as above to the point

$$\left[\frac{\partial X}{\partial \xi} + i\frac{\partial X}{\partial \eta}\right] \tag{2.5}$$

of the quadric Q_{n-2}.

The *Gauss map* G of S is the map of S_0 into Q_{n-2} defined locally by (2.5). Using the complex derivatives

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial \xi} - i\frac{\partial f}{\partial \eta}\right), \qquad \frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial \xi} + i\frac{\partial f}{\partial \eta}\right), \tag{2.6}$$

we may write the Gauss map in the form

$$G(z) = \left[\frac{\partial X}{\partial \bar z}\right]. \tag{2.7}$$

For historical reasons we work with the complex conjugate of this map

$$\bar G = \left[\frac{\partial X}{\partial z}\right] = \left[\frac{\partial X}{\partial \xi} - i\frac{\partial X}{\partial \eta}\right], \tag{2.8}$$

whose image also lies in Q_{n-2}. From (2.4) we have

$$ds^2 = \lambda^2|dz|^2 \quad \text{with } \lambda^2 = \left|\frac{\partial X}{\partial \xi}\right|^2 = \left|\frac{\partial X}{\partial \eta}\right|^2 = 2\left|\frac{\partial X}{\partial z}\right|^2, \tag{2.9}$$

where

$$\left|\frac{\partial X}{\partial z}\right|^2 = \sum_{k=1}^{n}\left|\frac{\partial x_k}{\partial z}\right|^2 = \frac{1}{4}\sum\left|\frac{\partial x_k}{\partial \xi} - i\frac{\partial x_k}{\partial \eta}\right|^2. \tag{2.10}$$

We also have the relationship

$$\left(\frac{4}{\lambda^2}\right)X_{z\bar z} = \Delta X = 2H, \tag{2.11}$$

where H is the mean curvature vector field of X, and Δ is the Laplace-Beltrami operator on S (see, for example, [15, §4]).

We may now formulate our basic question as follows. *Given a Riemann surface S_0, how can one characterize among all maps of S_0 into Q_{n-2} those which arise in the manner described as Gauss maps of surfaces S in $\mathbf{R}^n$?*

To answer the question, we will start by deriving necessary conditions on such a map, and then describe the extent to which they are also sufficient. We will also discuss uniqueness; that is, the degree to which s is determined by its Gauss map.

Locally, we may view the problem as simply one of existence and uniqueness of integrating factors. That is to say, given a map of S_0 into Q_{n-2}, we may represent it locally in the form $[\Phi]$, where $\Phi(z) = (\varphi_1, \cdots, \varphi_n) \in \mathbf{C}^n \setminus \{0\}$ satisfies

$$\sum_{k=1}^{n} \varphi_k^2(z) = 0. \tag{2.12}$$

We then look for $X(z) = (x_1, \cdots, x_n)$, such that (2.8) is satisfied. But that means

$$\frac{\partial X}{\partial z} = \psi \Phi \tag{2.13}$$

for some function ψ: $S_0 \to \mathbf{C}$. We note that the surface is regular wherever ψ does not vanish. From (2.11) and (2.13) we have

$$\frac{\lambda^2}{2} H = X_{z\bar{z}} = (\psi \Phi)_{\bar{z}} = \psi \Phi_{\bar{z}} + \psi_{\bar{z}} \Phi.$$

Using (2.9) we have $\frac{1}{2}\lambda^2 = |\partial X / \partial z|^2 = |\psi|^2 |\Phi|^2$, whence

$$|\Phi|^2 \bar{\psi} H = \Phi_{\bar{z}} + (\log \psi)_{\bar{z}} \Phi, \tag{2.14}$$

wherever $\psi \neq 0$.

Our first results follow from (2.14) by the application of two elementary lemmas.

Lemma 2.1. *Let D be a simply-connected domain in $\mathbf{C}$, and let Φ: $D \to \mathbf{C}^n$ be a C^1-mapping. Necessary and sufficient that there exist a mapping X: $D \to \mathbf{R}^n$ such that $X_z = \Phi$ is that Φ satisfy* $\operatorname{Im} \Phi_{\bar{z}} = 0$.

Remark. Lemma 2.1 holds also with the roles of z and $\bar{z}$ reversed.

Lemma 2.2. *Let W be a vector in $\mathbf{C}^n$ of the form $W = A + iB$, where A and B are nonzero real vectors satisfying* (2.2). *Let Π be the plane spanned by A and B. For any vector C in $\mathbf{R}^n$, let C^Π be the projection of C on Π. For any pair of vectors C, D in $\mathbf{R}^n$, let $Z = C + iD \in \mathbf{C}^n$, and define $Z^\Pi = C^\Pi + iD^\Pi$. Then*

$$Z^\Pi = \langle Z, W \rangle W / |W|^2 + \langle Z, \overline{W} \rangle \overline{W} / |W|^2,$$

where $\langle \, , \rangle$ denotes the usual Hermitian product on $\mathbf{C}^n$.

