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STRUCTURE THEOREMS ON RIEMANNIAN SPACES SATISFYING $R(X, Y) \cdot R = 0$. I. THE LOCAL VERSION

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Introduction

The curvature tensor R of a locally symmetric Riemannian space satisfies $R(X, Y) \cdot R = 0$ for all tangent vectors X and Y, where the linear endomorphism $R(X, Y)$ acts on R as a derivation. This identity holds in a space of recurrent curvature also.

The spaces with $R(X, Y) \cdot R = 0$ have been investigated first by E. Cartan [2] as these spaces can be considered as a direct generalization of the notion of symmetric spaces. Further on remarkable results were obtained by the authors A. Lichnerowicz [13], R. S. Couty [3], [4] and N. S. Sinjukov [19], [20], [21]. In one of his papers K. Nomizu [15] conjectered that an irreducible, complete Riemannian space with dim $\geqslant 3$ and with the above symmetric property of the curvature tensor is always a locally symmetric space. But this conjecture was refuted by H. Takagi [22] who constructed 3-dimensional complete irreducible nonlocally-symmetric hypersurfaces with $R(X, Y) \cdot R = 0$. These two papers were very stimulating for the further investigations. We also have to mention the following authors in this field: S. Tanno [23], [24], [25], K. Sekigawa [16], [17] and P. I. Kovaljev [9], [10], [11].

In the following we call a space satisfying $R(X, Y) \cdot R = 0$ a semi-symmetric space. The main purpose of this paper is to determine all semi-symmetric spaces in a structure theorem.

In §1 we give local decomposition theorems using the infinitesimal holonomy group, and in §2 we give some basic formulas. We would like to make it perfectly clear that the results of these chapters are concerning general Riemannian spaces, and not only semi-symmetric spaces. In §3 we construct several nonsymmetric semi-symmetric spaces and in §4 we show that every semi-symmetric space can be decomposed locally on an everywhere dense open subset into the direct product of locally symmetric spaces and of the spaces constructed in §3.

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1. Primitive holonomy groups and the local decomposition by means of infinitesimal holonomy group

Let (M^n, g) be a Riemannian space of class C^∞, whose curvature tensor field $R(X, Y)Z$ satisfies the so-called Bianchi-identities:

$$\begin{aligned}
&R(X, Y) = -R(Y, X),\\
&R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0,\\
&g(R(X, Y)Z, V) = -g(R(X, Y)V, Z),\\
&g(R(X, Y)Z, V) = g(R(Z, X)X, Y),\\
&(\nabla_X R)(Y, Z)V + (\nabla_Y R)(Z, X)V + (\nabla_Z R)(X, Y)V = 0;
\end{aligned}$$

the first four identities are called the first Bianchi-identities, and the 5th the second Bianchi identity.

For a point $p \in M^n$ let us consider the linear set h_p of skew-symmetric endomorphisms in the tangent euclidean space $(T_p(M), g_{|p})$ spanned by elements of the form $R_{|p}(X, Y)$, i.e.,

$$h_p := \left\{ \sum_i a_i R_{|p}(X_i, Y_i) \mid X_i, Y_i \in T_p(M^n), a_i \in \mathbf{R} \right\}. \tag{1.1}$$

The Riemannian space (M^n, g) is said to be semi-symmetric iff for it

$$\nabla_X \nabla_Y R - \nabla_Y \nabla_X R - \nabla_{[XY]} R = 0, \tag{1.2}$$

or by Ricci's identities, for every point $p \in M$ and $X, Y, U, V \in T_p(M)$,

$$\begin{aligned}
\left[R_{|p}(U, V), R_{|p}(X, Y)\right] &:= R_{|p}(U, V) \circ R_{|p}(X, Y) - R_{|p}(X, Y) \circ R_{|p}(U, V)\\
&= R_{|p}\left(R_{|p}(U, V)X, Y\right) + R_{|p}\left(X, R_{|p}(U, V)Y\right).
\end{aligned} \tag{1.3}$$

For a semi-symmetric space h_p is a Lie algebra with the bracket operation $[u, v] = u \circ v - v \circ u$. Furthermore if $\mathcal{H}_p$ denotes the connected subgroup of isometries in $T_p(M)$ determined by the Lie algebra h_p, then the curvature tensor $R_{|p}$ is invariant under the action of $\mathcal{H}_p$, i.e., for every $u \in \mathcal{H}_p$, $X, Y, Z \in T_p(M)$

$$(uR)(X, Y)Z := uR(u^{-1}X, u^{-1}Y)u^{-1}Z = R(X, Y)Z \tag{1.4}$$

holds. In this chapter we consider a general C^∞ Riemannian manifold. For such a space let $\bar{h}_p$ be the Lie algebra generated by h_p, and let $\mathcal{H}_p$ be the connected subgroup of isometries in $T_p(M)$ determined by $\bar{h}_p$. Then from the above considerations it obviously follows that a Riemannian space (M^n, g) is semi-symmetric iff for every point $p \in M$ the group $\mathcal{H}_p$ leaves the tensor $R(X, Y)Z$ invariantly.

In the general case the group $\mathcal{H}_p$ is called the *primitive holonomy group* at the point p. It is a subgroup of the infinitesimal holonomy group and thus also of the whole holonomy group. Let

$$T_p(M) = V_p^{(0)} + V_p^{(1)} + \cdots + V_p^{(r)} \tag{1.5}$$

be the irreducible decomposition of the tangent space with respect to $\mathcal{H}_p$. Thus the subspaces $V_p^{(i)}$ are invariant under the action of $\mathcal{H}_p$; they are pairwise orthogonal. Furthermore $\mathcal{H}_p$ acts on $V_p^{(0)}$ trivially, and its action is irreducible on $V_p^{(i)}$, $i > 0$.

Definition 1.1. The decomposition (1.5) is called the V-decomposition of the tangent space.

Because of the invariance of the subspaces $V_p^{(i)}$ we get

$$R(X, Y)X_i \in V_p^{(i)} \text{ for } X_i \in V_p^{(i)} \text{ and } X, Y \in T_p(M).$$

On the other hand if $i \neq j$, $X_i \in V_p^{(i)}$, $X_j \in V_p^{(j)}$, then

$$0 = g\big(R_{|p}(X, Y)X_i, X_j\big) = g\big(R_{|p}(X_i, X_j)X, Y\big),$$

and so $R_{|p}(X_i, X_j) = 0$. Thus for arbitrary vectors $X, Y \in T_p(M)$ we get

$$R_{|p}(X, Y) = \sum_{i=1}^{r} R_{|p}(X_i, Y_i), \tag{1.6}$$

where $X = \sum_{i=1}^{r} X_i$ and $Y = \sum_{i=1}^{r} Y_i$ are the decompositions of X and Y with respect to the V-decomposition (1.5). It is plain that for every vectors $X_p \in V_p^{(0)}$ and $X \in T_p(M)$ the equation $R_{|p}(X_0, X) = 0$ holds. Furthermore for $i \neq j$ we get

$$\begin{aligned} R_{|p}(X_i, Y_i)X_j = -R_{|p}(Y_i, X_j)X_i - R_{|p}(X_j, X_i)Y_i = 0,& \\ \text{if } X_i, Y_i \in V_p^{(i)}, X_j \in V_p^{(j)},& \end{aligned} \tag{1.7}$$

which means that the action of endomorphisms of the form $R_{|p}(X_i, Y_i)$, $X_i, Y_i \in V_p^{(i)}$, is trivial on the subspace $V_p^{(j)}$ (where $j \neq i$). Thus let $h_p^{(i)}$ be the Lie algebra of skew-symmetric endomorphisms in $V_p^{(i)}$ generated by the elements of the form

$$R_{|p}(X_i, Y_i)\colon V_p^{(i)} \to V_p^{(i)},\ X_i, Y_i \in V_p^{(i)}, \tag{1.8}$$

and let $\mathcal{H}_p^{(i)}$ be the connected subgroup of isometries in $V_p^{(i)}$ referring to $\bar{h}_p^{(i)}$. Then the following statement is a simple consequence of the above considerations.

Proposition 1.1. *For a Riemannian manifold (M^n, g) let*

$$T_p(M) = V_p^{(0)} + V_p^{(1)} + \cdots + V_p^{(r)}$$

be the V-decomposition of the tangent space $T_p(M)$, $p \in M$. Then $\mathcal{H}_p$ is the direct product

$$\mathcal{H}_p = \mathcal{H}_p^{(0)} \times \mathcal{H}_p^{(1)} \times \cdots \times \mathcal{H}_p^{(r)} \tag{1.9}$$

of normal subgroups, where $\mathcal{H}_p^{(i)}$ is trivial on $V_p^{(j)}$ for $i \neq j$, and is irreducible on $V_p^{(i)}$ for each $i = 1, \cdots, r$, and $\mathcal{H}_p^{(0)}$ consists only of the identity.

Now we state an important theorem.

Theorem 1.1 (*The first stability theorem*). *For a C^∞ Riemannian space (M^n, g) there can be chosen an everywhere dense open subset U of M, such that on every arcwise connected component U_α of U for a certain order $V_p^{(0)}$, $V_p^{(1)}, \cdots, V_p^{(r)}$, $p \in U_\alpha$, of the invariant subspaces the subspaces $V_p^{(i)}$, $p \in U_\alpha$, have constant dimension, and the distributions $p \to V_p^{(i)}$ are of class C^∞ for any index i. Such a differentiable decomposition $T(M) = V^{(0)} + V^{(1)} + \cdots + V^{(r)}$ of the tangent space is unique in the sense that if on an open set Q the decomposition $T(M) = V^{(0)} + V^{(1)} + \cdots + V^{(r)}$ is of class C^∞, then the C^∞ distributions $V^{(i)}$ are unique on Q up to an order.*

Proof. First we need some preparation.

Let $a_j^i(x)$, $x \in \mathbf{R}^k$ be a continuous field of symmetric $n \times n$-matrices on $\mathbf{R}^k$, and let $\lambda_1(x) \leqslant \lambda_2(x) \leqslant \cdots \leqslant \lambda_n(x)$ be the (real) eigenvalue functions of the matrix field. Then it is well known that for every position i the function $\lambda_i(x)$ is continuous in x. The same statement is true if $a_j^i(x)$ is skew-symmetric and continuous. In this case the eigenvalues are imaginary, i.e., they are of the form $\lambda_j(x) = \mu_j(x)\underline{i}$, $\mu_j(x) \in \mathbf{R}$. If we consider the ordered function system defined by $\mu_1(x) \leqslant \mu_2(x) \leqslant \cdots \leqslant \mu_n(x)$, then the real functions $\mu_i(x)$ are continuous in x. The same holds also in the case if we consider a polynomial field of the form

$$\lambda^n + a_1(x)\lambda^{n-1} + \cdots + a_n(x) = \varphi(\lambda, x)$$

on $\mathbf{R}^k$ such that the functions $a_i(x)$ are continuous in $x \in \mathbf{R}^k$, and all the roots are real (resp. imaginary) of the form $\lambda_j(x)$ (resp. $\lambda_j(x) = \mu_j(x)\underline{i}$), where these functions are defined by

$$\lambda_1(x) \leqslant \lambda_1(x) \leqslant \cdots \leqslant \lambda_n(x) \;(\text{resp. } \mu_1(x) \leqslant \mu_2(x) \cdots \mu_n(x)).$$

Then they are continuous in the variable x.

In the following lemma we examine the differentiability property of the eigenvalue functions.

Lemma 1.1. *Let $a_j^i(x)$ be a C^r field of symmetric matrices on $\mathbf{R}^k$, and let us define the i-th eigenvalue function $\lambda_i(x)$ by*

$$\lambda_1(x) \leqslant \lambda_2(x) \leqslant \cdots \leqslant \lambda_n(x).$$

Then there exists an everywhere dense open subset U of $\mathbf{R}^k$ on which every function $\lambda_i(x)$ is of class C^r. More precisely the function $\lambda_i(x)$ is of class C^r on the maximal everywhere dense open subset U^i of $\mathbf{R}^k$ on the arcwise connected components U^i_α of which the multiplicity of $\lambda_i(x)$ is constant.

Proof. Turning to the characteristic polynomial of $a^i_j(x)$ we must prove the following statement.

Let $\lambda^n + a_1(x)\lambda^{n-1} + \cdots + a_n(X) = \varphi(\lambda, x)$, $x \in \mathbf{R}^k$, be a polynomial field such that the functions $a_i(x)$ are of class C^r, and at every point $x \in \mathbf{R}^k$ the roots are real. Then the root function $\lambda_i(x)$, defined by

$$\lambda_1(x) \leqslant \lambda_2(x) \leqslant \cdots \leqslant \lambda_n(x),$$

are of class C^r on an everywhere dense open subset U of $\mathbf{R}^k$. More precisely $\lambda_i(x)$ is of class C^r on the maximal everywhere dense open subset U^i of $\mathbf{R}^k$ on the arcwise connected components of which the multiplicity of $\lambda_i(x)$ is constant.

It is obvious that the open sets U (resp. U^i) defined in the statement are everywhere dense in $\mathbf{R}^k$.

We shall prove by induction that the root $\lambda_i(x)$ is of class C^r on U^i_α which is an arcwise connected component of U^i. From this statement the others in the lemma follow evidently.

The case $n = 1$ is trivial.

For a natural number ρ let us assume that the statement is true for $n < \rho$, and let

$$\varphi(\lambda, x) = \lambda^\rho + a_1(x)\lambda^{\rho-1} + \cdots + a_\rho(x)$$

be an arbitrary polynomial field of grade ρ. We prove that the statement holds also for $\varphi(\lambda, x)$.

Let us consider a component U^i_α with respect to $\lambda_i(x)$. The multiplicity of $\lambda_i(x)$ is constant on U^i_α. If this multiplicity is greater than one, then $\lambda_i(x)$ is the root of the derived polynomial

$$\frac{1}{\rho}\frac{\partial\varphi}{\partial\lambda} = \lambda^{\rho-1} + \frac{\rho-1}{\rho}a_1(x)\lambda^{\rho-2} + \cdots + \frac{1}{\rho}a_{\rho-1}(x).$$

It is well known that the derived polynomial has only real roots. Furthermore it can be seen that the multiplicity of $\lambda_i(x)$ is constant on U^i_α with respect to the derived polynomial. Thus by induction $\lambda_i(x)$ is of class C^r on U^i_α.

Now let us consider the case where $\lambda_i(x)$ is only a simple root on U^i_α. For a vector $\underline{h}_i := (0,0,\cdots,0,h_i,0,\cdots,0)$ and a point $x \in U^i_\alpha$ we get

$$0 = \lim_{h_j\to 0} \frac{\varphi(\lambda_i(x+\underline{h}_j), x+\underline{h}_j) - \varphi(\lambda_i(x),x)}{h_j}$$

$$= \lim_{h_j\to 0}\left\{\frac{\lambda_i(x+\underline{h}_j)-\lambda_i(x)}{h_j} Z^{\rho-1}(\lambda_i,x,h_j)\right.$$

$$\left. + \sum_{m=0}^{\rho} \frac{a_m(x+\underline{h}_j)-a_m(x)}{h_j}\lambda_i^{\rho-m}(x+\underline{h}_j)\right\},$$

where the function $Z^{\rho-1}(\lambda_i, x, h_j)$ is defined by

$$Z^{\rho-1}(\lambda_i,x,h_j) := \sum_{r=1}^{\rho-1}\sum_{u=0}^{\rho-1-r} a_r(x)\lambda_i^u(x+\underline{h}_j)\lambda_i^{\rho-1-r-u}(x).$$

Thus we get

$$\lim_{h_j\to 0} Z^{\rho-1}(\lambda_i,x,h_j) = \frac{\partial\varphi}{\partial\lambda}(\lambda_i(x),x) = \rho\lambda_i^{\rho-1}(x) + \cdots + a_{\rho-1}(x).$$

Since $\lambda_i(x)$ is a simple root on U^i_α, the above limit $Z^{\rho-1}(\lambda_i,x,0)$ never vanishes on U^i_α. So we get that the limit

$$\lim_{h_j\to 0}\frac{\lambda_i(x+\underline{h}_j)-\lambda_i(x)}{h_j} = \frac{\partial\lambda_i}{\partial x^j}$$

exists on U^i_α and that

$$\frac{\partial\lambda_i}{\partial x^j} = \frac{1}{Z^{\rho-1}(\lambda_i,x,0)}\sum_{m=1}^{\rho}\frac{\partial a_m}{\partial x^j}\lambda_i^{\rho-m}(x).$$

It can be seen that the function $\partial\lambda_i/\partial x^j$ is continuous on U^i_α. By continuation of the above procedure we get that $\lambda_i(x)$ is of class C^r on U^i_α, thus proving our assertion. q.e.d.

Now we turn to the examination of the eigenvectors.

Lemma 1.2. *Let $a^i_j(x)$ be a symmetric $n\times n$-matrix field of class C^r on $\mathbf{R}^k$. Furthermore, let U be the maximal everywhere dense open subset of $\mathbf{R}^k$ on the arcwise connected components U_α of which the multiplicity of several eigenvalue functions $\lambda_i(x)$ is constant. Then arbitrary linearly independent eigenvectors $(e_{|p}, e_{2|p},\cdots,e_{n|p})$ at a fixed point $p\in U_\alpha$ can be extended to a C^r field $(e_1,\cdots,e_n)$ of linearly independent eigenvectors onto a whole neighborhood of p.*

More precisely let U^i be the maximal everywhere dense open subset of $\mathbf{R}^k$ on the arcwise connected components U^i_α of which the multiplicity of $\lambda_i(x)$ is *constant. Then any eigenvector* $e_{|p}$; $p \in U^i_\alpha$ *with eigenvalue* $\lambda_i(p)$ *can be extended to a* C^r *field* e_i *of eigenvectors with eigenvalue* $\lambda_i(x)$ *onto a neighborhood of* p.

Proof. We must prove only the last statement. Let U^i_α be a component of U^i on which the multiplicity of $\lambda_i(x)$ is constant, say m_i. Let $p \in U^i_\alpha$ be a point and let $e_{|p}$ be an eigenvector at p. The rank of the characteristic matrix

$$b^r_k := a^r_k - \lambda_i \delta^r_k$$

is $(n - m_i)$ on U^i_α. Let us assume (without the restriction of generality) that the submatrix

$$B = (b^r_k), \quad 1 \leqslant k, r \leqslant n - m_i$$

is a maximal nonsingular matrix at p. Then it is nonsingular in a neighborhood V of p. If $e^i_{|p}$ denotes the components of $e_{|p}$, then it is obvious that the vector field e defined by its components e^k by

$$\begin{aligned} e^k &:= -\sum_{r=1}^{n-m_i} (b^{-1})^k_r \sum_{l=n-m_i+1}^{n} b^r_l e^l_{|p} \quad &&\text{if } k = 1, \cdots, n - m_i; \\ &:= e^k_{|p} &&\text{if } k > n - m_i \end{aligned}$$

is of class C^r on V and is suitable for the lemma. q.e.d.

