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FOLIATIONS AND THE TOPOLOGY OF 3-MANIFOLDS

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1. Introduction

Given a compact connected oriented 3-manifold M with boundary ∂M, when does there exist a codimension-1 transversely oriented foliation $\mathcal{F}$ which is transverse to ∂M and has no Reeb components? If such an $\mathcal{F}$ exists, then ∂M necessarily is a (possibly empty) union of tori and M is either $S^2 \times S^1$ (and $\mathcal{F}$ is the product foliation) or irreducible. The first condition follows by Euler characteristic reasons while the latter follows from the work of Rosenberg [24] extending the work of Reeb [23] and Novikov [21]. Our main result says that such conditions are sufficient when $H_2(M, \partial M) \neq 0$.

If such a foliation $\mathcal{F}$ exists on M, then it follows from the work of Thurston [32] that any compact leaf L is a Thurston norm minimizing surface for the class $[L] \in H_2(M, \partial M)$. Our main result says that for a 3-manifold M satisfying the above necessary conditions any norm minimizing surface can be realized as a compact leaf of a foliation without Reeb components.

Theorem 5.5. *Let M be a compact connected irreducible oriented 3-manifold whose boundary ∂M is a (possibly empty) union of tori. Let S be any norm minimizing surface representing a nontrivial class $z \in H_2(M, \partial M)$. Then there exists foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ of M such that:*

(1) *for $i = 0, 1$, $\mathcal{F}_i \pitchfork \partial M$ and $\mathcal{F}_i | \partial M$ has no Reeb components,*

(2) *every leaf of $\mathcal{F}_0$ and $\mathcal{F}_1$ nontrivially intersects a closed transverse curve,*

(3) *S is a compact leaf of both $\mathcal{F}_0$ and $\mathcal{F}_1$,*

(4) *$\mathcal{F}_0$ is of finite depth,*

(5) *$\mathcal{F}_1$ is C^∞ except possibly along toral components of S.*

We now state some corollaries of the theorem.

Corollary 6.2. *Let L be an oriented nonsplit link in S^3. Then S is a surface of minimal genus for L if and only if there exists a C^∞ transversely oriented foliation*

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$\mathcal{F}$ of $S^3 - \mathring{N}(L)$ such that

(1) $\mathcal{F} \pitchfork \partial N(L)$ and $\mathcal{F}$ and $\mathcal{F} | \partial N(L)$ have no Reeb components,

(2) S is a compact leaf.

The $\Rightarrow$ direction follows from Theorem 5.5 and Thurston [32] proved the converse. Simple methods for explicitly constructing foliations for certain classes of knots and links, thereby computing their genera, can be found in [5], [6] and [7].

Corollary 6.5. *A nontrivial link L in S^3 is nonsplit if and only if L is the set of cores of Reeb components of some foliation $\mathcal{F}$ of S^3.*

The $\Rightarrow$ direction follows from Theorem 5.5. Novikov [21] proved the converse in 1965. We therefore answer the so-called "Reeb placement problem" of Laudenbach and Roussarie [16] who asked which links could be realized as cores of Reeb components of foliations of S^3. The holonomy of our foliations along the toral leaves is in general C^0. The C^∞ problem is open although it can be solved for the alternating knots, fibred knots, many other knots, and certain "sums" of such knots using the constructions [6]–[8]

Corollary 6.7. *Let R_i be a Seifert surface for the oriented link $L_i \subset S^3$ for $i = 1, 2$, and R be any Murasugi sum (or generalized plumbing) of R_1 and R_2 with $L = \partial R$. Then R is a minimal genus surface for the oriented link L if and only if each R_i is a minimal genus surface for the oriented link L_i.*

This generalizes the classical result due to Seifert in the 1930's that the connected sum of minimal genus surfaces is a surface of minimal genus.

Corollary 6.9. *Let M be a compact connected irreducible oriented 3-manifold whose boundary ∂M is a (possibly empty) union of incompressible tori, and $H_2(M, \partial M)$ is not generated by tori and annuli. Then there exists a C^∞ transversely oriented foliation $\mathcal{F}$ on M such that $\mathcal{F} \pitchfork \partial M$, $\mathcal{F} | \partial M$ has no Reeb components, and no leaf of $\mathcal{F}$ is compact.*

In particular we have

Corollary 6.11. *Let M be either a compact connected oriented 3-manifold whose interior has a complete hyperbolic metric and $H_2(M, \partial M) \neq 0$, or $M = S^3 - \mathring{N}(L)$ where L is a nonsplit nontrivial link in S^3. Then there exists a C^∞ transversely oriented foliation $\mathcal{F}$ of M such that $\mathcal{F}$ has no compact leaves, $\mathcal{F} \pitchfork \partial M$, and $\mathcal{F} | \partial M$ has no Reeb components.*

The conditions that ∂M be a union of incompressible tori and M be irreducible are necessary by Novikov's work. The question of whether a manifold possesses a C^∞ codimension-1 foliation without compact leaves has been precisely answered by the work of Thurston [31], Levitt [18], Wood [34], and Milnor [19] for circle bundles over surfaces and for most Seifert fibred spaces by [4]; see also [5]. The 2-dimensional homology of these spaces (except for trivial cases) is generated by tori and annuli. It would be interesting to finish the 'in between' case to completely answer this question for all 3-manifolds with $H_2(M, \partial M) \neq 0$.

Applying a criterion of Sullivan [30] we obtain

Corollary 6.12. *Suppose M is a compact connected oriented irreducible 3-manifold with boundary ∂M such that $\chi(\partial M) = 0$ and $H_2(M, \partial M)$ is not generated by tori and annuli. Then there exist a Riemannian metric and foliation $\mathfrak{F}$ on M such that $\mathfrak{F} \pitchfork \partial M$, and every leaf is minimal (i.e., mean curvature 0).*

Corollary 6.13. *Let M be a compact oriented 3-manifold. Let $p : \tilde{M} \to M$ be an n-fold covering map and let $z \in H_2(M) = H^1(M, \partial M)$ or $z \in H_2(M, \partial M) = H^1(M)$. Then $n(x(z)) = x(p^*(z))$ where $x(z)$ denotes the Thurston norm of z.*

The truth of Corollary 6.13 was conjectured by Thurston in [32].

Corollary 6.18. *Let M be a compact oriented 3-manifold. Then on $H_2(M)$ or $H_2(M, \partial M)$, $x_s = x = \frac{1}{2}g$ where x_s denotes the norm on homology based on singular surfaces, and g denotes the Gromov norm.*

The equality of the singular and Thurston norms was also conjectured by Thurston in [32]. Recall that the immersed genus of a knot K in S^3 is the smallest g such that K bounds a punctured immersed surface S of genus g, which is nonsingular along the boundary, i.e., $f: S \to S^3$ and $f^{-1}(K) = \partial S$. A special case of Corollary 6.18 is

Corollary 6.22. *If K is a knot in S^3, then the immersed genus of K equals the embedded genus of K.*

More generally we have

Corollary 6.23. *Let M be a compact oriented 3-manifold, S a compact oriented surface with connected boundary, and $f: S \to M$ a map such that $f | \partial S$ is an embedding and $f^{-1}(f(\partial S)) = \partial S$. Then there exists an embedded surface T in M such that $\partial T = \partial S$ and genus $T \leqslant$ genus S.*

In words, Corollary 6.18 says that given an immersed surface in M there exists an embedded surface of not larger "topological complexity" which represents the same homology class. Corollary 6.23 is exactly Dehn's lemma for higher genus surfaces. Papakyriakopoulos [22] asked about the truth of the higher genus Dehn's lemma in his 1957 paper giving the first correct proof of Dehn's lemma.

The paper is organized as follows. Basic definitions and notation are introduced in §2. In §3 we define the general sutured manifold operations and show that a sutured manifold satisfying certain hypotheses can be split in a nice way to yield a new one. In §4 we show that one can only 'nontrivially' decompose a sutured manifold a finite number of times. In §5 we show how a sutured manifold hierarchy yields a prescription to construct a foliation on a manifold, completing the proof of Theorem 5.5. In §6 we state and prove some corollaries of the theorem. To prove Corollary 6.18 we need

Theorem 7.1. *Let M be a closed oriented 3-manifold. Let $\mathcal{F}$ be a finite depth transversely oriented foliation without Reeb components. Let $f: S \to M$ be a map of a closed oriented surface $S \neq S^2$ such that the image of every homotopically nontrivial simple closed curve in S is homotopically nontrivial. Then $f \simeq g: S \to M$ where g is an immersion and $g(S) \pitchfork \mathcal{F}$ except for a finite number of circle and saddle tangencies.*

The proof of Theorem 7.1 is the content of §7. Roussarie [25] and Thurston [31] independently proved this result around 1971 for the case f was an embedding. When finite depth is replaced by C^∞, this result also follows from the work of Sullivan [30], Sacks-Uhlenbeck [26], Schoen-Yau [27] and Hass [12] using minimal surface techniques. The proof here is elementary and topological.

2. Preliminaries

If M is an oriented manifold, and S is an oriented codimension-1 submanifold, then they determine a well-defined field of unit normal vectors to S. Conversely a normal direction determines an orientation for S. Define the $+$ *side* (or $-$*side*) to be the side where the normal points out (in). If X is a set (or space), define $|X|$ to be the number of elements (or components) of X. If $\mathcal{F}$ is a codimension-1 foliation, then a transverse curve is a smooth curve which intersects the leaves of $\mathcal{F}$ transversely. The symbol $\pitchfork$ means transverse to. For ideas and definitions concerning 3-manifolds see Jaco [15] or Hempel [14]. For ideas and definitions concerning foliations see Lawson [17].

Notation 2.1. If R is a properly embedded compact oriented surface in a compact oriented manifold, then $[R]$ denotes the homology class which R represents. If S is a submanifold of M, then $N(S)$ denotes a product neighborhood of S in M. If R and S are oriented submanifolds of M, and $\dim R + \dim S = \dim M$, then $\langle\, ,\rangle$ denotes their algebraic intersection number. $\mathring{E}$ denotes the interior of E.

Definition 2.2. Let S be a compact oriented surface $S = \bigcup_{i=1}^n S_i$, S_i connected. Then define the *norm of* S to be

$$x(S) = \sum_{i|\chi(S_i)<0} |\chi(S_i)|\,.$$

In [32] Thurston defines a pseudonorm on $H_2(M, \partial M)$ and $H_2(M)$. We review the definition of the Thurston norm presenting it in a slightly more general context.

Definition 2.3. Let M be a compact oriented 3-manifold. Let K be a codimension-0 submanifold of ∂M. Let $z \in H_2(M, K)$. Define the *norm of z* to be

$x(z) = \mathrm{Min}\{x(S) \mid (S, \partial S)$ is a properly embedded surface in (M, K), and $[S] = z \in H_2(M, K)\}$.

Definition 2.4. Let S be a properly embedded oriented surface in the compact oriented 3-manifold M. Then S is *norm minimizing* in $H_2(M, K)$ if $\partial S \subset K$, S is incompressible, and $x(S) = x([S])$ for $[S] \in H_2(M, K)$.

Theorem 2.5 (*Thurston* [32]). *Let M be a compact oriented* 3*-manifold. Let $\mathcal{F}$ be a codimension-*1 *transversely oriented foliation without Reeb components of M such that $\mathcal{F}$ is transverse to ∂M. If R is a compact leaf, then R is norm minimizing* (*as an element of $H_2(M, \partial M)$*).

Definition 2.6. A sutured manifold (M, γ) is a compact oriented 3-manifold M together with a set $\gamma \subset \partial M$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$. Furthermore, the interior of each component of $A(\gamma)$ contains a *suture*, i.e., a homologically nontrivial oriented simple closed curve. We denote the set of sutures by $s(\gamma)$.

Finally every component of $R(\gamma) = \partial M - \mathring{\gamma}$ is oriented. Define $R_+(\gamma)$ (or $R_-(\gamma)$) to be those components of $\partial M - \mathring{\gamma}$ whose normal vectors point out of (into) M. The orientations on $R(\gamma)$ must be coherent with respect to $s(\gamma)$, i.e., if δ is a component of $\partial R(\gamma)$ and is given the boundary orientation, then δ must represent the same homology class in $H_1(\gamma)$ as some suture.

Example 2.7 (Figure 2.1).

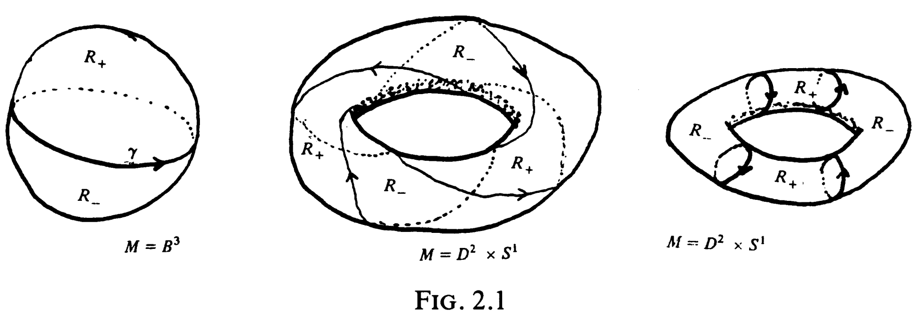


FIG. 2.1

We state a basic result, part (1) of which is classical, parts (2)–(5) are basically due to Novikov [21], [13]. Part (3) is Rosenberg's [24] significant improvement of Novikov's result $\pi_2(M) = 0$.

Theorem 2.8. *Let M be a compact oriented* 3*-manifold. Let $\mathcal{F}$ be a transversely oriented codimension-*1 *foliation of M such that $\mathcal{F}$ has no Reeb components, and $\mathcal{F} \pitchfork \partial M$. Then*:

(1) *∂M is a* (*possibly empty*) *union of tori*,

(2) $|\pi_1(M)| = \infty$,

(3) *M is either irreducible or $S^2 \times S^1$ with the product foliation,*

(4) *for every leaf L in $\mathcal{F}$ the map $\pi_1(L) \to \pi_1(M)$ is injective,*

(5) *no transverse closed curve is homotopically trivial.*

Definition 2.9. Let M be a compact oriented 3-manifold, and $\mathcal{F}$ a codimension-1 foliation. We say a leaf L is *depth* 0 if L is compact. Having defined the depth j ($\leq k$) leaves we say L is a depth $k+1$ leaf if $\bar{L} - L$ is a union of depth j ($\leq k$) leaves and contains some depth k leaf. $\mathcal{F}$ is *depth* k if $k = \max\{\text{depth } L \mid L \in \mathcal{F}\}$. In general the depth of a leaf or a foliation may not be defined.

For a good discussion of depth see Cantwell-Conlon [2]. Be aware that they call depth k, totally proper at level k. Note that if $\mathcal{F}$ is transversely oriented, and M is compact, then $\mathcal{F}$ is depth 0 if and only if M fibres over S^1.

Definition 2.10. A sutured manifold (M, γ) is *taut* if M is irreducible and $R(\gamma)$ is norm minimizing in $H_2(M, \gamma)$.

Definition 2.11. A transversely oriented codimension-1 foliation $\mathcal{F}$ on (M, γ) is *taut* if $\mathcal{F}$ is transverse to γ, tangent to $R(\gamma)$ with the normal direction pointing inward (outward) along $R_-(\gamma)$ ($R_+(\gamma)$), $\mathcal{F}|\gamma$ has no Reeb components, and each leaf intersects a transverse curve or properly embedded arc.

Theorem 2.12. *Let M be oriented. If (M, γ) has a taut foliation $\mathcal{F}$, then either (M, γ) is taut or $M = S^2 \times S^1$ or $S^2 \times I$, and $\mathcal{F}$ is the product foliation.*

Proof. This is basically Theorem 2.5 restated in the language of sutured manifolds. Thurston's proof extends to this more general setting. We actually only need the hypothesis that $\mathcal{F}$ has no Reeb components (instead of $\mathcal{F}$ being taut). q.e.d.

A goal of this paper is to prove the converse of Theorem 2.12 when $H_2(M, \gamma) \neq 0$.

3. Sutured Manifold Decomposition

Definition 3.1. Let (M, γ) be a sutured manifold, and S a properly embedded surface in M such that for every component λ of $S \cap \gamma$ one of (1)–(3) holds:

(1) λ is a properly embedded nonseparating arc in γ.

(2) λ is a simple closed curve in an annular component A of γ in the same homology class as $A \cap s(\gamma)$.

(3) λ is a homotopically nontrivial curve in a toral component T of γ, and if δ is another component of $T \cap S$, then λ and δ represent the same homology class in $H_1(T)$.

Then S defines a *sutured manifold decomposition*

$$(M,\gamma) \overset{S}{\rightsquigarrow} (M',\gamma')$$

where $M' = M - \mathring{N}(S)$ and

$$\gamma' = (\gamma \cap M') \cup N(S'_+ \cap R_-(\gamma)) \cup N(S'_- \cap R_+(\gamma)),$$
$$R_+(\gamma') = ((R_+(\gamma) \cap M') \cup S'_+) - \mathring{\gamma}',$$
$$R_-(\gamma') = ((R_-(\gamma) \cap M') \cup S'_-) - \mathring{\gamma}',$$

where S'_+ (S'_-) is that component of $\partial N(S) \cap M'$ whose normal vector points out of (into) M' (see Figure 3.1 for an example).

Definition 3.2. If $(M,\gamma) \overset{S}{\rightsquigarrow} (M',\gamma')$ is a sutured manifold decomposition, define

$$S_+ = S'_+ \cap R_+(\gamma'), \qquad S_- = S'_- \cap R_-(\gamma').$$

Remark 3.3. In words the sutured manifold (M',γ') is constructed by splitting M along S, creating $R_+(\gamma')$ by adding S'_+ to what was left of $R_+(\gamma)$ and creating $R_-(\gamma')$ by adding S'_-, to what was left of $R_-(\gamma)$. Finally one creates the annuli of γ' by "thickening" $R_+(\gamma') \cap R_-(\gamma')$.

Definition 3.4. A sutured manifold is *decomposable* if there is a sequence of *decompositions*

$$(M,\gamma) \overset{S_1}{\rightsquigarrow} (M_1,\gamma_1) \rightsquigarrow \cdots \overset{S_n}{\rightsquigarrow} (M_n,\gamma_n) = (S \times I, \partial S \times I)$$

where $R_+(\gamma_n) = S \times 1$.

Example.

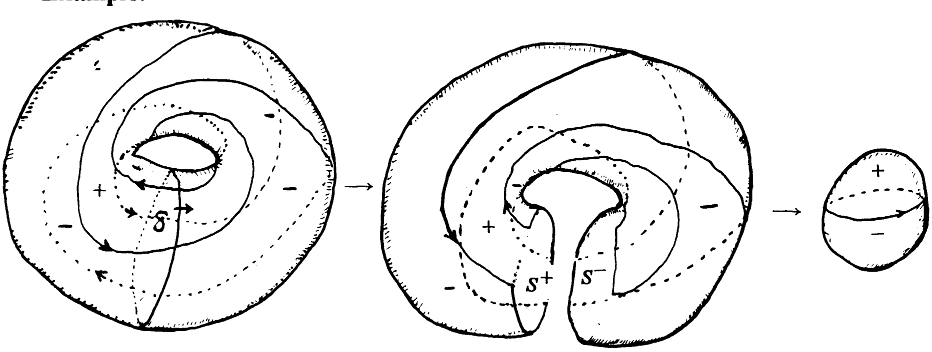


FIG. 3.1

Our goal now is to show that decompositions exist under certain conditions.

Lemma 3.5. *Let* $(M,\gamma) \overset{S}{\rightsquigarrow} (M',\gamma')$ *be a sutured manifold decomposition. If* (M',γ') *is taut, then either* (M,γ) *is taut or* $M = D^2 \times S^1$ *and* S *is a disc with* $\partial S \subset R(\gamma) = \partial M$.

Proof. If (M, γ) is not taut and either $R(\gamma)$ is compressible or M is reducible, then either $R'(\gamma)$ is compressible or M' is reducible. Otherwise there exists an embedded incompressible surface $T \subset M$ such that $\partial T = s(\gamma)$, $[T] = [R_+(\gamma)] \in H_2(M, \gamma)$, but $x(T) < x(R_+(\gamma))$; hence there exists an embedded incompressible $T' \subset M'$ (obtained by doing surgery with T and $R_+(\gamma)$ and compressing) such that $\partial T' = s(\gamma')$, $[T'] = [R_+(\gamma')] \in H_2(M', \gamma')$ and $x(T') < x(R_+(\gamma'))$.

Lemma 3.6. *Let T and R be properly embedded compact incompressible oriented surfaces in an oriented 3-manifold M with boundary ∂M. Suppose W is a union of disjoint annuli and tori in ∂M, $\partial T \cup \partial R \subset W$, $[T] = [R] \in H_2(M, W)$ and $x(R) > x(T)$. Then there exists an oriented properly embedded surface S in M such that $\partial S \subset W$ and*

(1) $[S] = [R] \in H_2(M, W)$,

(2) $x(S) < x(R)$,

(3) $R \cap S = \varnothing$.

Proof. Step 1. First perturb T so that $R \pitchfork T$. Since R and T restrict to the same homology class on each component of W, we can attach annuli to oppositely oriented parallel components of ∂T, isotope the resulting surface T_1, and perform compressions where necessary to conclude $x(T_1) \leqslant x(T)$, $[T_1] = [T]$, $\partial T_1 \cap \partial R = \varnothing$, and T_1 is incompressible. Let T_2 be an incompressible surface so that $x(T_2) \leqslant x(T_1)$, $[T_2] = [T_1]$, $\partial T_2 \cap \partial R = \varnothing$, and $|T_2 \cap R|$ is minimal over all surfaces satisfying the above conditions.

Step 2. No component of $R - T_2$ is a disc. Otherwise we can perform the appropriate compression to T_2 to get T_3 so that $x(T_3) \leqslant x(T_2)$, $[T_2] = [T_3]$, and $|T_3 \cap R| < |T_2 \cap R|$.

It follows from Step 2 and the incompressibility of R and T_2 that no component of $T_2 - R$ is a disc.