We apply Lemma 2.2 with $W = \Phi$ so that Π is the tangent plane to our surface, and with $Z = \Phi_{\bar{z}}$. We may write (2.12) as $\Phi \cdot \Phi \equiv 0$, and hence

$$0 = (\Phi \cdot \Phi)_{\bar{z}} = 2\Phi \cdot \Phi_{\bar{z}} = 2\langle \Phi_{\bar{z}}, \overline{\Phi} \rangle.$$

Thus by Lemma 2.2,

$$(\Phi_{\bar{z}})^\Pi = \eta \Phi, \tag{2.15}$$

where we denote

$$\eta = \Phi_{\bar z} \cdot \overline{\Phi}/|\Phi|^2. \tag{2.16}$$

Further, we denote by V the component of $\Phi_{\bar z}$ orthogonal to Π, so that

$$V = \Phi_{\bar z} - (\Phi_{\bar z})^{\Pi} = \Phi_{\bar z} - \eta\Phi. \tag{2.17}$$

Since the mean curvature vector H is orthogonal to the tangent plane Π, we obtain two equations by taking the tangent and normal components of (2.14):

$$|\Phi|^2 \bar\psi H = V, \tag{2.18}$$

$$(\log \psi)_{\bar z} = -\eta. \tag{2.19}$$

We are now in a position to formulate our first result.

Theorem 2.3. *Let S be a surface in $\mathbf{R}^n$ given locally by a conformal map X: $D \to \mathbf{R}^n$. Let Φ be the Gauss map of S in the sense of* (2.7) *and* (2.13). *Form the quantities η and V from Φ by* (2.16) *and* (2.17) *Then for every $z \in D$, we have*

$$V(z) = e^{i\alpha(z)}R(z), \tag{2.20}$$

where $R(z)$ is a real vector; furthermore, on the set where $V(z) \neq 0$, the function $\alpha(z)$ is uniquely defined modulo π, and satisfies

$$\alpha_{z\bar z} = \operatorname{Im}\{\eta_z\}. \tag{2.21}$$

Proof. Set $\rho = |\psi|$, and write

$$\psi = \rho e^{-i\alpha}, \tag{2.22}$$

where α is uniquely defined modulo 2π when $\rho \neq 0$. Then (2.18) becomes $V = \rho|\Phi|^2 e^{i\alpha}H$, which proves (2.20). When $V \neq 0$, we must have $\rho \neq 0$, and substituting (2.22) in (2.19) yields

$$(\log \rho)_{\bar z} - i\alpha_{\bar z} = -\eta, \tag{2.23}$$

so that

$$(\log \rho)_{z\bar z} = i\alpha_{z\bar z} - \eta_z. \tag{2.24}$$

Equation (2.21) follows from the fact that $(\log \rho)_{z\bar z}$ and $\alpha_{z\bar z}$ are both real. This proves Theorem 2.3.

There are several remarks to be made concerning the statement of Theorem 2.3.

First, the vanishing of the vector V at a point is equivalent, on the one hand by (2.18), to the vanishing of the mean curvature vector H, and on the other hand by (2.17), to the condition that $(\varphi_j/\varphi_k)_{\bar z} = 0, j = 1,\cdots,n$, for any k such that $\varphi_k \neq 0$. This latter condition means that the differential of the map Φ is complex linear at the point. One consequence is the result of Chern [3] that Φ is a holomorphic map if and only if S is minimal. For our purposes, however, the equivalence of the three pointwise conditions will be more important. Note that we may view the vanishing of V as a special case, and consider that generically $V \neq 0$ and (2.21) must be satisfied.

Second, when $V \neq 0$, condition (2.20) is equivalent to

$$(2.20') \qquad \operatorname{Im}\left\{v_j(z)\overline{v_k(z)}\right\} = 0, \qquad j = 1,\cdots,n, j \neq k,$$

for any fixed k with $v_k(z) \neq 0$, where $V = (v_1,\cdots,v_n)$. The $n-1$ equations of (2.20′) are not independent, since we know that $V \cdot \Phi = V \cdot \overline{\Phi} = 0$, and hence V lies in a space spanned by $n-2$ independent real vectors. But (2.20) (or (2.20′)) requires V to be a multiple of a single real vector, and hence involves $n-3$ independent conditions. In fact, (2.20) is no constraint in $\mathbf{R}^3$ (see [11]), and it reduces to a single equation in $\mathbf{R}^4$ (equation (3.7)).

A map into $G_{2,n}$ consists locally of $2(n-2)$ functions of a complex parameter z, while a conformal immersion consists of n real-valued functions of z which satisfy the two equations in (2.4). As pointed out to us by M. Gromov, one would therefore expect $2(n-2)-(n-2) = n-2$ independent conditions on a map into $G_{2,n}$ in order for it to be a Gauss map. Combining (2.20) and (2.21) gives precisely $n-2$ conditions.

We next note the important fact that the conditions (2.20) and (2.21) are purely statements about the Gauss map G; that is, they are expressed via (2.16) and (2.17) in terms of the components of a representation Φ of G in homogeneous coordinates, but they are independent of the particular representation. Specifically, one has the following lemma.

Lemma 2.4. *Given a map Φ: $D \to \mathbf{C}^n\backslash\{0\}$, set $\hat{\Phi} = f\Phi$ where f is a smooth nonvanishing complex function. Use* (2.16) *and* (2.17) *to define the quantities η, V in terms of Φ, and the corresponding qualities $\hat{\eta}$, $\hat{V}$ in terms of $\hat{\Phi}$. Then*

$$(2.25) \qquad \hat{V} = fV,$$

and on the set where V and $\hat{V}$ are nonzero, the functions α, $\hat{\alpha}$ defined by (2.20) *satisfy*

$$(2.26) \qquad \hat{\alpha}_{z\bar{z}} - \operatorname{Im}\{\hat{\eta}_z\} = \alpha_{z\bar{z}} - \operatorname{Im}\{\eta_z\}.$$

Proof. If we set $f = re^{i\theta}$, it follows directly from the definition that

$$(2.27) \qquad \hat{\eta} = \eta + f_{\bar{z}}/f = \eta + (\log f)_{\bar{z}} = \eta + (\log r)_{\bar{z}} + i\theta_{\bar{z}},$$

while on the set where $V \neq 0$,

$$(2.28) \qquad \hat{\alpha} = \alpha + \theta.$$

The result follows easily.