Similar statements can be proved also for a skew-symmetric matrix field $a^i_j(x)$. In this case the eigenvalues are of the form $\lambda_j(x) = \mu_j(x)\underline{i}$, where the functions $\mu_j(x)$ are defined by $\mu_1(x) \leqslant \mu_2(x) \leqslant \cdots \leqslant \mu_n(x)$. If the components of $a^i_j(x)$ are of class C^r, then on the maximal everywhere dense open subset U of $\mathbf{R}^k$ on the arcwise connected components of which the multiplicity of every eigenvalue $\lambda_i(x)$ is constant, the functions $\mu_j(x)$, $\lambda_j(x)$ are also of class C^r.

For a skew-symmetric matrix field $a^i_j(x)$ and for a point $x \in \mathbf{R}^k$ let

$$\mathbf{R}^n = u_0(x) + u_1(x) + \cdots + u_r(x) \quad \text{(direct sum)}$$

be Jordan decomposition of $\mathbf{R}^n$ with respect to $a^i_j(x)$. I.e., $u_0(x)$ is the maximal 0-space of $a^i_j(x)$. Furthermore the 2-dimensional subspaces $u_i(x)$, $i \geqslant 1$, are invariant under the action of $a^i_j(x)$, and this action is irreducible on $u_i(x)$, $i \geqslant 1$. Now we can formulate the following statement.

If $x \in U$, then the Jordan decomposition can be extended onto a neighborhood Q of x such that the distributions $p \to u_i(p)$ are of class C^r on Q.

The proof of this statement is the same as before.

Now we return to the examination of the curvature tensor in a Riemannian space. We examine it at a fixed point p.

By the Bianchi identities $g(R(X, Y)V, Z) = g(R(V, Z)X, Y)$ and $R(X, Y) = -R(Y, X)$ the curvature $R_{|p}$ can be considered as a symmetric bilinear map on the two-vectors $T_p(M) \wedge T_p(M)$. So the tensor with components $R^{ij}{}_{kl} := g^{ir}R_r{}^j{}_{kl}$ at p can be considered as a symmetric linear endomorphism

$$\bar{R}_{|p}: T_p(M) \wedge T_p(M) \to T_p(M) \wedge T_p(M)$$

of the two-vectors, where we consider the space $T_p(M) \wedge T_p(M)$ as an euclidean space with the induced inner product $\langle \, , \rangle_{|p}$ defined by

$$2\langle X \wedge Y, V \wedge Z \rangle_{|p} = g_{|p}(X, V) g_{|p}(Y, Z) - g_{|p}(X, Z) g_{|p}(Y, V).$$

Let us notice that the space $T_p(M) \wedge T_p(M)$ can be identified with the space $\mathfrak{A}_p$ of skew-symmetric linear endomorphisms in $T_p(M)$ in a natural manner. If we consider in $\mathfrak{A}_p$ the natural inner product $\langle \, , \rangle_{|p}$ defined by

$$\langle A, B \rangle_{|p} = -\mathrm{Trace}\, A \circ B, \quad A, B \in \mathfrak{A}_p,$$

then the identification $T_p(M) \wedge T_p(M) \to \mathfrak{A}_p$, $X \wedge Y \to \overline{X \wedge Y}$ is defined by

$$\langle \overline{X \wedge Y}, A \rangle_{|p} = g_{|p}(A(X), Y), \quad X, Y \in T_p(M), A \in \mathfrak{A}_p.$$

It is obvious that this identification is an isometry between the two euclidean spaces.

Now if w is an arbitrary two-vector, and $\bar{w} \in \mathfrak{A}_p$ denotes its image by the above identification, then let us consider a Jordan decomposition

$$(*) \qquad T_p(M) = u_0 + u_1 + \cdots u_r \quad \text{(direct sum)}$$

of $T_p(M)$ with respect to $\bar{w}$. Then in the two-dimensional subspaces u_i, $i > 0$, we can choose vectors X_i and Y_i such that the two-vector w is of the form

$$w = \sum_{i=1}^{r} X_i \wedge Y_i.$$

Such a decomposition of w is called the Darboux's decomposition of w. We mention that such a decomposition is not unique as it is shown by the following consideration.

If the multiplicity of a nonnull eigenvalue $\lambda_j = \mu_j \underline{i}$ is greater than one, say m_j, then there exist exactly m_j invariant subspaces $u_{i_1}, \cdots, u_{i_{m_j}}$, $i_k > 0$, in a Jordan decomposition $(*)$ such that $\bar{w}$ has just the eigenvalues $\pm\lambda_j = \pm\mu_j \underline{i}$ on u_{i_k}. It is obvious that the Jordan decomposition of the subspace

$$W_j := u_{i_1} + u_{i_2} + \cdots + u_{i_{m_j}}$$

is not unique. But if the eigenvalue $\lambda_j = \mu_j \underline{i} \neq 0$ is simple, then there is only one subspace u_j corresponding to λ_j (on which $\bar{w}$ has the eigenvalues $\pm\lambda_j$), and this subspace is unique for every Jordan decomposition $(*)$. *Thus if* $\lambda_j \neq 0$

is simple, then the two-vector $X_j \wedge Y_j$ *corresponding to* λ_j *is unique and always occurs in every Darboux decomposition of* w. This remark is important for the next lemma.

The number r in (*) is the *rank of* w.

Definition 1.2. The eigenvector $w \in T_p(M) \wedge T_p(M)$ of the symmetric endomorphism $\bar{R}_{|p}$ is said to be irreducible iff any Darboux normal form

$$w = \sum_{i=1}^{r} X_i \wedge Y_i$$

does not split into two non-trivial summands such that they are also eigenvectors of $\bar{R}_{|p}$.

Let w be an irreducible eigenvector of $\bar{R}_{|p}$ with nonnull eigenvalue $\lambda_j = \mu_j \underline{i}$. The skew-symmetric linear endomorphism $\bar{w}$ lies in the linear holonomy set h_p, defined before the theorem. Let us consider once more the decomposition (*). Since $\bar{w}$ leaves the subspaces $V_p^{(j)}$ invariant by Proposition 1.1 we can choose such a decomposition (*) that any subspace u_i, $i > 0$, is contained in one of the subspaces $V_p^{(j)}$. But the subspaces u_k, $k > 0$, must be contained in the same invariant subspace $V_p^{(j)}$, $j > 0$, because w is irreducible, and the subspaces u_k contained in the same space $V_p^{(j)}$ determine an eigenvector of $\bar{R}_{|p}$. As the subspaces u_q, $q > 0$, not contained in $V_p^{(j)}$ would determine another eigenvector, thus this part is null indeed. So we get that *for an irreducible eigenvector* $w \in T_p(M) \wedge T_p(M)$ *of* $\bar{R}_{|p}$ *with nonnull eigenvalue the nontrivial invariant subspace*

$$u_1 + u_2 + \cdots + u_r$$

of $\bar{w}$ *is contained in a single invariant subspace* $V_p^{(j)}$. This is an important property of the irreducible eigenvectors of $\bar{R}_{|p}$.

It is also evident that *one can choose a complete system of linearly independent irreducible eigenvectors of* $\bar{R}_{|p}$ *which form a basis in* $T_p(M) \wedge T_p(M)$. Let $\{w_1, w_2, \cdots, w_\rho, w_{\rho+1}, \cdots, w_{\binom{n}{2}}\}$ be such a system, and let us assume that just the first ρ vectors are corresponding to nonnull eigenvalues. For a vector w_k, $1 \leqslant k \leqslant \rho$, let us consider the decomposition (*), and the subspace u_0 let us denote it by W_k^0 and the subspace $u_1 + u_2 + \cdots + u_r$ by W_k^1. It is obvious that *we can construct the irreducible decomposition* $T_p(M) = V_p^{(0)} + V_p^{(1)} + \cdots + V_p^{(r)}$ *also in the following manner.*

Let us choose an arbitrary vector w_{k_1}, $1 \leqslant k_1 \leqslant \rho$, and let us consider its subspaces $W_{k_1}^0$ and $W_{k_1}^1$ constructed above. If for any w_i, $i \neq k_1$, the relation $W_{k_1}^1 \subseteq W_i^0$ holds, then obviously $W_{k_1}^1$ is one of the invariant subspaces $V_p^{(k)}$. If there exist vectors $w_{k_2}, w_{k_3}, \cdots, w_{k_l}$ such that $1 \leqslant k_i \leqslant \rho$ and $W_{k_1}^1 \nsubseteq W_{k_i}^0$ hold, then let us consider the subspace

$$W^1_{k_1} + W^1_{k_2} + \cdots + W^1_{k_l}.$$

Now we repeat the above examination. If for any w_i with $i \neq k_j$ the relation

$$W^1_{k_1} + W^1_{k_2} + \cdots + W^1_{k_l} \subseteq W^0_i$$

holds then $W^1_{k_1} + \cdots + W^1_{k_l}$ is just the subspace $V^{(k)}_p$. But if there exists such a W^0_i for which the above relation does not hold, then by extending the system $W^1_{k_1}, \cdots, W^1_{k_l}$ with the element W^1_i and continuing the procedure we get a maximal system $w_{k_1}, \cdots, w_{k_j}$ of eigenvectors such that for every index i, $1 \leqslant i \leqslant j$, there is another index i', such that $1 \leqslant i' < i$ and $W^1_{i'} \not\subseteq W^0_i$ holds. Furthermore for every index $i \neq k_1, \cdots, k_j$ we get the relation $W^1_{k_l} \subseteq W^0_i$. It is obvious that

$$V^{(k)}_p = W^1_{k_1} + W^1_{k_2} + \cdots + W^1_{k_j},$$

and thus we get one of the invariant subspaces.

By the continuation of this procedure we can construct all other invariant subspaces $V^{(j)}_p$.

We need this construction also in the following.

Lemma 1.3. *For a C^∞ Riemannian manifold (M^n, g) there can be chosen an everywhere dense open subset U^2 of M and on U^2 a system $\{w_1, w_2, \cdots, w_{\binom{n}{2}}\}$ of eigenvector functions of $\bar{R}$ such that they are of class C^∞, irreducible and linearly independent so that they form a basis in $T_p(M) \wedge T_p(M)$ for any point $p \in U^2$.*

Proof. Let U^1 be the maximal everywhere dense open subset of M on the connected component of which the eigenvalue functions $\lambda_i(x)$ of $\bar{R}$ have constant multiplicity. By Lemmas 1.1 and 1.2 the $\lambda_i(x)$'s are of class C^∞ furthermore any eigenvector $e_{i|p}$ at a point $p \in U^1$ with eigenvalue $\lambda_i(p)$ can be extended to a C^∞ field e_i of eigenvectors with eigenvalue λ_i onto a neighborhood of p.

Let m_1 be the maximum of rank of irreducible eigenvectors over U^1 and let $p \in U^1$, $w_{1|p} \in T_p(M) \wedge T_p(M)$ be an irreducible eigenvector, for which rank $(w_{1|p}) = m_1$ holds. Let us extend the eigenvector $w_{1|p}$ to a C^∞ field w_1 of eigenvectors onto a neighborhood of p.

Now we prove that in the case $m_1 > 1$ the w_1 is irreducible in a neighborhood Q^1 of p, and in the case $m_1 = 1$ $w_{1|p}$ can be extended also to a C^∞ irreducible eigenvector field.

Indeed, if $m_1 > 1$ then from the irreducibility of $w_{1|p}$ it follows that the nontrivial partial sums in any Darboux normal form of $w_{1|p}$ do not intersect the invariant subspaces of $\bar{R}_p$. But the whole set of real partial sums of the

$w_{1|p}$'s Darboux normal forms is a compact set in $T_p(M) \wedge T_p(M)$. So by the continuity of w_1 and of the invariant subspaces of $\bar{R}$ on U (see Lemma 1.2) we get that this compact set does not intersect these invariant subspaces in a neighborhood Q^1 of p as well. Thus w_1 is irreducible on Q^1, and the first statement is proved.

Now if $m_1 = 1$, then for every eigenvector $w_{1|p}$, $p \in U^1$, there exists a Darboux normal form which splits into the sum of eigenvectors of rank 1, and thus every irreducible eigenvector is of rank 1. In this case let $w_{1|p} = X_{1|p} \wedge Y_{1|p}$ be an arbitrary irreducible eigenvector of $\bar{R}_p$ at a point $p \in U^1$, and let us extend this vector to a C^∞ field of eigenvectors also. First we prove that around p the w_1 can be written in the form

$$(**) \qquad w_1 = X_1 \wedge Y_1 + w_1^*,$$

where $X_1 \wedge Y_1$ and w_1^* are of class C^∞, and furthermore $\overline{w^*_{1|p}} = 0$ holds. Indeed the skew-symmetric linear endomorphism $\bar{w}_{1|p} = \overline{X_{1|p} \wedge X_{2|p}}$ in $T_p(M)$ has only two nonnull eigenvalues, the values $\mu_1 \underline{i}$ resp. $-\mu_i \underline{i}$. Thus $\mu_1 \underline{i}$ is a simple eigenvalue at p, and it is a simple eigenvalue function in a neighborhood $Q^1 \subseteq U^1$ of p because of the continuity of the eigenvalues. For $q \in Q^1$ let $X_{1|q} \wedge Y_{1|q}$ be the uniquely determined two-vector corresponding to the simple eigenvalue $\mu_1(q)\underline{i}$. Thus the plane determined by $X_{1|q} \wedge Y_{1|q}$ is invariant under the action of $\bar{w}_{1|q}$, and in this plane $\bar{w}_{1|q}$ has the eigenvalues $\pm\mu_1(q)\underline{i}$. Since $\mu_1(p)\underline{i}$ is simple on Q^1, $X_1 \wedge Y_1$ is of class C^∞ on Q^1. So the decomposition $(**)$ is correct indeed.

Since $m_1 = 1$, for an arbitrary point $q \in Q^1$ let us consider such a Darboux normal form of $w_{1|q}$ which splits into the sum of eigenvectors or rank 1. Since the eigenvalue function $\mu_1 \underline{i}$ is simple on Q^1, by a previous remark the two-vector $X_{1|q} \wedge Y_{1|q}$ surely occurs in the considered Darboux normal form. Thus $X_{1|q} \wedge Y_{1|q}$ is an irreducible eigenvector for every point $q \in Q^1$, and the field $X_1 \wedge Y_1$ is of class C^∞. This proves the second statement above. In the next step we consider only the irreducible eigenvectors of $\bar{R}$, which lie over Q^1 and are linearly independent from w_1. Let m_2 be the maximum of the ranks of these eigenvectors, and let $w_{2|p}$, $p \in Q^1$, be an eigenvector for which $\operatorname{rank}(w_{2|p}) = m_2$ holds.

Let us extend $w_{2|p}$ to a C^∞ field w_2 of eigenvectors. It is linearly independent of w_1 in a neighborhood of p also, It can be proved as before that in the case $m_2 > 1$, w_2 is irreducible and linearly independent of w_1 in a neighborhood Q^2 of p, in the case $m_2 = 1$, $w_{2|p}$ can be extended to a C^∞ field of eigenvectors with rank 1 onto a whole neighborhood Q^2 of p. Continuing the process we get in the last $\binom{n}{2}$-th step a nonempty open subset $Q^{\binom{n}{2}}$ and on it C^∞ fields of linearly independent and irreducible eigenvectors denoted by $(w_1, w_2, \cdots, w_{\binom{n}{2}})$.

Now let us consider the closure of the open set $Q^{\binom{n}{2}}$ in M^n, and let us turn to its complement, i.e., to the open set

$$M \setminus \text{closure}\left(Q^{\binom{n}{2}}\right).$$

If this set is empty, then the proof is finished since $Q^{\binom{n}{2}}$ is suitable for the lemma. If $(M\text{-closure}(Q^{\binom{n}{2}})$ is not empty, then by the above considerations there can be chosen a suitable nonempty open subset $S^{\binom{n}{2}}$ in the set and on $S^{\binom{n}{2}}$ suitable C^∞ fields $(w_1, w_2, \cdots, w_{\binom{n}{2}})$ of irreducible eigenvectors. Continuing the process then by a standard application of the Zorn-lemma we get an everywhere dense open subset U^2 in M^n and on U^2 the system $(w_1, \cdots, w_{\binom{n}{2}})$ of irreducible C^∞ eigenvector fields of $\bar{R}$, which are suitable for the lemma. q.e.d.

Let us consider a system $\{U^2, (w_1, w_2, \cdots, w_{\binom{n}{2}})\}$ constructed in the previous lemma, and for a field w_i and $p \in U^2$ let us remember the subspaces $W^0_{|pi}$ and $W^1_{|pi}$, which are respectively the null-space and the nontrivial invariant subspace corresponding to $\bar{w}^i_{|p}$. The following lemma is obvious.

Lemma 1.4. *Let $\{U^2, (w_1, w_2, \cdots, w_{\binom{n}{2}})\}$ be as before. Then there exists a maximal everywhere dense open subset U^3 of U^2 such that on any connected component U^3_α of U^3 the subspaces*

$$W^l_{i_1} + W^l_{i_2} + \cdots + W^l_{i_s}, \quad l = 0, 1,$$

have constant dimension for any fixed sequence $i_1, i_2, \cdots, i_s$ of indices, and thus these distributions are of class C^∞.

Of course U^3 is an everywhere dense open subset also in M.

Now we finish the proof of Theorem 1.1.

Let n_1 be the maximum of the dimensions of the subspaces $V^{(l)}_p$, $l > 0$, on U^3, and let $p \in U^3$ be a point, and $V^{(k)}_p$ be a subspace for which $\dim V^{(k)}_p = n_1$ holds. Furthermore, let $(w_{k_1|p}, \cdots, w_{k_j|p})$ be the system corresponding to $V^{(k)}_p$ constructed before Lemma 1.3. Thus all the subspaces $W^1_{|pk_l}$ are contained in $V^{(k)}_p$, and furthermore for any index $1 \leqslant i \leqslant j$ there is an index $i' < i$ such that

$$(1) \qquad W^1_{|pi'} \not\subseteq W^0_{|pi},$$

and that

$$(2) \qquad W^1_{|pk_q} \subseteq W^0_{|pi}$$

for any index $i \neq k_l$.

By the continuity of subspaces W^k_i the relation (1) holds also in a neighborhood N^1 of p, and as the value $\dim(V^{(k)}_p)$ is maximum, also (2) must hold on N^1. So it is obvious that the distribution

$$W^1_{k_1} + W^1_{k_2} + \cdots + W^1_{k_j}$$

determines at every point $q \in N^1$ a complete invariant subspaces $V_q^{(k)}$, and besides $\dim V_q^{(k)} =$ constant the distribution $q \to V_q^{(k)}$ is of class C^∞ on N^1. Thus the distribution $V^{(k)}$ is suitable on N^1.

Now we continue the procedure.

Let n_2 be the maximum of the numbers

$$\dim\left(V_q^{(j)}\right), \quad \text{where } q \in N^1, j > 0 \text{ and } j \neq k,$$

and let $V_q^{(j)}$, $q \in N^1$, be such a subspace for which $j \neq k$ and $\dim(V_q^{(j)}) = n_2$ hold. We get as before that there is a unique C^∞ extension of $V_q^{(j)}$ to a distribution $V^{(j)}$ onto a nonempty open set $N^2 \subseteq N^1$. So in a finite number of steps we get a nonempty open set N^r and the suitable C^∞ distributions $V^{(0)}$, $V^{(1)}, V^{(2)}, \cdots, V^{(r)}$ on N^r spanning the tangent spaces of M at the points of N^r.