Step 3. Define $\varphi : M - T_2 \cup R \to \mathbf{Z}$ by

$$\varphi(t) = \langle \lambda, R \rangle - \langle \lambda, T_2 \rangle,$$

where $x \in M - (T_2 \cup R)$ is fixed, λ is some oriented path from x to t transverse to $T_2 \cup R$, and $\langle\, , \rangle$ denotes algebraic intersection number. φ is well defined, for if δ is any other path from x to T, then

$$\begin{aligned} 0 = \langle \lambda * \delta^{-1}, R \cup (-T_2) \rangle &= \langle \lambda * \delta^{-1}, R \rangle - \langle \lambda * \delta^{-1}, T_2 \rangle \\ &= (\langle \lambda, R \rangle - \langle \lambda, T_2 \rangle) - (\langle \delta, R \rangle - \langle \delta, T_2 \rangle), \end{aligned}$$

where $-T_2$ denotes T_2 oppositely oriented. By choosing x appropriately we can assume that $\varphi \geqslant 0$. If $T_2 \cap R \neq \varnothing$, then $\max \varphi \geqslant 2$. Let J be any connected region where φ takes on a maximal value. Then $\bar{J}$ has the property that the normal to $T_2 \cap \bar{J}$ points out of $\bar{J}$, and the normal to $R \cap \bar{J}$ points into $\bar{J}$.

If $x(\bar{J} \cap R) > x(\bar{J} \cap T_2)$, let $S' = (R - \bar{J} \cap R) \cup (\bar{J} \cap T_2)$. Let S be the surface S' perturbed slightly so that $S \cap R = \varnothing$, $[S] = [R]$, and $x(S) < x(R)$. Whence we are done.

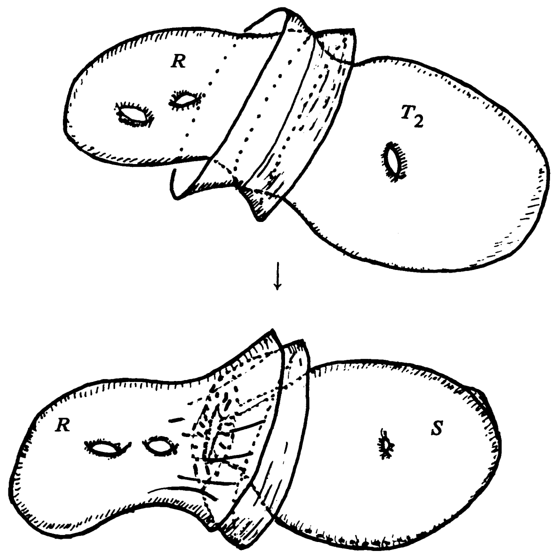


FIG. 3.2

If $x(\bar{J} \cap R) \leq x(\bar{J} \cap T_2)$, let $T_3 = (T_2 - \bar{J} \cap T_2) \cup (\bar{J} \cap R)$. Then T_3 can be isotoped slightly so that $|T_3 \cap R| \leq |T_2 \cap R|$, $x(T_3) \leq x(T_2)$, $[T_3] = [T_2]$, and no component of $T_3 \cup R - (T_3 \cap R)$ is a disc. If $\varphi_1 : M - T_3 \cup R \to \mathbf{Z}$ is defined analogously to φ, then either $\max \varphi_1 < \max \varphi$ or $\max \varphi_1 = \max \varphi$, and the number of regions where φ_1 is maximal is less than the number of regions where φ is maximal. The proof follows by induction.

Lemma 3.7. *Let (M, γ) be a sutured manifold such that $R_+(\gamma)$ and $R_-(\gamma)$ are norm minimizing in $H_2(M, \gamma)$ and γ is incompressible. Let N be the 3-manifold obtained by doubling M along $R(\gamma)$. Then $R_+(\gamma)$, $R_-(\gamma)$ are norm minimizing surfaces in $H_2(N, \partial N)$.*

Remark. N should be viewed as the union of the two sutured manifolds (M_1, γ_1) and (M_2, γ_2) where $M = M_1 = M_2$ and $R(\gamma) = R(\gamma_1) = -R(\gamma_2)$. By $R_+(\gamma) \subset N$ we mean the inclusion of $R_+(\gamma_1)$ into N.

Proof. We will prove that $R = R_+(\gamma_1) \cup R_-(\gamma_1)$ is a norm minimizing surface in N. The result will then follow from the elementary fact that if P is a norm minimizing surface, and Q is a union of components of P, then Q is a norm minimizing surface.

If R is not norm minimizing, then there exists a properly embedded oriented surface T such that $[T] = [R] \in H_2(N, \partial N)$ and $x(T) < x(R)$. Since $R_+(\gamma)$, $R_-(\gamma)$ and ∂N are incompressible, we can apply Lemma 3.6 to conclude that there exists an oriented properly embedded surface $S \subset N$ such that $[S] = [R]$, $x(S) < x(R)$, and $S \cap R = \varnothing$.

Define $\varphi : N - R \cup S \to \mathbf{Z}$ as in Lemma 3.6, i.e., fix $x \in N - R \cup S$, and define $\varphi(t) = \langle R, \gamma \rangle - \langle S, \gamma \rangle$ where γ is a path from x to t transverse to $R \cup S$. For simplicity pick a basepoint so that φ is nonnegative. φ defines a chain $d = \sum \varphi(W)\overline{W}$ such that $\partial d = S \cup (-R) \cup e$ where $e \subset \partial N$, and the sum is taken over all components W of $N - (R \cup S)$.

Suppose that V was a region of $N - R \cup S$ such that $\varphi(V)$ was maximal. Then since R separates N, $\overline{V} \subset M_1$ or $\overline{V} \subset M_2$. Furthermore, $(\partial\overline{V} \cap R) \subset R_+(\gamma)$ or $R_-(\gamma)$. So we can assume $\overline{V} \subset M_1$ and $\partial\overline{V} \cap R \subset R_+(\gamma)$. Now $\partial\overline{V} = A \cup B \cup C$ where A is a union of components of $R_+(\gamma)$, B is a union of components of S, and $C \subset \partial N$.

Since R_+ is norm minimizing in M_1, so is A; hence $x(A) \leqslant x(B)$. Thus if $\varphi(W) \leqslant 1$ for all components W of $N - (R \cup S)$, we conclude $x(S) \geqslant x(R)$ contradicting the hypothesis.

If $\varphi(V) > 1$, let $A \times I$ be a product neighborhood of A such that $A = A \times \frac{1}{2}$, $A \times [1/2, 1] \subset \overline{V}$, $A \times [0, 1/2] \cap \overline{V} = \varnothing$, and $x \notin A \times I$. Let $D = A \times 0$ (Figure 3.3) be oriented so that $[D] = [A] = [B] \in H_2(N, \partial N)$. Finally let $S_1 = (S - B) \cup D$.

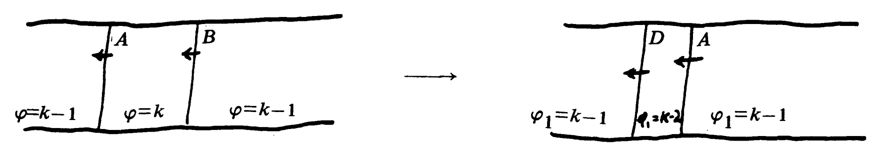


FIG. 3.3

We have $x(S_1) \leqslant x(S)$, $[S_1] = [S]$, $S_1 \cap R = \varnothing$ and a new map $\varphi_1 : N - R \cup S_1 \to \mathbf{Z}$ defined analogously to φ satisfying Maximum$(\varphi) \geqslant$ Maximum(φ_1) with equality holding only if the number of regions where the maximum is achieved is fewer under φ_1 than under φ. The result now follows by induction.

Lemma 3.8. *Let (M, γ) be a sutured manifold. If ∂M is not empty and not a union of 2-spheres, then there exists a class $z \in H_2(M, \partial M)$ such that $0 \neq \partial z \in H_1(\partial M)$ and the following hold.*

(1) *For each nonplanar component V of $R(\gamma)$ and each component λ of ∂V, $\langle z, \lambda \rangle = 0$.*

(2) *For each planar component V of $R(\gamma)$ there exist at most two components λ_1 and λ_2 of ∂V such that $\langle z, \lambda_i \rangle \neq 0$, $i = 1, 2$.*

Proof. For simplicity assume that no component of ∂M is a 2-sphere.

It follows from Poincaré duality and the long exact homology sequence that rank $H_2(M, \partial M) \geqslant \frac{1}{2}$rank $H_1(\partial M) = |\partial M| + \frac{1}{2}x(\partial M) = k$ and is generated by $z_1, \cdots, z_k, z_{k+1}, \cdots, z_l$ where $\partial z_1, \cdots, \partial z_k$ are linearly independent in $H_1(\partial M)$.

Let $\delta_1, \cdots, \delta_j$ be a maximal set of sutures in $s(\gamma)$ such that no component of $\partial M - \cup \delta_i$ is planar. Since $j \leqslant k - 1$, it follows that by taking a linear combination of the z_i's, $i \leqslant k$, there exists a $z \in H_2(M, \partial M)$ such that for $0 \leqslant i \leqslant j$ $\langle z, \delta_j \rangle = 0$ and $0 \neq \partial z \in H_1(\partial M)$.

Let S be a properly embedded oriented surface representing z in $H_2(\mathrm{M}, \partial \mathrm{M})$ such that $S \cap \cup_{i=1}^{j} \delta_i = \varnothing$. Let V be a component of $R(\gamma)$, J be the component of $\partial M - \cup_{i=1}^{j} \delta_i$ which contains V, $\bar{J}$ be the surface obtained by capping off the components of ∂J with discs, and W be the set of oriented curves $S \cap J$. If α is a component of ∂V, then $\langle z, \alpha \rangle = \langle W, \alpha \rangle$ where the latter algebraic intersection number is computed in $\bar{J}$. Hence if α is homotopically trivial in $\bar{J}$, then $0 = \langle W, \alpha \rangle = \langle z, \alpha \rangle$. If α is not homotopically trivial in $\bar{J}$, then $\bar{J} - \alpha$ is an annulus containing $\mathring{V}$. Hence V is planar, and there can be at most one other curve in ∂V homotopically nontrivial in $\bar{J}$.

Lemma 3.9. *Let V be a compact oriented surface with boundary ∂V.*

(i) *If $0 \neq \lambda \in H_1(V, \partial V)$, and $\langle \lambda, \alpha \rangle = 0$ for every component α of ∂V, then $\lambda = k[\mu]$ where μ is a simple closed curve and $k \in \mathbf{Z}$.*

(ii) *If V is planar, and $\langle \lambda, \alpha \rangle \neq 0$ for at most two components α of ∂V, then $\lambda = k[\mu]$ where μ is a simple arc.*

Proof. (i) Assume that λ is represented by a set δ of pairwise disjoint oriented simple closed curves with fewest number of components. If the closure of some component of $V - \delta$ did not intersect exactly two components of δ, then δ is not a smallest set (Figure 3.4). It is now evident that $\lambda = k[\mu]$ where μ is one component of λ.

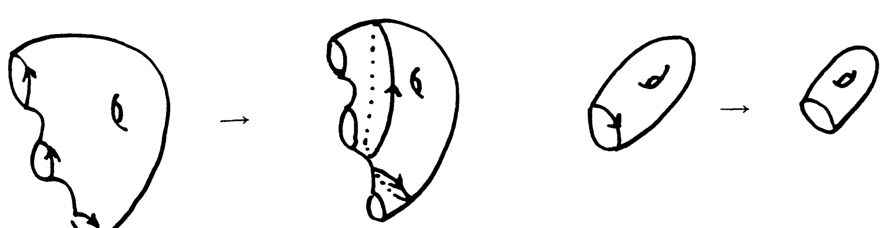


FIG. 3.4

(ii) Assume that λ is represented by a set δ of properly embedded pairwise disjoint oriented arcs δ so that if α is a component of ∂V, then $\langle \alpha, \delta \rangle = \pm|\alpha \cap \delta|$. Thus $\lambda = k[\mu]$ where μ is a component of δ and $k = |\alpha \cap \delta|$. q.e.d.

The proof of the following stronger lemma follows along the same lines as Lemma 3.9.

Lemma 3.10. *If V is a compact oriented surface, and δ is a set of properly embedded oriented curves such that $[\delta] \in H_1(V, \partial V)$ satisfies* (i) *or* (ii) *of Lemma* 3.9, *then there exists a sequence of sets of pairwise disjoint properly embedded oriented curves $\lambda_0, \lambda_1, \cdots, \lambda_p$ such that $\lambda_0 = \delta$, λ_p is a set of k parallel oriented properly embedded simple closed curves or arcs (depending on whether* (i) *or* (ii) *held), and $\partial W = \lambda_{i+1} \cup (-\lambda_i)$ for some compact subsurface (i.e., inherits orientation from V) W_i.*

Lemma 3.11. *Let (M, γ) be a taut sutured manifold. Let N be the manifold with boundary obtained by doubling M along $R(\gamma)$, and let $z \in H_2(N, \partial N)$. Then there exists an integer $n \geqslant 0$ and a properly embedded oriented surface T such that the following hold.*

(1) *$[T] = n[R] + z$.*

(2) *T is norm minimizing.*

(3) *If S is a surface obtained by doing cut and paste surgery to T and either $R_+(\gamma)$ or $R_-(\gamma)$, then S is norm minimizing, and each component of $S \cap \gamma$ satisfies one of the three conditions of Definition* 3.1 *where (M, γ) is viewed as being embedded in N.*

(4) *If V is a component of $R(\gamma)$, then no nontrivial subset of $V \cap T$ is homologically trivial in $H_1(V, \partial V)$. Furthermore, the following hold.*

(a) *If V is a component of $R(\gamma)$ such that $\langle z, \delta \rangle = 0$ for every component δ of ∂V, then $T \cap V$ is a set of k ($\geqslant 0$) parallel oriented simple closed homologically nontrivial curves.*

(b) *If V is a planar component of $R(\gamma)$ such that $\langle z, \delta \rangle \neq 0$ for exactly two components δ_1 and δ_2 of ∂V, then $T \cap V$ is a set of $|\langle z, \delta_1 \rangle|$ parallel oriented properly embedded arcs.*

Proof. By [31, Theorem 2] there exists an integer $m \geqslant 0$ such that for all $k \geqslant 0$

$$x[(m+k)[R(\gamma)] + z] = kx[R(\gamma)] + x[mR(\gamma) + z].$$

This statement can be deduced from the facts that the norm is linear on rays and is convex, i.e.,

$$x(k(z)) = kx(z), \qquad x(z_1) + x(z_2) \geqslant x(z_1 + z_2),$$

and takes on integer values on integer lattice points. Thurston more generally proves that the norm is the supremum of a finite number of linear forms.

Let T_1 be a norm minimizing surface representing $m[R(\gamma)] + z$. If A is any component of ∂N, then we can assume that $T_1 \cap A$ is a union of parallel oriented simple closed curves, since oppositely oriented components of $T_1 \cap A$ can be capped off by annuli. Furthermore, m can be chosen to be sufficiently large so that if α is a component of $\partial R(\gamma) \cap A$, then either $\langle \alpha, T_1 \rangle = |\alpha \cap T_1| \neq 0$ or $\alpha \cap T_1 = \varnothing$, and $[T_1 \cap A] = k[\alpha] \in H_1(A)$ for some $k > 0$. Since M is irreducible and T_1 and $R(\gamma)$ are incompressible, we can isotope T_1 so that the surface T_2 obtained by cutting and pasting r (> 0) copies of $R(\gamma)$ and T_1 contains no surfaces of positive Euler characteristic. Hence

$$\begin{aligned} x(T_2) &= |\chi(T_2)| = |r\chi(R(\gamma)) + \chi(T_1)| = rx(R(\gamma)) + x(T_1) \\ &= rx[R(\gamma)] + x[m[R(\gamma)] + z] = x(T_2). \end{aligned}$$

This shows in particular that every component of negative Euler characteristic of T_2 is incompressible. It follows from the previous paragraph that no component of T_2 is ∂-compressible.

Let us assume that T_2 was obtained by doing surgery where $r = 1$. If W is a subsurface of $R(\gamma)$ such that $\partial W - \partial N = \lambda_1 \cup \cdots \cup \lambda_k \cup (-\delta_1) \cup \cdots \cup (-\delta_l)$ where δ_i is a component of $T_1 \cap R(\gamma)$ and $\lambda_i \cap T_1 = \varnothing$, then T_2 can be isotoped slightly so that $T_2 \cap R(\gamma) = (T_1 \cap R(\gamma) - \cup \delta_i) \cup (\cup \lambda_i)$. This can be seen by performing the surgery in two steps. First do the surgery along the curves $T_1 \cap W$, and homotope the resulting surface P slightly so that $P \cap R(\gamma) = (T_1 \cap R(\gamma) - \cup \delta_i) \cup (\cup \lambda_i)$. Finally do surgery along the remaining curves to get T_3. It now follows from Lemma 3.10 that by surgering T_1 and r copies of $R(\gamma)$ we can obtain a surface T_3 so that (4) holds.

Let $T = T_3 - J$ where J is a maximal collection of components (necessarily tori) of T_3 such that $0 = [J] \in H_2(N, \partial N)$.

Lemma 3.12. *Let $(M, \gamma) \overset{J}{\rightsquigarrow} (M', \gamma')$ be a decomposition such that either J is a disc and $|J \cap s(\gamma)| = 2$ or J is an annulus with one component of ∂J lying in each of $R_+(\gamma)$ and $R_-(\gamma)$. Then (M, γ) is taut if and only if (M', γ') is taut.*

Theorem 3.13. *Let (M, γ) be a taut sutured manifold such that $H_2(M, \partial M) \neq 0$. Then there exists a decomposition $(M, \gamma) \overset{S}{\rightsquigarrow} (M_1, \gamma_1)$ such that (M_1, γ_1) is taut, S is connected, and $0 \neq [\partial S] \in H_1(\partial M)$ if $\partial M \neq \varnothing$. Furthermore, for a component V of $R(\gamma)$, $S \cap V$ is a union of k ($\geqslant 0$) parallel oriented nonseparating simple closed curves, if V is nonplanar, or arcs if V is planar.*

Proof. If M is closed let S be any norm minimizing surface. If $\partial M \neq \varnothing$, let $\bar{z}$ be a homology class obtained by applying Lemma 3.8 to (M, γ), P a properly embedded surface in M such that $[P] = \bar{z}$, N the 3-manifold obtained by doubling M along $R(\gamma)$, and P' the oriented surface in N obtained by doubling P along $\partial P - \mathring{\gamma}$. Let $z = [P'] \in H_2(N, \partial N)$, T be the properly embedded oriented surface obtained by applying Lemma 3.11 to z, T' be the surface obtained by doing surgery with T and $R(\gamma)$, $S' = T \cap M$, and S a component of S' such that $0 \neq [\partial S] \in H_1(\partial M)$. Consider the decompositions

$$(M, \gamma) \overset{S}{\rightsquigarrow} (M_1, \gamma_1), \quad (M, \gamma) \overset{S'}{\rightsquigarrow} (M', \gamma'), \quad (N, \partial N) \overset{T'}{\rightsquigarrow} (N', \delta').$$

It suffices to show (M_1, γ_1) is taut.

There exists a set J of properly embedded pairwise disjoint annuli and discs in N' satisfying the hypotheses of Lemma 3.12 such that the decomposition $(N', \delta') \overset{J}{\rightsquigarrow} (N'', \delta'')$ yields (M', γ') as a component of (N'', δ''). By Lemma 3.12 (N'', δ''); hence (M', γ') is taut. This is schematically pictured in Figure 3.5.

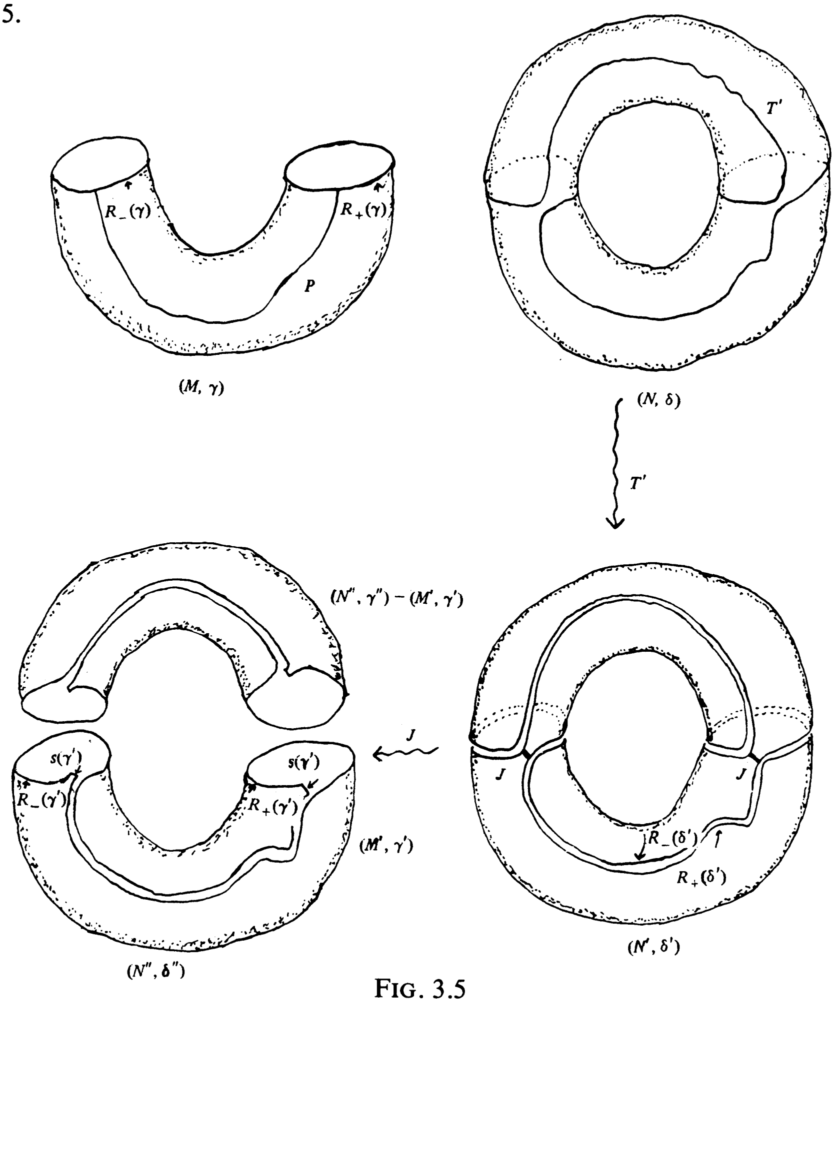


FIG. 3.5

To complete the proof apply Lemma 3.5 to the commutative diagram:

$$\begin{tikzcd}(M,\gamma) \arrow[rr, dashed, "S'"] \arrow[dr, "S"'] & & (M',\gamma') \\ & (M_1,\gamma_1) \arrow[ur, dashed, "S-S'"'] & \end{tikzcd}$$

4. Sutured manifold hierarchies

Definition 4.1. A *sutured manifold hierarchy* is a sequence of decompositions

$$(M_0,\gamma_0) \overset{S_1}{\rightsquigarrow} (M_1,\gamma_1) \rightsquigarrow \cdots \overset{S_n}{\rightsquigarrow} (M_n,\gamma_n)$$

where $(M_n,\gamma_n) = (R\times I, \partial R\times I)$ and $R_+(\gamma_n) = R\times 1$.