We may note that Lemma 2.4 can be used to give an alternative proof of Theorem 2.3. Namely, by (2.13) we can let ψ play the role of f in Lemma 2.4, and set

$$\partial X/\partial z = \hat{\Phi}. \tag{2.29}$$

Then by (2.11),

$$\hat{\Phi}_{\bar{z}} = \frac{\lambda^2}{2} H. \tag{2.30}$$

Since H is a real normal vector, it follows that

$$\hat{\eta} \equiv 0, \tag{2.31}$$

$$\hat{V} = \frac{\lambda^2}{2} H. \tag{2.32}$$

Thus $\hat{\alpha} \equiv 0$, so that (2.20) and (2.21) follow trivially for $\hat{\Phi}$ from (2.31) and (2.32). But by (2.25) and (2.26) they must also hold for Φ.

We next explore the extent to which a conformally immersed surface is determined by its Gauss map. For minimal surfaces, the answer is well known (see, for example, Chern and Osserman [4]). Any oriented minimal surface S in $\mathbf{R}^n$ may be represented by a conformal map X: $S_0 \to \mathbf{R}^n$, where $X = \operatorname{Re}\{(\int \alpha_1, \cdots, \int \alpha_n)\}$, and $\alpha_1, \cdots, \alpha_n$ are holomorphic differentials on the Riemann surface S_0. The Gauss map is determined by Φ: $S_0 \to Q_{n-2}$, where locally $\Phi = (\varphi_1, \cdots, \varphi_n)$, with $\alpha_k = \varphi_k\, dz$. If $\psi(z)$ is a nonvanishing holomorphic function on S_0, then setting $\hat{\Phi} = \psi(\varphi_1, \cdots, \varphi_n)$ defines the same Gauss map and corresponds to the surface $\hat{X} = \operatorname{Re}(\int \hat{\alpha}_1, \cdots, \int \hat{\alpha}_n)$, with $\hat{\alpha}_k = \psi \alpha_k$. Thus the set of all minimal surfaces with the same Gauss map as S is in one-to-one correspondence with the set of nonvanishing holomorphic functions ψ on S_0.

It turns out that minimal surfaces are totally idiosyncratic in this respect. Every other surface is essentially uniquely determined by its Gauss map.

Theorem 2.5. *Let a surface S be defined by a conformal immersion X: $S_0 \to \mathbf{R}^n$ of a Riemann surface S_0. If the mean curvature of S is different from zero at some point, then S is determined up to a similarity transformation of $\mathbf{R}^n$ by its Gauss map G; specifically, if Y: $S_0 \to \mathbf{R}^n$ is another conformal immersion inducing the same Gauss map G: $S_0 \to Q_{n-2}$, then there exist a real constant c and a constant vector X_0 such that $X = cY + X_0$.*

Proof. Given the maps X, Y: $S_0 \to \mathbf{R}^n$, set $Y_z = \Phi$, in terms of a local conformal parameter z on S_0. Setting

$$\alpha_k = \varphi_k\, dz = \frac{\partial y_k}{\partial z}\, dz,$$

we see that the α_k are globally defined differentials of type $(1,0)$ on S_0. Similarly, we may define

$$\beta_k = \hat{\varphi}_k\, dz = \frac{\partial x_k}{\partial z}\, dz.$$

The fact that X and Y induce the same Gauss map is equivalent to the existence of a nonvanishing complex function ψ on S_0 such that $\beta_k = \psi\alpha_k$, $k = 1,\cdots,n$. In terms of any local coordinate z, this means $\partial x_k/\partial z = \psi\varphi_k$, or $X_z = \psi\Phi$. We may define η and V in terms of Φ by (2.16) and (2.17). From (2.19) we have $(\log\psi)_{\bar z} = -\eta$. But since Y satisfies the equation $Y_z = \hat{\psi}\Phi$, $\hat{\psi} \equiv 1$, we also have $-\eta = (\log\hat{\psi})_{\bar z} \equiv 0$. It follows that $\log\psi$ is a holomorphic function of z, and hence ψ is a global holomorphic function on S_0. Since $X_z = \psi Y_z$, we see that two surfaces with the same Gauss map are always related by a holomorphic function, just as in the special case of minimal surfaces, described above. We next note that by (2.18), applied to the two surfaces, we have

$$|\Phi|^2\bar{\psi}H = V = |\Phi|^2\bar{\hat{\psi}}\hat{H}.$$

Suppose now we have a point z_0 where $H \neq 0$. It then follows that $V \neq 0$ and $\hat{H} \neq 0$ at the point, and of course, the same is true in some neighborhood U of z_0. But since $\hat{\psi} \equiv 1$, V is a nonzero real vector at each point of U, and hence ψ is real on U. But a holomorphic function which is real on an open set must be constant. Hence $\psi \equiv c$, a real constant, and in each parameter neighborhood, $X_z = cY_z$. It follows that $X - cY$ is a constant vector, which proves the theorem.

Corollary 1. *Let S and $\hat{S}$ be two surfaces immersed in $\mathbf{R}^n$, not both minimal. Suppose there is a conformal map $f\colon S \to \hat{S}$ which preserves tangent planes:*

$$T_pS = T_{f(p)}\hat{S} \quad \textit{for all } p \in S.$$

Then f is the restriction to S of a similarity transformation of $\mathbf{R}^n$.