Now if the open set

$$U^3 \setminus \text{closure}(N^r)$$

is not empty, then we chose in it a suitable nonempty open set endowed with suitable C^∞ fields of distributions $V^i, \cdots$, etc. Again with a standard application of the Zorn-lemma we get a suitable open everywhere dense set U of U^3 which is suitable also for the theorem. As we have seen in the above proof the differentiable decomposition

$$T(U) = V^{(0)} + V^{(1)} + \cdots + V^{(r)}$$

is unique on U up to an order. Thus the proof of Theorem 1.1 is complete.

In the following proposition the formula of the form $\nabla_{V^{(i)}} V^{(j)} \subseteq V^{(k)}$ means that for any C^∞ vector field X_l with $X_l(p) \in V_p^{(l)}$, $p \in U$, the vector field $\nabla_{X_i} X_j$ belongs to $V^{(k)}$.

Proposition 1.2. *The following formulas hold on the everywhere dense open subset U defined in Theorem* 1.1:

$$\begin{aligned} &\nabla_{V^{(0)}} V^{(0)} \subseteq V^{(0)}, \quad \nabla_{V^{(0)}} V^{(i)} \subseteq V^{(i)}, \quad \nabla_{V^{(i)}} V^{(i)} \subseteq V^{(0)} + V^{(i)}, \\ &\nabla_{V^{(i)}} V^{(0)} \subseteq V^{(0)} + V^{(i)}, \quad \nabla_{V^{(i)}} V^{(j)} \subseteq V^{(j)} \quad \text{if } i \neq j, i;\ j \neq 0. \end{aligned} \tag{1.10}$$

Proof. In the following the symbols $X_i, Y_i, Z_i, \cdots$, etc. stand for vector fields on U, which have their values in $V^{(i)}$. Now if $i \neq 0, j \neq 0$ and $i \neq j$ hold, then by the second Bianchi identity we get

$$(\nabla_{X_i} R)(Y_j, Z_j)Q_j = -(\nabla_{Y_j} R)(Z_j, X_i)Q_j - (\nabla_{Z_j} R)(X_i, Y_j)Q_j,$$

and thus by (1.6),

$$\begin{aligned} \nabla_{X_i}\{R(Y_j, Z_j)Q_j\} = {} & R(\nabla_{X_i} Y_j, Z_j)Q_j + R(Y_j; \nabla_{X_i} Z_j)Q_j + R(Y_j, Z_j)\nabla_{X_i} Q_j \\ & + R(\nabla_{Y_j} Z_j, X_i)Q_j + R(Z_j, \nabla_{Y_j} X_i)Q_j \\ & + R(\nabla_{Z_j} X_i, Y_j)Q_j + R(X_i, \nabla_{Z_j} Y_j)Q_j. \end{aligned}$$

By Proposition 1.1 every term on the right side of the above equation belongs to $V^{(j)}$, so

$$\nabla_{X_i}\{R(Y_j, Z_j)Q_j\}_{|p} \in V_p^{(j)}, \quad p \in U.$$

But the primitive holonomy group $\mathcal{H}_p^{(j)}$ acts on $V_p^{(j)}$ irreducibly, so from the definition it follows that around every point $p \in U$ the vector fields of the form $R(Y_j, Z_j)Q_j$ span the distribution $V^{(j)}$. Thus for every C^∞ vector field X_j

$$\nabla_{X_i} X_{j|p} \in V_p^{(j)}, \quad p \in U$$

holds, which proves the last formula.

Now if $i \neq 0$, then we get the following from the second Bianchi identity:

$$\begin{aligned}\nabla_{X_0}\{R(Y_i, Z_i)Q_i\} = {} & R(\nabla_{X_0}Y_i, Z_i)Q_i + R(Y_i, \nabla_{X_0}Z_i)Q_i \\ & + R(Y_i, Z_i)\nabla_{X_0}Q_i + R(Z_i, \nabla_{Y_i}X_0)Q_i + R(\nabla_{Z_i}X_0, Y_i)Q_i.\end{aligned}$$

By the same argument as before, the relation $\nabla_{X_0}X_{i|p} \in V_p^{(i)}, p \in U$, holds.

Finally if $i \neq j$ and $i, j \neq 0$, then by the above relations we have

$$\begin{aligned} g(\nabla_{X_0}Y_0, X_i) &= -g(Y_0, \nabla_{X_0}X_i) = 0, \\ g(\nabla_{X_i}Y_0, X_j) &= -g(Y_0, \nabla_{X_i}X_j) = 0, \\ g(\nabla_{X_i}Y_i, X_j) &= -g(Y_i, \nabla_{X_i}X_j) = 0, \end{aligned}$$

i.e., $\nabla_{X_0}Y_{0|p} \in V_p^{(0)}$, $\nabla_{X_i}Y_{0|p} \in V_p^{(0)} + V_p^{(i)}$, $\nabla_{X_i}Y_i \in V_p^{(0)} + V_p^{(i)}$, which proves the proposition completely.

Corollary. *The C^∞ distributions $V^{(0)}$ and $V^{(0)} + V^{(i)}$ are involutive on U, and their integral manifolds are totally geodesic. Moreover, the integral manifolds of $V^{(0)}$ are of zero curvature.*

The last statement was proved also by S. S. Chern and N. H. Kuiper [12].

In the following we give another so called Z-decomposition of a Riemannian space different from the V-decomposition.

Let us consider again the everywhere dense open set U in M corresponding to Theorem 1.1, and let $V^{(i)}$, $i \neq 0$, be a fixed distribution on U. For a point $p \in U$ let us consider the subspace $Z_p^{(i)}$ in $T_p(U)$ spanned by the vectors of the forms

$$(1.11) \qquad X_{1|p}, \nabla_{X_1}X_{2|p}, \nabla_{X_1}\nabla_{X_2}X_{3|p}, \cdots, \nabla_{X_1}, \cdots, \nabla_{X_l}X_{l+1|p}, \cdots, \text{etc.},$$

where the vector fields X_l are C^∞ around p and belong to $V^{(i)}$. In this manner we construct the subspaces $Z_p^{(i)}$ only for the indices $i > 0$. By definition let $Z_p^{(0)}$ be the complete subspace in $T_p(M)$ which is totally orthogonal to the subspace

$$Z_p^{(1)} + Z_p^{(2)} + \cdots + Z_p^{(r)}.$$

Definition 1.3. The subspace $V_p^{(0)}$ is called the null-space of the curvature at p, and the subspace $Z_p^{(0)}$ is the so called total null-space of curvature at p. The number $\nu(p) := \dim V_p^{(0)}$ is called the index of nullity of the curvature at p, and the number $u(p) := \dim T_p(M) - \dim V_p^{(0)}$ is called the index of non-nullity of curvature at p, i.e.,

$$u(p) = \dim V_p^{(1)} + \cdots + \dim V_p^{(r)}.$$

It is obvious that the relations $Z_p^{(0)} \subseteq V_p^{(0)}$ and $V_p^{(i)} \subseteq Z_p^{(i)}$, $i > 0$, hold for every point $p \in U$.

Theorem 1.2 (*The second stability theorem*). *The subspaces $Z_p^{(i)}$, $p \in U$, are pairwise orthogonal, and there can be chosen an everywhere dense open subset G of M^n (which is also a subset of U) on the arcwise connected components of which the subspaces $Z_p^{(i)}$ for any index i have constant dimension and the distributions $p \to Z_p^{(i)}$ are of class C^∞ on G.*

Proof. By the definition it is sufficient to prove the orthogonality for the cases $i, j \geqslant 1$ only. The vector fields

$$X_1, X_2, \cdots, X_k, \cdots, \text{ etc. (resp. } Y_1, Y_2, \ldots, Y_l, \ldots, \text{etc.)}$$

of class C^∞ stand for fields tangent to $V^{(i)}$ (resp. $V^{(j)}$).

At first we prove a lemma.

Lemma 1.5. *The vector fields of the form*

$$\nabla_{X_1}\nabla_{Y_1}\nabla_{Y_2} \cdots \nabla_{Y_k} Y_{k+1}$$

are tangent to $Z^{(j)}$, i.e., for $i \neq j$, $i, j \neq 0$ the relation $\nabla_{V^i} Z^j \subseteq Z^j$ holds.

Proof. We can prove the statement by induction. By $R(X_1, Y_1)Y_2 = 0$ we get

$$\nabla_{X_1}\nabla_{Y_1}Y_2 = \nabla_{Y_1}\nabla_{X_1}Y_2 + \nabla_{[X_1 Y_1]}Y_2 = \nabla_{Y_1}Y_2^* + \nabla_{Y_1^*}Y_2 - \nabla_{X_1^*}Y_2,$$

where the fields $Y_2^* := \nabla_{X_1}Y_2$ and $Y_1^* := \nabla_{X_1}Y_1$ are tangent to $V^{(j)}$, and the field $X_1^* := \nabla_{Y_1}X_1$ is tangent to $V^{(i)}$. Thus the first two terms above are tangent to $Z^{(j)}$, and the third term is tangent to $V^{(j)}$. So $\nabla_{X_1}\nabla_{Y_1}Y_{2|p} \in Z_p^{(j)}$ holds indeed.

Now we consider the general case.

From $R(X_1, Y_1) = 0$ we can write

$$\nabla_{X_1}\nabla_{Y_1}\nabla_{Y_2} \cdots \nabla_{Y_k}Y_{k+1} = \nabla_{Y_1}\nabla_{X_1}\nabla_{Y_2} \cdots \nabla_{Y_k}Y_{k+1} + \nabla_{[X_1 Y_1]}\nabla_{Y_2} \cdots \nabla_{Y_k}Y_{k+1}.$$

As the vector field $\nabla_{X_1}Y_1 := Y_1^*$ is tangent to $V^{(j)}$, and the field $\nabla_{Y_1}X_1 := X_1^*$ to $V^{(i)}$, by the induction hypothesis it can be proved that the vector field $\nabla_{X_1}\nabla_{Y_1} \cdots \nabla_{Y_k}Y_{k+1}$ is tangent to $Z^{(j)}$. q.e.d.

Secondly we prove that the vector fields of the form

$$\nabla_{Y_1}\nabla_{Y_2}\cdots\nabla_{Y_k}Y_{k+1},\quad k \geqslant 0,$$

are orthogonal to $V^{(i)}$. We prove this statement by induction.

For $k = 0$ and $k = 1$ the statement is evident from (1.10).

Now if the statement is true for fields of the form $\nabla_{Y_2}\cdots\nabla_{Y_k}Y_{k+1}$, then by the induction hypotheses we get

$$g(X, \nabla_{Y_1}\nabla_{Y_2}\cdots\nabla_{Y_k}Y_{k+1}) = -g(\nabla_{Y_1}X, \nabla_{Y_2}\cdots\nabla_{Y_k}Y_{k+1}) = 0,$$

where the field X is tangent to $V^{(i)}$, and thus the field $\nabla_{Y_1}X$ is also tangent to $V^{(i)}$. This equation proves the statement.

Using induction again we can prove that the vector fields of the form $\nabla_{X_1}\nabla_{X_2}\cdots\nabla_{X_k}X_{k+1}$ are orthogonal to the fields of the form $\nabla_{Y_1}\cdots\nabla_{Y_l}Y_{l+1}$.

For $k = 0$ the proof is given above. Now if the fields of the form $\nabla_{X_2}\cdots\nabla_{X_k}X_{k+1}$ are orthogonal to the field of the form $\nabla_{Y_1}\nabla_{Y_2}\cdots\nabla_{Y_l}Y_{l+1}$, then by this induction hypothesis and Lemma 1.5 we get

$$g(\nabla_{X_1}\nabla_{X_2}\cdots\nabla_{X_k}X_{k+1}, \nabla_{Y_1}\cdots\nabla_{Y_l}Y_{l+1})$$
$$= -g(\nabla_{X_2}\cdots\nabla_{X_k}X_{k+1}, \nabla_{X_1}\nabla_{Y_1}\cdots\nabla_{Y_l}kY_{l+1}) = 0,$$

as the field $\nabla_{X_1}\nabla_{Y_1}\cdots\nabla_{Y_l}Y_{l+1}$ is tangent to $Z^{(j)}$. This proves the first part of the theorem completely.

The last part of the theorem is obvious, because every distribution $Z^{(i)}$ is spanned by the differentiable vector fields of the form (1.11). Thus for a $Z^{(i)}$ the maximal open set G^i on the arcwise connected components of which the function $\dim(Z_p^{(i)})$ is constant is everywhere dense in U and so also in M^n. It is evident, that on the everywhere dense open set

$$G := G^1 \cap G^2 \cap \cdots \cap G^r$$

the distributions $p \to Z_p^{(i)}$ are of class C^∞, which proves the theorem completely.

Definition 1.4. The decomposition

$$T(G) = Z^{(0)} + Z^{(1)} + \cdots + Z^{(r)}$$

is called the Z-decomposition of the tangent space over the set G.

Proposition 1.3. *The distributions $Z^{(i)}$ are totally parallel on G. Thus they are involutive, and the integral manifolds are totally geodesic.*

Proof. In Lemma 1.5 we have seen the relations $\nabla_{V^{(i)}}Z^{(j)} \subseteq Z^{(j)}$ only for the cases $i, j \neq 0$. Thus we must prove the relations $\nabla_{V^{(i)}}Z^{(0)} \subseteq Z^{(0)}$ for $i > 0$, and $\nabla_{V^{(0)}}Z^{(j)} \subseteq Z^{(j)}$ for $j \geqslant 0$.

The first formula is obvious, as

$$g(\nabla_{V^{(i)}}Z^{(0)}, Z^{(j)}) = -g(Z^{(0)}, \nabla_{V^{(i)}}Z^{(j)}) = -g(Z^{(0)}, Z^{(j)}) = 0 \quad \text{for } j > 0.$$

The formula $\nabla_{V^{(0)}} Z^{(j)} \subseteq Z^{(j)}, j \geqslant 1$, can be proved in a similar manner as the corresponding formula in Lemma 1.5. In this case

$$\nabla_X \nabla_{Y_1} \cdots \nabla_{Y_l} Y_{l+1} = \nabla_{Y_1} \nabla_X \nabla_{Y_2} \cdots \nabla_{Y_l} Y_{l+1} + \nabla_{[X, Y_1]} \nabla_{Y_2} \cdots \nabla_{Y_l} Y_{l+1}$$

holds, where X is tangent to $V^{(0)}$. As $[X, Y_1]$ is tangent to $V^{(0)} + V^{(j)}$ using induction we get the above relation.

Finally for formula $\nabla_{V^{(0)}} Z^{(0)}$ we get

$$g(\nabla_{V^{(0)}} Z^{(0)}, Z^{(j)}) = -g(Z^{(0)}, \nabla_{V^{(0)}} Z^{(j)}) = g(Z^{(0)}, Z^{(j)}) = 0,$$

which proves the assertion completely. q.e.d.

For a point $p \in G$ let $M_0, M_1, \cdots, M_r$ be the integral manifolds of the distributions $Z^{(0)}, Z^{(1)}, \cdots, Z^{(r)}$ respectively through the point p. From the above proposition we get evidently that around the point p the Riemannian manifold can be considered as the direct product of Riemannian spaces $M_0, M_1, \ldots, M_r$, and this decomposition is unique up to the order. It can also be seen that M_0 is of null curvature, $M_1, M_2, \cdots, M_r$ are irreducible, and even also the infinitesimal and the local holonomy groups act in this space irreducibly (for details see [7, p. 182]). It is also obvious that the action of the infinitesimal holonomy group is trivial on $Z^{(0)}$. Thus the above local decomposition of a Riemannian space can be considered as a decomposition using the infinitesimal holonomy group and also as a decomposition using the local holonomy group.

Definition 1.5. A Riemannian manifold (M^n, g) is called a simple leaf if at any point the V-decomposition of the tangent space is of the form

$$T_p(M) = V_p^{(0)} + V_p^{(1)},$$

i.e., there is at most a single invariant subspace on which the primitive holonomy group $\mathcal{H}_p$ acts irreducibly.

A simple leaf is said to be infinitesimally irreducible if at least at one of its points the infinitesimal holonomy group acts irreducibly, or equivalently at least at one of its points the Z-decomposition contains only the space $T_p(M) = Z_p^{(1)}$.

So we can state

Theorem 1.3 (*The local decomposition using the infinitesimal or local holonomy group*). *For any C^∞ Riemannian space (M^n, g) there exists an everywhere dense open subset G such that around every point $p \in G$ the space can be decomposed into a direct product of Riemannian manifolds in the form*

$$M_0 \times M_1 \times \cdots \times M_r,$$

where M_0 is a zero curvature and furthermore the manifolds M_i, $i > 0$, are infinitesimally irreducible simple leafs.

2. Basic formulas

Let us examine a C^∞ Riemannian manifold on the everywhere dense open subset U on which the V-decomposition of tangent space is of class C^∞. The dimension of the null-spaces $V_p^{(0)}$ on an arcwise connected component U_δ of U is constant, say ν. As the integral manifolds of $V^{(0)}$ are of zero curvature, around every point $p \in U_\delta$ we can choose a system $\{\underline{m}_1, \underline{m}_2, \cdots, \underline{m}_\nu\}$ of C^∞ pairwise orthogonal unit vector fields which are tangent to $V^{(0)}$ and further satisfy

$$\nabla_{\underline{m}_\beta}\underline{m}_\gamma = 0, \quad 1 \leqslant \beta, \gamma \leqslant \nu. \tag{2.1}$$

In the following the symbols X, Y stand always for C^∞ vector fields tangent to $V^{(1)} + V^{(2)} + \cdots + V^{(r)}$.

For a field $\underline{m}_\beta$ let us consider the derived tensor field $\nabla_X \underline{m}_\beta$, which can be written in the form

$$\nabla_X \underline{m}_\beta = B_\beta(X) + \sum_{\gamma=1}^{r} M_\beta^\gamma(X)\underline{m}_\gamma, \tag{2.2}$$

where $B_\beta(X) \in V^{(1)} + V^{(2)} + \cdots + V^{(r)}$.

$B_{\beta|p}$ is a linear endomorphism in the space $V_{|p}^{(1)} + \cdots + V_p^{(r)}$ which we extend onto the whole tangent space $T_p(M)$ in such a way that it has the value zero on the subspaces $V_p^{(0)}$. On the other hand $M_\beta^\gamma(X)$ is a covariant vector in $V^{(1)} + V^{(2)} + \cdots + V^{(r)}$, which we also extend similarly onto $T_p(M)$ by $M_\beta^\gamma(\underline{m}_\gamma) = 0$.

From (1.10) it is obvious that the endomorphisms $B_{\alpha|p}$ leave the subspaces $V_p^{(j)}, j \geqslant 0$, invariantly, and furthermore the skew-symmetry

$$M_\alpha^\beta(X) = g(\nabla_X \underline{m}_\alpha, \underline{m}_\beta) = -g(\underline{m}_\alpha, \nabla_X \underline{m}_\beta) = -M_\beta^\alpha(X) \tag{2.3}$$

also holds.

Definition. The tensor fields B_α as well as M_α^β are called second fundamental forms corresponding to the system $\{\underline{m}_1, \underline{m}_2, \cdots, \underline{m}_\nu\}$.