Theorem 4.2. *Every connected taut sutured manifold (M,γ), where M is not a rational homology sphere containing no essential tori, has a sutured manifold hierarchy such that $S_i \cap \partial M_{i-1} \neq \varnothing$ if $\partial M_{i-1} \neq \varnothing$, and for every component V of $R(\gamma_i)$, $S_{i+1}\cap V$ is a union of k ($\geqslant 0$) parallel oriented nonseparating simple closed curves or arcs.*

Proof. If M is closed, then a decomposition $(M,\varnothing) \overset{S}{\rightsquigarrow} (M_1,\gamma_1)$ yields a taut decomposition if and only if S is norm minimizing. Therefore either $H_2(M)\neq 0$ or M contains an essential torus.

We first define the notion of *complexity* of taut sutured manifolds and induct on the complexity.

Definition 4.3. The complexity $c(M,\gamma)$ of the taut sutured manifold (M,γ) is given by the 4-tuple (c_1,c_2,c_3,c_4) (c_i may be denoted $c_i(M,\gamma)$) where c_1 is a nonnegative integer, c_2 is a 6-tuple of nonnegative integers with the dictionary ordering, and c_3 and c_4 are finite, possibly empty, sets of positive integers. The values of c_3, c_4 are ordered as follows. If $A=\{a_1,\cdots,a_n\}$ and $B=\{b_1,\cdots,b_m\}$ are two sets of positive integers with $a_1\geqslant a_2\geqslant\cdots\geqslant a_n$, $b_1\geqslant b_2\geqslant\cdots\geqslant b_m$, then $A<B$ if for some j, $a_i=b_i$ for $i<j$ and either $a_j<b_j$ or $n=j<m$ holds. The set of complexities is given the dictionary ordering, i.e., $(c_1,c_2,c_3,c_4)<(c_1',c_2',c_3',c_4')$ if $c_i=c_i'$, $i<j$, and $c_j<c_j'$.

Proposition 4.4. *Let α_i, $i\in\mathbf{Z}$, be possible complexities of sutured manifolds so that $\alpha_1\geqslant\alpha_2\geqslant\cdots$. Then there exists an n so that $\alpha_n=\alpha_{n+j}$ for all $j\geqslant 0$.* q.e.d.

We need the following theorem to allow us to define $c_1(M,\gamma)$. It was originally basically proven by Haken. This version is due to Jaco [15].

Theorem 4.5. *Let $M = M_0$ be a Haken 3-manifold. Then there exists a smallest integer m, called the height of M, so that any sequence $M_0, M_1, \cdots, M_k$ satisfies $k \leqslant m$ where $M_{i+1} = M_i - \mathring{N}(S_i)$ for some nonboundary parallel, properly embedded, incompressible, boundary incompressible surface $S_i \neq D^2$.*

Definition 4.6. Let (M, γ) be a taut sutured manifold not containing the component $(D^2 \times S^1, \partial D^2 \times S^1)$. Let $D = \{D_1, \cdots, D_n\}$ be a set of property embedded discs in M so that the following holds.

(1) $D_i \pitchfork s(\gamma)$.

(2) $M - \bigcup \mathring{N}(D_i)$ is a union of ∂-incompressible 3-manifolds $M_1, \cdots, M_k$ and 3-balls $B_1, \cdots, B_l$. Furthermore, if V is a component of ∂M_i, then $V \cap (\bigcup N(D_i))$ has at most one component, and if $V = \partial B_i$, then $V \cap (\bigcup N(D_i))$ has exactly three or zero components, unless V intersects a unique $N(D_i)$, and $|V \cap N(D_i)| = 2$.

Order $M_1, \cdots, M_k$, and fix r so that $j \leqslant r$ if and only if M_j is homeomorphic to $P \times I$ for some closed surface P. Let

$$\hat{D} = \{\hat{D}_1, \cdots, \hat{D}_s\} = \{D_i \in D \mid N(D_i) \cap \partial M_j \neq \varnothing \text{ for some } j \leqslant k\}.$$

$\hat{D}$ consists of those discs which split off the "handlebody part" of M from the "nonhandlebody part" of M. We say that D is a set of *complexity discs*.

Definition 4.7. $n_3^D = \{a_1, a_2, \cdots, a_s\}$ where a_i is the number of components of $\hat{D}_i \cap s(\gamma)$.

$$n_4^D = \{b_1\} \cup \{b_2\} \cup \cdots \cup \{b_n\} \quad \text{where } b_i = \begin{cases} |D_i \cap s(\gamma)| & \text{if } D_i \cap s(\gamma) > 2, \\ \varnothing & \text{if } D_i \cap s(\gamma) = 2. \end{cases}$$

D is a set of *minimal complexity* discs if D minimizes (n_3^E, n_4^E) where E ranges over all sets of complexity discs, and the 2-tuple is given the dictionary ordering.

Definition 4.8. Let D be a set of minimal complexity discs

$$C_1(M, \gamma) = \sum_{i=r+1}^{k} \text{length } M_i,$$

$$C_2(M, \gamma) = \left(\left| \left(\bigcup_{r+1}^{k} \partial M_i \right) \cap \left(\bigcup_{i=1}^{n} N(D_i) \right) \right|, \right.$$

$$\left| \left\{ M_i \mid i \leqslant r, \left| M_i \cap \bigcup_{i=1}^{n} N(D_i) \right| = 2 \right\} \right|,$$

$$\left| \left\{ M_i \mid i \leqslant r, \left| M_i \cap \bigcup_{i=1}^{n} N(D_i) \right| = 1 \right\} \right|,$$

$$\left|\left\{V \subset \partial M_j \mid j > r,\ V \cap s(\gamma) \neq \varnothing\right\}\right|,$$

$$\left|\left\{M_j \mid j \leqslant r,\ \left|\left\{V \subset \partial M_j \mid V \cap s(\gamma) \neq \varnothing\right\}\right| = 2\right\}\right|,$$

$$\left.\left|\left\{M_j \mid j \leqslant r,\ \left|\left\{V \subset \partial M_j \mid V \cap s(\gamma) \neq \varnothing\right\}\right| = 1\right\}\right|\right),$$

$$C_3(M,\gamma) = n_3^D, \qquad C_4(M,\gamma) = n_4^D.$$

Remarks 4.9. (1) $C_1(M,\gamma)$ and the first 3 components of $C_2(M,\gamma)$ depend only on the topology of M.

(2) The last 3 components of $C_2(M,\gamma)$, $C_3(M,\gamma)$ and $C_4(M,\gamma)$ measure the complexity of the sutures in M.

(3) In words if $C_2(M,\gamma) = (a_1, a_2, a_3, a_4, a_5, a_6)$ then a_1 (a_4) is the number of components of ∂M_i, $i \geqslant r+1$, which nontrivially intersect $\cup N(D_i)$ ($s(\gamma)$), a_2 (a_5) is the number of product components of $\cup M_i$ such that both boundary components nontrivially intersect $\cup N(D_i)$ ($s(\gamma)$), etc.

(4) If (M,γ) is taut, connected and Haken, and $C(M,\gamma) = (0,(0,0,0,0,0,0),\varnothing,\varnothing)$, then $(M,\gamma) = (P \times I, \partial P \times I)$ where $R_+(\gamma) = P \times 1$, and P is a compact surface. Define $C(D^2 \times S^1, \partial D^2 \times S^1) = (0,(0,0,0,0,0,0),\varnothing,\varnothing)$. The complexity measures how far a structured manifold is from being a product. Discs intersecting $s(\gamma)$ twice correspond to where the sutured manifold is locally a product, hence do not count in the definition of n_4^D.

Definition 4.10. Let (M,γ) be a taut sutured manifold, and let S be a maximal set of pairwise disjoint properly embedded annuli and discs in M such that the following hold.

(1) If $A \in S$ is an annulus, then one component of ∂A is contained in $R_+(\gamma)$ and the other component is contained in $R_-(\gamma)$.

(2) If $A_1, A_2 \in S$ are annuli, then A_1 and A_2 are not parallel in M (i.e., bound (annulus) $\times I$), and for $* = +, -$, $i = 1,2$, $A_i \cap R_*(\gamma)$ are not parallel in the same component of $R_*(\gamma)$.

(3) If $D \in S$ is a disc, then $|D \cap s(\gamma)| = 2$.

(4) If $D_1, D_2 \in S$ are discs, then D_1 and D_2 are not parallel in M.

Let $(\overline{M}, \overline{\gamma})$ be the sutured manifold obtained from (M,γ) by decomposing along S and throwing away product sutured manifold components. Then $(\overline{M}, \overline{\gamma})$ is called the *reduced sutured manifold* of (M,γ).

$(\overline{M}, \bar{\gamma})$ is well defined and is the sutured manifold obtained by excising all the "product parts" of (M, γ). $(\overline{M}, \bar{\gamma})$ is taut by Lemma 3.12.

Definition 4.11. Define the *reduced complexity* $\overline{C}(M, \gamma)$ of the taut sutured manifold (M, γ) by $\overline{C}(M, \gamma) = C(\overline{M}, \bar{\gamma})$.

The notions of reduced complexity and reduced sutured manifold will not be used in this paper; however, they are essential in other contexts where one needs to measure how far certain 3-manifolds with trivial suturing are from being a product.

We are now ready for the proof of Theorem 4.2.

Step 1. Decompose (M, γ) along discs intersecting $s(\gamma)$ twice to get, by Lemma 3.12, a taut sutured manifold (M_1, γ_1) with $C(M_1, \gamma_1) \leqslant C(M, \gamma)$, so that if $D = \{D_1, \cdots, D_n\}$ is a set of minimal complexity discs in (M_1, γ_1), then $|D_i \cap s(\gamma_1)| \geqslant 4$.

Proof. Since ∂M is compact, and decomposing raises $\chi(\partial M)$ by 2, we can only do a finite number of such decompositions.

Step 2. Let S be a surface so that the decomposition $(M_1, \gamma_1) \overset{S}{\rightsquigarrow} (M_2, \gamma_2)$ yields a taut decomposition. Then there exists a commutative diagram

$$\begin{tikzcd}
(M_1, \gamma_1) \arrow[r, rightsquigarrow, "S"] \arrow[rr, rightsquigarrow, bend right, "S_1"'] & (M_2, \gamma_2) \arrow[r, rightsquigarrow, "F"] & (M_3, \gamma_3)
\end{tikzcd}$$

such that F is a union of disjoint discs $\{F_j\}$ where $|F_j \cap s(\gamma_2)| = 2$, $[S] = [S_2] \in H_2(M_1, \partial M_2)$ and the following four conditions hold.

Condition 1. S_2 is transverse to $\cup D_i$ and intersects $\cup D_i$ only in arcs. Furthermore, each component of $S_2 \cap A(\gamma_1)$ intersects each component of $(\cup D_i) \cap (A(\gamma_1))$ at most once.

Condition 2. No component E of $R(\gamma_1) - S_2$ is a disc satisfying that $\overline{E} \cap S_2$ is connected and nontrivial.

Condition 3. There exist tubular neighborhoods $N(D_i) = D_i \times I$ of D_i so that the following hold.

(a) If W is a component of $M_1 - \cup \mathring{N}(D_i)$, and T is a component of $S_2 \cap W$, then either

(i) $S_2 = T$ is boundary parallel in W, or

(ii) T is not boundary parallel in W, or

(iii) T is parallel to $P \subset \partial W - \cup \mathring{N}(D_i)$, i.e., there exists an embedding $\varphi : (T \times I, T \cap (\cup N(D_i)) \times I) \to (W, W \cap (\cup N(D_i)))$ such that $\varphi(T \times 0) = T$ and $\varphi(T \times 1) = P$.

(b) $N(D_i) \cap \gamma_1 = (D_i \cap \gamma_1) \times I$, $N(D_i) \cap \partial S_2 = (D_i \cap \partial S_2) \times I$, and $N(D_i) \cap S_2$ can be described as follows:

There exists numbers $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_n = 1$ so that $S_2 \cap D_i \times \alpha_j$ is a set of properly embedded pairwise disjoint arcs $\lambda_{j_1}, \cdots, \lambda_{j_k}$ and each component of $S_2 \cap (D_i \times [\alpha_j, \alpha_{j+1}])$ is either of the form $\lambda_i \times [\alpha_j, \alpha_{j+1}]$ or is a saddle as in Figure 4.1.

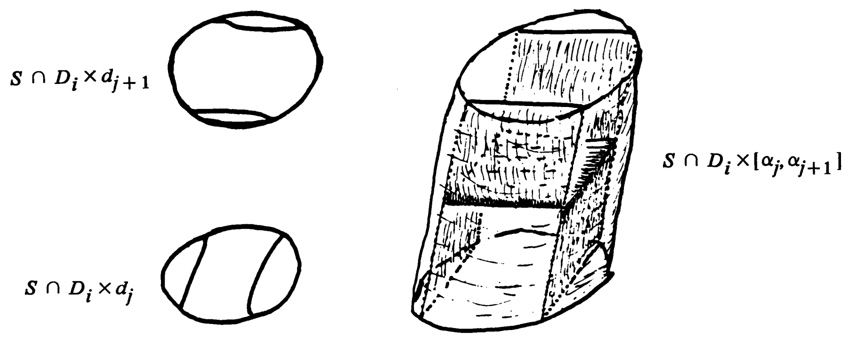


FIG. 4.1

Condition 4. No component λ of $D_i \times \alpha_j \cap S_2$ occurs as in Figure 4.2 where ▬ denotes a component of $\alpha_j \times D_i \times A(\gamma_1)$, the arrows denote the normal orientation to λ, $\pm$ denotes $\partial(D_i \cap \alpha_j) \cap R_\pm(\gamma_1)$, and the shaded area is a disc E in $D_i \times \alpha_j$.

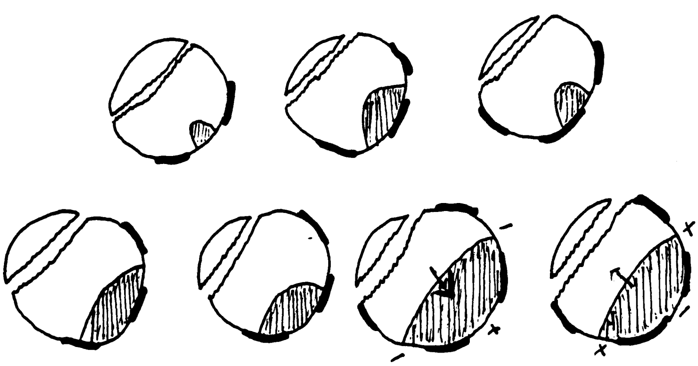


FIG. 4.2

Proof. The proof is by induction on $|S \cap (\cup D_i)|$. Isotope S so that Condition 1 holds (for $S = S_2$). If Condition 2 failed (for $S = S_2$) then S can be replaced by S_1 as in Figure 4.3 because $(M_1, \gamma_1) \overset{S_1}{\rightsquigarrow} (M_2, \gamma_2)$. The important point is that given S with normal vector oriented as in Figure 4.3 we are forced to have $A \subset R_+(\gamma_1)$ and $B \subset R_-(\gamma_1)$. If the reverse were true, the decomposition $(M_1, \gamma_1) \overset{S}{\rightsquigarrow} (M_2, \gamma_2)$ would yield a nontaut sutured manifold.

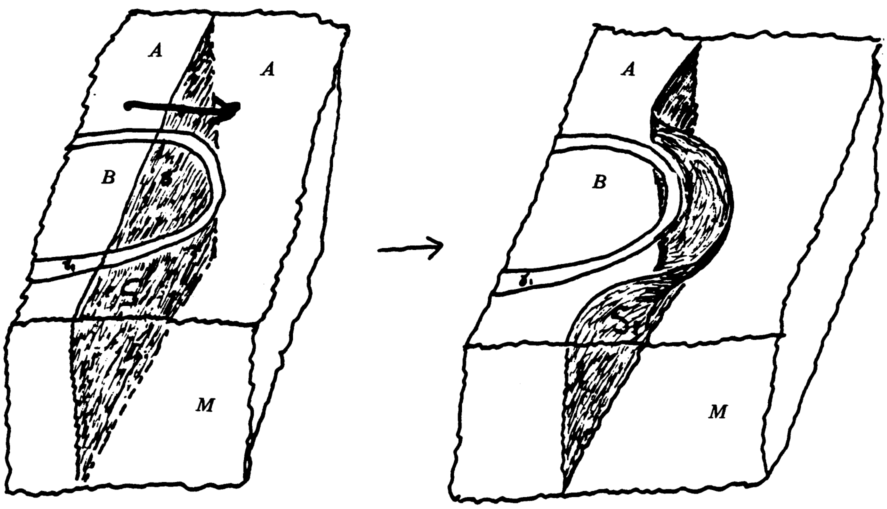


FIG. 4.3

Assuming that Conditions 1 and 2 hold for S, then isotope S as follows to satisfy Condition 3.

First find a small tubular neighborhood $N(D_i) \approx D_i \times I$ so that $N(D_i) \cap \gamma_1 = (D_i \cap \gamma_1) \times I$. Let W be a component of $M - \bigcup \mathring{N}(D_i)$, and T a component of $S \cap W$.

If T is boundary parallel in W, and none of 3(a) holds, then isotope T to lie close to ∂W. By pushing 'saddles' into $\bigcup N(D_i)$ (Figure 4.4) the resulting surface T' satisfies (iii).

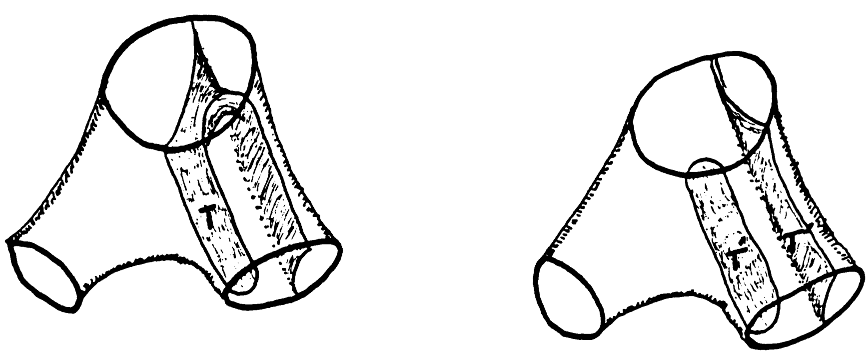


FIG. 4.4

Now consider a component T of $S \cap N(D_i)$. Since $N(D_i)$ is a 3-ball and S is incompressible, T is a properly embedded disc. By first isotoping T to lie near $\partial(N(D_i))$ it is evident that T can be isotoped to look like a union of saddles near $D \times 1$ and $D \times 0$ and to look like $\lambda \times I$ elsewhere, where λ is a set of properly embedded arcs. Finally one perturbs T so that multipronged saddles appear as two pronged saddles at discrete levels.

If one of the bad cases of Condition 4 occurred, then E would correspond to a properly embedded disc F' in M_2, which intersected $s(\gamma_2)$ twice. By letting S_1 be the surface in M_1 obtained by performing a boundary compression along E to S we get the commutative diagram:

$$\begin{tikzcd}
(M_1, \gamma_1) \arrow[r, squiggly, "S"] \arrow[rr, squiggly, bend right, "S_1"'] & (M_2, \gamma_2) \arrow[r, squiggly, "F'"] & (M_2', \gamma_2')
\end{tikzcd}$$

Observe that $|S_1 \cap (\cup D_i)| < |S \cap (\cup D_i)|$ and

$$[S] = [S_1] \in H_2(M_1, \partial M_1).$$

By induction there exists a commutative diagram

$$\begin{tikzcd}
(M_1, \gamma_1) \arrow[r, squiggly, "S"] \arrow[rr, squiggly, bend right, "S_1"'] \arrow[rrr, squiggly, bend right=50, "S_2"'] & (M_2, \gamma_2) \arrow[r, squiggly, "F'"] & (M_2', \gamma_2') \arrow[r, squiggly, "F''"] & (M_3, \gamma_3)
\end{tikzcd}$$

where $[S] = [S_2] \in H_2(M_1, \partial M_1)$, S_2 satisfies Conditions 1–4 and each component of F'' intersects $s(\gamma_2')$ in two points. The result follows by letting $F = F' \cup F''$.

Step 3. We show $C(M_2, \gamma_2) < C(M_1, \gamma_1)$ where $(M_1, \gamma_1) \overset{S_2}{\rightsquigarrow} (M_2, \gamma_2)$, (M_2, γ_2) is taut, S_2 satisfies Conditions 1–4 of Step 2, some component of S_2 is not boundary parallel, and (M_1, γ_1) satisfies Step 1. For simplicity denote S_2 by S, and assume that it is connected.

Case 1. S is closed.

Proof. If S is not boundary parallel in $M - \cup \mathring{N}(D_i)$, then $C_1(M_2, \gamma_2) < C_1(M_1, \gamma_1)$. If S is boundary parallel in $M_1 - \cup \mathring{N}(D_i)$, then $C_1(M_2, \gamma_2) = C_2(M_1, \gamma_1)$ and $C_2(M_2, \gamma_2) < C_2(M_1, \gamma_1)$.

Case 2. $\partial S \neq \varnothing, S \cap (\cup D_i) = \varnothing$.

Proof. S is contained in some component W of $M_1 - (\cup N(D_i))$.

(a) $W \neq B^3$.

(i) If S is boundary parallel, then $C_1(M_1, \gamma_1) = C_1(M_2, \gamma_2)$, but $C_2(M_1, \gamma_1) > C_2(M_2, \gamma_2)$.