Corollary 2. *Under the hypotheses of Corollary* 1, *if for some point $p \in S$, either df_p is an isometry, or else the mean curvature of S at p and that of $\hat{S}$ at $f(p)$ are equal and nonzero, then f must be a translation*; *in particular S and $\hat{S}$ are congruent.*

We note that Corollary 1 is basically a statement of Theorem 2.5. Corollary 2 follows immediately, since each of the further assumptions implies that the constant c in the theorem must equal 1.

As noted in the introduction, a weaker form of Corollary 2 for surfaces in $\mathbf{R}^3$ was proved by Abe and Erbacher [1], who also obtained an analogous result for hypersurfaces in $\mathbf{R}^n$ (see also Weiner [20], [21] for other uniqueness results).

Concerning Corollary 1, we note that not only is the result not true for minimal surfaces, as remarked earlier, but in a sense the exact opposite is true for minimal surfaces in $\mathbf{R}^3$. Essentially *any* two minimal surfaces in $\mathbf{R}^3$ are related by a conformal map preserving tangent planes. In fact, let S, $\hat{S}$ be any two minimal surfaces immersed in $\mathbf{R}^3$, and let $p \in S$, $\hat{p} \in \hat{S}$ be any pair of points where the Gauss curvature is different from zero. (Recall that the Gauss curvature can vanish at most at isolated points on a minimal surface.) Make a preliminary rotation so that the tangent planes T_pS and $T_{\hat{p}}\hat{S}$ coincide. Then the Gauss maps G: $S \to \Sigma$ and $\hat{G}$: $\hat{S} \to \Sigma$ are conformal maps in neighborhoods of p and $\hat{p}$, where Σ is the unit sphere. It follows that the map $f = \hat{G}^{-1} \circ G$ is a conformal map of some neighborhood of p on S onto a neighborhood of $\hat{p}$ on $\hat{S}$, and by its definition the map f preserves tangent planes at every point.

We now turn to the question of the extent to which the two necessary conditions (2.20), (2.21) for a map Φ to be a Gauss map are also sufficient. We note that (2.20) must hold everywhere and is trivial when $V = 0$, while (2.21) is only meaningful on the set where $V \neq 0$, so that α is well defined.

Theorem 2.6. *Let S_0 be a simply-connected Riemann surface, and let G: $S_0 \to Q_{n-2}$ be a map into the complex quadric. Represent G locally by a map Φ into $\mathbf{C}^n \backslash \{0\}$ in the sense that*

$$\bar{G} = [\Phi]. \tag{2.33}$$

Define η and V in terms of Φ by (2.16) *and* (2.17).

Case 1. *If $V \equiv 0$ on S_0, then G is the Gauss map of a minimal surface in $\mathbf{R}^n$ provided S_0 is not compact.*

Case 2. *If V never vanishes on S_0, then a sufficient condition for G to be the Gauss map of a surface S given by a conformal map X: $S_0 \to \mathbf{R}^n$ is that the necessary conditions* (2.20), (2.21) *of Theorem* 2.3 *be satisfied, where $R(z)$ is a nonzero real vector for each z.*

Remark. By Lemma 2.4, the conditions (2.20), (2.21), as well as the condition $V = 0$ of Case 1, even through expressed in terms of Φ, depend only on G (i.e., the projective class of Φ).

Proof. In both cases we have to solve both a local problem and a global problem. The local problem may be stated as follows: let D be a domain on S parametrized by the disk $|z| < 1$, such that on D the map G may be represented as in (2.33) by

$$\Phi(z) = (\varphi_1(z), \cdots, \varphi_n(z)).$$

We want a map X: $D \to \mathbf{R}^n$ and a function ψ: $D \to \mathbf{C}\backslash\{0\}$, satisfying (2.13). We have seen that the function ψ must satisfy (2.19). Note that this last equation always has a nonvanishing solution. We may first solve for Ψ the equation $\Psi_{\bar z} = -\eta$ in the disk $|z| < 1$ (see, for example, [13, Theorem 1.2.2]), and then set $\psi = e^{\Psi}$. We now treat the two cases separately.

Case 1. The condition $V \equiv 0$ means (by (2.17)) that $\Phi_{\bar z} \equiv \eta\Phi$. Using a solution ψ of (2.19), we obtain

$$(\psi\Phi)_{\bar z} = \psi\Phi_{\bar z} + \psi_{\bar z}\Phi = \psi(\Phi_{\bar z} + (\log\psi)_{\bar z}\Phi) = \psi(\Phi_{\bar z} - \eta\Phi) \equiv 0.$$

Thus all the components $\psi\varphi_k$ of the map $\hat\Phi = \psi\Phi$ are holomorphic in D, and Case 1 corresponds precisely to the condition that the map $G\colon S_0 \to Q_{n-2}$ be anti-holomorphic. Hence it is locally the Gauss map of a minimal surface S defined by

$$X = \operatorname{Re}\int \hat\Phi\, dz = \operatorname{Re}\int \psi\Phi\, dz.$$

To solve the global problem in Case 1, we note that not all the functions φ_k can vanish identically; say $\varphi_1 \not\equiv 0$. Then $\hat\varphi_1 = \psi\varphi_1$ is holomorphic and has well defined zeros of well-defined orders (independent of the particular representation Φ of $\bar G$). Assuming S_0 is not compact, it is conformally either the disk or the plane. We may then find a globally defined holomorphic function f on S_0 whose zeros coincide, in position and order, with those of φ_1; that is, whenever $\bar G$ is represented locally by Φ, f/φ_1 is smooth and nonzero. Furthermore, the functions φ_k/φ_1 are globally defined meromorphic functions on S_0. It follows that the map $\bar G$ may be globally represented on S_0 by

$$\tilde\Phi = \left(f, f\frac{\varphi_2}{\varphi_1}, \cdots, f\frac{\varphi_n}{\varphi_1}\right).$$

The surface $X = \operatorname{Re}\int \tilde\Phi\, dz$, where S_0 corresponds to $|z| < R \leqslant \infty$, solves the global problem in Case 1.