Let us define the tensor fields M^α, $\alpha = 1, 2, \cdots, \nu$ of type $(0,2)$ by the following formulas:

$$\begin{aligned} M^\alpha(X, Y) &:= -g(B_\alpha(X), Y), \\ M^\alpha(X, \underline{m}_\beta) = M^\alpha(\underline{m}_\beta, X) &= M^\alpha(\underline{m}_\beta, \underline{m}_\gamma) := 0, \end{aligned} \tag{2.3'}$$

where the C^∞ vector fields X, Y are tangent to $V^{(1)} + \cdots + V^{(r)}$.

With the help of these tensor fields we introduce a covariant derivative denoted by $\tilde{\nabla}$. It is defined uniquely by the following formulas:

$$\tilde{\nabla}_X Y := \nabla_X Y - \sum_{\alpha=1}^{\nu} M^\alpha(X, Y)\underline{m}_\alpha, \tag{2.4}$$

$$\tilde{\nabla}_X \underline{m}_\alpha := \sum_{\beta=1}^{\nu} M_\alpha^\beta(X)\underline{m}_\beta = \nabla_X \underline{m}_\alpha - B_\alpha(X), \tag{2.5}$$

$$\tilde{\nabla}_{\underline{m}_\alpha} X = \nabla_{\underline{m}_\alpha} X, \tag{2.6}$$

$$\tilde{\nabla}_{\underline{m}_\alpha} \underline{m}_\beta = \nabla_{\underline{m}_\alpha} \underline{m}_\beta = 0, \tag{2.7}$$

where X and Y are tangent to $V^{(1)} + \cdots + V^{(r)}$. It is clear that the vector fields $\tilde{\nabla}_X Y$ (resp. $\tilde{\nabla}_X \underline{m}_\alpha$) are just the projections of $\nabla_X Y$ (resp. $\nabla_X \underline{m}_\alpha$) onto $V^{(1)} + \cdots + V^{(r)}$ (resp. $V^{(0)}$).

It is also clear that $\tilde{\nabla}$ is metrical, i.e., $\tilde{\nabla}$ satisfies $\tilde{\nabla} g = 0$, but it has torsion. Let us denote its curvature tensor field by $\tilde{R}(X, Y)Z$. Moreover we introduce the operation $[\widetilde{X, Y}]$ defined by

$$[\widetilde{X, Y}] := \tilde{\nabla}_X Y - \tilde{\nabla}_Y X, \tag{2.8'}$$

where $X_{|p}, Y_{|p} \in V_p^{(1)} + \cdots + V_p^{(r)}$.

Proposition 2.1. *The second fundamental forms satisfy the following so-called first basic formulas*:

$$(2.8)\quad (\tilde{\nabla}_X B_\alpha)(Y) - (\tilde{\nabla}_Y B_\alpha)(X) + \sum_{\beta=1}^{\nu} \left(M_\alpha^\beta(Y) B_\beta(X) - M_\alpha^\beta(X) B_\beta(Y) \right) = 0,$$

$$(2.9)\quad dM_\alpha^\beta(X, Y) + \sum_{\gamma=1}^{\nu} M_\alpha^\gamma(Y) \wedge M_\gamma^\beta(X) + \frac{1}{2}\left[M^\beta(X, B_\alpha(Y)) - M^\beta(Y, B_\alpha(X)) \right] = 0,$$

$$(\nabla_{\underline{m}_\alpha} B_\beta)(X) + B_\beta \circ B_\alpha(X) = 0, \tag{2.10}$$

$$(\nabla_{\underline{m}_\alpha} M_\beta^\gamma)(X) + M_\beta^\gamma(B_\alpha(X)) = 0, \tag{2.11}$$

$$\nabla_{\underline{m}_\alpha} \tilde{\nabla}_X Y = \tilde{\nabla}_X \nabla_{\underline{m}_\alpha} Y + \tilde{\nabla}_{\nabla_{\underline{m}_\alpha}(X)} Y - \tilde{\nabla}_{B_\alpha(X)} Y - \sum_{\beta=1}^{\nu} M_\alpha^\beta(X) \nabla_{\underline{m}_\beta} Y, \tag{2.12}$$

i.e., $\tilde{R}(\underline{m}_\alpha, X)Y = 0$,

(2.13)

$$R(X, Y)Z = \tilde{R}(X, Y)Z + \sum_{\gamma=1}^{\nu} M^\gamma(Y, Z) B_\gamma(X) - M^\gamma(X, Z) B_\gamma(Y),$$

where d is the exterior derivative, the symbol $\wedge$ denotes the skew-product, and the C^∞ vector fields X, Y, Z are tangent to $V^{(1)} + \cdots + V^{(r)}$.

Proof. We get the first two equations from the equation $R(X, Y)\underline{m}_\alpha = 0$ as follows. Let us consider

$$\nabla_X\nabla_Y\underline{m}_\alpha = \nabla_X\left\{B_\alpha(Y) + \sum_{\beta=1}^{\nu} M_\alpha^\beta(Y)\underline{m}_\beta\right\}$$

$$= \tilde{\nabla}_X B_\alpha(Y) + \sum_{\beta=1}^{\nu}\left\{M^\beta(X, B_\alpha(Y)) + \nabla_X\left(M_\alpha^\beta(Y)\right)\right.$$

$$\left. + \sum_{\gamma=1}^{\nu} M_\alpha^\gamma(Y)M_\gamma^\beta(X)\right\}\underline{m}_\beta + \sum_{\gamma=1}^{\nu} M_\alpha^\gamma(Y)B_\gamma(X),$$

$$-\nabla_{[X,Y]}\underline{m}_\alpha = -B_\alpha(\widetilde{[X, Y]}) - \sum_{\beta=1}^{\nu} M_\alpha^\beta(\widetilde{[X, Y]})\underline{m}_\beta.$$

Substituting these in the formula

$$R(X, Y)\underline{m}_\alpha = \nabla_X\nabla_Y\underline{m}_\alpha - \nabla_Y\nabla_X\underline{m}_\alpha - \nabla_{[X,Y]}\underline{m}_\alpha = 0,$$

we get (2.8) and (2.9).

Formulas (2.10) and (2.11) follow in a similar manner from the equation $R(\underline{m}_\alpha, X)\underline{m}_\gamma = 0$, and (2.12) comes from $R(\underline{m}_\alpha, X)Y = 0$.

Finally we get (2.13) from the formula

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

by substituting in it formulas (2.4)–(2.7). The detailed computations are left to the reader.

Proposition 2.2. *The curvature tensor field R satisfies the following so called second basic formulas*:

$$(2.14) \qquad (\nabla_{\underline{m}_\alpha}R)(X, Y) = R(Y, B_\alpha(X)) + R(B_\alpha(Y), X),$$

$$(2.15) \qquad (\nabla_{X_i}R)(X_j, Y_j)Z_j = 0 \quad \textit{for } i \neq j \textit{ and } i, j \neq 0,$$

$$\sigma R(X_j, Y_j)B_\alpha(Z_j) = 0 \ (\textit{cyclic sum}),$$

where X, Y are tangent to $V^{(1)} + \cdots + V^{(r)}$, X_i is tangent to $V^{(i)}$, and X_j, Y_j, Z_j are tangent to $V^{(j)}$.

Proof. These formulas follow from the second Bianchi identity, as it can be seen from the following:

$$(\nabla_{\underline{m}_\alpha}R)(X, Y)V = -(\nabla_X R)(Y, \underline{m}_\alpha)V - (\nabla_Y R)(\underline{m}_\alpha, X)V$$

$$= R(Y, B_\alpha(X))V + R(B_\alpha(Y), X)V,$$

$$(\nabla_{X_i}R)(X_j, Y_j)Z_j = -(\nabla_{X_j}R)(Y_j, X_i)Z_j - (\nabla_{Y_j}R)(X_i, X_j)Z_j = 0. \quad \text{q.e.d.}$$

One of the most important consequences of the basic formulas is that the second fundamental forms and the curvature tensor are analytic fields along the integral manifolds of the null-space $V^{(0)}$.

To prove this statement let us consider a simply connected integral manifold N of the distribution $V^{(0)}$. We consider on N the vector fields $\underline{m}_1,\cdots,\underline{m}_\nu$ constructed in (2.1). Let $(u^1, u^2,\cdots,u^\nu)$ be the coordinate neighborhood according to $\underline{m}_1, \underline{m}_2,\cdots,\underline{m}_\nu$ so that $\underline{m}_\alpha = \partial/\partial u^\alpha$. Let us denote the origin of this coordinate neighborhood by 0.

Let us notice that the whole tangent space $T(M)$ can be parallelized over N. Indeed, since $R(\underline{m}_\alpha, \underline{m}_\beta) = 0$, the differential equations

$$\nabla_{\underline{m}_\alpha} X = 0, \quad \alpha = 1,\cdots,\nu,$$

are completely integrable. So there can be chosen a system $(\underline{v}_1, \underline{v}_2,\cdots,\underline{v}_{n-\nu})$ of C^∞ pairwise orthogonal unit vector fields on N, which are tangent to $V^{(1)} + \cdots + V^{(r)}$ and are totally parallel on N, i.e., $\nabla_{\underline{m}_\alpha} \underline{v}_j = 0$. It is obvious that the system

$$\underline{v}_1, \underline{v}_2,\cdots,\underline{v}_{n-\nu}, \underline{m}_1,\cdots,\underline{m}_\nu \tag{2.16}$$

defines an orthonormal basis in the whole tangent space $T(M)$ at every point of N. With the help of this parallelization we can identify two arbitrary tangent spaces $T_p(M)$ and $T_q(M)$ $(p, q \in N)$ over N.

Let us consider a geodesic $\varphi(s)$ of N of the form

$$\varphi(S) = \left(u^1(s),\cdots,u^\nu(s)\right) = \left(sa^1,\cdots,sa^\nu\right), \quad \sum_{\alpha=1}^{\nu} (a^\alpha)^2 = 1,$$

where $a^i \in \mathbf{R}$ are constant. $\varphi(s)$ goes through the origin and is parametrized by the arc length. Now if we introduce the field L of linear endomorphisms along $\varphi(s)$ defined by

$$L = \sum_{\alpha=1}^{\nu} a^\alpha B_\alpha, \tag{2.18}$$

then by the basic formula $\nabla_{\underline{m}_\alpha} B_\beta + B_\beta \circ B_\alpha = 0$ we get

$$\nabla_{\dot\varphi} B_\beta = \sum_{\gamma=1}^{\nu} a^\gamma \nabla_{\underline{m}_\gamma} B_\beta = -B_\beta \sum_{\gamma=1}^{\nu} a^\gamma B_\gamma = -B_\beta \circ L, \tag{2.19}$$

$$\nabla_{\dot\varphi} L = \sum_{\gamma=1}^{\nu} a^\gamma \nabla_{\dot\varphi} B_\gamma = -L^2. \tag{2.20}$$

It can be proved by induction that for the ρ-th derivative $\nabla^\rho_{\dot\varphi} B_j$ and $\nabla^\rho_{\dot\varphi} L$ we have

$$\nabla^\rho_{\dot\varphi} B_\gamma = (-1)^\rho \rho! B_\gamma \circ L^\rho, \tag{2.21}$$

(2.22) $$\nabla_{\varphi}^{\rho} L = (-1)^{\rho} \rho! L^{\rho+1}.$$

Now let us consider the analytic field $A(s)$ of endomorphisms along $\varphi(s)$ defined by

(2.23) $$A(s) = \sum_{\rho=0}^{\infty} (-1)^{\rho} L_{|0}^{\rho} s^{\rho}.$$

By the parallelization (2.16) the endomorphism $A(s)$ can be considered as an endomorphism over $\varphi(s)$ in $T_{\varphi(s)}(M)$. From (2.21) it is clear that the analytic field $B_{\gamma}(s)$ along $\varphi(s)$ defined by

(2.24) $$B_{\gamma}(s) := B_{\gamma|0} \circ A(s),$$

$B_{\gamma|0}$ being considered by the parallelization (2.16) over $\varphi(s)$, satisfies the differential equation (2.19) and $B_{\gamma}(0) = B_{\gamma|0}$. But the solutions of (2.19) are uniquely determined by the initial values. Thus $B_{\gamma}(s)$ is just the second fundamental form B_{γ} at $\varphi(s)$.

Let us write (2.24) in a more attractive form. As $u^{\alpha}(s) = sa^{\alpha}$ along $\varphi(s)$, we have

$$A(u^1,\cdots,u^{\nu}) = A(u^1(s),\cdots,u^{\nu}(s)) = \sum_{\rho=0}^{\infty} (-1)^{\rho} s^{\rho} \left(\sum_{\gamma=1}^{\nu} a^{\gamma} B_{\gamma|0} \right)^{\rho}$$

(2.25) $$= \sum_{\rho=0}^{\infty} (-1)^{\rho} \left(\sum_{\gamma=1}^{\nu} u^{\gamma} B_{\gamma|0} \right)^{\rho}$$

$$= \sum_{\varepsilon=0}^{\infty} \sum_{\substack{\rho_1 + \cdots + \rho_{\nu} = \varepsilon \\ \rho_{\alpha} \geqslant 0}} (-1)^{\varepsilon} (u^1)^{\rho_1} \cdots (u^{\nu})^{\rho_{\nu}} A^{\varepsilon}_{\rho_1 \cdots \rho_{\nu}}.$$

By definition

(2.26) $$A^{\varepsilon}_{\rho_1 \cdots \rho_{\nu}} := \sum B_{\alpha_1|0} \circ B_{\alpha_1|0} \circ \cdots \circ B_{\alpha_{\varepsilon}|0},$$

where the sum contains the part of the form $B_{\alpha_1|0} \circ \cdots \circ B_{\alpha_{\varepsilon}|0}$ in which B_{γ} occurs exactly ρ_{γ}-times, and $A^{\varepsilon}_{\rho_1 \cdots \rho_{\nu}}$ is considered by (2.16) over the point parametrized by $(u^1, u^2, \cdots, u^{\nu})$.

It is clear that the tensor field A is an analytic field on N. So by (2.24) we have

(2.27) $$B_{\gamma}(u^1, u^2, \cdots, u^{\nu}) = B_{\gamma|0} \circ A(u^1, \cdots, u^{\nu}),$$

where $B_{\gamma|0}$ is considered with the parallelization (2.16) over the point parametrized by $(u^1, u^2, \cdots, u^{\nu})$.

The tensor fields $M_\alpha^\beta(X)$ and $R(X,Y)Z$ can be discussed in a similar manner. From (2.11) it follows that

$$\nabla_{\dot\varphi} M_\alpha^\beta = -M_\alpha^\beta \circ L, \quad \nabla_{\dot\varphi}^\rho M_\alpha^\beta = (-1)^\rho \rho! M_\alpha^\beta \circ L^\rho, \tag{2.28}$$

so that, similarly as before,

$$M_\alpha^\beta(u^1,\cdots,u^\nu) = M_{\alpha|0}^\beta \circ A(u^1,\cdots,u^\nu). \tag{2.29}$$

Finally from (2.14) we get

$$(\nabla_{\dot\varphi} R)(X,Y) = -R(L(X),Y) - R(X,L(Y)), \tag{2.30}$$

and therefore

$$(\nabla_{\dot\varphi}^\rho R)(X,Y) = (-1)^\rho \rho! \sum_{\varepsilon=0}^{\rho} R(L^{\rho-\varepsilon}(X), L^\varepsilon(Y)). \tag{2.31}$$

Thus R is of the form

$$R_{|(u^1,u^2,\cdots,u^\nu)}(X,Y)Z = R_{|0}\big(A_{|(u^1,\cdots,u^\nu)}(X), A_{|(u^1,\cdots,u^\nu)}(Y)\big)Z, \tag{2.32}$$

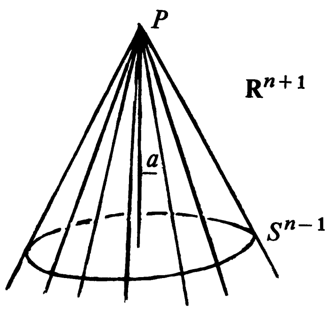
where $R_{|0}$ is considered over $(u^1, u^2,\cdots,u^\nu)$ by (2.16). Hence we have

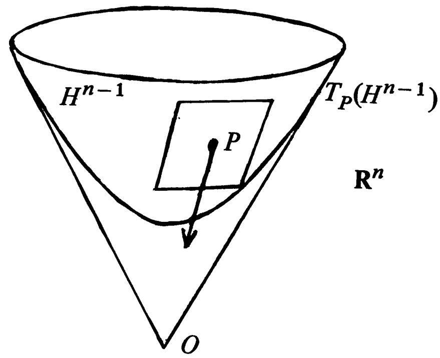
Proposition 2.3. *The second fundamental forms B_γ and M_α^β and the curvature tensor R are analytic fields along any integral manifold $V^{(0)}$. More precisely they are respectively of the form* (2.27), (2.29) *and* (2.32), *where the analytical endomorphism field A is defined by* (2.25) *and* (2.26).

3. The construction of simple semi-symmetric leafs

From the definitions it is clear that by the direct product of semi-symmetric Riemannian spaces we get again semi-symmetric spaces. Furthermore if we decompose a semi-symmetric space with the method described in Theorem 1.3, then it is obvious that every single simple leaf of the decomposition is semi-symmetric. By this reason we examine here simple semi-symmetric leafs; more precisely we construct several simple semi-symmetric leafs. In the next section we shall show that all these leafs constructed here will form a complete list of nonsymmetric simple semi-symmetric leafs. This observation leads us to state a local structure theorem on semi-symmetric spaces.

A. Elliptical and hyperbolical cones. The definition of elliptical cones is well known. Let us consider in $\mathbf{R}^n$ an $(n-1)$-dimensional hypersphere, denoted by S^{n-1}, and let $\underline{a}$ be the axis of S^{n-1} in $\mathbf{R}^{n+1}$ with an arbitrary point $P \in \underline{a}$, $P \notin \mathbf{R}^n$. Then the elliptical cone is inscribed as hypersurface in $\mathbf{R}^{n+1}$ by the half straight lines starting from P and crossing the points of S^{n-1}.





For the construction of hyperbolical cone let us consider in $\mathbf{R}^n$ a symmetric nondegenerate inner product $(,)$ with Lorentz signature. On the set of vectors X for which $(X, X) < 0$ holds we construct a positive definite Riemannian metric. For a negative number $-a^2$ the vectors X with $(X, X) = -a^2$ form a hypersurface which we denote by H^{n-1}. On the tangent spaces of H^{n-1} the indefinite inner product $(,)$ induces a positive definite inner product which defines a hyperbolic metric on H^{n-1}. Let us consider this positive definite inner product $\langle , \rangle$ on the tangent spaces of hypersurfaces H^{n-1}, and let us define the positive definite inner product for a vector X pointing from a point P to the origin O in such way that X is orthogonal to $T_p(H^{n-1})$ and its inner product with itself is the positive number $-b^2(X, X)$ for some positive constant b^2. It is clear that $\langle , \rangle_{|p}$ defines a positive definite Riemannian metric inside of a cone. We call this metric space a hyperbolic cone.