(ii) If S is not boundary parallel, and $W \neq P \times I$, then $C_1(M_2, \gamma_2) < C_1(M_1, \gamma_1)$. This follows from the general fact that if W is a compact irreducible 3-manifold with incompressible boundary, and S is an incompressible surface, then S is either boundary parallel or one can obtain, by performing boundary compressions, a boundary incompressible surface S'. Therefore

$$\text{height } W - \mathring{N}(S) = \text{height } W - \mathring{N}(S') < \text{height } W.$$

(iii) If S is not boundary parallel, and $W = P \times I$, then

$$C_2(M_2, \gamma_2) < C_2(M_1, \gamma_1), \qquad C_1(M_2, \gamma_2) = C_1(M_1, \gamma_1).$$

(b) If $W = B^3$, then S is a disc E parallel to some D_i. By Step 1 we conclude $|S \cap s(\gamma)| \geq 4$. Let $E = D - D_i$ where D was our original set of minimal complexity discs. Then some subset F of E gives a set of complexity discs for (M_2, γ_2). We now conclude $C_3(M_2, \gamma_2) < C_3(M_1, \gamma_1)$ or $C_3(M_2, \gamma_2) = C_3(M_1, \gamma_1)$ and $C_4(M_2, \gamma_2) < C_4(M_1, \gamma_1)$. In both cases $C_i(M_2, \gamma_2) \leq C_i(M_1, \gamma_1)$, $i = 1, 2$.

Case 3. $S \cap (\cup \hat{D}_i) \neq \varnothing$.

Proof. (a) If some component of $S \cap (M_1 - \cup \mathring{N}(D_i))$ was not boundary parallel in a nonproduct component of $M_1 - \cup \mathring{N}(D_i)$, then $C_1(M_1, \gamma_1) > C_1(M_2, \gamma_2)$.

(b) If (a) does not hold, then $C_i(M_1, \gamma_1) \geq C_i(M_2, \gamma_2)$, $i = 1, 2$. We now show $C_3(M_2, \gamma_2) < C_3(M_1, \gamma_1)$.

Let $\hat{D}_i \in \hat{D}$, and $N(\hat{D}_i) = \hat{D}_i \times [0, 1]$ be the standard neighborhood of $\hat{D}_i$ described in Condition 4. Let $E = \cup_{i, \alpha_j} \hat{D}_i \times \alpha_j$ where α_j is as in Condition 3(b). Let $F = E \cap M_2$. One can now find a set of complexity discs G for (M_2, γ_2) such that $\hat{G} \subset F$. The key point is that if F' is a component of $\hat{D}_i \times \alpha_j \cap M_2$, then

$$|F' \cap s(\gamma_2)| < |\hat{D}_i \cap s(\gamma_1)|.$$

To see this when $\hat{D}_i \cap S$ is connected with $\hat{D}_i \cap M_1 = E_1 \cup E_2$, note

(1) $|E_1 \cap s(\gamma_2)| + |E_2 \cap s(\gamma_2)| = |\hat{D}_i \cap s(\gamma_1)| + 2$,

(2) $|E_i \cap s(\gamma_2)| = 2$ if and only if one of the seven cases listed in Condition 4 holds.

It now follows that $C_3(M_2, \gamma_2) < C_3(M_1, \gamma_1)$.

Case 4. $S \cap (\cup \hat{D}_i) = \varnothing$ and $S \cap (\cup D_i) \neq \varnothing$.

Proof. Since S is connected, S must lie in the "handlebody" part of M_1. We certainly have $C_i(M_1, \gamma_1) \geq C_2(M_2, \gamma_2)$ for $i \leq 3$. We prove that either $C_4(M_2, \gamma_2) < C_4(M_1, \gamma_1)$ or $C_3(M_2, \gamma_2) < C_3(M_1, \gamma_1)$.

Let $N(D_i) = D_i \times I$ denote a standard neighborhood described by Condition 3. Let $F = \{D_i \times \alpha_j \cap M_2\}$, and let B be a ball component of $M_1 - (\bigcup N(D_i))$. Since each component of $S \cap B$ is "boundary parallel" in $B - (\bigcup N(D_i))$, every component of $B - (\mathring{N}(S) \cup \mathring{N}(F))$ intersects $N(F)$ in 2 or 3 components. If $V = D_i \times [\alpha_j, \alpha_{j+1}]$, and $S \cap V$ contains a saddle, then $V - (\mathring{N}(S) \cup \mathring{N}(F))$ is a union of $p + 2$ balls $B_1, B_2, \cdots, B_{p+2}$ with $|B_i \cap \mathring{N}(F)| = 3$ for $i \leqslant 2$ and $|B_i \cap N(F)| = 2$ for $i > 2$. It follows that some subset G of F is a set of complexity discs.

The proof of Case 4 is easily deduced from the following facts (compare Case 3) about G:

(1) $\hat{G} \subset \hat{D}$.

(2) If $F_1, \cdots, F_j = N(D_i) \cap G$, then for $l = 1, 2, \cdots, j$, $|F_l \cap s(\gamma_2)| < |D_i \cap s(\gamma_1)|$.

Step 4.

Lemma 4.12. *Let* $(N, \delta) \overset{S}{\rightsquigarrow} (N', \delta')$ *be a sutured manifold decomposition so that* (N, δ) *and* (N', δ') *are taut. Let* (N'', δ'') *be obtained from* (N', δ') *by decomposing along a set of discs* D *each intersecting* $s(\delta)$ *twice so that no nontrivial compressing disc in* N'' *intersects* $s(\delta'')$ *twice. Then the following hold:*

(1) $C(N'', \delta'') \leqslant C(N, \delta)$.

(2) *If no nontrivial compressing disc in* N *intersects* $s(\delta)$ *twice, and some component of* S *is not boundary parallel, then* $C(N'', \delta'') < C(N, \delta)$.

Proof. We have the commutative diagram

$$\begin{tikzcd}
(N, \delta) \arrow[r, squiggly, "S"] \arrow[d, squiggly, "D_0"] & (N', \delta') \arrow[d, squiggly, "D_3"] \arrow[dr, squiggly, "D"] & \\
(N_1, \delta_1) \arrow[r, squiggly, "S'"] & (N_2, \delta_2) \arrow[r, squiggly, "D_2"] & (N'', \delta'')
\end{tikzcd}$$

where D_0 (D_2) is a maximal set of pairwise disjoint nonparallel discs each intersecting $s(\delta)$ $(s(\delta_2))$ twice, and S' is the surface obtained by modifying the surface $T = N_1 \cap S$ in N_1 to satisfy Conditions 1–4 of Step 2. From the modifications one observes that there exists a set D_3 of discs in N' each intersecting $s(\delta')$ twice so that the diagram commutes. It follows from Lemma 3.12 and Step 3 that $C(N, \delta) \geqslant C(N_1, \delta_1) \geqslant C(N_2, \delta_2) \geqslant C(N'', \delta'')$ with $C(N_1, \delta_1) = C(N_2, \delta_2)$ only if each component of S was boundary parallel. To prove (2) observe that $(N, \delta) = (N_1, \delta_1)$.

Step 5. *Proof of Theorem* 4.2. Let S be the surface obtained by applying Theorem 3.13 to (M_1, γ_1), and let S_2 be the surface obtained by applying Step 2 to S. Perform the decompositions $(M_1, \gamma_1) \overset{S}{\rightsquigarrow} (M_2, \gamma_2) \overset{D}{\rightsquigarrow} (M_3, \gamma_3)$, where D

is a maximal set of nonparallel discs each intersecting $s(\gamma_2)$ twice. By Lemma 3.12, (M_3, γ_3) is taut, and by Lemma 4.12, $C(M_3, \gamma_3) < C(M_1, \gamma_1)$.

It follows by induction that M has a sutured manifold hierarchy satisfying the conclusions of Theorem 4.2 with one exception. Some decomposing surface S_i may be a disc such that $|S_i \cap s(\gamma_{i-1})| = 2$, and S_i separates a component of $\partial R(\gamma_{i-1})$. Since this can be rectified by applying Lemma 5.4, the result follows.

5. The construction

Theorem 5.1. *Suppose M is connected, and (M, γ) has a sutured manifold hierarchy*

$$(M, \gamma) = (M_0, \gamma_0) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1) \rightsquigarrow \cdots \overset{S_n}{\rightsquigarrow} (M_n, \gamma_n)$$

so that no component of $R(\gamma_i)$ is a compressing torus. Then there exist transversely oriented foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ of M such that the following hold.

(1) *$\mathcal{F}_0$ and $\mathcal{F}_1$ are tangent to $R(\gamma)$.*

(2) *$\mathcal{F}_0$ and $\mathcal{F}_1$ are transverse to γ.*

(3) *If $H_2(M, \gamma) \neq \varnothing$, then every leaf of $\mathcal{F}_0$ and $\mathcal{F}_1$ nontrivially intersects a transverse closed curve or a transverse arc with endpoints in $R(\gamma)$. However, if $\varnothing \neq \partial M = R_+(\gamma)$ or $R_-(\gamma)$, then this holds only for interior leaves.*

(4) *There are no 2-dimensional Reeb components on $\mathcal{F}_i | \gamma$, $i = 0, 1$.*

(5) *$\mathcal{F}_1$ is C^∞ except possibly along toral components of $R(\gamma)$ or S_1 if $\partial M = \varnothing$.*

(6) *$\mathcal{F}_0$ is of finite depth.*

Remarks 5.2. (1) If some component of $R(\gamma_i)$ was a compressible torus, then the construction described below would yield a foliation with Reeb components.

(2) If each S_i is a disc, then by the construction of [6], $\mathcal{F}_0$ could be made to be of depth at most 1.

Corollary 5.3. *Let (M, γ) be a sutured manifold such that $M \neq B^3$ or $S^2 \times S^1$ and $H_2(M, \gamma) \neq 0$, then the following are equivalent.*

(a) *(M, γ) has a sutured manifold decomposition so that no component of $R(\gamma_i)$ is a compressible torus.*

(b) *(M, γ) has a taut foliation $\mathcal{F}$.*

(c) *(M, γ) is taut.*

Proof. (a) $\Rightarrow$ (b) by Theorem 5.1, (b) $\Rightarrow$ (c) by Theorem 2.12, and (c) $\Rightarrow$ (a) by Theorem 4.2.

Lemma 5.4. *Let* $(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$ *be a decomposition operation. Then there exists a commutative diagram of sutured manifold decompositions*

$$\begin{tikzcd}
(M,\gamma) \arrow[rrr, rightsquigarrow, "S"] \arrow[d, rightsquigarrow, "S_1"] & & & (M',\gamma') \\
(M_1,\gamma_1) \arrow[r, rightsquigarrow, "S_2"] & (M_2,\gamma_2) \arrow[r, rightsquigarrow] & \cdots \arrow[r, rightsquigarrow, "S_n"] & (M_n,\gamma_n) \arrow[u, rightsquigarrow, "S_{n+1}"']
\end{tikzcd}$$

so that if V is a component of $R(\gamma_{i-1})$, then either $S_i \cap V$ is a set of parallel nonseparating oriented simple closed curves or arcs or $\partial V \neq \varnothing$ and $S_i \cap V$ is a set of oriented properly embedded arcs such that $|\lambda \cap S_i| = |\langle \lambda, S_i \rangle|$ for each component λ of ∂V. If S is a disc with $|S \cap s(\gamma)| = 2$, then the former holds for all i.

Proof.

Case 1. S is an annulus such that $\partial S = \delta_1 \cup \delta_2$ where δ_1 and δ_2 lie in different components of $R(\gamma_{i-1})$.

Proof. If δ_i is homologically trivial in $H_2(R, \partial R)$, then there exists a connected subsurface $W_i \subset R$ so that $\partial W_i = \lambda_i \cup (-\delta_i)$, where $\lambda_i \subset \partial\gamma$, and $-\delta_i$ denotes δ_i oppositely oriented. If δ_i is homologically nontrivial, let $W_i = \varnothing$. Let $S_i = W_2 \cup W_1 \cup S$, pushed slightly to be properly embedded in $M - \partial\gamma$. If $\lambda_1 \cup \lambda_2 \neq \varnothing$, then $\lambda_1 \cup \lambda_2$ should be isotoped to lie parallel to some sutures. In any case one can find $S_2, \cdots, S_k$ so that the conclusions of the lemma are satisfied. These $S_j, j \geqslant 2$, are annuli and discs.

Case 2. S is a disc satisfying $|S \cap s(\gamma)| = 2$.

Proof. Proceed analogously as in Case 1.

General case.

Proof. Let V be a component of $R(\gamma_{i-1})$, and let $\delta = V \cap S$. Then as in Lemma 3.10 there is a sequence of property embedded simple pairwise disjoint arcs and closed curves $\delta = \delta_0, \delta_1, \cdots, \delta_r$, where $\delta_i \cup (-\delta_{i-1}) \cup \lambda_i = \partial W_i$, $\lambda_i \subset \partial\gamma$, and W is a codimension-0 subsurface of V such that no proper subset T of W_i satisfies $\partial T \subset (-\delta_{i-1}) \cup \gamma$, and further that the set δ_r satisfies the conclusions of the lemma with respect to V.

Let $T_0 = S$, and having constructed T_{i-1} let T' be $T_{i-1} \cup W_i$ pushed into M so as to be properly embedded in M and $\partial T_i' \cap V = \delta_i$. Obtain T_i by doing cut and paste surgery to eliminate double curves. Finally isotope T_i as in Figure 5.1

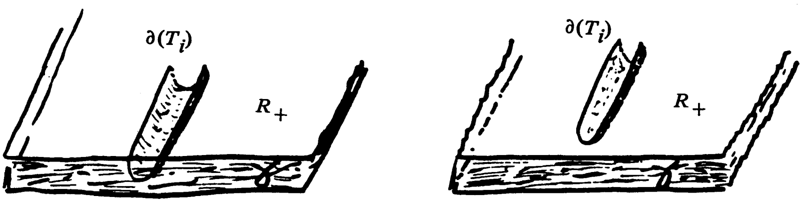


FIG. 5.1

to eliminate trivial intersections with γ. There is now a commutative diagram

$$\begin{tikzcd}
(M,\gamma) \arrow[rrr, rightsquigarrow, "T_0"] \arrow[d, rightsquigarrow, "T_1"] & & & (M',\gamma') \\
(M_1,\gamma_1) \arrow[r, rightsquigarrow, "S_2"] & (M_2,\gamma_2) \arrow[r, rightsquigarrow] & \cdots \arrow[r, rightsquigarrow] & (M_t,\gamma_t) \arrow[u, rightsquigarrow, "S_{t+1}"']
\end{tikzcd}$$

so that each S_i is either an annulus with ∂S_i lying in distinct components of $R(\gamma_{i-1})$ or S_i is a disc intersecting $s(\gamma_{i-1})$ exactly twice. One now applies Cases 1 and 2 to these decompositions. By continuing in this manner with $T_2,\cdots,T_k$ and repeating this procedure for each component of $R(\gamma)$ we complete the proof.

Proof of 5.1 (*Outline*). (1) Apply Lemma 5.4 to each term of the original decomposition of (M,γ) to get a new decomposition. Assume that each decomposing surface is connected.

(2) Use the new decomposition as a prescription to construct foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ satisfying Conditions 1, 2, and 4. These foliations will satisfy Condition 3 if there does not exist a subset T of $\cup S_i$ which is a union of tori and $O = [T] \in H_2(M, \partial M)$. $\mathcal{F}_1$ will satisfy Condition 5 if no S_i ($i > 1$ if $\partial M = \varnothing$, $i \geqslant 1$ if $\partial M \neq \varnothing$) is a torus. $\mathcal{F}_0$ will satisfy Condition 6 if for every component V of $R(\gamma_{i-1})$ with $S_i \cap \partial V \neq \varnothing$ then $V \cap S_i$ is a union of parallel oriented nonseparating simple curves. One observes from the construction that these foliations have no Reeb components.

(3) Apply Theorem 2.12 to conclude $R_+(\gamma)$ and $R_-(\gamma)$ are norm minimizing.

(4) Apply Theorems 3.13 and 4.2 to (M,γ) if $H_2(M,\gamma) \neq 0$, or to (M_1,γ_1) if $H_2(M,\gamma) = 0$ to construct a new sutured manifold decomposition of (M,γ) so that no decomposing surface S_i is a torus unless that $\partial M = \varnothing$, $H_2(M)$ is generated by tori and $i = 1$, and further that if V is a component of $R(\gamma_{i-1})$, then $V \cap S_i$ consists of k ($\geqslant 0$) parallel oriented nonseparating simple curves.

(5) Apply the construction of (2) to the decomposition of (4) to yield the desired foliations.

The construction (*Proof of* (2)). Let

$$(M, \gamma) \overset{T_1}{\rightsquigarrow} (M_1, \gamma_1) \overset{T_2}{\rightsquigarrow} \cdots \overset{T_k}{\rightsquigarrow} (M_k, \gamma_k)$$

be the decomposition yielded by Step 1.

By definition $(M_k, \gamma_k) = (S \times I, \partial S \times I)$ with $R_+(\gamma_k) = \partial S \times 1$. Let $\mathfrak{F}_0^k = \mathfrak{F}_1^k$ be the product foliation.

Induction hypothesis. (i) Foliations $\mathfrak{F}_0^i$ and $\mathfrak{F}_1^i$ have been constructed on (M_i, γ_i) satisfying the results of Theorem 5.1 except possibly parts (3), (5) and (6).

(ii) $\mathfrak{F}_0^i$ and $\mathfrak{F}_1^i$ satisfy (3) if $\partial M_j \neq R_+(\gamma_j)$ or $R_-(\gamma_j)$ for each $j \geqslant i$. In particular if $\cup_{j=i+1}^k T_j$ contains no tori and $\partial M_i \neq R_+(\gamma_i)$ or $R_-(\gamma_i)$, then (3) holds.

(iii) $\mathfrak{F}_1$ is C^∞ except possibly along toral components of $\cup_{j=i+1}^k T_j \cup R(\gamma_i)$. Furthermore, if δ is a curve on a nontoral component of $R(\gamma_i)$, and $f\colon [0, a) \to [0, b)$ is a representative of the germ of the holonomy map around δ for the foliation $\mathfrak{F}_1^i$, then

$$\frac{d^n f}{dt^n}(0) = \begin{cases} 1, & i = 1, \\ 0, & i > 1. \end{cases}$$

(iv) $\mathfrak{F}_0^i$ is of finite depth if $V \cap S_{j-1}$ is a union of parallel oriented simple curves for each component V of $R(\gamma_j)$ with $S_{j-1} \cap \partial V \neq \varnothing$.

(v) $\mathfrak{F}_0^i$ and $\mathfrak{F}_1^i$ have no Reeb components.

To construct $\mathfrak{F}_0^{i-1}$ there are three cases to consider. Checking that the induction hypothesis hold for each case is routine.

Case 1. $\partial T_i \cap T(\gamma_{i-1}) = \varnothing$.

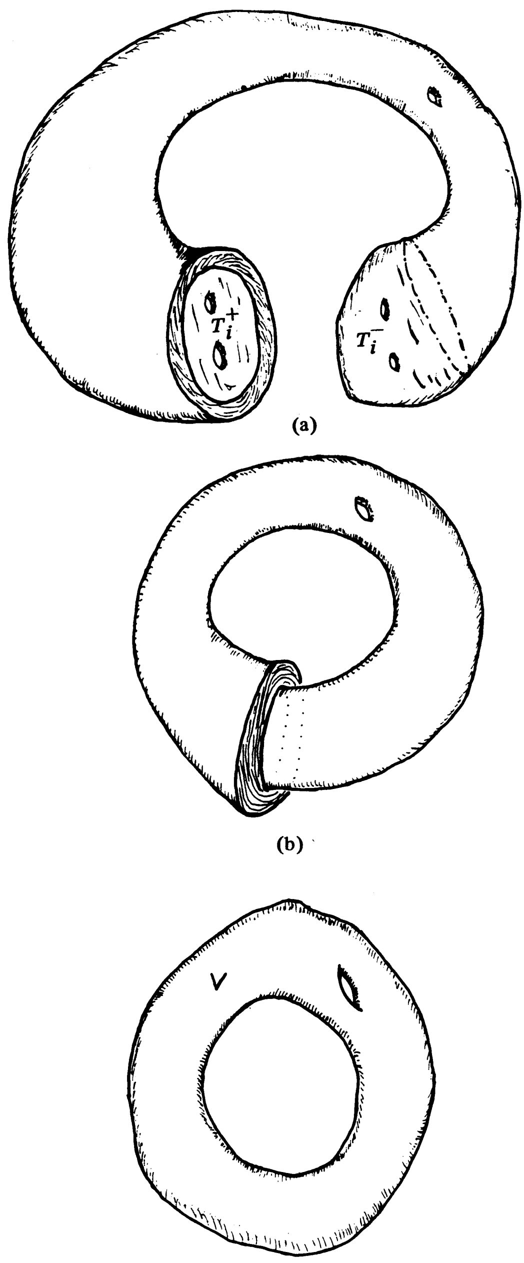
Proof. (M_{i-1}, γ_{i-1}) is obtained by gluing T_i^+ to T_i^-. Let $\mathfrak{F}_0^{i-1}$ and $\mathfrak{F}_1^{i-1}$ be the respective foliations induced by $\mathfrak{F}_0^i$ and $\mathfrak{F}_1^i$ on (M_{i-1}, γ_{i-1}).

Case 2. ∂T_i is contained in a component V of $R(\gamma_i)$.

Proof. By hypothesis $T_i \cap V$ is a union of K parallel oriented homologically nontrivial curves. For simplicity we will assume that ∂T_i is connected and contained in $R_-(\gamma_{i-1})$. See Figure 5.2(a) for a view of that part of ∂M_i which contains $T_i^+ \cap T_i^-$.

Let Q be the manifold obtained by gluing $T_i^+ \subset R_+(\gamma_i)$ to $T_i^- \subset R_-(\gamma_i)$ (Figure 5.2(b)). Let $\mathfrak{F}_0'$ be the foliation obtained by extending $\mathfrak{F}_0^i$ to Q. Q is homeomorphic to M_{i-1} and should be thought of as being embedded in M_{i-1} so that $M_{i-1} - Q \subset N(V)$.

To construct $\mathfrak{F}_0^{i-1}$ extend the foliation $\mathfrak{F}_0^i$ to M_{i-1} by spiraling the leaves in towards V to get (M_{i-1}, γ_{i-1}) (Figure 5.2(c)). It follows that depth $\mathfrak{F}_0^{i-1} =$ depth $\mathfrak{F}_0^i + 1$. This operation of spiraling is analogous to the procedure of extending a 1-dimensional foliation defined on a subset of an annulus to the



(c)

FIG. 5.2

annulus (Figure 5.3). More formally let A be the annular component of γ_i such that $\partial T_i^- \subset A$. Let $f\colon I \to I$ be the holonomy map of $\mathcal{F}_0^i | A$. Let δ be the simple closed curve $T_i \cap V$, and let λ be a simple closed curve in V whose geometric intersection number with δ is 1. Let $\lambda \times I$ be a tubular neighborhood of λ in V.