Case 2. We start again with the local problem. Writing the integration factor as $\psi = \rho e^{i\theta}$, (2.19) takes the form

$$-\eta = (\log\psi)_{\bar z} = (\log\rho)_{\bar z} + i\theta_{\bar z}. \tag{2.34}$$

By Lemma 2.1 we can solve (2.13) for X in the simply-connected domain D provided

$$\operatorname{Im}\{\psi\Phi\}_{\bar z} \equiv 0. \tag{2.35}$$

But using (2.19), (2.17), and (2.20), we find

$$\begin{aligned}\operatorname{Im}\{\psi\Phi\}_{\bar z} &= \operatorname{Im}\{\psi\Phi_{\bar z} + \psi_{\bar z}\Phi\} = \operatorname{Im}\{\psi(\Phi_{\bar z} - \eta\Phi)\}\\ &= \operatorname{Im}\{\rho e^{i\theta}V\} = \rho\operatorname{Im}\{e^{i(\theta+\alpha)}\}R,\end{aligned}$$

where R is a real vector. Thus (2.35) is equivalent to

$$\theta + \alpha \equiv 0 \pmod{\pi}. \tag{2.36}$$

We may therefore write (2.34) as

$$(\log \rho)_{\bar{z}} = -\eta - i\theta_{\bar{z}} = -\eta + i\alpha_{\bar{z}}. \tag{2.37}$$

Note that the right-hand side is determined by Φ on the set where $V \neq 0$. Again by Lemma 2.1 we can solve (2.37) for ρ in D provided

$$0 = \operatorname{Im}\{(-\eta + i\alpha_{\bar{z}})_z\} = -\operatorname{Im}\{\eta_z\} + \alpha_{z\bar{z}},$$

which is precisely (2.21). Thus the condition $V \neq 0$ allows one to define the function α by (2.20) which determines θ by (2.36) and ρ via (2.37) in such a fashion that the function $\psi = \rho e^{i\theta}$ satisfies (2.34) and the integrability condition (2.35). Hence the local problem has a solution.

To obtain the global solution in Case 2, we note first that the condition $V \neq 0$ means (by (2.18)) that the surface obtained locally via the argument above has nonvanishing mean curvature. It then follows from Theorem 2.5 that the surface so obtained is unique up to similarity transformation. Thus if we fix any point p_0 of S_0, and define a surface S in a neigborhood of p_0 with the prescribed Gauss map, then S can be uniquely continued along every path starting at p_0 by piecing together local solutions. Finally, since S_0 is simply connected, the monodromy theorem guarantees that the resulting surface is independent of path and is globally well defined over S_0. This proves the theorem.

An obvious question which remains is what can be said concerning existence, if the vector V constructed from the map Φ vanishes somewhere, but not identically? To answer, we note first that on any connected component of the set S' where $V \neq 0$, we have by unique continuation a well-defined surface if and only if there are no "periods" which arise from multiple connectivity. Next, this surface must extend smoothly to points where $V = 0$ on the boundary of S'. Finally, if the complement of S' has a nonempty interior, one must be able to choose an integrating factor ψ in a way that involves finding a holomorphic function whose boundary values are prescribed in terms of the given surface on the closure of S'. The desired surface S exists if and only if all these conditions can be met, but clearly they are not of a directly verifiable nature as in the two cases of Theorem 2.6.

As a final outcome of our approach, we give a representation theorem for surfaces in $\mathbf{R}^n$ whose mean curvature vector is nonzero. This representation may be viewed as complementing the one we have given earlier [9, Theorem 3.1] for minimal surfaces in $\mathbf{R}^n$. In a sense, it may also be considered as a generalization of the representation theorem of Kenmotsu [13] for surfaces in $\mathbf{R}^3$, the main difference being that Kenmotsu uses both the Gauss map and the mean curvature function, whereas, as a consequence of our uniqueness theorem above, we may consider the representation to be described totally by the Gauss map.

Theorem 2.7. *Let S be a simply-connected surface in $\mathbf{R}^n$ whose mean curvature vector is never zero, and let the Gauss map of S not Q_{n-2} be represented locally by a map $\Phi(z)$ into $\mathbf{C}^n\backslash\{0\}$ in terms of a conformal parameter z; that is,*

$$[\Phi(z)] = \left[\frac{\partial X}{\partial z}\right]. \tag{2.38}$$

If we write (2.38) *in the form*

$$\frac{\partial X}{\partial z} = \psi(z)\Phi(z), \tag{2.39}$$

where ψ is a nonvanishing complex function, then ψ is determined up to a real multiplicative constant by Φ, and the map $X(z)$ is determined up to homothety and translation by (2.39); *that is, the gradient of each coordinate function x_k is given by* (2.39), *and hence x_k is determined up to an additive constant. We may write*

$$x_k = \int \operatorname{Re}\{2\psi\phi_k\, dz\}, \qquad k = 1,\cdots,n, \tag{2.40}$$

where the integrals are taken from a fixed point to a variable point and are independent of path. Furthermore, if we start with an arbitrary map of a simply-connected Riemann surface S_0 into the Grassmannian, represented by a map Φ into $\mathbf{C}^n\backslash\{0\}$, and Φ satisfies the necessary conditions of Theorem 2.3 *for a Gauss map with $V \neq 0$, then there is a nonvanishing complex function ψ uniquely determined by Φ up to a multiplicative constant such that the differentials* $\operatorname{Re}\{\psi\phi_k\, dz\}$, $k = 1,\cdots,n$, *are all exact, and equations* (2.40) *define a conformal immersion of S_0 into $\mathbf{R}^n$ as a surface S whose Gauss map is given by Φ.*