All these cones are semi-symmetric simple leafs. Indeed, in both cases the primitive holonomy group $\mathcal{H}_p$ is isomorphic to $SO(n-1)$, and leaves the tangent spaces $T_p(S^{n-1})$ and $T_p(H^{n-1})$ invariant. On the other hand the curvature tensor on these tangent subspaces is of the following form:

$$R(X, Y)Z = \kappa(g(X, Z)Y - g(Y, Z)X),$$

thus R remains invariant under the action of $\mathcal{H}_p$. This property proves the semi-symmetricity of the cones.

It is obvious that the nullspace $V^{(0)}$ is 1-dimensional at every point, and is spanned by the vector pointing to the vertex of the cones. It is also clear that all these spaces are infinitesimally irreducible. Moreover these spaces are not complete but are maximal in the following sense.

Definition 3.1. An arcwise connected Riemannian space M^n is said to be maximal if there does not exist another arcwise connected Riemannian space M'^n such that M^n is isometric to a real subset of M'^n.

B. Kaehlerian cones. We define these spaces in §4. These cones are the complex analogues of the above real cones.

C. Spaces foliated with $(n-2)$-dimensional Euclidean spaces. In the following let (M^n, g) be a simple leaf with index of nullity $\nu(p)$ equal to $(n-2)$ at every point of the manifold. Then considering the V-decomposition the tangent space is of the form

$$T(M) = V^{(0)} + V^{(1)},$$

where $V^{(1)}$ is of dimension 2 at every point, and $V^{(0)}$ is of dimension $(n-2)$. As these dimension numbers are constant the distributions $V^{(0)}$ and $V^{(1)}$ are of class C^∞ on the whole manifold M^n.

It is clear that all these spaces are semi-symmetric. Indeed in this case $\mathcal{H}_p$ is isomorphic to $SO(2)$, and the curvature tensor is of the form

$$R_{|p}(X, Y)Z = \kappa\big(g_{|p}(X, Z)Y - g_{|p}(Y, Z)X\big), \quad X, Y, Z \in V_p^{(1)}$$

on $V_p^{(1)}$, $p \in M$. Thus $\mathcal{H}_p$ leaves the curvature tensor invariant at every point, but this property guarantees the semi-symmetricity of the space. This fact motivates the following definition, since the $(n-2)$-dimensional integral manifolds of $V^{(0)}$ are Euclidean subspaces.

Definition 3.2. A space (M^n, g) with $\nu(p) = n-2$, $p \in M^n$, is called a space foliated with $(n-2)$-dimensional Euclidean spaces.

In the following we construct the metric of these spaces.

Let us consider in such a space a local system $(\underline{m}_1, \underline{m}_2, \cdots, \underline{m}_{n-2})$ of vector fields considered in the previous section in the formula (2.1). So these unit vector fields are pairwise orthogonal. Furthermore they are tangent to $V^{(0)}$ and satisfy $\nabla_{\underline{m}_\alpha}\underline{m}_\gamma = 0$. We consider the unit vector fields $\underline{v}_1, \underline{v}_2$ constructed in (2.16) such that the vector fields

$$\underline{v}_1, \underline{v}_2, \underline{m}_1, \underline{m}_2, \cdots, \underline{m}_{n-2} \tag{3.1}$$

are pairwise orthogonal and that $\nabla_{\underline{m}_\alpha}\underline{v}_j = 0$ holds. Then the fields $\underline{v}_1$ and $\underline{v}_2$ span just the subspace $V^{(1)}$ at every point of the manifold. Finally let $\underline{x}_1, \underline{x}_2$ be C^∞ vector fields such that in the system

$$\underline{x}_1, \underline{x}_2, \underline{m}_1, \underline{m}_2, \cdots, \underline{m}_{n-2} \tag{3.2}$$

the elements are linearly independent, and for their Lie derivatives

$$[\underline{x}_1, \underline{x}_2] = [\underline{x}_i, \underline{m}_\gamma] = 0$$

holds. It is obvious that such fields $\underline{x}_1$ and $\underline{x}_2$ exist. Since $[\underline{m}_\alpha, \underline{m}_\gamma] = 0$ holds, there exists a coordinate neighborhood of the form

$$\left(x^1, x^2, u^2, \cdots, u^{n-2}\right) \tag{3.3}$$

in M^n such that

$$\underline{x}_i = \partial/\partial x^i, \quad \underline{m}_\alpha = \partial/\partial u^\alpha.$$

In the sequel we write the metrical tensor field of the considered spaces in these coordinate neighborhoods in a characteristic form.

At first we introduce some important differential equations. The vector fields $\underline{v}_1$ and $\underline{v}_2$ can be written in the form

$$\underline{v}_i = \sum_{r=1}^{2} \varphi_i^r \underline{x}_r + \sum_{\alpha=1}^{n-2} \gamma_i^\alpha \underline{m}_\alpha, \tag{3.4}$$

where $\det \varphi := \varphi_1^1\varphi_2^2 - \varphi_1^2\varphi_2^1 \neq 0$, and thus also

$$\underline{x}_1 = \frac{1}{\det \varphi}\left(\varphi_2^2\underline{v}_1 - \varphi_1^2\underline{v}_2 + \sum_{\alpha=1}^{n-2}\left(\varphi_1^2\gamma_2^\alpha - \varphi_2^2\gamma_1^2\right)\underline{m}_\alpha\right), \tag{3.5}$$

$$\underline{x}_2 = \frac{1}{\det \varphi}\left(\varphi_1^1\underline{v}_2 - \varphi_2^1\underline{v}_1 + \sum_{\alpha=1}^{n-2}\left(\varphi_2^1\gamma_1^\alpha - \varphi_1^1\gamma_2^\alpha\right)\underline{m}_\alpha\right). \tag{3.6}$$

Now let us compute the fields $[\underline{m}_\gamma, \underline{v}_i]$ from the above formulas. So we have

$$\begin{aligned}[\underline{m}_\gamma, \underline{v}_i] = {} & \frac{1}{\det \varphi}\left(\varphi_2^2\frac{\partial\varphi_i^1}{\partial u^\gamma} - \varphi_2^1\frac{\partial\varphi_i^2}{\partial u^\gamma}\right)\underline{v}_1 + \frac{1}{\det \varphi}\left(\varphi_1^1\frac{\partial\varphi_i^2}{\partial u^\gamma} - \varphi_1^2\frac{\partial\varphi_i^1}{\partial u^\gamma}\right)\underline{v}_2 \\ & + \sum_{\alpha=1}^{n-2}\left\{\frac{\partial\gamma_i^\alpha}{\partial u^\gamma} + \frac{1}{\det\varphi}\left(\frac{\partial\varphi_i^2}{\partial u^\gamma}\varphi_2^1 - \frac{\partial\varphi_i^1}{\partial u^\gamma}\varphi_2^2\right)\gamma_1^\alpha \right. \\ & \left. + \frac{1}{\det\varphi}\left(\frac{\partial\varphi_i^1}{\partial u^\gamma}\varphi_1^2 - \frac{\partial\varphi_i^2}{\partial u^\gamma}\varphi_1^1\right)\gamma_2^\alpha\right\}\underline{m}_\alpha.\end{aligned} \tag{3.7}$$

On the other hand

$$[\underline{m}_\gamma, \underline{v}_i] = \nabla_{\underline{m}_\gamma}\underline{v}_i - \nabla_{\underline{v}_i}\underline{m}_\gamma = -B_\gamma(\underline{v}_i) - \sum_{\alpha=1}^{n-2} M_\gamma^\alpha(\underline{v}_i)\underline{m}_\alpha, \tag{3.8}$$

so introducing the components

$$B_\gamma(\underline{v}_i) = \sum_{r=1}^{2} B_{\gamma i}^{\ r}\underline{v}_r, \; M_\gamma^\alpha(\underline{v}_i) = M_{\gamma^i}^\alpha, \tag{3.9}$$

and comparing the coefficients in (3.7) and (3.8) we have the differential equations

$$\begin{aligned}&\frac{\partial\varphi_i^r}{\partial u^\gamma} + B_{\gamma i}^{\ s}\varphi_s^r = 0, \\ &\frac{\partial\gamma_i^\alpha}{\partial u^\gamma} + B_{\gamma i}^{\ s}\gamma_s^\alpha + M_{\gamma i}^\alpha = 0.\end{aligned} \tag{3.10}$$

We mention here the basic formulas (2.10) and (2.11), which in this formalism take the following form:

$$(3.11)\qquad \begin{aligned} &\frac{\partial B_{\alpha k}^{\,l}}{\partial u^{\gamma}} + B_{\alpha s}^{\,l} B_{\gamma k}^{\,s} = 0,\\ &\frac{\partial M_{\alpha l}^{\beta}}{\partial u^{\gamma}} + M_{\alpha s}^{\beta} B_{\gamma l}^{\,s} = 0. \end{aligned}$$

We state some further important equations. The vector field $[\underline{v}_1, \underline{v}_2]$ can be written as

$$(3.12)\qquad [\underline{v}_1, \underline{v}_2] = \nabla_{\underline{v}_1}\underline{v}_2 - \nabla_{\underline{v}_2}\underline{v}_1 = -\lambda_1\underline{v}_1 + \lambda_2\underline{v}_2 + \sum_{\alpha=1}^{n-2}\left(B_{\alpha 2}^{\,1} - B_{\alpha 1}^{\,2}\right)\underline{m}_{\alpha},$$

where the functions λ_1 and λ_2 are defined by means of the covariant derivative $\tilde{\nabla}$ by the formulas

$$(3.13)\qquad \tilde{\nabla}_{\underline{v}_1}\underline{v}_1 = \lambda_1\underline{v}_2,\ \tilde{\nabla}_{\underline{v}_2}\underline{v}_2 = \lambda_2\underline{v}_1,\ \tilde{\nabla}_{\underline{v}_1}\underline{v}_2 = -\lambda_1\underline{v}_1, \tilde{\nabla}_{\underline{v}_2}\underline{v}_1 = -\lambda_2\underline{v}_1.$$

These formulas are satisfied indeed, because $\tilde{\nabla}$ is metrical, i.e., $\tilde{\nabla} g = 0$ holds. Considering the functions λ_1 and λ_2 we mention that the basic formula (2.12) is equivalent to the differential equations

$$(3.14)\qquad \begin{aligned} &\frac{\partial \lambda_1}{\partial u^{\gamma}} = -B_{\gamma 1}^{\,1}\lambda_1 + B_{\gamma 1}^{\,2}\lambda_2,\\ &\frac{\partial \lambda_2}{\partial u^{\gamma}} = B_{\gamma 2}^{\,1}\lambda_1 - B_{\gamma 2}^{\,2}\lambda_2. \end{aligned}$$

Let us notice that the vector field $[\underline{v}_1, \underline{v}_2]$ can be computed with the help of formulas (3.4), (3.5) and (3.6). The result is:

$$(3.15)\qquad \begin{aligned} [\underline{v}_1, \underline{v}_2] = &-\left\{\frac{1}{\det\varphi}\varphi_2^r\frac{\partial \det\varphi}{\partial x^r} - \frac{\partial \varphi_2^r}{\partial x^r} + \gamma_1^{\alpha}B_{\alpha 2}^{\,1} - \gamma_2^{\alpha}B_{\alpha 1}^{\,1}\right\}\underline{v}_1\\ &+\left\{\frac{1}{\det\varphi}\varphi_1^r\frac{\partial \det\varphi}{\partial x^r} - \frac{\partial \varphi_1^r}{\partial x^r} + \gamma_2^{\alpha}B_{\alpha 1}^{\,2} - \gamma_1^{\alpha}B_{\alpha 2}^{\,2}\right\}\underline{v}_2\\ &+\sum_{\alpha=1}^{n-2}\left\{\gamma_1^{\alpha}\left(\frac{1}{\det\varphi}\varphi_2^r\frac{\partial \det\varphi}{\partial x^r} - \frac{\partial \varphi_2^r}{\partial x^r}\right)\right.\\ &\qquad -\gamma_2^{r}\left(\frac{1}{\det\varphi}\varphi_1^r\frac{\partial \det\varphi}{\partial x^r} - \frac{\partial \varphi_1^r}{\partial x^r}\right) - \gamma_1^{\rho}M_{\rho 2}^{\alpha} + \gamma_2^{\rho}M_{\rho 1}^{\alpha}\\ &\qquad \left. +\varphi_1^r\frac{\partial \gamma_2^{\alpha}}{\partial x^r} - \varphi_2^r\frac{\partial \gamma_1^{\alpha}}{\partial x^r}\right\}\underline{m}_{\alpha}. \end{aligned}$$

Comparing the coefficients in (3.15) and (3.12) gives

$$\lambda_1 = \frac{1}{\det\varphi}\varphi_2^r\frac{\partial \det\varphi}{\partial x^r} - \frac{\partial\varphi_2^r}{\partial x^r} + \gamma_1^\rho B_{\rho 2}^{\,1} - \gamma_2^\rho B_{\rho 1}^{\,1}, \tag{3.16}$$

$$\lambda_2 = \frac{1}{\det\varphi}\varphi_1^r\frac{\partial \det\varphi}{\partial x^r} - \frac{\partial\varphi_1^r}{\partial x^r} + \gamma_2^\rho B_{\rho 1}^{\,2} - \gamma_1^\rho B_{\rho 2}^{\,2}, \tag{3.17}$$

$$\begin{aligned}&\varphi_1^r\frac{\partial\gamma_2^\alpha}{\partial x^r} - \varphi_2^r\frac{\partial\gamma_1^\alpha}{\partial x^r} + \gamma_1^\alpha\left(\frac{1}{\det\varphi}\varphi_2^r\frac{\partial\det\varphi}{\partial x^r} - \frac{\partial\varphi_2^r}{\partial x^r}\right)\\ &-\gamma_2^\alpha\left(\frac{1}{\det\varphi}\varphi_1^r\frac{\partial\det\varphi}{\partial x^r} - \frac{\partial\varphi_1^r}{\partial x^r}\right) - \gamma_1^\rho M_{\rho 2}^\alpha + \gamma_2^\rho M_{\rho 1}^\alpha + B_{\alpha 1}^{\,2} - B_{\alpha 2}^{\,1} = 0.\end{aligned} \tag{3.18}$$

Now let us derive partially the above equation (3.18) with respect to the variable u^γ. Then by (3.10) and (3.11) we get

$$\begin{aligned}&\varphi_1^r\frac{\partial M_{\gamma 2}^\alpha}{\partial x^r} - \varphi_2^r\frac{\partial M_{\gamma 1}^\alpha}{\partial x^r} + M_{\gamma 1}^\alpha\left(\frac{1}{\det\varphi}\varphi_2^r\frac{\partial\det\varphi}{\partial x^r} - \frac{\partial\varphi_2^r}{\partial x^r}\right)\\ &-M_{\gamma 2}^\alpha\left(\frac{1}{\det\varphi}\varphi_1^r\frac{\partial\det\varphi}{\partial x^r} - \frac{\partial\varphi_1^r}{\partial x^r}\right) + B_{\gamma 1}^{\,1}B_{\alpha 2}^{\,1} + B_{\gamma 1}^{\,2}B_{\alpha 2}^{\,2}\\ &-B_{\gamma 2}^{\,2}B_{\alpha 1}^{\,2} - B_{\gamma 2}^{\,1}B_{\alpha 1}^{\,1} + M_{\gamma 2}^\rho M_{\rho 1}^\alpha - M_{\gamma 1}^\rho M_{\rho 2}^\alpha = 0.\end{aligned} \tag{3.19}$$

Finally substituting (3.16) and (3.17) into (3.14) yields

$$\begin{aligned}&\varphi_1^r\frac{\partial B_{\gamma 2}^{\,1}}{\partial x^r} - \varphi_2^r\frac{\partial B_{\gamma 1}^{\,1}}{\partial x^r} - B_{\gamma 2}^{\,l}\left(\frac{1}{\det\varphi}\varphi_l^r\frac{\partial\det\varphi}{\partial x^r} - \frac{\partial\varphi_l^r}{\partial x^r}\right)\\ &-B_{\gamma 1}^{\,2}\left(\frac{1}{\det\varphi}\varphi_1^r\frac{\partial\det\varphi}{\partial x^r} - \frac{\partial\varphi_1^r}{\partial x^r}\right) + B_{\gamma 1}^{\,1}\left(\frac{1}{\det\varphi}\varphi_2^r\frac{\partial\det\varphi}{\partial x^r} - \frac{\partial\varphi_2^r}{\partial x^r}\right)\\ &+B_{\gamma 2}^{\,2}\left(\gamma_2^\rho B_{\rho 1}^{\,1} - \gamma_1^\rho B_{\rho 2}^{\,1}\right) + B_{\gamma 1}^{\,2}\left(\gamma_1^\rho B_{\rho 2}^{\,2} - \gamma_2^\rho B_{\rho 1}^{\,2}\right) - M_{\gamma 1}^\rho B_{\rho 2}^{\,1} + M_{\gamma 2}^\rho B_{\rho 1}^{\,1} = 0,\end{aligned} \tag{3.20}$$

$$\begin{aligned}&\varphi_1^r\frac{\partial B_{\gamma 2}^{\,2}}{\partial x^r} - \varphi_2^r\frac{\partial B_{\gamma 1}^{\,2}}{\partial x^r} + B_{\gamma 1}^{\,l}\left(\frac{1}{\det\varphi}\varphi_l^r\frac{\partial\det\varphi}{\partial x^r} - \frac{\partial\varphi_l^r}{\partial x^r}\right)\\ &+B_{\gamma 2}^{\,1}\left(\frac{1}{\det\varphi}\varphi_2^r\frac{\partial\det\varphi}{\partial x^r} - \frac{\partial\varphi_2^r}{\partial x^r}\right) - B_{\gamma 2}^{\,2}\left(\frac{1}{\det\varphi}\varphi_1^r\frac{\partial\det\varphi}{\partial x^r} - \frac{\partial\varphi_1^r}{\partial x^r}\right)\\ &-B_{\gamma 1}^{\,1}\left(\gamma_1^\rho B_{\rho 2}^{\,2} - \gamma_2^\rho B_{\rho 1}^{\,2}\right) - B_{\gamma 2}^{\,1}\left(\gamma_2^\rho B_{\rho 1}^{\,1} - \gamma_1^\rho B_{\rho 2}^{\,1}\right) + M_{\gamma 2}^\rho B_{\rho 1}^{\,2} - M_{\gamma 1}^\rho B_{\rho 2}^{\,2} = 0.\end{aligned} \tag{3.21}$$

The differential equations (3.10), (3.11), (3.18), (3.19), (3.20), (3.21) play a basic role in the following considerations.

Without the description of the details of the computations we mention that deriving (3.18), (3.19), (3.20) and (3.21) corresponding to the variable u^ε we do not have new independent equations. More precisely let us denote the left side of (3.18), (3.19), (3.20), (3.21) by Γ^α, $\mathfrak{M}_\gamma^\alpha$, $\mathfrak{B}_\gamma^1$, $\mathfrak{B}_\gamma^2$. Then using the formulas (3.10), (3.11) we have

$$\frac{\partial \mathfrak{B}_\gamma^i}{\partial u^\varepsilon} = -(\operatorname{Trace} B_\varepsilon)\mathfrak{B}_\gamma^i - B_{\gamma l}{}^i\mathfrak{B}_\varepsilon^l, \tag{3.22}$$

$$\frac{\partial \Gamma^\alpha}{\partial u^\varepsilon} = -\mathfrak{M}_\varepsilon^\alpha - (\operatorname{Trace} B_\varepsilon)\Gamma^\alpha, \tag{3.23}$$

$$\frac{\partial \mathfrak{M}_\gamma^\alpha}{\partial u^\varepsilon} = -(\operatorname{Trace} B_\varepsilon)\mathfrak{M}_\gamma^\alpha. \tag{3.24}$$

The details of the computations are left to the reader.