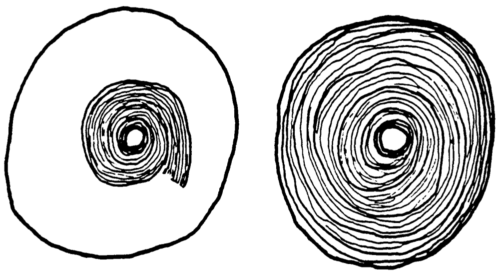


FIG. 5.3

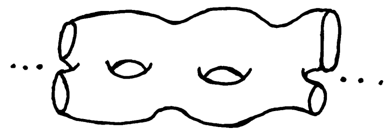


FIG. 5.4

Construct a foliation on $V \times I \cong V \times [-\infty, \infty]$ as follows.

(A) Give $(V - (\lambda \times (0, 1))) \times [-\infty, \infty]$ the product foliation.

(B) Give $V \times [-\infty, \infty]$ the induced foliation $\mathcal{F}^1$ by identifying $(\lambda, 0, t)$ to $(\lambda, 1, [t] + f(t - [t]))$ where [ ] denotes the greatest integer function.

(C) Let $\mathcal{F}^2 = \mathcal{F}^1 | (V - \delta \times (0, 1)) \times [-\infty, \infty]$.

(D) Let $\mathcal{F}^3$ be the induced foliation on $V \times [-\infty, \infty]$ obtained by identifying $(\delta, 0, t)$ with $(\delta, 1, t + 1)$ on the $\mathcal{F}^2$ foliated manifold $(V - \delta \times (0, 1)) \times [-\infty, \infty]$.

Let μ be the circle $(\delta, 0, 0) \subset V \times [-\infty, \infty]$. The leaf L of $\mathcal{F}^3$ which contains μ is homeomorphic to the infinite ladder (Figure 5.4). Its ends spiral (limit) towards $V \times \pm\infty$. Let $\tilde{L}$ be μ together with those points of L lying on the + side of μ. Let $Z \subset V \times [-\infty, \infty]$ be the set of points in $\tilde{L}$ plus those points (x, t) whose normal ray $(x, (t, -\infty])$ intersects $\tilde{L}$ nontrivially. Z is topologically $V \times I$ and geometrically diffeomorphic to $M_{i-1} - \mathring{Q}$ where $V \times 0$ is the unique compact leaf of $\mathcal{F}^3 | Z$, and $V \times I$ is the union of a twice punctured surface contained in $\tilde{L}$ and an annulus transverse to $\mathcal{F}^3 | Z$. The holonomy along the transverse annulus equals f. By gluing Z into $M_{i-1} - \mathring{Q}$ we get the desired foliation $\mathcal{F}_0^{i-1}$.

Construction of $\mathcal{F}_1^{i-1}$. Glue T_i^+ to T_i^-, and extend $\mathcal{F}_1^i$ to $\mathcal{F}^1$ on the resulting manifold Q. Let f be the holonomy of the transverse annulus.

(1) If the holonomy map f along the transverse annulus is the identity then by repeating the above construction in an appropriate way one extends $\mathcal{F}^1$ to $\mathcal{F}_1^{i-1}$ as desired. If f is not the identity, the above construction yields only a C^0 foliation.

(2) If V is a torus, we apply the previous construction and must be satisfied with $\mathfrak{F}_1^{i-1}$ having C^0 holonomy maps along V.

(3) If $\partial V \neq \varnothing$, then by pushing the holonomy to the boundary as follows we can assume that (1) holds. Construct a codimension-1 foliation on $(S^1 \times I) \times I$ so that the leaves are transverse to the $S^1 \times I \times t$ factors and the holonomy is f^{-1}. Construct the manifold Q' by gluing $(S^1 \times I) \times 0$ to a collar neighborhood of a component of ∂V. Finally attach a band $(I \times I) \times I$ with the product foliation to Q' so that $I \times I \times 0$ glues to V, $0 \times I \times I$ glues to the transverse annulus, and $1 \times I \times I$ glues to $(S^1 \times I) \times I$.

(4) If V is a surface of genus > 1, and $f \neq \mathrm{id}$, then we need the following theorem whose genesis is due to the work of Mather and Thurston.

Theorem (*Mather-Sergeraert-Thurston* [28]). *If $f: I \to I$ is a C^∞ diffeomorphism satisfying*

$$\frac{d^n f}{dt^n}(\alpha) = \begin{cases} 1, & n = 1, \\ 0, & n > 1, \end{cases}$$

for $\alpha \in \{0, 1\}$, then there exists C^∞ diffeomorphisms $c_i, b_i: I \to I$, $i = 1, \cdots, n$, satisfying the above conditions so that

$$f \circ [c_1, b_1] \circ [c_2, b_2] \circ \cdots \circ [c_n, d_n] = \mathrm{Id}.$$

Now let Q_1 be obtained by attaching thick bands B_1 and C_1 to ∂Q. (Compare Figures 5.2(b) and 5.5). Extend $\mathfrak{F}^1$ and $\mathfrak{F}^2$ defined on Q_1 by foliating these bands so that the holonomy along B_1 (or C_1) is b_1 (or c_1). One now observes that the holonomy along the new transverse annulus is $fc_1b_1c_1^{-1}b_1^{-1} = f \circ [c_1, b_1]$. By repeating this procedure n times one gets a foliation $\mathfrak{F}^{n+1}$ on Q^n, whose holonomy along the transverse annulus is trivial.

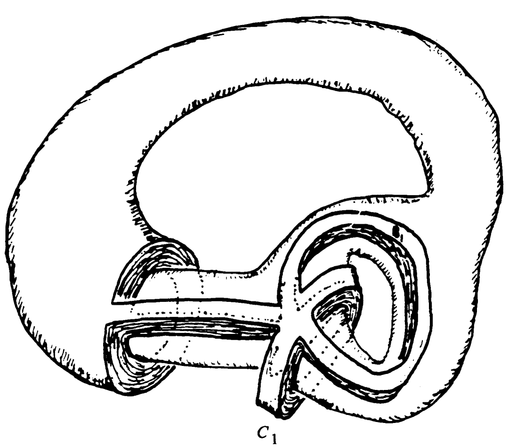


FIG. 5.5

We have now reduced to (1).

Case 3. $\partial T_i \cap \gamma_{i-1} \neq \varnothing$, and ∂T_i is connected (Figures 5.6 and 5.7).

(A) ∂T_i^+ is a union of arcs contained in $\partial \gamma_i$ and properly embedded arcs in $R_+(\gamma_i)$ (Figure 5.6(a)). A similar situation holds for ∂T_i^-. Perform the diffeomorphism on M_i (Figure 5.6(b)) obtained by "extending" pieces of γ_i which contain $\partial T_i^+ \cup \partial T_i^-$. Finally glue T_i^+ to T_i^- (Figure 5.6(c)) to create the manifold Q. This gluing is analogous to the operation of stacking chairs on top of each other. As before Q is homeomorphic to M_{i-1} and should be thought of as lying in $M_{i-1} - N(R(\gamma_{i-1}))$. For $j = 0, 1$, define $\mathcal{F}_j' = \mathcal{F}_j^i$ extended to Q. $\mathcal{F}_j^{i-1}$ is constructed by extending $\mathcal{F}_j'$ to M_{i-1} (Figure 5.6(d)).

(B) Let V be a component of $R(\gamma_{i-1})$ such that $T_i \cap V \neq \varnothing$. Then $P = N(V) \cap Q$ appears as in Figure 5.7(a), that is, P is homeomorphic to $V \times I$ and $V \times 1 = J \cup (\mu_1 \times I) \cup \cdots \cup (\mu_n \times I)$ where J is tangent to $\mathcal{F}_j'$, $\mu_m \times 0$, $m = 1, \cdots, n$, is properly embedded in both $V \times 1$ and the leaf L of $\mathcal{F}_j'$ which contains J, $\mu_m \times 1 \subset \partial L$ is properly embedded in $V \times 1$, and $\mathcal{F}_j' \mid \mu_m \times I$ has the product foliation (Figure 5.8).

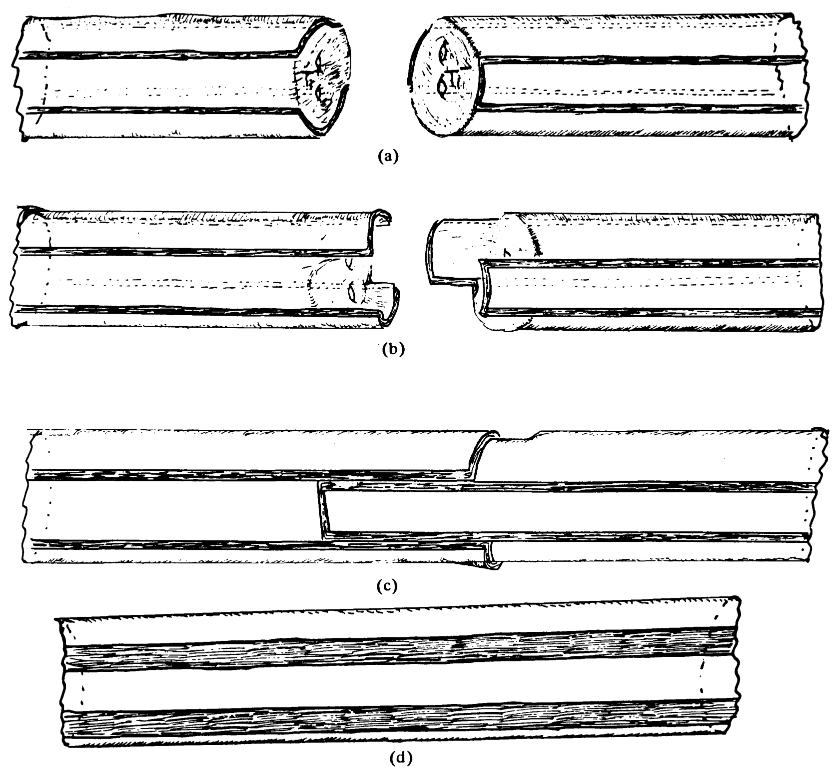
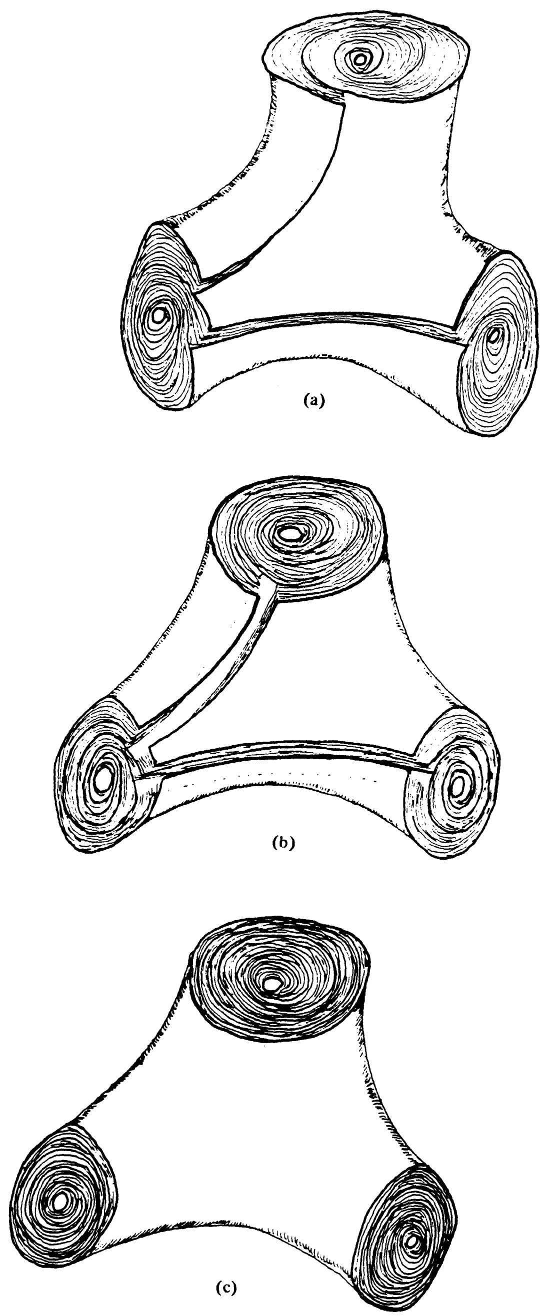


FIG. 5.6



(a)

(b)

(c)

FIG. 5.7

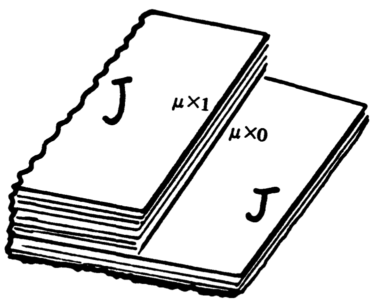


FIG. 5.8

(C) Let $Q_1 = Q \cup_{J'\times 0} J' \times I$ (Figure 5.7(b)) where $J' = J - N(\bigcup_{m=1}^{n}(\mu_m \times 0))$. Q_1 looks like (M_{i-1}, γ_{i-1}) with ditches $\beta_m \times \mathring{I} \times \mathring{I}$, $m = 1, 2, \cdots, n$, drilled out, where $\beta_m \times 0 \times 0$ is identified with $\mu_m \times 0$ and $\beta_m \times I \times 0 \subset L$. By giving $(\beta_m \times I) \times I$ the product foliation and gluing it into Q_1 (Figure 5.7(c)) we get the desired foliations $\mathfrak{F}_0^{i-1}$ and $\mathfrak{F}_1^{i-1}$. To construct $\mathfrak{F}_1^{i-1}$ near V perform the appropriate smooth gluing.

(D) How to construct $\mathfrak{F}_0^{i-1}$ near V.

View $J' \times I$ as $J' \times [1, \infty]$ and $\beta_m \times I \times I$ as $\beta_m \times I \times [0, \infty]$. Glue $\beta_m \times 0 \times [0, \infty]$ into Q_1 by identifying $\beta_m \times 0 \times [0, 1]$ with $\mu_m \times [0, 1]$, and $\beta_m \times 0 \times [1, \infty]$ with $\alpha_m \times [1, \infty]$, where $\alpha_m = \mu_m \times 1 \subset J'$, and the identification map is the identity on $[1, \infty]$. Glue $\beta_m \times 1 \times [0, \infty]$ into Q_1 by identifying $\beta_m \times 1 \times [0, \infty]$ with $\alpha'_m \times [1, \infty]$, where $\alpha'_m = (N(\mu_m \times 0) \cap J') - \alpha_m$, and the identification map f on the second factor is $f(x) = x + 1$.

It is easy to check that when depth $\mathfrak{F}_0^1$ is defined, and the μ_m's are parallel, then depth $\mathfrak{F}_0^{i-1} = \text{depth}\,\mathfrak{F}_0^i + 1$. In general when the μ_m's are not parallel, there is no way to glue in $\beta \times I \times I$ so that $\mathfrak{F}_0^{i-1}$ has finite depth even when depth $\mathfrak{F}_0^i$ is finite.

(E) Extend the foliation to the rest of $N(R(\gamma_{i-1}))$ as in parts (B), (C) and (D).

General Case. Glue T_i^+ to T_i^-, and extend $\mathfrak{F}_j^i$ to $\mathfrak{F}_j'$, $j = 0, 1$, on Q. Apply Cases 2 and 3 where appropriate to extend these foliations to M_{i-1}.

Theorem 5.5. *Let M be a compact connected irreducible oriented 3-manifold whose boundary ∂M is a (possibly empty) union of tori. Let S be a norm minimizing surface representing a nontrivial class $z \in H_2(M, \partial M)$. Then there exist transversely oriented foliations $\mathfrak{F}_0$ and $\mathfrak{F}_1$ of M such that:*

(1) *for $i = 0, 1$, $\mathfrak{F}_i \pitchfork \partial M$ and $\mathfrak{F}_i | \partial M$ has no Reeb components,*

(2) *every leaf of $\mathfrak{F}_0$ and $\mathfrak{F}_1$ nontrivially intersects a transverse closed curve,*

(3) *$\mathfrak{F}_0$ is of finite depth,*

(4) *$\mathfrak{F}_1$ is C^∞ except possibly along toral components of S,*

(5) *S is a compact leaf of both $\mathfrak{F}_0$ and $\mathfrak{F}_1$.*

Proof. Consider the decomposition $(M, \partial M) \overset{S}{\rightsquigarrow} (M_1, \gamma_1)$. Since S is norm minimizing, (M_1, γ_1) is taut and $H_2(M_1, \gamma_1) \neq 0$. Apply Theorem 4.2 to (M_1, γ_1) to extend the above decomposition to a sutured manifold decomposition. Apply Theorem 5.1 to this decomposition to yield the foliations $\mathcal{F}_0'$, $\mathcal{F}_1'$ on (M_1, γ_1). Identify S^+ to S^-, and extend $\mathcal{F}_0'$ and $\mathcal{F}_1'$ to M to get the desired foliations.

6. Consequences

Definition 6.1. A surface of minimal genus for an oriented link L in S^3 is an oriented embedded surface S in S^3 containing no closed components, whose oriented boundary is L and $\chi(S) \geqslant \chi(T)$ for any other surface T satisfying the above conditions.

This definition generalizes the notion of a minimal genus surface for a knot in S^3 to surfaces spanning oriented links in S^3.

Recall that L is a *nonsplit* link in S^3 if there exists no embedded $S^2 \subset S^3$ such that $S^2 \cap L = \varnothing$ but each component of $S^3 - S^2$ intersects L nontrivially. Equivalently $\pi_2(S^3 - L) = 0$.

Corollary 6.2. *Let L be an oriented nonsplit link in S^3. Then S is a surface of minimal genus for L if and only if there exists a C^∞ transversely oriented foliation $\mathcal{F}$ of $S^3 - \mathring{N}(L)$ such that:*

(1) *$\mathcal{F} \pitchfork \partial N(L)$ and $\mathcal{F} | \partial N(L)$ has no Reeb components,*

(2) *$\mathcal{F}$ has no Reeb components,*

(3) *S is a compact leaf.*

Proof. $\Rightarrow$: Apply Theorem 5.5. to $S^3 - \mathring{N}(L)$.

$\Leftarrow$: Apply Thurston's Theorem 2.5.

Remark 6.3. Explicit constructions of foliations in the complement of alternating and arborescent (i.e., algebraic in the sense of Conway [3] and Bonahon-Siebenmann [1]) can be found in [6] and [7]. For knots (links) of less than 11 (10) crossings see [6].

Definition 6.4. A core of a Reeb component $V = D^2 \times S^1$ is a smooth simple closed curve δ in V isotopic to $t \times S^1$ for some $t \in D^2$.

Corollary 6.5. *A nontrivial link L in S^3 is nonsplit if and only if L is the set of cores of Reeb components of some foliation $\mathcal{F}$ of S^3.*

Proof. $\Rightarrow$: Apply Theorems 4.2 and 5.1 to the sutured manifold $(S^3 - \mathring{N}(L), \gamma)$ where $R_+(\gamma) = \partial(N(L))$ to construct a foliation $\mathcal{F}$ of $S^3 - \mathring{N}(L)$ containing no Reeb components such that $\partial(N(L))$ is a compact leaf. Extend $\mathcal{F}$ to S^3 by gluing in Reeb components.

$\Leftarrow$: This was proven by Novikov [21] in 1965.

Remarks 6.6. (1) Corollary 6.5 answers the so-called "Reeb placement problem" of Laudenbach and Roussarie [16] which asked what links in S^3 are cores of Reeb components.

(2) Corollary 6.5 is a special case of the following general result. A link L in a 3-manifold M has an irreducible ∂-incompressible complement if and only if L is the set of cores of Reeb components of some foliation $\mathfrak{F}$ on M. The proof is as in Corollary 6.5.

(3) The holonomy along the toral leaves of the foliations constructed in Corollary 6.5 may be C^0. It follows from [6], [7] and [8] that for fibred knots, knots of fewer than 10 crossings, the arborescent knots, connected sums of the above knots and more generally knots which are boundaries of Murasugi sums of certain spanning surfaces for many classes of links are cores of Reeb components of C^∞ foliations. It is an open and very interesting problem to determine exactly which links in a 3-manifold are cores of Reeb components of C^∞ foliations.

Corollary 6.7. *Let S_i be a Seifert surface for the oriented link $L_i \subset S^3$ for $i = 1, 2$, and S be any Murasugi sum or generalized plumbing (see [20] or [8] for a definition) of S_1 and S_2 with $L = \partial S$. Then S is a minimal genus surface for the oriented link L if and only if each S_i is a minimal genus surface for the oriented link L_i.*

Proof. If S_i is a surface of minimal genus for the oriented nonsplit link L_i, then by Corollary 6.2 there exists a transversely oriented foliation $\mathfrak{F}_i$ of $S^3 - \mathring{N}(L_i)$ without Reeb components such that S_i is a compact leaf. If S is a Murasugi sum of S_1 and S_2, and $L = \partial S$, then it follows from [8] that there exists a transversely oriented foliation $\mathfrak{F}$ of $S^3 - \mathring{N}(L)$ such that $\mathfrak{F}$ has no Reeb components, and S is a compact leaf. It follows by Thurston's Theorem 2.5 that S is a surface of minimal genus. Conversely in [9] we show how to construct a foliation $\mathfrak{F}_i$ on $M - \mathring{N}(L_i)$ with S_i a compact leaf and no Reeb components when given a foliation F without Reeb components on $M - \mathring{N}(L)$ with S as a compact leaf. q.e.d.

The process of "plumbing" and "deplumbing" foliations in fact preserves the best qualities of foliations, e.g., all leaves compact, the foliation is depth 1, the foliation is smooth, or all leaves intersect transverse closed curves. See [8] and [9].

Remark 6.8. This corollary generalizes the classical fact due to Seifert in the 1930's that the connected sum of minimal genus surfaces is minimal genus. An elementary (nonfoliations) proof of the $\Rightarrow$ part of Corollary 6.7 can be found in [8].