Proof. The principal observation is that given any Φ satisfying the conditions (2.20), (2.21) of a Gauss map with $V \neq 0$, then there is an essentially unique function ψ making the differentials on the right of (2.40) exact, and, furthermore, ψ may be explicitly determined from Φ. Namely, from Φ we construct η and V by (2.16) and (2.17). Then ψ is of the form $\rho e^{-i\alpha}$, where α is the argument of any nonzero component of V, by virtue of (2.20), and $\log \rho$ is determined up to an additive constant by the fact that its gradient is explicity given in terms of α and η by (2.37). Then the proof of Theorem 2.6 shows that there is a real vector function X satisfying (2.39), or equivalently (2.40).

There are several remarks to be made concerning the representation (2.40).

First, if the ϕ_k are holomorphic functions, so that $[\Phi]$ represents a holomorphic map into Q_{n-1}, and if ψ is an arbitrary holomorphic function, then (2.40) represents a minimal surface in $\mathbf{R}^n$. In that case, we may write it in the more familiar form

$$x_k = \operatorname{Re} \int 2\psi\phi_k \, dz, \tag{2.41}$$

since the integrals in question are independent of path. In the general case, the integrals in (2.41) are not path independent, but their real parts are, which allows for the representation (2.40).

Second, since the surface S in the above theorem is essentially determined by its Gauss map, one should be able to determine associated geometric quantities, such as the mean curvature vector H. In fact, H is easily given, since by (2.18) we have

$$H = V/\bar{\psi}|\Phi|^2, \tag{2.42}$$

where ψ and V are determined by Φ. Viewing the same equation in the other direction, if H is assumed known, or given in advance, then the integrating factor ψ can be described explicitly from H and Φ (without integrations) by

$$\bar{\psi} = v_k/h_k|\Phi|^2, \tag{2.43}$$

where h_k is any nonzero component of H, and v_k the corresponding component of V. For surfaces in $\mathbf{R}^3$ we may write $H = hN$ where N is the unit normal, and h the scalar mean curvature function. Then one may write ψ explicitly from (2.43) in terms of h and the Gauss map. That is the approach of Kenmotsu [13].

3. Surfaces in $\mathbf{R}^4$

We begin with some basic facts concerning $Q_2 \subset \mathbf{CP}^3$. The map φ from $\mathbf{C} \times \mathbf{C}$ given by

$$\varphi\colon (w_1, w_2) \mapsto (1 + w_1w_2, i(1 - w_1w_2), w_1 - w_2, -i(w_1 + w_2)) \tag{3.1}$$

has the property that $\varphi^2 = \Sigma_{k=1}^4 \varphi_k^2 = 0$. Hence $[\varphi]$ takes values in $Q_2 = \{[Z] \in \mathbf{CP}^3 \,|\, Z^2 = 0\}$. On $\varphi(\mathbf{C} \times \mathbf{C})$, φ^{-1} is given by

$$(z_1, z_2, z_3, z_4) \mapsto (w_1, w_2) = \left(\frac{z_3 + iz_4}{z_1 - iz_2}, \frac{-z_3 + iz_4}{z_1 - iz_2}\right). \tag{3.2}$$

It can be shown that $[\varphi]$ is a biholomorphic map from $\mathbf{C} \times \mathbf{C}$ into Q_2 which, when we consider $(w_1, w_2) \in \mathbf{C} \times \mathbf{C}$ as homogeneous coordinates on $\mathbf{CP}^1 \times \mathbf{CP}^1$, extends to a biholomorphic map of $\mathbf{CP}^1 \times \mathbf{CP}^1$ onto Q_2. If we consider $\mathbf{CP}^3$ with the Fubini-Study metric of constant holomorphic curvature 2, the induced metric on Q_2, expressed in terms of the coordinates (w_1, w_2), has the form

$$ds^2 = \frac{2|dw_1|^2}{\left(1+|w_1|^2\right)^2} + \frac{2|dw_2|^2}{\left(1+|w_2|^2\right)^2},$$

from which we conclude that Q_2 is the product of two standard spheres of constant Gauss curvature equal to 2. (For details, see Hoffman and Osserman, [9, §2].)

We now use our earlier results to derive explicit necessary conditions on the Gauss map of an oriented immersed surface in $\mathbf{R}^4$. Let $S \subset \mathbf{R}^4$ be such a surface whose Gauss map is given by $G\colon S \to Q_2$. We write Q_2 as a product of spheres $Q_2 = S_1 \times S_2$, where each S_k is a standard sphere of radius $1/\sqrt{2}$. Denote by π_k the projection of Q_2 onto S_k, $k = 1, 2$. Using the (extended) complex variable w_k to parametrize S_k, where w_k is given in terms of homogeneous coordinates on $\mathbf{C}P^3$ by (3.2), the conjugate $\bar{G}$ of the Gauss map may be expressed in terms of a local conformal parameter z on S by the pair of functions $w_1 = f_1(z)$, $w_2 = f_2(z)$. We may then write

$$\bar{G}(z) = [\Phi(z)], \tag{3.3}$$

where by (3.1),

$$\begin{aligned}\Phi(z) &= \varphi(f_1(z), f_2(z)) \\ &= \big(1 + f_1(z)f_2(z), i(1 - f_1(z)f_2(z)), f_1(z) - f_2(z), \\ &\qquad -i(f_1(z) + f_2(z))\big).\end{aligned} \tag{3.4}$$