Now on the 2-dimensional sub-plane defined by

$$(x^1x^2) := (x^1, x^2, 0, \cdots, 0),$$

let us consider arbitrary C^∞ functions $\varphi_j^i(x^1, x^2)$, $1 \leqslant i;\ j \leqslant 2$, such that $\det \varphi := \varphi_1^1 - \varphi_1^2\varphi_2^1 \neq 0$. Henceforth let us consider (3.18), (3.19), (3.20), (3.21) on this plane as differential equations containing the unknown functions γ_i^α, $M_{\alpha i}^\beta$, $B_{\alpha i}^{\ j}$. Let $c^2(s)$ be the maximal integral curve of the vector field $\varphi_2^1\underline{x}_1 + \varphi_2^2\underline{x}_2$ on the plane through the origin 0. From the Picard-Lindelöff theorem on differential equations it is clear that for arbitrary C^∞ functions $\gamma_1^\alpha(x^1, x^2)$, $M_{\alpha 1}^\beta(x^1, x^2)$, $B_{\alpha 1}^1(x^1, x^2)$, $B_{\alpha 1}^2(x^1, x^2)$ on (x^1, x^2) and for arbitrary C^∞ functions $\gamma_2^\alpha(s)$, $M_{\alpha 2}^\beta(s)$, $B_{\alpha 2}^1(s)$, $B_{\alpha 2}^2(s)$ on $c^2(s)$ there exists a uniquely determined system $(\gamma_i^\alpha(x^1, x^2), M_{\alpha i}^\beta(x^1, x^2), B_{\alpha i}^j(x^1, x^2))$ of solutions of (3.18), (3.19), (3.20) and (3.21) on the plane (x^1, x^2) with the above prescribed initial conditions.

Let us consider the differential equations (3.10) on an integral manifold N of $V^{(0)}$. In a similar manner as in Proposition 2.3 we get that the solutions of (3.10) are analytic on N and they are of the form

$$\varphi_i^r = \varphi_{s|0}^r A_i^s(u^1, \cdots, u^{n-2}),$$
$$\gamma_i^\alpha = \gamma_{s|0}^\alpha A_i^s(u^1, \cdots, u^{n-2}) - u^\beta M_{\beta s|0}^\alpha A_i^s(u^1, u^2, \cdots, u^{n-2}).$$

According to these facts we have

Theorem 3.1. *In a coordinate neighborhood* (3.3), *the metrical tensor field of a space foliated with* $(n-2)$*-dimensional Euclidean spaces is of the form*

$$g_{11} = g(\underline{x}_1, \underline{x}_2) = \frac{1}{(\det\varphi)^2}\left((\varphi_2^2)^2 + (\varphi_1^2)^2 + \sum_{\alpha=1}^{n-2}(\varphi_1^2\gamma_2^\alpha - \varphi_2^2\gamma_1^\alpha)^2\right),$$

$$g_{22} = g(\underline{x}_2, \underline{x}_2) = \frac{1}{(\det\varphi)^2}\left((\varphi_1^1)^2 + (\varphi_2^1)^2 + \sum_{\alpha=1}^{n-2}(\varphi_2^1\gamma_1^\alpha - \varphi_1^1\gamma_2^\alpha)^2\right),$$

$$g_{12} = g_{21} = g(\underline{x}_1, \underline{x}_2)$$

$$(3.25)\qquad = \frac{1}{(\det\varphi)^2}\left(-\varphi_2^1\varphi_2^2 - \varphi_1^1\varphi_1^2 + \sum_{\alpha=1}^{n-2}(\varphi_1^2\gamma_2^\alpha - -\varphi_2^2\gamma_1^\alpha)(\varphi_2^1\gamma_1^\alpha - \varphi_1^1\gamma_2^\alpha)\right),$$

$$g_{1\alpha} = g(\underline{x}_1, \underline{m}_{\alpha-2}) = \frac{1}{\det\varphi}(\varphi_1^2\gamma_2^{\alpha-2} - \varphi_2^2\gamma_1^{\alpha-2}) \quad \text{if } \alpha \geqslant 3,$$

$$g_{2\alpha} = g(\underline{x}_2, \underline{m}_{\alpha-2}) = \frac{1}{\det\varphi}(\varphi_2^1\gamma_1^{\alpha-2} - \varphi_1^1\gamma_2^{\alpha-2}) \quad \text{if } \alpha \geqslant 3,$$

$$g_{\alpha\beta} = \delta_{\alpha\beta}\ (\textit{Kronecker } \delta\textit{-function}) \quad \text{if } \alpha, \beta \geqslant 3,$$

where the functions $B_{\alpha i}^j$, $M_{\alpha i}^\beta$, φ_i^k, γ_i^α *are of the form*

$$(3.26)\qquad B_{\alpha i}^j = B_{\alpha s}^j(x^1, x^2)A_i^s(x^1, x^2, u^1, \cdots, u^{n-2}),$$

$$(3.27)\qquad M_{\alpha i}^\beta = M_{\alpha s}^\beta(x^1, x^2)A_i^s(x^1, x^2, u^1, \cdots, u^{n-2}),$$

$$(3.28)\qquad \varphi_i^k = \varphi_s^k(x^1, x^2)A_i^s(x^1, x^2, u^1, \cdots, u^{n-2}),$$

$$(3.29)\qquad \gamma_i^\alpha = \gamma_s^\alpha(x^1, x^2)A_i^s(x^1, x^2, u^1, \cdots, u^{n-2}) - u^\beta M_{\beta s}^\alpha(x^1, x^2)A_i^s(x^1, \cdots, u^{n-2}),$$

the functions $B_{\alpha i}^j(x^1, x^2)$, $M_{\alpha i}^\beta(x^1, x^2)$, $\varphi_i^k(x^1, x^2)$ *and* $\gamma_i^\alpha(x^1, x^2)$ *are solutions of* (3.18), (3.19), (3.20) *and* (3.21) *on the sub-plane* $(x^1, x^2) = (x^1, x^2, 0, \cdots, 0)$, *and*

$$(3.30)\qquad \begin{aligned} A_i^s(x^1, x^2, u^1, \cdots, u^{n-2}) &= \left[\sum_{\rho=0}^{\infty}\left(\sum_{\alpha=1}^{n-2} - u^\alpha B_\alpha(x^1, x^2)\right)^\rho\right]_i^s \\ &= \sum_{\varepsilon=0}^{\infty}\ \sum_{\substack{\rho_1+\cdots+\rho_{n-2}=\varepsilon \\ \rho_\alpha\geqslant 0}} (-1)^\varepsilon\left[A_{\rho_1\cdots\rho_{n-2}}^\varepsilon(x^1, x^2)\right]_i^s (u^1)^{\rho_1}\cdots(u^\nu)^{\rho_\nu}, \end{aligned}$$

with

$$(3.31)\qquad \left[A_{\rho_1\cdots\rho_{n-2}}^\varepsilon(x^1, x^2)\right]_i^s := \sum_{l_1\cdots l_{\varepsilon-1}} B_{\alpha_1 l_1}^{\ s}(x^1, x^2)B_{\alpha_2 l_2}^{l_1}\cdots B_{\alpha_\varepsilon i}^{l_{\varepsilon-1}}(x^1, x^2),$$

where the parts in the sum contain the matrix $B_{1i}^{\ j}$ *just* ρ_1*-times,* $\cdots$*,etc., the matrix* $B_{(n-2)i}^{\ \ \ \ \ \ j}$ *just* $\rho_{(n-2)}$*-times.*

The form (3.25) *of the metric is a characteristic for a space foliated with* $(n-2)$*-dimensional Euclidean spaces. This means that a space with a metric of the form* (3.25) *is always a space foliated with* $(n-2)$*-dimensional Euclidean spaces.*

Let us consider an arbitrary coordinate neighborhood $(x^1, x^2, u^1, \cdots, u^{n-2})$. *Then for arbitrary prescribed* C^∞ *functions* $\varphi_j^i(x^1, x^2)$ $(\det \varphi \neq 0)$, $\gamma_1^\alpha(x^1, x^2)$, $M_{\alpha 1}^\beta(x^1, x^2)$, $B_{\alpha 1}^{\ i}(x^1, x^2)$ *on the plane* $(x^1, x^2) = (x^1, x^2, 0, \cdots, 0)$, *and for arbitrary* C^∞ *functions* $\gamma_2^\alpha(s)$, $M_{\alpha 2}^\beta(s)$, $B_{\alpha 2}^{\ 2}(s)$ *on the coordinate line* $x^2(s)$ *through the origin there exists exactly one space foliated with* $(n-2)$*-dimensional Euclidean spaces on* $(x^1, x^1, u^1, \cdots, u^{n-2})$ *with the prescribed initial conditions. The metric is defined in such a way that the system* $(\underline{v}_1, \underline{v}_2, \underline{m}_1, \underline{m}_2, \cdots, \underline{m}_{n-2})$ *defined in* (3.4) *is orthonormal in the space.*

Proof. We need to prove only the characteristic property of the above metric.

Let g_{ij} be a Riemannian metric satisfying (3.25)–(3.31). First we prove that the functions γ_i^α, φ_i^j, $M_{\alpha i}^\beta$ and $B_{\alpha i}^{\ j}$ satisfy the differential equations (3.18)–(3.21) not only on the sub-plane $(x^1, x^2) = (x^1, x^1, 0, \cdots, 0)$ but also on the whole coordinate neighborhood $(x^1, x^1, u^1, \cdots, u^{n-2})$.

In fact, by (3.26) $-$ (3.31) we get (3.10) and (3.11). Thus these functions are analytic along the integral manifolds N of the distribution $V^{(0)}$. Let us substitute these functions into the left side of (3.18)–(3.21) which we denote by

$$\Gamma^\alpha, \mathfrak{M}_\alpha^\beta, \mathfrak{B}_\alpha^1, \mathfrak{B}_\alpha^2.$$

It is obvious that all these functions are analytic along the integral manifolds of $V^{(0)}$. As (3.10)–(3.11) hold, we get (3.22), (3.23), (3.24). With the help of these equations we can prove (by induction) that not only the functions Γ^α, $\mathfrak{M}_\alpha^\beta$, $\mathfrak{B}_\alpha^i$ but also their derivatives of arbitrary orders (with respect to the variable $(u^1, \cdots, u^{n-2})$ vanish at $(x^1, x^2, 0, \cdots, 0)$. As these functions are analytic in the variable $(u^1, u^2, \cdots, u^{n-2})$, $\Gamma^\alpha = 0$, $\mathfrak{M}_\alpha^\beta = 0$, $\mathfrak{B}_\alpha^i = 0$ hold everywhere. This proves the first statement and we get that (3.10), (3.11), and (3.18)–(3.21) are satisfied on the whole coordinate neighborhood.

Now let us introduce the vector fields $\underline{v}_1$ and $\underline{v}_2$ by the formula (3.4) where $\underline{m}_i := \partial/\partial u^i$. Then the fields $(\underline{v}_1, \underline{v}_2, \underline{m}_1, \cdots, \underline{m}_{n-2})$ form an orthonormal basis at every point. We define the covariant derivative ∇ by the following formulas:

$$\nabla_{\underline{m}_\alpha} \underline{m}_\beta := 0, \quad \nabla_{\underline{m}_\alpha} \underline{v}_i = 0, \quad \nabla_{\underline{v}_1} \underline{v}_1 = \lambda_1 \underline{v}_2 - \sum_{\alpha=1}^{n-2} B_{\alpha 1}^{\ 1} \underline{m}_\alpha,$$

$$\nabla_{\underline{v}_2}\underline{v}_2 = \lambda_2\underline{v}_1 - \sum_{\alpha=1}^{n-2} B_{\alpha 2}^{2}\underline{m}_\alpha, \quad \nabla_{\underline{v}_1}\underline{v}_2 = -\lambda_1\underline{v}_1 - \sum_{\alpha=1}^{n-2} B_{\alpha 1}^{2}\underline{m}_\alpha,$$

$$\nabla_{\underline{v}_2}\underline{v}_1 = -\lambda_2\underline{v}_2 - \sum_{\alpha=1}^{n-2} B_{\alpha 2}^{1}\underline{m}_\alpha, \quad \nabla_{\underline{v}_j}\underline{m}_\alpha = \sum_{k=1}^{2} B_{\alpha j}^{k}\underline{v}_k + \sum_{\beta=1}^{n-2} M_{\alpha j}^{\beta}\underline{m}_\beta,$$

where the functions λ_1 and λ_2 are defined by (3.16) and (3.17). It is obvious that ∇ is metrical, i.e., $\nabla g = 0$ holds.[1] But by (3.7)–(3.10), (3.12)–(3.17) and (3.20)–(3.21) we get that the torsion of ∇ vanishes, i.e., ∇ is the Levi-Civita connection of the space. So we must show that $\nu(\rho) = n - 2$ is satisfied for ∇.

Let us denote the curvature of ∇ by $R(X, Y)Z$. We must prove that

$$R(\underline{m}_\gamma, \underline{m}_\rho) = 0, \quad R(\underline{m}_\gamma, \underline{v}_j) = 0.$$

The first equation is obvious as $\nabla_{\underline{m}_\gamma}\underline{m}_\rho = 0$, $\nabla_{\underline{m}_\gamma}\underline{v}_j = 0$. It can be seen that the second equation is equivalent to the basic formulas (2.8)–(2.12) in Proposition 2.1. But (3.11), (3.20), (3.21) are equivalent to (2.10), (2.11) and (2.12). Thus we must prove only (2.8) and (2.9). But these equations can be obtained by substituting into the Bianchi identity

$$[\underline{m}_i, [\underline{v}_1, \underline{v}_2]] = -[\underline{v}_1, [\underline{v}_2, \underline{m}_i]] - [\underline{v}_2, [\underline{m}_i, \underline{v}_1]]$$

the previously proved formulas

$$[\underline{v}_1, \underline{v}_2] = \nabla_{\underline{v}_1}\underline{v}_2 - \nabla_{\underline{v}_2}\underline{v}_1 = -\lambda_1\underline{v}_1 + \lambda_2\underline{v}_2 + \sum_{\alpha=1}^{n-2}\left(B_{\alpha 2}^{1} - B_{\alpha 1}^{2}\right)\underline{m}_\alpha,$$

$$[\underline{v}_i, \underline{m}_\alpha] = \nabla_{\underline{v}_i}\underline{m}_\alpha - \nabla_{\underline{m}_\alpha}\underline{v}_i = \sum_{k=1}^{2} B_{\alpha i}^{k}\underline{v}_k + \sum_{\beta=1}^{n-2} M_{\alpha i}^{\beta}\underline{m}_\beta.$$

Thus the space is foliated with $(n - 2)$-dimensional Euclidean space, and the proof is finished.

4. The main theorems and the local structure theorem

In the sequel we use a theorem of B. Kostant which we describe in the following. (This result was not published by B. Kostant but can be found in [18, p. 230].) We mention also the fundamental ideas of holonomy systems developed by J. Simons [18].

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[1] This statement can be proved by showing the formula

$$g(\nabla_X Z, Y) = \tfrac{1}{2}\{X \cdot g(Y, Z) - Y \cdot g(Z, X) + Z \cdot g(X, Y) \\ -g(Z, [X, Y]) - g(X, [Z, Y]) - g(Y, [Z, X])\}$$

with the help of (3.7)–(3.21) with respect to the fields $\underline{v}_1, \underline{v}_2, \underline{m}_1, \cdots, \underline{m}_{n-1}$.

Let V^n be an n-dimensional real vector space with a positive definite inner product $\langle\,,\rangle$. A tensor $R(X,Y)Z$ of type (1.3) over V^n is called a curvature operator whenever the following hold:

$$
\begin{aligned}
&R(X,Y) = -R(X,Y),\\
&R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0,\\
&\langle R(X,Y)Z, V\rangle = -\langle R(X,Y)V, Z\rangle,\\
&\langle R(X,Y)Z, V\rangle = \langle R(Z,V)X, Y\rangle.
\end{aligned}
\tag{4.1}
$$

Let R be a curvature operator on $(V^n, \langle\,,\rangle)$, and G some compact group of orthogonal endomorphisms of V^n with Lie algebra $\mathcal{G}$. G is called a holonomy group of R if $R(X,Y) \in \mathcal{G}$ for all $X, Y \in V^n$.

A triple $S = [V^n, R, G]$ consisting of a Euclidean space V^n, a curvature operator R, and a connected holonomy group G is called a holonomy system. The holonomy system S is said to be symmetric if $g(R) = R$ for all $g \in G$, where the action of g on R is defined by

$$(g(R))(X,Y) := gR(g^{-1}X, g^{-1}Y)g^{-1}.$$

The system S is said to be irreducible iff G acts on V irreducibly.

Let us denote the Lie algebra of all skew-symmetric endomorphisms of V by $\mathcal{A}$. On $\mathcal{A}$ there exists a natural negative definite inner product defined by

$$\langle A, B\rangle := \operatorname{Trace} AB, \quad A, B \in \mathcal{A}.$$

Let $\mathcal{G}$ be the Lie algebra of some compact subgroup of orthogonal endomorphisms in V, and let $K(\,,)$ denote the Killing form of $\mathcal{G}$. Since the form is negative semidefinite, the bilinear form $K(\,,) + \langle\,,\rangle$ is negative definite on $\mathcal{G}$. Thus there is a nonsingular transformation $T\colon \mathcal{G} \to \mathcal{G}$ such that

$$K(A,B) + \langle A, B\rangle = \langle A, T(B)\rangle. \tag{4.2}$$

It is clear that T is symmetric, i.e.,

$$\langle A, T(B)\rangle = \langle B, T(A)\rangle. \tag{4.3}$$

From (4.2) and (4.3) we get that all the eigenvalues of T are positive real numbers.

There is a natural identification $X \wedge Y \to (X \wedge Y)$ of $\wedge^2 V$ with $\mathcal{A}$ defined by

$$\langle A, (X \wedge Y)\rangle = \langle A(X), Y\rangle.$$

Finally let $P\colon \mathcal{A} \to \mathcal{G}$ be the projection of $\mathcal{A}$ onto $\mathcal{G}$ via $\langle\,,\rangle$. Then the above mentioned theorem of B. Kostant is as follows.

Theorem 4.1 (*B. Kostant*). *Let $S = [V^n, R, G]$ be an irreducible symmetric holonomy system. Then there is a constant γ such that*

$$R(X,Y) = \gamma(T^{-1} \circ P)((X \wedge Y)).$$

A simple consequence of Kostant's theorem is the following, which is also mentioned in [18, p. 232].