Corollary 6.9. *Let M be a compact irreducible connected oriented 3-manifold such that its boundary ∂M is a (possibly empty) union of tori, and $H_2(M, \partial M)$ is* *not generated by tori and annuli (i.e., $x(z) \neq 0$ for some $z \in H_2(M, \partial M)$). Then there exists a C^∞ transversely oriented foliation $\mathfrak{F}$ of M such that $\mathfrak{F} \pitchfork \partial M$, $\mathfrak{F} | \partial M$ has no 2-dimensional Reeb components, and no leaf of $\mathfrak{F}$ is compact.*

Proof. Step 1. There exists a transversely oriented foliation $\mathfrak{F}_1$ of M such that $\mathfrak{F}_1 | \partial M$ has no Reeb components, and $\mathfrak{F}_1$ has a finite number of compact connected leaves $T_1, \cdots, T_k$ such that $\chi(T_i) < 0$ for each i.

Step 2. Foliation $\mathfrak{F}_1$ can be perturbed slightly in a small neighborhood of the compact leaves to eliminate them.

Proof of Step 1.

(1) **Definition.** Let

$$(M, \partial M) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1) \rightsquigarrow \cdots \overset{S_n}{\rightsquigarrow} (M_n, \gamma_n)$$

be a sequence of sutured manifold decompositions. Define $E_0 = \partial M$. Define E_i to be the union of those components of γ_i which are contained in E_{i-1} after gluing S_i^+ to S_i^-. In other words E_i consists of those components of $E_{i-1} - \mathring{N}(S_i)$ which are annuli and tori.

(2) Let

$$(M, \partial M) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1) \rightsquigarrow \cdots \overset{S_m}{\rightsquigarrow} (M_m, \gamma_m)$$

be a sutured manifold hierarchy. Let $\mathfrak{F}_1$ be the foliation constructed by applying Theorem 5.1 to the decomposition. If Lemma 5.4 was not invoked in this construction, then $\mathfrak{F}_1$ has a finite number of compact leaves, and each compact leaf corresponds to a unique component of some S_i. Conversely a component T of S_i gives rise to a compact leaf if and only if $\partial T \subset E_{i-1}$.

(3) **Induction hypothesis.** A decomposition

$$(M, \partial M) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1) \rightsquigarrow \cdots \overset{S_n}{\rightsquigarrow} (M_n, \gamma_n)$$

has been constructed such that (M_n, γ_n) is taut, Lemma 5.4 need not be applied to any term of the sequence, every annular component T of S_i, $1 \leqslant i \leqslant n$, satisfies $\partial T \not\subset E_{i-1}$, $0 \neq [T] \in H_2(M_{i-1}, \partial M_{i-1})$ for each component T of S_i, and if A is a properly embedded homologically nontrivial (in $H_2(M_n, \partial M_n)$) annulus such that $\partial A \subset E_n$, then ∂A is contained in toral components of γ_n.

(4) **Induction step.**

Case 1. $(M_n, \gamma_n) = (M, \partial M)$. By hypothesis there exists a norm minimizing connected surface T in M such that $\chi(T) < 0$. Let $A_1, \cdots, A_s$ be a maximal collection of pairwise disjoint homologically nontrivial annuli disjoint from T no two of which are parallel, and for each A_i there exists a component V of ∂M

such that $V \cap A_i \neq \varnothing$ and $V \cap T \neq \varnothing$. The finiteness of r follows from the fact due to Haken [14] that a 3-manifold can have only a finite number of disjoint nonparallel incompressible surfaces. By taking the right number of parallel copies of T and each of the A_i's and orienting the A_i's appropriately and attaching annuli to oppositely oriented components of ∂T and $\partial(\cup A_i)$ one constructs a surface S_1 satisfying the induction hypothesis. S_1 is norm minimizing because $[S_1] = k[T] + [\alpha]$ where $x[\alpha] = 0$. Hence $x(S_1) = x(k[T]) = |\chi(S_1)|$.

Case 2. $\gamma_n \neq \partial M_n$.

Lemma 6.10. *For each component V of E_n there exists some nontrivial simple closed curve $\delta \subset V$ such that any homologically nontrivial annulus A with $\partial A \subset E_n$ satisfies $\langle [A \cap V], \delta \rangle = 0$.*

Proof. If not the existence of annuli A_1 and A_2 such that $\langle [A_1 \cap V], [A_2 \cap V] \rangle \neq 0$ and $\partial(A_1 \cup A_2) \subset E_n$ together with the irreducibility of M_n would imply that M_n could be obtained by taking a regular neighborhood of $A_1 \cup V \cup W$ and gluing on a solid torus to one boundary component. W (possibly empty) is a component of E_n distinct from V satisfying $W \cap A_1 \neq \varnothing$. We conclude $\gamma_n \supset E_n = \partial M_n$, contradicting the hypothesis. q.e.d.

If there exists a nontrivial compressing disc D such that $D \cap s(\gamma_n) = 2$, then let $S_{n+1} = D$. (If D separated a component of $R(\gamma_n)$, then replace this term by the sequence obtained by invoking Lemma 5.4.) If no such D exists continue as follows. Let $V_1, \cdots, V_t$ be the toral components of E_n. For each V_i choose a simple closed curve λ_i such that $|\lambda_i \cap \delta_i| = 1$. One now applies the method of Lemma 3.8 to find a homology class $z \in H_2(M_n, \partial M_n)$ satisfying the results of Lemma 3.8 and further satisfying $\langle z, \lambda_i \rangle = 0$ for all i. Applying the proof of Theorem 3.13 to this class z one finds a surface S_{n+1} such that the decomposition $(M_n, \gamma_n) \overset{S_{n+1}}{\rightsquigarrow} (M_{n+1}, \gamma_{n+1})$ satisfies the results of Theorem 3.13 and $\langle S_{n+1}, \lambda_i \rangle = 0$ for all i. This decomposition extends our sequence to a new one satisfying the induction hypothesis.

The sequence must eventually terminate by Lemma 4.12. Applying Theorem 5.1 to this sutured manifold decomposition yields the desired foliation.

Proof of Step 2. Let L be a compact leaf. By construction there exists a simple closed curve or properly embedded arc α in L such that a foliated neighborhood of the "$-$side" of L is diffeomorphic to a neighborhood of $L \times 0$ where $L \times [-1, 0]$ is foliated as follows. Give $L \times [-1, 0]$ the product foliation. Let $f: [-1, 0] \to [-1, 0]$ be the appropriate (depending on $\mathfrak{F}$) C^∞ diffeomorphism such that $f(t) > t$ for $t \notin \{0, -1\}$ and

$$\frac{d^n f}{dt^n}(0) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

The desired foliation is the induced foliation on

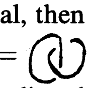
$$\frac{L \times [-1,0] - (\alpha \times \mathring{I}) \times [-1,0]}{(x,0,t) \sim (x,1,f(t))}, \qquad x \in \alpha,$$

where $\alpha \times I$ is a product neighborhood of α in L. Similarly $\mathcal{F}_1$ spirals in towards L along a simple closed curve or arc β on the "+side" of L. Let λ and δ be simple closed curves in L such that $|\lambda \cap \alpha| = 1$, $|\delta \cap \beta| = 1$, and let W be a simple closed curve or arc such that $0 \neq [W] \in H_1(L, \partial L)$, $W \cap \lambda = \varnothing$, and $|\langle W, \alpha \rangle| + |\langle W, \beta \rangle| = 0$ if W is a closed curve. Notice that such a W cannot be found on a torus or annulus.

It follows that $\mathcal{F}_1 | (W \times I) \times I$ has the product foliation for some sufficiently small neighborhood of W where $(W \times I) \times 0 \subset L$. By excising $W \times I \times I$ and regluing in a smooth fashion so as not to cancel the holonomy of $\mathcal{F}_1$ in a neighborhood of δ, one can construct a C^∞ transversely oriented foliation $\mathcal{F}_2$ with one fewer compact leaf. By performing this modification on each compact leaf one achieves the desired foliation.

Corollary 6.11. *Let M either be a compact 3-manifold with boundary ∂M, whose interior has a complete hyperbolic metric and $H_2(M, \partial M) \neq 0$ or $S^3 - \mathring{N}(L)$ where L is a nonsplit nontrivial link in S^3. Then there exists a C^∞ transversely oriented foliation $\mathcal{F}$ of M such that $\mathcal{F}$ has no compact leaves, $\mathcal{F} \pitchfork \partial M$, and $\mathcal{F} | \partial M$ has no Reeb components.*

Proof. If $\mathring{M}$ is hyperbolic it follows from Thurston [33] that it is atoroidal, acylindrical, and irreducible, hence M satisfies the hypothesis of Corollary 6.9.

If $M = S^3 - \mathring{N}(L)$ where L is nonsplit and nontrivial, then M is irreducible and either $x(z) \neq 0$ for some $z \in H_2(M, \partial M)$ or $L =$ , whence $M = T^2 \times I$. In the latter case one constructs the foliation directly. In the former case M satisfies the hypothesis of Corollary 6.9. q.e.d.

See the remarks following the statement of Corollary 6.11 in the introduction.

Corollary 6.12. *Suppose M is a compact irreducible 3-manifold, ∂M is a (possibly empty) union of tori, and $H_2(M, \partial M)$ is not generated by tori and annuli. Then there exists a Riemannian metric and foliation $\mathcal{F}$ on M such that $\mathcal{F} \pitchfork \partial M$, and every leaf is minimal, i.e., has mean curvature zero.*

Proof. Sullivan [30] has shown that the leaves of a transversely oriented C^∞ foliation are all minimal for some Riemannian metric on M if and only if every leaf intersects a closed transverse curve. The corollary follows from Corollary 6.9.

Corollary 6.13. *Let M be compact and orientable. Let $p: \tilde{M} \to M$ be an n-fold covering map, and let $z \in H_2(M) = H^1(M, \partial M)$ or $z \in H_2(M, \partial M) = H^1(M)$. Then $n(x(z)) = x(p^*(z))$.*

Proof. **Case 1.** M is closed, orientable and irreducible.

Proof. Let S be a norm minimizing surface representing $z \in H_2(M) = H^1(M)$. By Theorem 5.5 there exists a transversely oriented foliation $\mathcal{F}$ without Reeb components such that S is a compact leaf. Then $p^*(\mathcal{F})$ has no Reeb components and $p^{-1}(S)$ is a compact leaf. Hence $p^{-1}(S)$ is norm minimizing. Since $\chi(p^{-1}(S)) = n\chi(S)$, the result follows.

Case 2. M is compact, oriented and irreducible, and ∂M is incompressible.

Lemma 6.14. *Let M be compact, oriented and irreducible with incompressible ∂M. Let S be a closed surface. Then S is norm minimizing in M, representing a class in $H_2(M)$ if and only if S is norm minimizing in $D(M)$.*

Proof. $\Rightarrow$: Let $\mathcal{T}\colon D(M) \to D(M)$ be the doubling involution, and let R be an incompressible surface such that $[R] = [S] \in H_2(D(M))$. View $D(M) = M_1 \cup M_2$ with $S \subset M_1$. Let $R_i = R \cap M_i$ and $\bar{R} = R_1 \cup \mathcal{T}(-R_2)$. Then $\bar{R}$ is an immersed surface with only double curves of self-intersection. Hence there exists an embedded surface T such that $[T] = [\bar{R}] = [S] \in H_2(M)$ and $x(S) \leqslant x(T) \leqslant x(R)$.

$\Leftarrow$: Immediate.

Lemma 6.15. *Let M be a compact oriented irreducible 3-manifold with boundary ∂M incompressible. Then S is a norm minimizing surface in M representing a class in $H_2(M, \partial M)$ if and only if $D(S)$ is a norm minimizing surface in $D(M)$.*

Proof. $\Rightarrow$: Let T be an incompressible surface such that $[T] = [D(S)] \in H_2(D(M))$. Isotope T so that no component of $T \cap M_1$ or $T \cap M_2$ is a disc. Since $[T \cap M_1] = [S] \in H_2(M_1, \partial M_1)$ and $[T \cap M_2] = [\mathcal{T}S] \in H_2(M_2, \partial M_2)$, and S and $\mathcal{T}S$ are norm minimizing, we conclude $x(D(S)) \leqslant x(T)$.

$\Leftarrow$: Immediate.

Proof of Case 2. Consider the induced map $D(\tilde{M}) \overset{Dp}{\to} D(M)$. It follows from Lemma 6.14 (Lemma 6.15) that $p^{-1}(S)$ is norm minimizing in $\tilde{M}$ representing a class of $H_2(\tilde{M})$ ($H_2(\tilde{M}, \partial\tilde{M})$) if and only if $(Dp)^{-1}S$ ($(Dp)^{-1}(D(S))$) is norm minimizing in $D\tilde{M}$. The result now follows from Case 1.

Case 3. General case. M is a compact oriented 3-manifold.

One can obtain a 3-manifold N which is compact oriented irreducible and ∂-incompressible by splitting M along a set J of disjoint properly embedded spheres and discs, capping off the resulting spherical boundary components with 3-balls and throwing away the fake 3-spheres. Let $q\colon \tilde{N} \to N$ be the induced covering map.

Any norm minimizing surface T for a class in $H_2(N, \partial N)$ (or $H_2(N)$) when regarded in M is norm minimizing. Hence by Case 2 the surface $q^{-1}(T)$ when regarded in $\tilde{M}$ is norm minimizing. Since any $z \in H_2(M, \partial M)$ (or $H_2(M)$)

satisfies $z = [T] + \alpha$ where $x(\alpha) = 0$ and T is a norm minimizing surface lying in N, we conclude $p^*(z) = p^*([T]) + p^*(\alpha)$ where $x(p^*[T]) = nx(z)$ and $x(p^*(\alpha)) = 0$. The result now follows from the fact that $x(\beta) = x(\beta + \lambda)$ if $x(\lambda) = 0$.

Remark. Corollary 6.12 was conjectured by Thurston in [32].

Definition 6.16. Let M be a compact oriented 3-manifold. Let $z \in H_2(M, \partial M)$ (or $H_2(M)$). Define

$$x_s(z) = \inf\left\{ \tfrac{1}{n}x(T) \,|\, f\colon T \to M \text{ and } f_*[T] = n \right.$$
$$\left. \text{where } f \text{ is a proper map of a compact oriented surface} \right\}.$$

x_s is the *singular norm* of $H_2(M, \partial M)$ (or $H_2(M)$).

Definition 6.17. Let M be a compact manifold. Let $z \in H_k(M; \mathbf{R})$ or $H_k(M, \partial M; \mathbf{R})$. Then the *Gromov norm* $g(z)$ of z is defined by

$$g(z) = \inf\left\{ \sum |a_i| \left[\sum a_i\sigma_i \right] = z \text{ where } \sum a_i\sigma_i \text{ is a singular chain} \right\}.$$

Corollary 6.18. *Let M be a compact oriented 3-manifold. Then on $H_2(M)$ or $H_2(M; \partial M)$,*

$$x_s = x = \tfrac{1}{2}g,$$

where x denotes the Thurston norm, x_s is the norm based on singular surfaces, and g denotes the Gromov norm.

Proof. $x_s = \frac{1}{2}g$. Gromov proved (see [33, §6]) that for hyperbolic k manifolds,

$$g([M, \partial M]) = \frac{\text{Volume } M}{\text{Volume } \sigma},$$

where σ is the volume of the largest hyperbolic k-simplex. For connected surfaces S of negative Euler characteristic we conclude

$$g([S, \partial S]) = \frac{2\pi|\chi(S)|}{\pi} = 2x(S).$$

For S of nonnegative Euler characteristic $g([S, \partial S]) = 2x(S) = 0$. Therefore if $kz \in H_2(M)$ or $H_2(M, \partial M)$ is represented by the singular surface S, then $g(z) \leqslant 2x(S)/k$; hence $\frac{1}{2}g(z) \leqslant x_s(z)$ for $z \in H_2(M, \partial M; \mathbf{Q})$. Since g, x_s are continuous functions, $\frac{1}{2}g \leqslant x_s$ on $H_2(M; \mathbf{R})$ or $H_2(M, \partial M; \mathbf{R})$.

Conversely, if $z \in H_2(M, \partial M; \mathbf{R})$, and $z = [\Sigma\, a_i\sigma_i]$, $a_i \in Z$, then by pasting together singular simplices one obtains a proper map $f\colon S \to M$ such that $f_*[S, \partial S] = z$; hence by Gromov, $x_s(z) \leqslant x(S) \leqslant \Sigma |a_i|$. For any singular cycle $z = [\Sigma\, a_i\sigma_i]$ and any $\epsilon > 0$ there exists a cycle $z' = [\frac{1}{n}(\Sigma\, m_i\lambda_i)]$ such that z, z' are ϵ close in $H_2(M, \partial M; \mathbf{R})$, and $m_i, n \in Z$ such that $\Sigma |m_i/n| \leqslant \Sigma |a_i| + \epsilon$.

The result $x_s(z) \leq \frac{1}{2}g(z)$ for $z \in H_2(M, \partial M; \mathbf{R})$ (and similarly $z \in H_2(M; \mathbf{R})$) now follows from the continuity of x_s and g.

To prove $x_s = x$ it suffices to show that for every compact oriented surface S and every proper map $f: S \to M$, $x(f_*[S]) \leq x(S)$.

Case 1. M is closed oriented and irreducible.

Proof. The idea of the proof is similiar to the idea of the proof given by Thurston showing that compact leaves of taut foliations are norm minimizing.

Let $f: S \to M$.

(A) it suffices to assume that if λ is a homotopically nontrivial simple closed curve in S, then $f(\lambda)$ is homotopically nontrivial in M.

(B) Find a norm minimizing surface T representing $f_*[S]$.

(C) Apply Theorem 5.5 to find a transversely oriented foliation $\mathfrak{F}$ of finite depth having no Reeb components such that T is a compact leaf.

Definition 6.19. Let $f: S \to M$ be an immersion of a surface in the $\mathfrak{F}$ foliated 3-manifold. f has *center tangency* at $p \in S$ if in some coordinate system, where leaves of $\mathfrak{F}$ restrict to subsets of horizontal planes, f maps a neighborhood of p into $\mathbf{R}^3$ so that $f(p)$ is a strict local maximum. f has a *saddle tangency* at $p \in S$ if in some coordinate system f maps a neighborhood of p into $\mathbf{R}^3$ as a saddle and f has a *circle tangency* along a simple closed curve γ if for each $x \in \gamma$ there exist a local coordinate system and a neighborhood U of X so that $f(U)$ appears as a ridgetop (Figure 6.1), where the arc of tangency lies in $f(\gamma)$. The rim of a volcano can be thought of as a circle tangency. In general $f(\gamma)$ is only an immersed curve on some leaf.

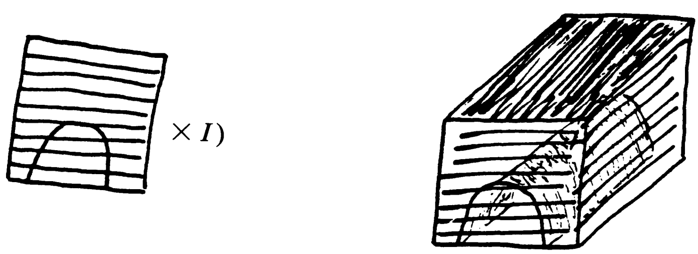


FIG. 6.1

(D) f is homotopic to an immersion $g: S \to M$ which has only circle and saddle tangencies.

Proof. The case when f is an embedding was proven independently by Thurston [31] and Roussarie [25]. An elementary topological proof is given in §7.

If F is smooth, and each leaf intersects a transverse closed curve (instead of F being finite depth), then there exists a proof of Theorem 7.1 due to Sullivan

[30], Sacks-Uhlenbeck [26], Schoen-Yau [27] and Hass [12] using minimal surface techniques. The idea is to use $[S_n]$ to find a Riemannian metric v making the leaves minimal, use [27] or [26] to show that f is homotopic to an immersion such that $g(S)$ is an immersed minimal surface in the v metric, and use [12] to observe that $g(S)$ is either contained in a leaf or transverse to F except at saddle tangencies (basically because if $g(S)$ met a leaf at a center, then $g(S)$ would not have mean curvature 0).

Definition 6.20. Let x be a saddle tangency. Let ($\vec{n}(x)$ resp. $\vec{N}(x)$) denote the unit normal to $g(S)$ (resp. $\mathfrak{F}$) at $g(x)$. Then define

$$\sigma(x) = \begin{cases} +1, & \text{if } n(x) = N(x), \\ -1, & \text{if } n(x) = -N(x). \end{cases}$$

(E) Let $E \in H^2(M)$ be the Euler class to the 2-plane bundle over M of tangent planes to $\mathfrak{F}$. By [32]

$$|E(g_*[S])| = \left| \sum_{\substack{x \text{ saddle} \\ \text{tangency}}} \sigma(x) \right| \leqslant \#\text{ saddle tangences} = |\chi(S)|.$$

Since T is a leaf, $|E([T])| = |\chi(T)|$. Since $[T] = g_*[S]$, we conclude

$$x(S) = |\chi(S)| \geqslant |\chi(T)| = x\big(f_*[S]\big).$$

Case 2. M is compact oriented and irreducible, and ∂M is incompressible.

Proof. (A) $f_*[S]$ represents a class in $H_2(M)$.

Consider

$$S \xrightarrow{f} M = M_1 \to D(M) = M_1 \cup M_2.$$

By Case 1 there exists an embedded T in $D(M)$ such that $[T] = f_*[S]$ and $x(T) \leqslant x(S)$. Since we can assume by Lemma 6.14 $f(S) \cap M_2 = \varnothing$, (A) follows by Case 1.

(B) $f_*[S]$ represents a class in $H_2(M, \partial M)$.

Consider $D(f)\colon D(S) \to D(M)$. By Case 1 there exists an oriented embedded $T \subset D(M)$ such that $(Df)_*[D(S)] = [T]$, $x(T) \leqslant x(D(S))$, and no component of $T - \partial M$ is a disc. Since $[T \cap M_i] = f_*[S] \in H_2(M_i, \partial M_i)$, the results follows.

Case 3. M is compact and oriented.

Proof. Let J be a collection of disjoint discs and spheres such that the manifold N obtained by splitting M along J and capping off spherical boundary components, and throwing away fake S^3's is compact, irreducible and ∂ incompressible. Since we can assume $f(S) \pitchfork J$ it follows $f^{-1}(J)$ is a union of disjoint simple closed curves; hence by pinching off these curves we can assume that $f(S) \cap J = \varnothing$. By Case 2, $f_*[S]$ can be represented by an embedded surface T with $x(T) \leqslant x(S)$, $[T] = f_*[S] \in H_2(N, \partial N)$ (or $H_2(N)$). Since $H_2(M, \partial M)$ is generated by $H_2(N, \partial N)$ (or $H_2(N)$) and components of J, the result follows.