In order to state the main theorem of this section, we need to introduce certain auxiliary functions derived from the functions f_1, f_2 describing the Gauss map:

$$F_i = \frac{(f_i)_{\bar z}}{1+|f_i|^2}, \quad \hat{F}_i = \frac{(f_i)_z}{1+|f_i|^2}, \qquad i = 1, 2, \tag{3.5}$$

and on the set $\{z \,|\, (f_i)_{\bar z} \neq 0\}$,

$$T_i = \left[(f_i)_{z\bar z}/(f_i)_{\bar z} - 2\bar{f}_i\hat{F}_i\right]_{\bar z}, \qquad i = 1, 2. \tag{3.6}$$

Theorem 3.1. *Let S be an oriented surface immersed in $\mathbf{R}^4$ whose Gauss map G is given locally by* (3.4) *via a pair of functions $f_1(z)$, $f_2(z)$, where z is a local conformal parameter on S. Then*

$$|F_1| \equiv |F_2|, \tag{3.7}$$

$$\operatorname{Im}\{T_1 + T_2\} = 0 \tag{3.8}$$

wherever $(f_1)_{\bar z} \neq 0$.

Remark. By straightforward computations it can be shown that $|F_k|$ and T_k are smooth functions, even at points where $w_k = \infty$, and also that (3.7) and (3.8) are independent of the choice of local parameter z. Furthermore, since our arguments here are all local, we may make a preliminary orthogonal transformation of $\mathbf{R}^4$ so that f_1 and f_2 are both finite in a given neighborhood. In such a neighborhood, the conditions $(f_i)_{\bar{z}} = 0$ is equivalent to $F_i = 0$, and by (3.7), $(f_1)_{\bar{z}} = 0$ if and only if $(f_2)_{\bar{z}} = 0$.

Proof. We apply Theorem 2.3. We first express the function

$$\eta = \Phi_{\bar{z}} \cdot \overline{\Phi}/\|\Phi\|^2 \tag{3.9}$$

and the complex vector-valued function

$$V = \Phi_{\bar{z}} - \eta\Phi \tag{3.10}$$

in terms of the functions f_j and F_j. Since

$$\Phi_{\bar{z}} = (f_1)_{\bar{z}}(f_2, -if_2, 1, -i) + (f_2)_{\bar{z}}(f_1, -if_1, -1, -i) \tag{3.11}$$

and

$$\|\Phi\|^2 = 2\left(1 + |f_1|^2\right)\left(1 + |f_2|^2\right), \tag{3.12}$$

it follows that

$$\eta = \bar{f}_1 F_1 + \bar{f}_2 F_2. \tag{3.13}$$

We now define a complex vector-valued function $A(z)$ by

$$A = \left(f_2 - \bar{f}_1, -i(f_2 + \bar{f}_1), 1 + \bar{f}_1 f_2, -i(1 - \bar{f}_1 f_2)\right), \tag{3.14}$$

and verify directly that

$$V = F_1 A - F_2 \overline{A}. \tag{3.15}$$

From (3.14) it is immediate that $A^2 \equiv 0$. Moreover,

$$\|A\|^2 = 2\left(1 + |f_1|^2\right)\left(1 + |f_2|^2\right) = \|\Phi\|^2, \tag{3.16}$$

so that A can never vanish. Letting $V = (V_1, \cdots, V_4)$ and $A = (A_1, \cdots, A_4)$ we compute

$$\begin{aligned} V_j \overline{V}_k &= \left(F_1 A_j - F_2 \overline{A}_j\right)\left(\overline{F}_1 \overline{A}_k - \overline{F}_2 A_k\right) \\ &= |F_1|^2 A_j \overline{A}_k + |F_2|^2 \overline{A}_j A_k - 2\operatorname{Re}\left(F_1 \overline{F}_2 A_j A_k\right) \\ &= \left(|F_1|^2 + |F_2|^2\right)\operatorname{Re}\left(A_j \overline{A}_k\right) - 2\operatorname{Re}\left(F_1 \overline{F}_2 A_j A_k\right) \\ &\quad + i\left(|F_1|^2 - |F_2|^2\right)\operatorname{Im}\left(A_j \overline{A}_k\right). \end{aligned} \tag{3.17}$$

It follows from (3.17) and (3.16) (and the fact that $A^2 = \Sigma_{i=j}^4 A_j^2 \equiv 0$) that

$$\|V\|^2 = \sum_{j=1}^4 V_j\bar{V}_j = \left(|F_1|^2 + |F_2|^2\right)\|A\|^2 \tag{3.18}$$
$$= 2\left(|F_1|^2 + |F_2|^2\right)\left(1 + |f_1|^2\right)\left(1 + |f_2|^2\right).$$

From (3.18) we can conclude that $V(z) = 0$ if and only if $F_1(z) = F_2(z) = 0$ or equivalently $(f_1)_{\bar{z}}(z) = (f_2)_{\bar{z}}(z) = 0$. In particular, wherever $V(z) = 0$, (3.7) is satisfied. The condition (2.20) of Theorem 2.3, namely that V be equal to a real vector-valued function multiplied by a nonzero complex function wherever $V(z) \neq 0$, is equivalent to the requirement that the 4×4 Hermitian matrix $V^T\bar{V}$ be a real matrix. The components $V_j\bar{V}_k$ are given in (3.17), and it follows from (3.17) that $V^T\bar{V}$ is a real matrix if and only if

$$\left(|F_1|^2 - |F_2|^2\right)\mathrm{Im}\left(A_j\bar{A}_k\right) = 0, \qquad j, k = 1,\cdots,4. \tag{3.19}$$

Clearly (3.7) implies (3.19).