Theorem 4.2. *Let $S = [V^n, R, G]$ and $S' = [V^n, R', G]$ be two symmetric and irreducible holonomy systems with the same vector space V and holonomy group G. Then there exist a nonzero real number such that $R = cR'$.*

For a curvature operator $R(X, Y)Z$ with components $R_i{}^k{}_{kl}$ let $R_{ij} := R_i{}^s{}_{sj}$ be the Ricci tensor, and $\mathcal{R} := R_{ij}{}^{ij} = -R_i{}^i$ be the Riemannian curvature scalar. If the system $S = [V^n, R, G]$ is irreducible and symmetric, then the Ricci tensor is the multiple of the inner product, i.e.,

$$R_{ij} = \kappa g_{ij},$$

where g_{ij} denotes the components of $\langle\, , \rangle$. Thus we have $\mathcal{R} = -n\kappa$. Let us denote the eigenvalues of T by $\lambda_1^*, \lambda_2^*, \cdots, \lambda_r^*$ where r is the dimension of $\mathcal{G}$. Then by Theorem 4.1 we get that for an irreducible symmetric holonomy system $[V^n, R, G]$ the formula

$$\mathcal{R} = 2\gamma\left(\sum_{i=1}^{\dim \mathcal{G}} 1/\lambda_i^*\right) \tag{4.4}$$

holds. Furthermore if we consider the curvature operator R by the Bianchi identities $\langle R(X, Y)Z, V\rangle = -\langle R(X, Y)V, Z\rangle$, $\langle R(X, Y)Z, V\rangle = \langle R(Z, V)X, Y\rangle$ as a symmetric linear endomorphism on $\mathcal{G}$, then its nonzero eigenvalues are $\lambda_j = \gamma/\lambda_j^*, j = 1, \cdots, r$, i.e., by (4.4) we have

$$\lambda_j = \frac{\mathcal{R}}{2(\sum_{i=1}^r 1/\lambda_i^*)}\frac{1}{\lambda_j^*}, \quad j = 1, 2, \cdots, r = \dim \mathcal{G}. \tag{4.5}$$

As the eigenvalues λ_i^* of T are positive, and γ in Kostant's theorem is nonzero, we have

Lemma 4.1. *The Riemannian curvature scalar $\mathcal{R}$ of an irreducible symmetric holonomy system $[V^n, R, G]$, $R \neq 0$, never vanishes.*

In the following considerations we need the following lemma. If B is a linear endomorphism in a real Euclidean vector space $(V, \langle\, , \rangle)$, then its transposition B^T is defined by $\langle X, B(Y)\rangle = \langle B^T(X), Y\rangle$.

Lemma 4.2. *Let $\mathcal{G}$ be an irreducible sub-Lie algebra of the skew-symmetric linear endomorphisms in an Euclidean real vector space $(V^n, \langle\, , \rangle)$, $n > 2$. Furthermore let B be a linear endomorphism in V^n for which*

$$u \circ B = -B^T u \text{ for all } u \in \mathcal{G}. \tag{4.6}$$

Then $B = 0$, or $n = 2m$ and $B = b\mathcal{F}$, where $b \in \mathbf{R}$, $\mathcal{F}^2 = -\mathrm{id}$, and $\mathcal{F}$ commutes with every element of $\mathcal{G}$, and thus $\mathcal{F}$ is skew-symmetric.

Proof. Let us suppose that $B \neq 0$, and let $\lambda \neq 0$ be an (in general complex) eigenvalue of B. If $V^c = V + \underline{i}V$ denotes the complexification of V, then let us extend the endomorphisms B and all $u \in \mathcal{G}$ to complex linear endomorphisms of V^c. Let $V_\lambda \subseteq V^c$ be the complete invariant subspace corresponding to eigenvalue λ, i.e., V_λ contains the vectors $X \in V^c$ for which $(B - \lambda I)^n X = 0$, where I denotes the identity endomorphism. For every $u \in \mathcal{G}$ and $X \in V_\lambda$ we get

$$0 = u(B - \lambda I)^n X = (-1)^n (B^T + \lambda I)^n u(X),$$
$$(B^T + \lambda I)^n u(V_\lambda) = 0.$$

Thus $u(V_\lambda)$ is contained in the complete invariant subspace of B^T corresponding to its eigenvalue $-\lambda$. But it is well known that this subspace is equal to $V_{-\bar{\lambda}}$ of B. Thus it follows that the values $-\lambda$, $-\bar{\lambda}$ are also eigenvalues of B, and

$$u(V_\lambda) \subseteq V_{-\bar{\lambda}} \tag{4.7}$$

for every $u \in \mathcal{G}$. Therefore the real and the imaginary parts of the subspace

$$V_\lambda + V_{\bar{\lambda}} + V_{-\lambda} + V_{-\bar{\lambda}}$$

are invariant for every $u \in \mathcal{G}$. As B is an extension of a real linear endomorphism, the above subspace has always nontrivial real and also nontrivial imaginary parts. Therefore because of the irreducibility

$$\begin{aligned} &\operatorname{Re}(V_\lambda + V_{\bar{\lambda}} + V_{-\lambda} + V_{-\bar{\lambda}}) \equiv V, \\ &\operatorname{Im}(V_\lambda + V_{-\lambda} + V_{-\lambda} + V_{-\bar{\lambda}}) \equiv \underline{i}V. \end{aligned} \tag{4.8}$$

We examine three cases:

(1) The eigenvalue λ is imaginary, i.e., $\lambda = c\underline{i}$, $-\lambda = -c\underline{i}$, where $c \in \mathbf{R}$ and $c > 0$.

(2) The eigenvalues λ and $-\lambda$ are real numbers.

(3) The eigenvalues λ, $\bar{\lambda}$, $-\lambda$, $-\bar{\lambda}$ are different complex numbers.

First we prove that cases (2) and (3) do not occur. Indeed, in case (3) let us denote the subspace $V_\lambda + V_{\bar{\lambda}}$ by W_λ, and in case (2), W_λ denotes the subspace V_λ. Then by (4.7), $u(W_\lambda) \subseteq W_{-\lambda}$ for every $u \in \mathcal{G}$. Now if $v \in \mathcal{G}$ is another element, then

$$[u, v]W_\lambda = (u \circ v - v \circ u)W_\lambda \subseteq W_{-\lambda},$$

as $\mathcal{G}$ is a Lie algebra. But on the other hand

$$uvW_\lambda \subseteq uW_{-\lambda} \subseteq W_\lambda,\ vuW_\lambda \subseteq vW_{-\lambda} \subseteq W_\lambda.$$

Thus $[u, v]W_\lambda \subseteq W_\lambda$, and so

$$[u, v]W_\lambda \subseteq W_\lambda \cap W_{-\lambda} = 0,$$
$$[u, v]W_{-\lambda} \subseteq W_\lambda \cap W_{-\lambda} = 0.$$

But the real projection of $W_\lambda + W_{-\lambda}$ is V; thus $[u, v] = 0$, i.e., $\mathcal{G}$ is an abelian Lie algebra. But this is impossible because $n > 2$ and $\mathcal{G}$ is irreducible. This proves the above statement.

Now let us consider case (1) when λ is imaginary, i.e., when $\lambda = c\underline{i}$. Then let V_λ^1 be the subspace for which $(B - \lambda I)V_\lambda^1 = 0$ holds. As λ is an eigenvalue of B, V_λ^1 is nontrivial. On the other hand for every element $u \in \mathcal{G}$ we get

$$0 = u(B - \lambda I)V_\lambda^1 = -(B^T + \lambda I)u(V_\lambda^1).$$

So as in (4.7) the relation

$$u(V_\lambda^1) \subseteq V_{-\lambda}^1 = \bar{V}_\lambda^1, u(V_{-\lambda}^1) \subseteq V_{-\lambda}^1 \tag{4.9}$$

follows, and thus by the irreducibility of $\mathcal{G}$ we get $\operatorname{Re} V_\lambda^1 = \operatorname{Re} V_{-\lambda}^1 = V$.

But on V_λ^1 we have

$$B(X) = \lambda X, X \in V_\lambda^1,$$

and so by (4.9) for every element $u \in \mathcal{G}$ we get

$$Bu(X) = \lambda u(X) = u(\lambda X) = uB(X), X \in V_\lambda^1,$$

i.e., B and u commute on V_λ^1. But $\operatorname{Re} V_\lambda^1 = V$, thus $Bu = uB$ on V, and by (4.6) B is skew-symmetric. As the skew-symmetric B commutes with the element of the irreducible skew-symmetric Lie algebra $\mathcal{G}$, by a well-known theorem of linear algebra [7, p. 278] we get $n = 2m$, $B = b\mathcal{F}$, $b \in \mathbf{R}$, $\mathcal{F}^2 = -I$ and $\mathcal{F}\mathcal{G} = \mathcal{G}\mathcal{F}$. This proves the lemma completely. q.e.d.

Now let us consider a simple semi-symmetric leaf (M, g) and the V-decomposition of its tangent bundle $T(M)$, which we denote by $T(M) = V^{(0)} + V^{(1)}$. Furthermore let $\underline{m}_1, \underline{m}_2, \cdots, \underline{m}_\nu$ be the vector fields constructed in §2, and let B_α, $\alpha = 1, \cdots, \nu$ be the second fundamental forms.

Lemma 4.3. *The covariant derivative $\nabla_{\underline{m}_\alpha} R$ of the curvature tensor in a simple semi-symmetric leaf is the multiple of itself, i.e.,*

$$\nabla_{\underline{m}_\alpha} R = -2\mu_\alpha R \tag{4.10}$$

where μ_α is a C^∞ function on the manifold.

Proof. Let $c(s)$ be an integral geodesic of the vector field $\underline{m}_\alpha$, and let $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_u$ be linearly independent parallel vector fields along $c(s)$ such that they span the subspaces $V_{c(s)}^{(1)}$. Let $B_{\alpha l}^k(s)$ (resp. $R_{jkl}^i(s)$) be the component of B_α (resp. R) corresponding to the above basis. Let us consider these components as functions of the arc length s.

As we have seen in Proposition 2.3, these components are analytic functions of s and

$$R_{jkl}^i(s) = R_{jsv|0}^i A_k^s(s) A_l^r(s), \tag{4.11}$$

where

$$A^q_l(s) := \sum_{k=1}^{\infty} (-1)^k s^k \left[B^k_{\alpha|0}\right]^q_l \tag{4.12}$$

is an analytic function.

Let $A(s)$ be the linear endomorphism in the tangent space $V^{(1)}_{c(0)}$ with matrix $A^i_j(s)$ in the basis $\{\underline{x}_1(0), \underline{x}_2(0), \cdots, \underline{x}_u(0)\}$. Now let τ^s_0 be the parallel displacement along $c(s)$ from $c(0)$ to $c(s)$. Then by (4.11) we get

$$\tau_s^0 R(\tau^s_0 X, \tau^s_0 Y)\tau^s_0 Z = R_{|c(0)}\big(A_{|s}(X), A_{|s}(Y)\big)Z, \quad X, Y, Z \in V^{(1)}_{c(0)}. \tag{4.13}$$

This means that the parallel moved Riemann curvature tensor $\tau_s^0 R$ from $c(s)$ to $c(0)$ along $c(s)$ generates the same Lie algebra as $R_{|0}$, i.e.,

$$\begin{aligned} \mathcal{H}_{c(0)} &= \left\{R_{|0}(X, Y) \mid X, Y \in T_{c(0)}(M)\right\} \\ &= \left\{\left(\tau_s^0 R\right)(X, Y) \mid X, Y \in T_{c(0)}(M)\right\}. \end{aligned}$$

It is also true that $\tau_s^0 R$ is invariant under the action of $\mathcal{H}_{c(0)}$ (as $\mathcal{H}_{c(0)} = \tau_s^0 \mathcal{H}_{c(s)} \tau^s_0$), so the holonomy systems

$$\left[T_{c(0)}(M), R_{|c(0)}, \mathcal{H}_{c(0)}\right], \quad \left[T_{c(0)}(M), \tau_s^0 R_{|c(s)}, \mathcal{H}_{c(0)}\right]$$

are symmetric. By Theorem 4.2 we get that there exists a real number $z(s)$ such that $\tau_s^0 R_{|c(s)} = z(s) R_{|c(0)}$. Thus

$$\begin{aligned} \nabla_{\underline{m}_\alpha} R_{|c(0)} &= \lim_{s \to 0} \frac{\tau_s^0 R_{|c(s)} - R_{|c(0)}}{s} = \left(\lim_{s \to 0} \frac{z(s) - z(0)}{s}\right) R_{|c(0)} \\ &= -2\mu_\alpha R_{|c(0)}, \\ \dot{z}(s) &= -2\mu_\alpha. \end{aligned} \tag{4.14}$$

It can be seen that $z(s)$ is of class C^∞, so that the lemma is proved. q.e.d.

Next we prove that for a simple semi-symmetric leaf with index of non-nullity greater than 2, all the second fundamental forms B_α are of the form $B_\alpha = \lambda_\alpha \mathcal{F} + \mu_\alpha I$, where λ_α, μ_α are C^∞ functions.

Lemma 4.4. *Let (M^n, g) be a simple semi-symmetric leaf with index of non-nullity $u(p)$ greater than 2. Then its second fundamental forms B_α are of the form $B_\alpha = \mu_\alpha I$ on $V^{(1)}$, or $u = 2m$, and B_α is of the form $B_\alpha = \lambda_\alpha \mathcal{F} + \mu_\alpha I$ on $V^{(1)}$ with $\mathcal{F}^2 = -I$, where λ_α and μ_α are C^∞ functions. In the last case $\mathcal{F}$ is uniquely determined, and is independent of the choice of the system $\underline{m}_1, \underline{m}_2, \cdots, \underline{m}_\nu$ and index α. Furthermore $\mathcal{F}$ is a skew-symmetric endomorphism field which commutes with the primitive holonomy group $\mathcal{H}_p$ at every point of the manifold.*

Proof. From Lemma 4.3 and equation (2.14) it follows that

$$-2\mu_\alpha R(X, Y) = R(B_\alpha(Y), X) + R(Y, B_\alpha(X)),$$

so that

$$R((B_\alpha - \mu_\alpha I)X, Y) + R(X, (B_\alpha - \mu_\alpha I)Y) = 0.$$

By the Bianchi identities we get

$$\begin{aligned} 0 &= g(R((B_\alpha - \mu_\alpha I)X, Y)U, V) + g(R(X, (B_\alpha - \mu_\alpha I)Y)U, V) \\ &= g(R(U, V)(B_\alpha - \mu_\alpha I)X, Y) + g(R(U, V)X, (B_\alpha - \mu_\alpha I)Y), \end{aligned}$$

and therefore

$$g(R(U, V)(B_\alpha - \mu_\alpha I)X, Y) = g(R(U, V)(B_\alpha - \mu_\alpha I)Y, X).$$

This means that for any vectors U, V the linear endomorphism $R(U, V) \circ (B_\alpha - \mu_\alpha I)$ is symmetric. But $R(U, V)$ is skew-symmetric, so

$$\begin{aligned} R(U, V) \circ (B_\alpha - \mu_\alpha I) &= [R(U, V) \circ (B_\alpha - \mu_\alpha I)]^T \\ &= -(B_\alpha - \mu_\alpha I)^T \circ R(U, V). \end{aligned}$$

Since the skew-symmetric linear endomorphisms of the form $R(U, V)$ form an irreducible Lie algebra, from Lemma 4.2 it follows that the linear endomorphism $B_\alpha - \mu_\alpha I$ is either null, (i.e., $B_\alpha - \mu_\alpha I$ on $V^{(1)}$), or $n = 2m$ and

$$B_\alpha - \mu_\alpha I = \lambda_\alpha \mathcal{F}, \lambda_\alpha(p) \in \mathbf{R}, \mathcal{F}^2 = -I,$$

where $\mathcal{F}$ is skew-symmetric and commutes with every element of the primitive holonomy group.

We prove that $\mathcal{F}$ is uniquely determined and independent of the index α and the choice of $\underline{m}_1, \underline{m}_2, \cdots, \underline{m}_\nu$. Indeed, in this case the holonomy system $S_p = [V_p^{(1)}, R_{|p}, \mathcal{H}_p]$, $p \in M$, is an irreducible symmetric holonomy system. Furthermore the isometry $\mathcal{F}$ commutes with $\mathcal{H}_p$ and satisfies $\mathcal{F}^2 = -I$. Thus by a well-known theorem of the symmetric spaces (see [6, p. 302, Proposition 4.2]) the system S_p is a Hermitian symmetric holonomy system. But for an irreducible Hermitian symmetric system the group $\mathcal{H}_p$ always contains a nontrivial 1-dimensional center (see [6, p. 310, Theorem 6.1]). Thus there is an element $\mathcal{F}^* \in \mathcal{H}_p$ commuting with $\mathcal{H}_p$. As $\mathcal{F}^*$ is skew-symmetric and $\mathcal{H}_p$ is irreducible, $\mathcal{F}^{*2} = -I$ holds [7, p. 278]. Since $\mathcal{F}$ is skew-symmetric and also $\mathcal{F}\mathcal{F}^* = \mathcal{F}^*\mathcal{F}$, we have $\mathcal{F} = \pm\mathcal{F}^*$, [7, p. 278], which proves the last statement in the lemma.

Lemma 4.5. *Let (M^n, g) be an infinitesimally irreducible simple semi-symmetric leaf with the index of non-nullity $u(p)$ greater then 2. Then its index of nullity $\nu(p)$ is 0, 1 or 2.*

Proof. Let $T(M) = V^{(0)} + V^{(1)}$ be the decomposition of the tangent space, and for the vector fields X, Y pointing in $V^{(1)}$ let $\mathrm{pr}_0(\nabla_X Y)$ be the perpendicular projection of $\nabla_X Y$ into the subspace $V^{(0)}$.

First we prove that at every point $p \in M$ the vectors $\mathrm{pr}_0(\nabla_X X)$, $X(p) \in V_p^{(1)}$, span at most a 1-dimensional subspace in $V_p^{(0)}$, $p \in M$. Indeed, let $X_1(p)$ and $X_2(p)$ be two orthogonal unit vectors in $V_p^{(1)}$, and let $\{X_1(p), X_2(p), \cdots, X_u(p)\}$ be an orthonormed basis in $V_p^{(1)}$. If the vectors $\mathrm{pr}_0(\nabla_{X_1}X_1)$ and $\mathrm{pr}_0(\nabla_{X_2}X_2)$ were not equal, then let $m_{1|p}$ be such a unit vector in the plane of vectors $\mathrm{pr}_0(\nabla_{X_1}X_1)$ and $\mathrm{pr}_0(\nabla_{X_2}X_2)$ for which the value $g(\underline{m}_{1|p}, \nabla_{X_1}X_1)$ is no equal to $g(\underline{m}_{1|p}, \nabla_{X_2}X_2)$. Such a vector can be chosen as it can be seen from Fig. 1. Now let us extend $\underline{m}_{1|p}$ to a suitable vector field $\underline{m}_1$ described in §2. Then the second fundamental from B_1 at p would be not of the form μI or $\lambda\mathcal{F} + \mu I$. Indeed its matrix considered in the basis $\{X_1(p), X_2(p), \cdots, X_u(p)\}$ would have two distinct values in its diagonal, namely, the values $g(B_1(X_1), X_1) = -g(\nabla_{X_1}X_1, \underline{m}_1)$ and $g(B_1(X_2), X_2) = -g(\nabla_{X_2}X_2, \underline{m}_1)$. Since $\mathcal{F}$ is skew-symmetric, the diagonal contains distinct elements. Thus B_1 is not of the form μI or $\lambda\mathcal{F} + \mu I$. This contradicts the previous lemma, so $\mathrm{pr}_0(\nabla_{X_1}X_1) = \mathrm{pr}_0(\nabla_{X_2}X_2)$ must hold.