Remarks 6.21. (1) The equality of the singular and Thurston norms was conjectured by Thurston in [32].

(2) Corollary 6.18 is a generalization of Dehn's lemma and the loop and sphere theorems to higher genus surfaces. How such theorems generalize was asked by Papakyriakopoulos [22] in 1957. In particular he asked about the relationship between the immersed genus and the genus of a knot.

Definition. The *genus* of a knot K in S^3 is the smallest g such that K bounds a punctured embedded surface of genus g. The *immersed genus* of a knot in S^3 is the smallest g such that K bounds a punctured immersed surface S of genus g which is nonsingular along the boundary, i.e. $f: S \to S^3$ and $f^{-1}(K) = \partial S$.

Corollary 6.22. If K is a knot in S^3, then the immersed genus equals the embedded genus. More generally if K is nontrivial and $f: T \to S^3 - \mathring{N}(K)$ is a proper map of an oriented surface no component of which is closed then $x(T) \geqslant (2g - 1)|n|$ where $f_*[T] = [n] \in H_2(S^3 - \mathring{N}(K),\ \partial N(K)) = \mathbf{Z}$. q.e.d.

More generally we have

Corollary 6.23. *Let M be a compact oriented 3-manifold, S a compact oriented connected surface with connected boundary, $f: S \to M$ a map such that $f|\partial S$ is an embedding and $f^{-1}(f(\partial S)) = \partial S$. Then there exists a compact embedded oriented surface T in M such that $\partial T = \partial S$ and genus $T \leqslant$ genus S.*

Proof. If genus $S = 0$, apply the classical Dehn's lemma. Otherwise let P be the manifold $M - \mathring{N}(\partial S)$ and let Q be the manifold obtained by doubling P along $\partial(N(\partial S))$. There is a natural map $g: \bar{S} \to Q$ where $\bar{S}$ is the double of S, $g|S = f$ and $g(S)$ is invariant under the doubling involution. Apply Corollary 6.18 to obtain an embedded norm minimizing surface $\bar{T}$ in Q representing the class $g_*[\bar{S}] \in H_2(Q)$. By pairing off oppositely oriented annuli we can assume that $\bar{T} \cap \partial(N(\partial S))$ is a simple closed curve. Our desired T is that component of $\bar{T} \cap P$ which intersects $\partial(N(\partial S))$ nontrivially. q.e.d.

This is exactly Dehn's lemma for higher genus surfaces.

7. The homotopy theorem

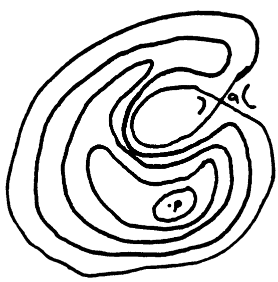
Theorem 7.1. *Let $M \neq S^2 \times S^1$ be a closed oriented 3-manifold. Let $\mathfrak{F}$ be a finite depth, transversely oriented foliation without Reeb components of M. Let $f_0: S \to M$ be a map of a closed oriented surface $S \neq S^2$ such that the image of* *every homotopically nontrivial simple closed curve in S is homotopically nontrivial. Then $f_0 \simeq g: S \to M$ where g is an immersion, and $g(S) \pitchfork \mathfrak{F}$ except for a finite number of circle and saddle tangencies.*

Proof. Suppose depth $\mathfrak{F} = K$, and for simplicity assume that the number of leaves of depth $< K$ is finite, if $K > 0$, since any depth K foliation can be "approximated" by such a foliation. In fact the depth K foliations constructed in §5 satisfy this condition.

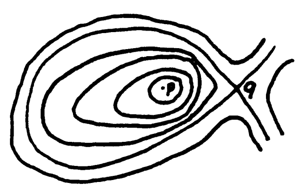
(1) By [14] f_0 is homotopic to an immersion f_1. By perturbing f_1 slightly we can assume that f_2 is an immersion with transverse self-intersections, $f_2(S) \pitchfork \mathfrak{F}$ except for a finite number of center and saddle tangencies, and that these points of tangency, triple points of $f_2(S)$, and points of double curves which are tangent to $\mathfrak{F}$ all lie on distinct depth K leaves.

(2) The general philosophy of the proof is to compress the centers until they either cancel with saddles or force other types of simplifications.

Let p be a center. As in Roussarie [25] ther exists a neighborhood U of p in S such that the induced foliation on U is one of the types exhibited in Figure 7.1 where $f_0(p)$ is a saddle tangency.



Type II



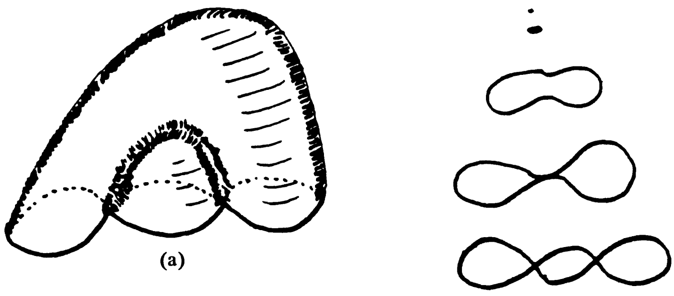
Type I

FIG. 7.1

As indicated in Figure 7.1 a neighborhood of p is foliated by a family of 'circles' indexed by $[0, 1]$ where the initial circle γ_0 is just the constant map to p, and the limiting circle is an embedded smooth curve with a corner at q when p is type I and is a smooth curve with two corners at q when p is type II. These facts follow from the Reeb stability theorem, Theorem 2.14 and the fact that a limit of compact leaves is a compact leaf [23]. One should think of $f_2 \circ \gamma: I \times S^1 \to M$ as a regular homotopy from γ_1 to γ_2. Each γ_i is homotopically trivial in some leaf L_i. For i sufficiently small there exists an immersion $h_i: D^2 \to L_i$ such that $f_2 \circ \gamma_i = h_i | \partial D^2$. For all $i \in [0, 1]$ we can define maps $h_i: D^2 \to L_i$ so that $f_2 \circ \gamma_i = h_i | \partial D^2$ by flowing discs along rays normal to $\mathfrak{F}$.

(3) The first obstruction to nicely squashing this center is that for some $i < 1$, $f_2 \circ \gamma_i$ does not bound an immersed disc in L_i. This is exemplified in

Figure 7.2 where we see the image of a neighborhood of p. Figure 7.2(b) shows the image of some of the γ_i's on the leaves containing them. If this occurs, and γ_j is the first circle such that $f(\gamma_j)$ does not bound an immersed disc, we define a homotopy $f: [2, 3] \times S \to M$ in 4 steps as follows.



(a) (b)

FIG. 7.2

(A) Define a homotopy $f: [2, 2\,1/4] \times S \to M$ which is fixed outside of $S - \gamma(0, j + \epsilon)$, i.e., $f(t, x) = f(t', x)$ for $x \in S - \gamma(0, j + \epsilon)$, by "pushing down" on the center to create a map which has two branched points and elsewhere is an immersion (Figure 7.3(a)).

Up to smoothing of corners $f_{2\,1/4}(S) = f_2(S - \gamma[0, j + \epsilon]) \cup h_{j+\epsilon}(D^2)$. The process of creating the branched points is similar to the homotopy from Figure 7.4(a) to Figure 7.4(b).

(B) Define $f: [2\,1/4, 2\,1/2] \times S \to M$ to be a homotopy fixed outside of the preimage of the double curves of $f_{2\,1/4}(S)$. This homotopy can be thought of as the operation of putting ones finger on one of the branch points and pushing (compare Figures 7.5(a) and 7.5(b)) it along a double curve until we get a very short double curve whose endpoints are branch points. Note that the two branch poinnts bound an immersed double curve γ in M; however, the path traversed by the finger may be quite different than the path described by γ. This is because when one traverses through a triple point, one changes the old double point locus. If B was an open set in M such that $B \cap f_{2\,1/4}(S)$ looks like Figure 7.2, then $B \cap f_{2\,1/2}(S)$ would appear as in Figure 7.3(b). By a very short double curve d we mean that there exists a neighborhood C of d such that $C \cap f_{2\,/12}(S)$ looks exactly like Figure 7.3(c). In particular there exists a simple closed curve δ on S lying close to d such that $f_{2\,1/2}(\delta) = \lambda$ is a simple closed curve bounding a disc in M. Hence δ bounds a disc D in S.

(C) Since $\mathfrak{F}$ has no Reeb components and $M \neq S^2 \times S^1$. $\pi_2(M) = 0$. Define a homotopy $f: [2\,1/2, 2\,3/4] \times M$ fixed outside of D so that $f_{2\,3/4}(D)$ lies very close to d. If Figure 7.3(b) denoted $B \cap f_{2\,1/2}(S)$, then $B \cap f_{2\,3/4}(S)$ would appear as Figure 7.3(d) or 7.3(e) depending on which side of λ, $f_{2\,1/2}(D)$ lay on.

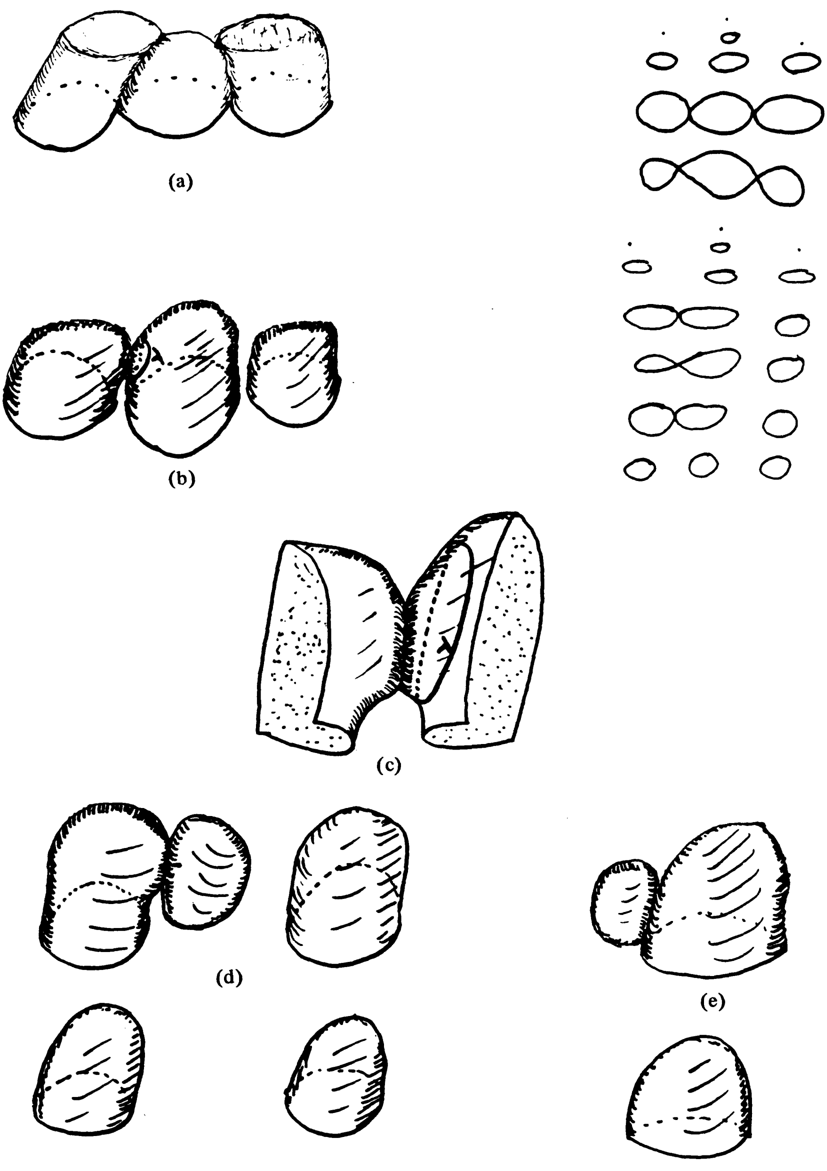
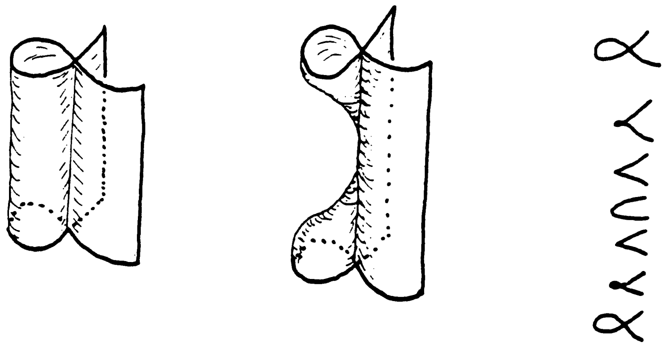
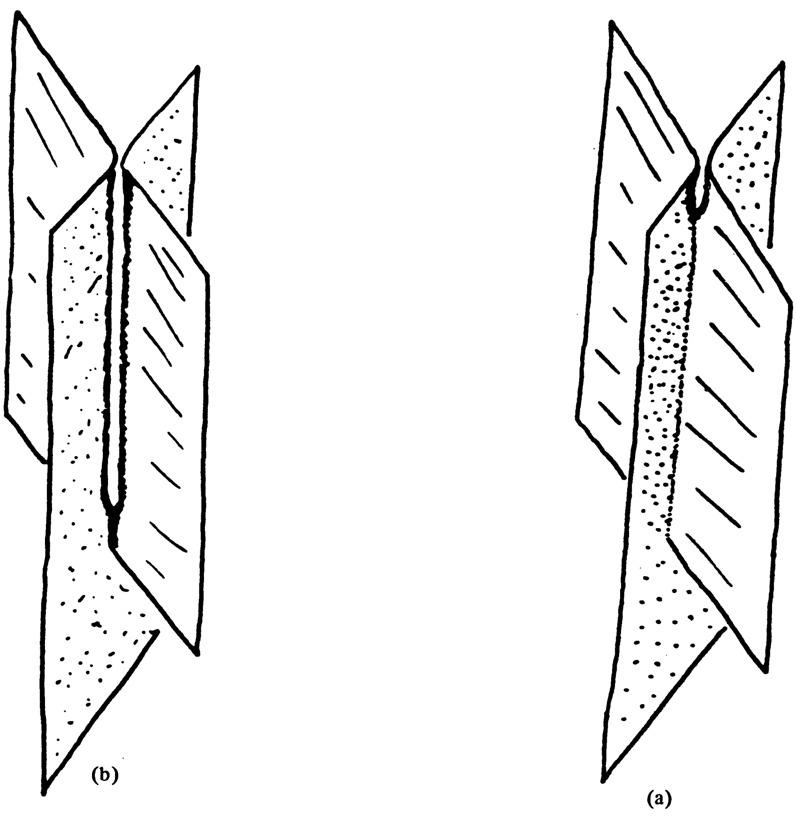


FIG. 7.3



(a) (b)

FIG. 7.4



(b) (a)

FIG. 7.5

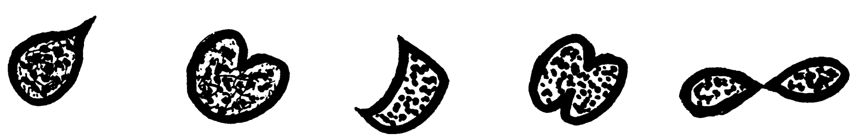
(D) Define $f[2\,3/4, D] \times S \to M$ to be the map which sucks in the disc completely and eliminates the branch points (Figure 7.3(f)).

This homotopy $f\colon [2, 3] \times S \to M$ should be done in such a way that center and saddle tangencies, triple points, and double curve tangencies lie on different depth K leaves.

(4) One now considers a center P of $f_3(S)$ and the corresponding curves γ_i, $i \in [0, 1]$. If for some $i < 1$, $f_3 \circ \gamma_i$ does not bound an immersed disc, then one performs a homotopy as in (3). After a finite number of such homotopies each center will satisfy the property that the γ_i's, $i < 1$, bound immersions. If not then a contradiction can be arrived at as in (10).

(5) Let P be a type I center with saddle q. We say that p is type Ia if $\sigma(p) = \sigma(q)$ (σ was defined in §6), and p is type Ib otherwise. Since each γ_i, $i < 1$, bounds an immersion and since tangencies of double curves cannot lie on L_1, we conclude that γ_1 bounds in some leaf a pinched immersed disc (Figure 7.6(a)). Typical examples of images of type I centers are given in Figure 7.7. Notice that the image of a small neighborhood of p was removed in the picture of the type Ib center.

If f_n was the last function defined, and p is type Ia, then let $f\colon [n, n + 1] \times S \to M$ be the homotopy which 'cancels' the center with the saddle. Notice that $f_{n+\alpha}(S)$ is approximately $f_n(S - \gamma[0, \alpha]) \cup h_\alpha(D)$ for $0 \leqslant \alpha \leqslant 1$.



Ia	Ib	IIa	IIb_1	IIb_2
a)	b)	c)	d)	e)

FIG. 7.6

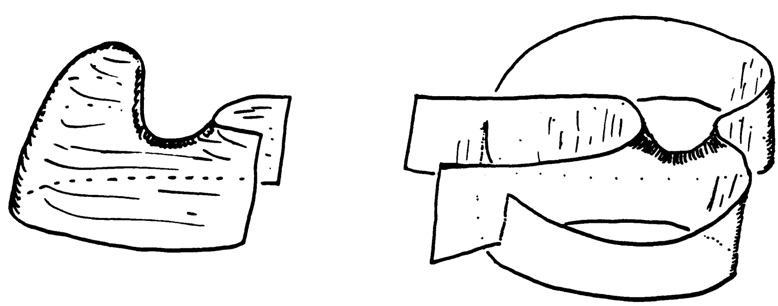


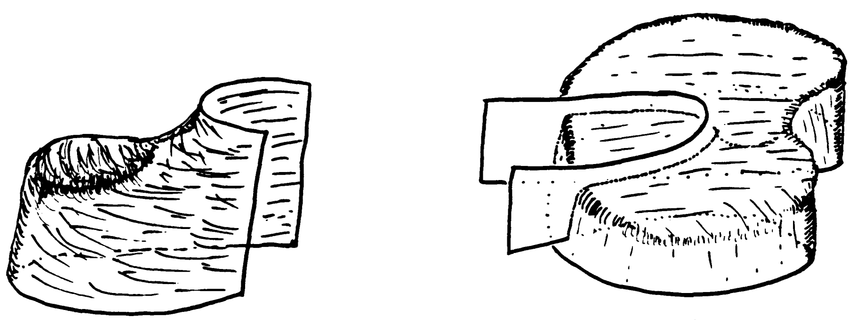
FIG. 7.7

If p is a type Ib center, then define $f[n, n+1] \times S \to M$ as the composite of the following 3 homotopies.

(A) Define $f\colon [n, n+1/3] \times S \to M$ to be the homotopy which compresses the center so that it lies just above the saddle as in Figure 7.8(b), i.e., $f_{n+1/3}(S) \approx f_n(S - \gamma[0, 1-\epsilon]) \cup h_{1-\epsilon}(D)$.

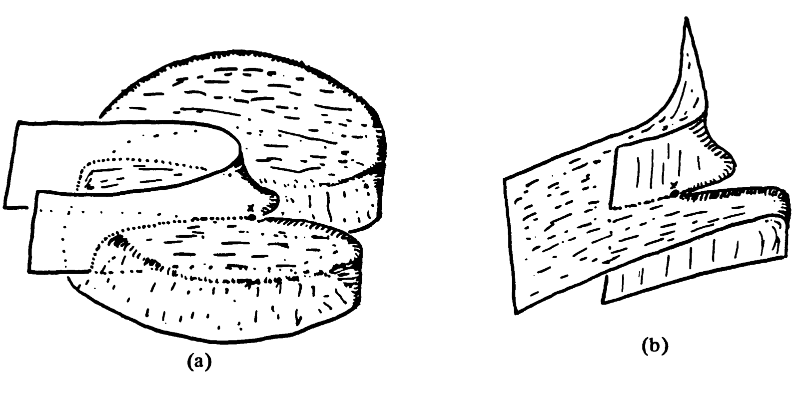
(B) Define $f[n+1/3, n+2/3] \times S \to M$ to be the map where the disc $h_{1-\epsilon}(D)$ is pushed slightly below the level of the saddle simultaneously performing the homotopy of Figure 7.4 to create a branched immersion (Figure 7.9(a)) with 2 branch points. Figure 7.9(b) gives a close-up view of one of the branch points. The inverse of this homotopy can be thought of as the map which coalesces the two branch points.

(C) We proceed as in parts 4(D) and 4(C) to define $f\colon [n+2/3, n+1] \times S \to M$ as the homotopy which creates a very short double curve, sucks in a disc homotoping the short double curve and its branch points away.



(a) (b)

FIG. 7.8



(a) (b)

FIG. 7.9

(6) By doing the above procedure we may have created many new centers and thus be forced back to (4). The methods of (10) will show that we can have only a finite number of homotopies of types already described.

(7) Suppose p is type II center with saddle q, and γ_i bounds an immersed disc in L_i for $i < 1$. Then the limiting curve γ_1 bounds either a doubly pinched immersed disc (Figure 7.6(c), (d)) or a squeezed immersed disc (Figure 7.6(e)). If the former occurs, then p is of type IIa or type IIb_1 depending on whether or not $\sigma(p) = \sigma(q)$. If γ_1 bounds a squeezed disc, then we conclude that $\sigma(p) \neq \sigma(q)$, and we say p is of type IIb_2.

An example of a type IIa center is given in Figure 7.10 and a type IIb_2 in Figure 7.11. Figure 7.12 shows $f_m(U)$ where U is a neighborhood of γ_1 when p is type IIb. Figure 7.13 shows a close-up view of the saddle at $f(q)$.