We will show that (3.19) implies (3.7). Assume (3.7) fails at some point $z_0 \in D$ where $V(z_0) \neq 0$. Then at z_0 (3.19) is equivalent, since $V^T\bar{V}$ is Hermitian symmetric, to the six equations

$$\mathrm{Im}\left(A_j(\bar{z}_0)\bar{A}_k(z_0)\right) = 0, \qquad 1 \leqslant j < k \leqslant 4. \tag{3.20}$$

We will now use (3.14) to show that (3.20) is not possible. We first have from (3.14),

$$\mathrm{Im}\left(A_1\bar{A}_2\right) = |f_2|^2 - |f_1|^2, \quad \mathrm{Im}\left(A_3\bar{A}_4\right) = 1 - |f_1|^2|f_2|^2. \tag{3.21}$$

Hence (3.20) implies that

$$|f_1(z_0)| = |f_2(z_0)| = 1. \tag{3.22}$$

Using (3.22) together with (3.14) we can write the other four equations of (3.20) in the form

$$\begin{aligned}
\mathrm{Im}\left(A_1\bar{A}_3\right) &= 2\,\mathrm{Im}(f_1 + f_2) = 0,\\
\mathrm{Im}\left(A_1\bar{A}_4\right) &= 2\,\mathrm{Re}(f_2 - f_1) = 0,\\
\mathrm{Im}\left(A_2\bar{A}_3\right) &= -2\,\mathrm{Re}(f_1 + f_2) = 0,\\
\mathrm{Im}\left(A_2\bar{A}_4\right) &= 2\,\mathrm{Im}(f_2 - f_1) = 0 \quad \text{at } z_0.
\end{aligned} \tag{3.23}$$

But (3.23) implies $f_1(z_0) + f_2(z_0) = f_2(z_0) - f_1(z_0) = 0$. This clearly contradicts (3.22). We conclude that (3.19) implies (3.7), and hence (3.19) is equivalent to (3.7) wherever $V(z) \neq 0$. This concludes the proof of the first part of the theorem.

In order to establish (3.8) we first derive an expression for the argument of V. Wherever $V(z) \neq 0$ we may write as in (2.20),

$$V(z) = e^{i\alpha(z)}R(z), \tag{3.24}$$

where $R(z)$ is a real nonzero vector. From (3.15) and (3.7) we deduce

$$\bar{F}_1\bar{F}_2V = |F_1|^2\bar{F}_2A - |F_2|^2\bar{F}_1\bar{A} = |F_1|^2(\bar{F}_2A - \bar{F}_1\bar{A}) = -|F_1|^2\bar{V}. \tag{3.25}$$

Since we are assuming that $V \neq 0$, we have some nonzero component V_j of V, which satisfies, by (3.25),

$$\bar{F}_1\bar{F}_2V_j = -|F_1|^2\bar{V}_j. \tag{3.26}$$

By (3.24), $\arg V_j = \alpha \pmod \pi$. Also we have seen from (3.18), $V \neq 0$ implies $F_1 \neq 0$ and $F_2 \neq 0$. By (3.5),

$$\arg F_k = \arg(f_k)_{\bar{z}}, \qquad k = 1,2. \tag{3.27}$$

Thus (3.26) implies

$$-(\arg f_1)_{\bar{z}} - (\arg f_2)_{\bar{z}} + \alpha = -\alpha \pmod \pi,$$

or

$$\alpha = \frac{1}{2}[(\arg f_1)_{\bar{z}} + (\arg f_2)_{\bar{z}}] \quad \left(\operatorname{mod} \frac{\pi}{2}\right). \tag{3.28}$$

From (3.28) one has

$$\begin{aligned}\alpha_{z\bar{z}} &= \frac{1}{2}\operatorname{Im}\left\{\left[\frac{(f_1)_{z\bar{z}}}{(f_1)_{\bar{z}}}\right]_{\bar{z}} + \left[\frac{(f_2)_{z\bar{z}}}{(f_2)_{\bar{z}}}\right]_{\bar{z}}\right\} \\ &= \frac{1}{2}\operatorname{Im}\{T_1 + T_2\} + \operatorname{Im}\{(\bar{f}_1\hat{F}_1)_{\bar{z}} + (f_2F_2)_{\bar{z}}\},\end{aligned} \tag{3.29}$$

where we have used the definition (3.6) of T_1, T_2. On the other hand, a computation shows that

$$(\bar{f}_k\hat{F}_k)_{\bar{z}} - (\bar{f}_kF_k)_z = |\hat{F}_k|^2 - |F_k|^2,$$

and therefore

$$\operatorname{Im}\{(\bar{f}_k\hat{F}_k)_{\bar{z}}\} = \operatorname{Im}\{(\bar{f}_kF_k)_z\}, \qquad k = 1,2. \tag{3.30}$$

Finally, from (3.13),

$$\operatorname{Im}\{\eta_z\} = \operatorname{Im}\{(\bar{f}_1F_1)_z + (\bar{f}_2F_2)_z\}. \tag{3.31}$$

Combining (3.29), (3.30), (3.31), we obtain

$$\alpha_{z\bar{z}} - \operatorname{Im}\{\eta_z\} = \frac{1}{2}\operatorname{Im}\{T_1 + T_2\}. \tag{3.32}$$

But our basic integrability condition (2.21) asserts that the left-hand side of (3.32) vanishes. We now see that for surfaces in $\mathbf{R}^4$ condition (2.21) reduces to (3.8). This completes the proof of the theorem.

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