(8) If p is a center of type IIb_2, then γ_1 is a composite of two simple closed homotopically trivial curves λ_1, λ_2. Each λ_i must bound a disc D_i in S. Assume $D = D_1 \supset D_2$. If f_m has been the last function defined, then let $f[m, m+1] \times S \to M$ be the homotopy fixed outside of D so that $f_{m+1}(D)$ lies near the

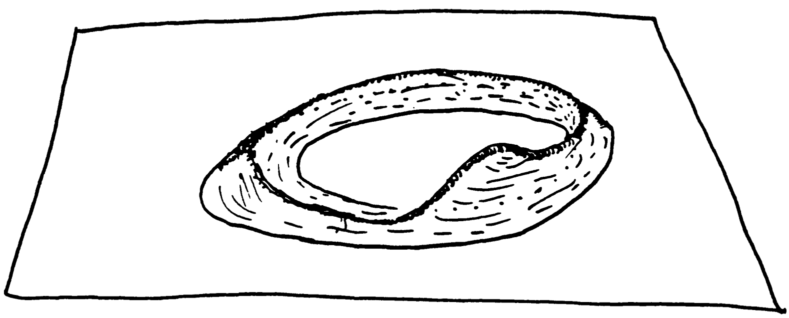


FIG. 7.10

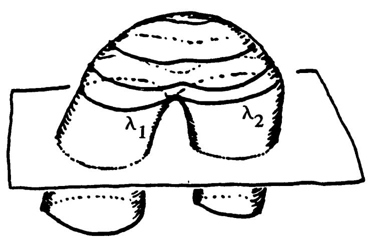


FIG. 7.11

immersed disc which λ_1 bounds in L_1. A view of $f_{m+1}(S)$ near q will appear as in Figure 7.14.

If p is a center of type IIb_1, perform a homotopy $f\,[m, m+1] \times S \to M$ as a composite of the following 3 homotopies.

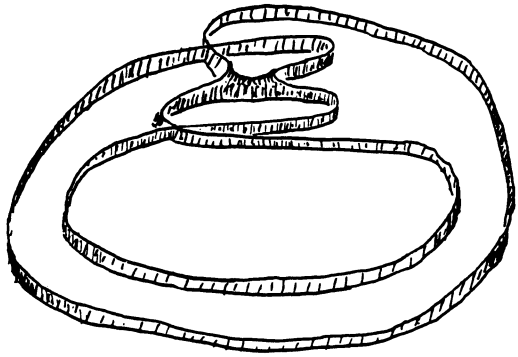


FIG. 7.12

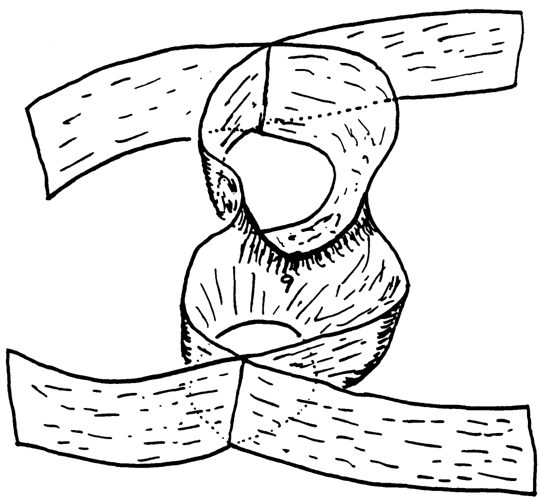


FIG. 7.13

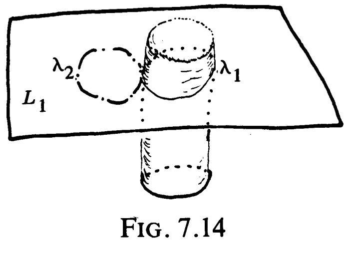


FIG. 7.14

(A) Define $f\colon [m, m + 1/3] \times S \to M$ to be the homotopy which compresses the center so that it lies in a leaf just above q, i.e., $f_{m+1/3}(S) \approx f_m(S - \gamma[0, 1 - \epsilon]) \cup h_{1-\epsilon}(D)$. Push the disc $h_{1-\epsilon}(D)$ normally to $\mathcal{F}$ simultaneously performing the homotopy of Figure 7.4 to create a branched immersion with two branch points. If B is a neighborhood in M which intersects $f_m(S)$ as in Figure 7.12 (recall that some surface has been excised), then $B \cap f_{n+1/3}(S)$ will appear as in Figure 7.15. (Note the branch points x_1 and x_2.) In that picture we have excised a disc E for greater clarity. It may be helpful to think of the inverse of the homotopy. First glue back the disc E and perform the homotopy of Figure 7.4 to coalesce the branch points with the newly created double curve passing through the disc E (Figure 7.16). Notice that $f_{m+1/3}(S)$ has a circle tangency.

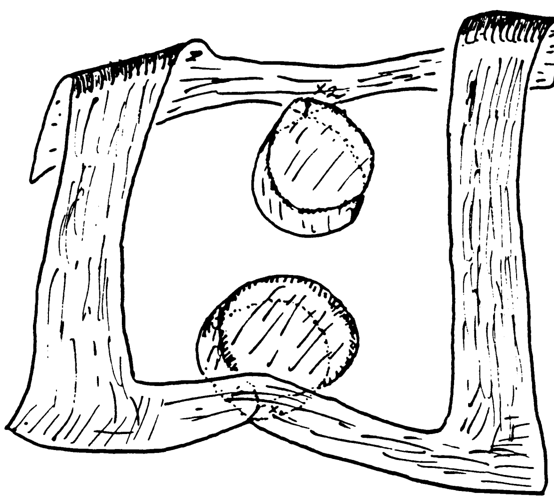


FIG. 7.15

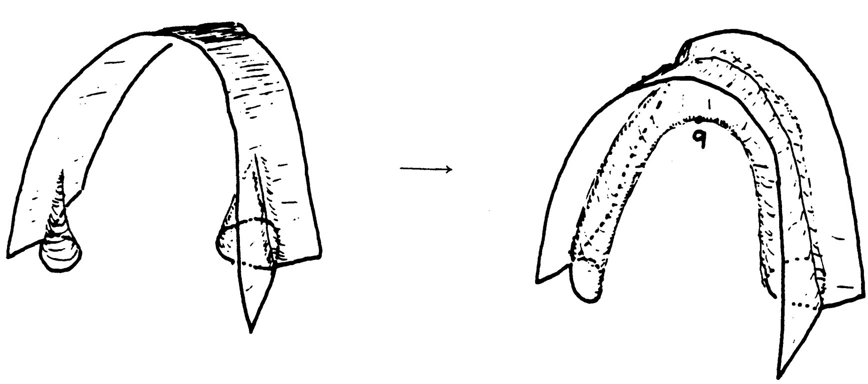


FIG. 7.16

(B) Define $f[m+1/3, m+2/3] \times S \to M$ as in (7)(C) and (7)(D) to create a short double curve and suck in a disc to eliminate the branch points.

(C) If after (B) the circle tangency is still there, perturb $f_{m+2/3}$ slightly to get a type IIa center. If the circle tangency has been destroyed in (B), then it may be necessary to perturb $f_{m+2/3}$ to make sure all the centers lies on different leaves, etc.

(9) Suppose that after finitely many homotopies we are only left with type IIa centers. Then by cancelling them with their saddles, to create circle tangencies, we complete the homotopy. Let $g: S \to M$ be the resulting map.

(10) We now show that there can only be a finite number of homotopies of types already described. We first give the proof in the case where depth $\mathcal{F} = 0$.

(A) Let T be a leaf of $\mathcal{F}$ satisfying $T \pitchfork f_2(S)$ and $T \cap$ (triple points of $f_2(S)) = \emptyset$. Then T has a tubular neighborhood $V = T \times [-1, 1]$ such that $\mathcal{F}| V$ has the product foliation and $V \cap f_2(S) = (f_2(S) \cap T) \times [-1, 1]$.

By checking what happens as we perform one of the previous homotopies we conclude that for each $n \geqslant 2$, $f_n(S) \cap V = (f_n(S) \cap T) \times [-1, 1]$, and either $f|[n, n+1]$ is *fixed on* V (i.e., $f_t(x) = f_{t'}(x)$ if $(f_t(x) \cup f_{t'}(x)) \cap V \neq \emptyset$) or $f_n(S) \cap T \underset{\neq}{\supset} f_{n+1}(S) \cap T$ and $\alpha_{n+1} < \alpha_n$ where

$$\alpha_j = |\text{Double points of } f_j(S) \cap T| + |f_j(S) \cap T - \text{Double points of } f_j(S) \cap T|.$$

Therefore if N is sufficiently large, then $f|[n, n+1]$ is fixed on V for $n \geqslant N$.

(B) Let $V' = T \times [-1/2, 1/2]$. Let $W = M - \mathring{V}'$. Then $W \cong T \times [2, 3]$ with the product foliation. Let n be a normal vector field to $\mathcal{F}| W$. Assume that except for 'sharp corners' $f_N(S)$ is composed of horizontal and vertical pieces, i.e., for almost all $x \in S$ there exists a neighborhood U of x such that $f_N(U)$ is tangent to $\mathcal{F}$, or $f_N(U)$ is contained in a union of normal curves through a curve on some leaf. For example Figure 7.17(a) is Figure 7.2 viewed as a union of horizontal and vertical pieces. Figure 7.17(b) shows precisely how to construct 7.17(a).

Let $\pi: V' \to T \times (2 + 1/2) = T'$ be the normal projection onto T'. Let P be the projection of all the vertical pieces of $f_N(S) \cap V$. Then P is a graph on T' with complementary regions $Y_1, \cdots, Y_r$. Let $y_i \in Y_i$ be an interior point and $y = \cup y_i$. Let $\beta_n = (a_n, b_n)$, where $a_n = |f_n^{-1}\pi^{-1}(y)|$, and b_n is the number of 'centers' of $f_n(S)$, i.e., b_n is the number of local maximal plateaus. Ordering such 2-tuples by the dictionary ordering we claim $\beta_{m+1} < \beta_m$ for all $m > N$. This can be seen by considering what happens to horizontal regions during homotopies. Any homotopy involving the compression of a center such that some γ_i does not bound an immersion or homotopies involving type Ib, IIb, or IIb_2 centers must reduce the first component of β. Any other homotopy involving the cancellation of a center or saddle must reduce the second component of β

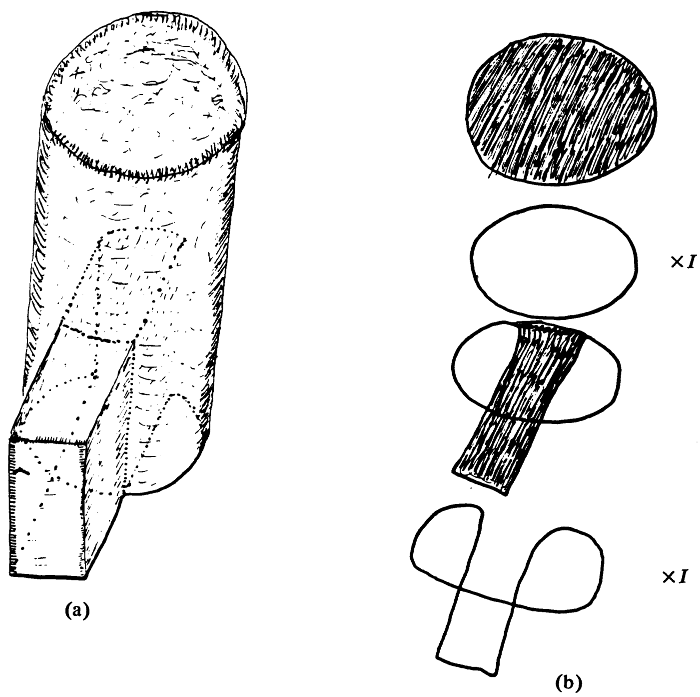


FIG. 7.17

while the first component is fixed. Figures 7.17 and 7.18 exemplify the case when some γ_i does not bound an immersion. If B is a region in M such that $f_n(S) \cap B$ is given as in Figure 7.17(a), then Figure 7.18(b) shows both the projection P' of the vertical regions of $f_n(S) \cap B$ onto T' and the number of horizontal pieces which project into each component of $T' - P'$. Figure 7.18(a) shows $f_{n+1/4}(S) \cap B$ and Figure 7.18(c) indicates the number of horizontal pieces of $f_{n+1/4}(S) \cap B$ which project into each component of $T' - P'$. Notice how vertical regions with opposite oriented normal vectors have cancelled each other out. The other cases are similar. This completes the proof when depth $\mathcal{F} = 0$.

(11) Suppose depth $\mathcal{F} = k > 0$. For simplicity assume that the number of depth $l < k$ leaves are finite. Let T_i be the union of the depth i leaves if $i < k$, and let T_k be a finite set of depth k leaves such that the quotient map

$$P : \text{depth } k \text{ leaves} - T_k \to \text{Space of } (\text{depth } k \text{ leaves} - T_k)$$

is a fibration over $(0, 1)$ and that for all $j > 2$, $T_k \cap [(\text{triple points of } f_j(S)) \cup (\text{tangencies of double curves of } f_j(S) \text{ with } \mathcal{F}) \cup (\text{points of tangency of } f_j(S) \cap \mathcal{F})] = \varnothing$.

(A) Let V be a small neighborhood of T_0 homeomorphic to $T_0 \times I$ such that $V \cap f_2(S) = (f_2(S) \cap T_0) \times I$. Furthermore, if $W = K \times I$ is a component of V and if $K \times i$, $i = 0, 1$, is component of ∂W, then $K \times i$, a surface with corners, is a union of an annulus A_i transverse to $\mathscr{F}$ and a twice punctured surface B_i contained in some component of T_1 (Figure 7.19). Let $B = B_0 \cup B_1$.

Equip W with the 'canonical' normal vector field to $\mathscr{F}$, and let $A_0 \cup A_1$ be tangent to normal curves. Let J_i be the leaf of $\mathscr{F}| W$ which contains B_i. If W is sufficiently small, then J_i is half of an infinite cyclic covering space of K (Figure 7.20).

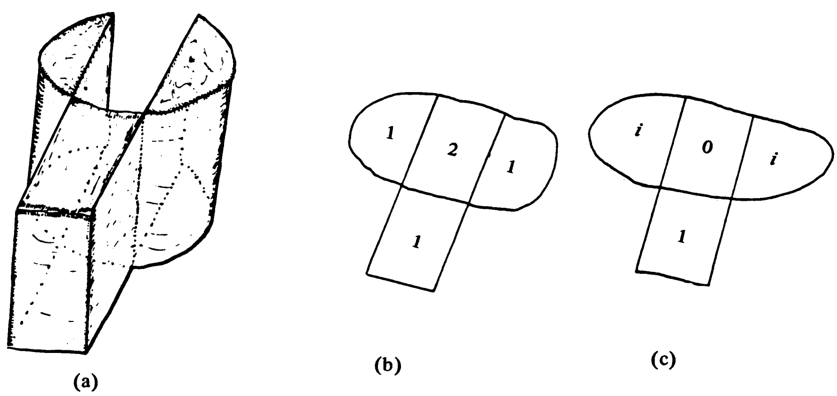


FIG. 7.18

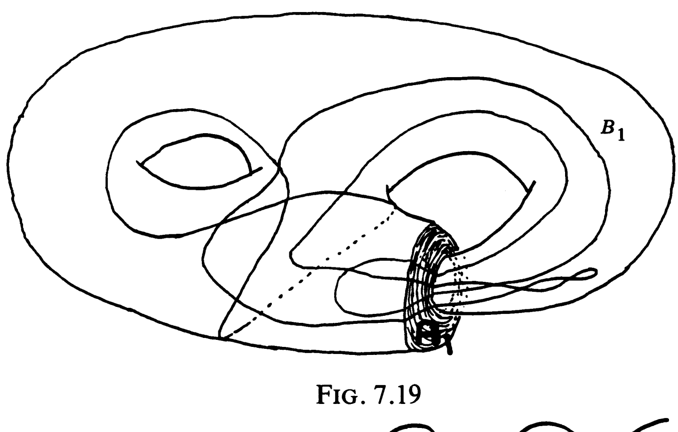


FIG. 7.19

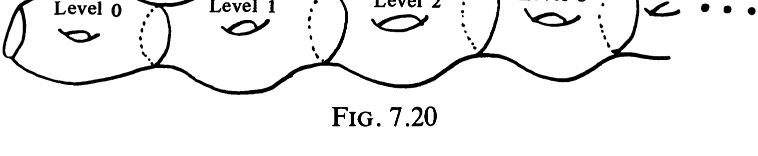


FIG. 7.20

Let $x \in W - K$. Then we say that x is at level j if the normal ray through x not passing through K intersects $L_0 \cup L_1$ $j+1$ times. Let $U_i = W - \{x \in W | \text{level } x < i\}$. U_i qualitatively looks like W except that a certain number of levels have been peeled away.

(B) We show that there exist N_1, N_2 sufficiently large such that any homotopy $f\colon [n, n+1] \times S \to M$, $n > N_1$, is fixed on U_{N_2}. Let i be the first integer so that $f\colon [i-1, i] \times S \to M$ is not fixed on W. Then one of the following holds.

(i) $c(f_i(S) \cap B) < c(f_{i-1}(S) \cap B)$, where

$$c\big(f_j(S) \cap B\big) = |f_j(S) \cap B - \text{Double points of } f_j(S) \cap B| + |\text{Double points of } f_j(S) \cap B|.$$

(ii) f is fixed on B.

View each $f\colon [j, j+1] \times S \to M$ as a composite of three homotopies, one that compresses a disc, one that does surgery along double curves (i.e., creates a short double curve with branch point end points), and one that sucks in a disc. Each time a compression of a disc involves B there exist a $j \in (i-1, i)$ and a maximal disc D in S such that $f_j(D)$ is an immersed disc lying in some leaf of $\mathcal{F}$ such that $f_j(D) \cap B \neq \varnothing$. If $f_j(\partial D) \cap B = \varnothing$, then $f_j(D) \cap \partial B \neq \varnothing$ which implies that some component of ∂B is null homotopic, so that K is compressible, and hence $\mathcal{F}$ has Reeb components by Novikov [21]. If $f_j(\partial D) \cap B \neq \varnothing$, then $c(f_{j+\epsilon}(S) \cap B) < c(f_{j-\epsilon}(S) \cap B)$. If the creation of a short double curve involves B, then the number of double points of $f_i(S) \cap B$ is less than the number of double points of $f_{i-1}(S) \cap B$ while the first component of the complexity does not rise. Since $M - N(B)$ is irreducible, the process of sucking in a disc can be made to avoid B if (i) does not hold. We now conclude that there exists an N such that if $n > N$, then every homotopy $f[n, n+1] \times S \to M$ will be fixed on B.

It is easy to check that for each n there exists an r_n such that $f_t(S) \cap U_{r_n}$ is vertical for $2 \leq t \leq n$, $t \in \mathbf{Z}$. As in (10), define P to be the normal projection of the vertical part of $f_N(S) \cap W$ into K. P is a graph with complimentary regions $Y_1, \cdots, Y_r$. Pick $y_i \in Y_i$ and let $y = \cup y_i$. For $n > N$, define $\beta_n = (a_n, b_n)$ where $a_n = |f_n^{-1}\pi^{-1}(y)|$, and b_n is the number of 'centers' of $f(S)$ lying in W. By applying the methods of (10) to W we conclude that if $n > N$, either $r_n = r_{n+1}$ and $f[n, n+1] \times S \to M$ is fixed on U_{r_n}, or $\beta_{n+1} < \beta_n$.

The proof of (B) now follows.

(C) Suppose that there exist a neighborhood X_j of $\cup_{i=0}^{j} T_j$ and an N such that $f[n, n+1] \times S \to M$ is fixed on X_j if $n \geq N$.

There is a finite union of compact incompressible surfaces $K_1, \cdots, K_r$ contained in T_{j+1} such that $K_1 \cup, \cdots, \cup K_r \cup X_j \supset T_{j+1}$ and $\partial K_i \subset X_j$. Let V (Figure 7.21) be a small neighborhood of $K = \bigcup K_i$ homeomorphic to $L \times I$ such that:

(i) $V \cap f_N(S) = (f_N(S) \cap L) \times I$,

(ii) $\partial V = B_0 \cup B_1 \cup A \cup E$ where $A \cup E \pitchfork \mathfrak{F}$, $E = \partial K \times I \subset X_j$, $B = B_0 \cup B_1$ are contained in depth $\min(j+2, k)$ leaves, and $A = \lambda \times [0, 1]$ where λ is a possibly empty union of arcs and simple closed curves.

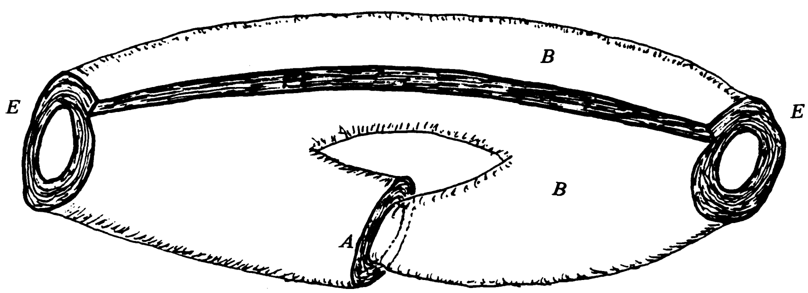


FIG. 7.21

Let L_i be the leaves of $\mathfrak{F} | V$ which contain B_i, $i = 0, 1$. As in (B), define the levels of $V - K$ and the neighborhoods U_i of K. Note that $E \subset X_j$; hence any homotopy $f[n, n+1] \times S \to M$ $(n > N)$ is fixed on a neighborhood of E. By mimicking the proof in (B) we conclude that there exist N_1, N_2 such that $n > N_1$ implies $f[n, n+1] \times S \to M$ is fixed on U_{N_2}. Let $X_{j+1} = U_{N_2} \cup X_j$.

(D) The set of depth k leaves is a union of a finite number of components $H_1, \cdots, H_n$ such that for each i the quotient map

$$p : H_i \to \text{Space of leaves of } \mathfrak{F} | H_i$$

is a fibration over S^1, and $H_i \cap T_k$ contains a leaf L_i. Hence $H_i = (L_i \times I)/[(x, 0) \sim (h_i(x), 1)]$ and $H_i - L_i = L_i \times (0, 1)$. By the compactness of M there exist $a, b \in (0, 1)$ and a compact incompressible $Q_i \subset L_i$ such that $L_i \times (0, 1) - (\mathring{Q}_i \times (a, b)) \subset X_{k-1}$. Hence any homotopy $f: [n, n+1] \times S \to M$ for n sufficiently large must be carried in the interior of the product $Q \times [a, b]$. By applying the methods of (10) we conclude that for n sufficiently large $f[n, n+1] \times S \to M$ is fixed on $Q_i \times [a, b]$. By applying this to each $Q_i \times [a, b]$ the proof of Theorem 7.1 is concluded.

Remark. By generalizing the above procedure (see [25]) it is not difficult to show that if every leaf intersects a closed transversal, then $f \simeq g : S \to M$ such that g is an immersion, and either $g(S)$ is contained in a leaf or $g(S) \pitchfork \mathfrak{F}$ except along a finite number of saddle tangencies.

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