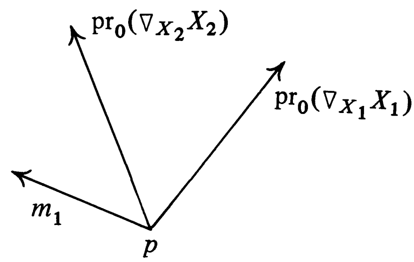

FIG. 1.

Now if the unit vectors $X_1(p)$ and $X_2(p)$ are not perpendicular, then let X_3 be a unit vector in $V_p^{(1)}$, which is perpendicular to both X_1 and X_2. Thus $\mathrm{pr}_0(\nabla_{X_1}X_1) = \mathrm{pr}_0(\nabla_{X_3}X_3) = \mathrm{pr}_0(\nabla_{X_2}X_2)$, which proves the above statement.

If all the second fundamental forms B_α are of the form $B_\alpha = \mu_\alpha I$, then it is obvious that all the vectors $\mathrm{pr}_0(\nabla_X Y)$, $X_{|p}$, $Y_{|p} \in V_p^{(1)}$, span at most a 1-dimensional subspace $\mathfrak{M}_p$ in $V_p^{(0)}$. First we consider this case and prove that there exists a unit vector field $\underline{m}_1$, pointing in $\mathfrak{M}$ such that $\underline{m}_1$ is totally parallel on the integral manifolds of $V^{(0)}$. So it can be extended to a suitable vector-field-system $\underline{m}_1, \underline{m}_2, \cdots, \underline{m}_k$ constructed in §2.

Indeed let A be an $u(p)$-dimensional submanifold in M such that $T_q(A) + V_q^{(0)} = T_q(M)$, $q \in A$. Then A intersects every integral-manifold of $V^{(0)}$ in a sufficiently small open subset at most at one point. Let $\underline{m}_1(q), \underline{m}_2(q), \cdots, \underline{m}_\nu(q)$, $q \in A$, $\underline{m}_\alpha(q) \in V_q^{(0)}$, be perpendicular C^∞ unit vector fields on A such that $\underline{m}_1(q)$ is pointing in $\mathfrak{M}_{|q}$ if $q \in A$. Let us extend these vector fields to a suitable vector-field-system $\underline{m}_1, \underline{m}_2, \cdots, \underline{m}_\nu$ onto a neighborhood of A as in §2, and let us denote the second fundamental forms by B_α. Then by $\nabla_{\underline{m}_\alpha} B_\gamma = -B_\gamma \circ B_\alpha$ we get that $B_\alpha = 0$ for $\alpha \geq 2$, because $B_\alpha(q) = 0$ ($\alpha \geq 2$) at A, and B_α is uniquely determined by their initial values. Thus the vector field $\underline{m}_1$ points always in $\mathfrak{M}$, and for X, Y pointing in $V^{(1)}$ we get

$$\nabla_X Y = \tilde{\nabla}_X Y - \mu g(X, Y)\underline{m}_1.$$

Now from (2.8) it follows that $\mu(M_\alpha^1(Y)X - M_\alpha^1(X)Y) = 0$ for $\alpha \geq 2$, so that $M_\alpha^1 = 0$. Hence $g(\nabla_X \underline{m}_1, \underline{m}_\alpha) = 0$, $X(p) \in V_p^{(1)}$. From this we get that the vector fields

$$\nabla_{X_1}\nabla_{X_2} \cdots \nabla_{X_l} X_{l+1}, \quad X_i(p) \in V_p^{(1)},$$

span just the $(u(p) + 1)$-dimensional distribution $V^{(1)} + \mathfrak{M}$. As M is infinitesimally irreducible, $V^{(0)} = \mathfrak{M}$ and $\mu(p) = 1$ in the considered case.

Secondly let us consider the case in which among the second fundamental forms there exists a field of the form $B_\alpha = \lambda_\alpha \mathfrak{F} + \mu_\alpha I$, where $\lambda_\alpha \neq 0$, and therefore also $u(p) = \dim V_p^{(1)} = 2m$ and $\mathfrak{F}^2 = -I$ hold. Let us choose an orthonormal basis of the form $\{X_1, Y_1 = \mathfrak{F}(X_1), X_2, Y_2 = \mathfrak{F}(X_1), \cdots, X_m, Y_m = \mathfrak{F}(X_m)\}$ in $V_p^{(1)}$. As we have seen, the vectors $\mathrm{pr}_0(\nabla_{X_i} X_i)$ and $\mathrm{pr}_0(\nabla_{Y_i} Y_i)$ span a one-dimensional subspace in $V_p^{(0)}$, $p \in M$. Now we prove that $\mathrm{pr}_0(\nabla_{X_i} Y_j) = \mathrm{pr}_0(\nabla_{Y_j} X_i) = \mathrm{pr}_0(\nabla_{X_i} X_j) = \mathrm{pr}_0(\nabla_{Y_i} Y_j) = 0$ if $i \neq j$, and the vectors $\mathrm{pr}_0(\nabla_{X_i} Y_i)$, $i = 1, 2, \cdots, m$, span a 1-dimensional subspace in $V_p^{(0)}$, $p \in M$. More precisely

$$\mathrm{pr}_0(\nabla_{X_i} Y_i) = \mathrm{pr}_0(\nabla_{X_j} Y_j),$$
$$\mathrm{pr}_0(\nabla_{X_i} Y_i) = -\mathrm{pr}_0(\nabla_{Y_i} X_i).$$

The statement $\mathrm{pr}_0(\nabla_{X_i} Y_j) = \mathrm{pr}_0(\nabla_{Y_j} X_i) = \mathrm{pr}_0(\nabla_{X_i} X_j) = \mathrm{pr}_0(\nabla_{Y_i} Y_j) = 0$ is trivial because all the second fundamental forms B_α are of the form $B_\alpha = \lambda_\alpha \mathfrak{F} + \mu_\alpha I$, and thus, for example, $g(\nabla_{X_i} Y_j, \underline{m}_\alpha) = -g(B_\alpha(X_i), Y_j) = 0$ for $i \neq j$. Furthermore we prove the above two equations by indirect method. If they do not hold, one could choose a unit vector $\underline{m}$ in $V_p^{(0)}$ such that

$$g(\nabla_{X_i} Y_i, \underline{m}) \neq g(\nabla_{X_j} Y_j, \underline{m}) \quad \text{for some } i \neq j \text{ or}$$
$$g(\nabla_{X_i} Y_i, \underline{m}) \neq -g(\nabla_{Y_i} X_i, \underline{m}) \quad \text{for some } i.$$

In both cases the second fundamental form B corresponding to $\underline{m}$ would be not of the form $\lambda \mathfrak{F} + \mu I$. This is impossible, thus the above equations are satisfied. So in this case the dimension of the space spanned by the vectors $\mathrm{pr}_0(\nabla_X Y)$, $X(p), Y(p) \in V_p^{(1)}, p \in M$, is at most 2 at every point.

It can be proved in the same way as before that in the case $\dim \mathfrak{M}_p = 1$ (resp. $\dim \mathfrak{M}_p = 2$) one can choose vector fields $\underline{m}_1, \underline{m}_2, \cdots, \underline{m}_\nu$ suitable for §2 and $\underline{m}_1$ (resp. $\underline{m}_1$ and $\underline{m}_2$) pointing in $\mathfrak{M}$. Now we consider the case in which $\dim \mathfrak{M}_p = 1$, and $B_1 = \lambda_1 \mathcal{F} + \mu_1 I$ where $\lambda_1 \neq 0$. In this case also $B_\alpha = 0$ for $\alpha \geqslant 2$. From (2.8) we get on $V^{(1)}$

$$M_\alpha^1(X)B_1(Y) - M_\alpha^1(Y)B_1(X) = 0 \quad \text{for } \alpha \geqslant 2,$$

and thus $M_\alpha^1 = 0$, because B_1 is nonsingular. Because of the irreducibility we get $\mathfrak{M} = V^{(0)}$, and thus $\nu(p) = 1$ in the considered case.

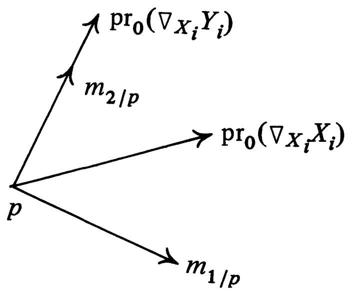

FIG. 2

Finally let us consider the case where $\dim \mathfrak{M}_p = 2$. Then the vectors $\mathrm{pr}_0(\nabla_{X_i} X_i)$ and $\mathrm{pr}_0(\nabla_{X_j} Y_j)$ point in distinct directions and span the plane $\mathfrak{M}_p$. In this plane let $\underline{m}_{2|p}$ be the unit vector pointing in the direction of $\mathrm{pr}_0 \nabla_{X_i} Y_i$, and let $\underline{m}_{1|p}$ be in $\mathfrak{M}_p$ the unit vector which is perpendicular to $\underline{m}_{2|p}$. Let us extend these vectors to suitable vector fields $\underline{m}_1$, $\underline{m}_2$ in such a way that they point everywhere in $\mathfrak{M}$. Furthermore let $\underline{m}_1, \underline{m}_2, \cdots, \underline{m}_\nu$ be the suitable vector-field-system. Then $B_\alpha = 0$ for $\alpha \geqslant 3$, and we get

$$B_{1|p} = \mu_1(p)I, B_{2|p} = \lambda_2(p)\mathcal{F} + \mu_2(p)I, \tag{4.15}$$

where $\mu_1(p) \neq 0$ and $\lambda_2(p) \neq 0$. For $\alpha \geqslant 3$, from (2.8) we have on $V^{(1)}$

$$M_\alpha^1(X)B_1(Y) + M_\alpha^2(X)B_2(Y) = M_\alpha^1(Y)B_1(X) + M_\alpha^2(Y)B_2(X),$$

and thus

$$\left(\sum_{\rho=1}^{2} \mu_\rho(p) M_{\alpha|p}^\rho(Y) \right) X + \lambda_2(p) M_{\alpha|p}^2(Y) \mathcal{F}_{|p}(X) = 0.$$

As X and $\mathcal{F}(X)$ point in distinct directions and $\mu_1(p) \neq 0$, $\lambda_2(p) \neq 0$, $M_\alpha^2 = M_\alpha^1 = 0$ for $\alpha \geqslant 3$. We get (by the irreduciblity) in the same way as above that in this case $\mathfrak{M} = V^{(0)}$ and $\nu(p) = 2$, which prove the lemma.
q.e.d.

Lemma 4.6. *If $\nu(p) = 2$ holds in the above lemma, then the fields* $\mathrm{pr}_0(\nabla_{X_i} X_i)$ *and* $\mathrm{pr}_0(\nabla_{X_i} Y_i)$ *span two orthogonal* 1*-dimensional subspaces* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$ *in* V^0 *respectively. The fundamental forms* $\tilde{B}_1$, $\tilde{B}_2$ *with respect to these directions are of the form* $\tilde{B}_1 = \mu I$, $\tilde{B}_2 = \mu \mathfrak{F}$. *The distribution* $V^1 + \mathfrak{M}_2$ *is involutive, and the functions* κ, μ *are constant on the integral manifolds of it. In the case* $\nu(p) = 1$ *the fundamental form* B *is of the form* $B = \mu I$.

Proof. Let us start the proof with the assumption $\nu(p) = 2$ on an open set M. As we have seen in Lemma 4.5, in this case $\dim V_p^{(0)} = 2$, $p \in M$, and at least one of the second fundmental forms is of the form

$$B_\alpha = \lambda_\alpha \mathfrak{F} + \mu_\alpha I \quad \text{with } \lambda_\alpha \neq 0.$$

Let us keep the notation of the previous lemma, and at a fixed point $p \in M$ let us consider the two unit vectors $\underline{m}_{1|p}$ and $\underline{m}_{2|p}$ as in Fig. 2. Then let us extend these two vectors to suitable fields $\underline{m}_1$, $\underline{m}_2$. If $\underline{m}_\alpha{}^i$ denotes the components of $\underline{m}_\alpha$, then the Ricci tensor of the space is of the form

$$R_{jl} = \kappa\left(g_{jl} - \sum_{\alpha=1}^{2} \underline{m}_{\alpha j}\underline{m}_{\alpha l}\right), \tag{4.16}$$

where $\underline{m}_{\alpha j} := \underline{m}_\alpha{}^i g_{ij}$, because the primitive holonomy group acts on V_q^1, $q \in M$, irreducibly and leaves the Ricci tensor invariantly. From this we get

$$\begin{aligned}\nabla_i R_{jl} = (\nabla_i \kappa)&\left(g_{jl} - \sum_{\alpha=1}^{2} \underline{m}_{\alpha j}\underline{m}_{\alpha l}\right)\\ &-\kappa \sum_{\alpha=1}^{2} \left(\underline{m}_{\alpha l}\nabla_i \underline{m}_{\alpha j} + \underline{m}_{\alpha j}\nabla_i \underline{m}_{\alpha l}\right),\end{aligned} \tag{4.17}$$

and so

$$\nabla_i R^i{}_l = \nabla_l \kappa - \sum_{\alpha=1}^{2} \underline{m}_{\alpha l}\underline{m}_\alpha{}^s \nabla_s \kappa - \kappa \sum_{\alpha=1}^{2} \underline{m}_{\alpha l}\nabla_s \underline{m}_\alpha{}^s, \tag{4.18}$$

as $\underline{m}_\alpha{}^s \nabla_s \underline{m}_{\alpha l} = 0$.

On the other hand using the second Bianchi identity

$$\nabla_s R_j{}^s{}_{lk} + \nabla_l R_j{}^s{}_{ks} + \nabla_k R_j{}^s{}_{sl} = 0,$$

and the contraction $j \to l$ we get

$$-2R^s{}_{k|s} + (n-2)\nabla_k \kappa = 0. \tag{4.19}$$

From (4.19) and (4.18) it follows that

$$(n-4)\nabla_k \kappa + 2 \sum_{\alpha=1}^{2} \left(\underline{m}_\alpha{}^s \nabla_s \kappa + \kappa \nabla_s \underline{m}_\alpha{}^s\right)\underline{m}_{\alpha k} = 0. \tag{4.20}$$

We can see that the covector field $\nabla_k \kappa$ never vanishes. For if $\nabla_k \kappa_{|p} = 0$ at the point p, then by (4.20) we have $\kappa_{|p} \nabla_s \underline{m}_1{}^s{}_{|p} = \kappa_{|p} \operatorname{Trace} B_1(p) = 0$. But from (4.15) it follows that $\operatorname{Trace} B_1(p) = (n-2)\mu_1(p) \neq 0$ so $\kappa_{|p} = 0$ which is impossible by Lemma 4.1. As p is arbitrary, $\nabla_k \kappa$ never vanishes on M.

Let us consider the hypersurface H through p, on which κ has constant value. If X lies in $V^{(1)}$, then $X^l \nabla_k \kappa = 0$ by (4.20), which proves that $V^{(1)}$ is always tangent to H. Now if X and Y are differentiable vector fields on H, which lie in $V^{(1)}$, then $[X, Y]$ is also tangent to H. But by (4.15) we get at p

$$[X, Y]_{|p} = [\widetilde{X, Y}]_{|p} + 2\lambda_2(p) g(\mathcal{F}(Y), X) \underline{m}_{2|p}.$$

Thus $\underline{m}_2$ is also tangent to H and $\underline{m}_2(\kappa) = 0$ at p. Thus $V^1 + \mathfrak{M}_2$ is involutive indeed.

Let $c(s)$ be the integral geodesic of $\underline{m}_1$ through the point p. We shall show that along $c(s)$ the vector $\underline{m}_2(s)$ always points into $T(H)$, and thus $\nabla_{\underline{m}_2} \kappa = 0$ along $c(s)$.

If $B_\alpha = \lambda_\alpha \mathcal{F} + \mu_\alpha I$, then from $\nabla_{\underline{m}_\alpha} B_\alpha = -B_\alpha^2$ we get

$$(\nabla_{\underline{m}_\alpha} \lambda_\alpha)\mathcal{F} + \lambda_\alpha(\nabla_{\underline{m}_\alpha} \mathcal{F}) + (\nabla_{\underline{m}_\alpha} \mu_\alpha) I = -2\lambda_\alpha \mu_\alpha \mathcal{F} + (\lambda_\alpha^2 - \mu_\alpha^2) I.$$

Since $\mathcal{F}$ is skew-symmetric, $\nabla_{\underline{m}_\alpha} \mathcal{F}$ is also so, and by the left side of the above equation, $\nabla_{\underline{m}_\alpha} \mathcal{F}$ is a multiple of $\mathcal{F}$. Thus $\nabla_{\underline{m}_\alpha} \mathcal{F}$ is a field of the form $c\mathcal{F}$. But $\mathcal{F}$ is of constant norm, and so it is parallel, i.e.,

$$\nabla_{\underline{m}_\alpha} \mathcal{F} = 0. \tag{4.21}$$

Thus

$$\nabla_{\underline{m}_\alpha} \lambda_\alpha = -2\lambda_\alpha \mu_\alpha, \quad \nabla_{\underline{m}_\alpha} \mu_\alpha = \lambda_\alpha^2 - \mu_\alpha^2. \tag{4.22}$$

So along $c(s)$

$$\frac{d\lambda_1}{ds} = -2\lambda_1 \mu_1. \tag{4.23}$$

Since $\lambda_1(0) = 0$ at p by (4.15), by (4.23) we have $\lambda_1(s) = 0$ along $c(s)$. Thus the vectors $\mathrm{pr}_0 \nabla_{X_i} Y_i$ in Fig. 2 are always perpendicular to $\underline{m}_1$ along $c(s)$ and so $\nabla_{\underline{m}_2} \kappa = 0$ along $c(s)$.

On the other hand by (4.20) we get

$$\mu_1 = -\nabla_{\underline{m}_1} \kappa / 2\kappa.$$

Since $[\underline{m}_1, \underline{m}_2] = 0$, $\nabla_{\underline{m}_2} \mu_1 = 0$ along $c(s)$. From $\nabla_{\underline{m}_\alpha} B_\beta = -B_\beta \circ B_\alpha$ it follows that

$$\nabla_{\underline{m}_2}(\lambda_1 \mathcal{F} + \mu_1 I) = -(\lambda_1 \mu_2 + \mu_1 \lambda_2)\mathcal{F} + (\lambda_1 \lambda_2 - \mu_1 \mu_2) I,$$

i.e.,

$$(4.24)\qquad \begin{aligned}\nabla_{\underline{m}_2}\lambda_1 &= -(\lambda_1\mu_2 + \mu_1\lambda_2),\\ \nabla_{\underline{m}_2}\mu_1 &= \lambda_1\lambda_2 - \mu_1\mu_2.\end{aligned}$$

Furthermore $\lambda_1 = \nabla_{\underline{m}_2}\mu_1 = 0$ and $\mu_1 \neq 0$ along $c(s)$. So by (4.24), $\mu_2 = 0$ along $c(s)$. Thus $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are orthogonal indeed. Let $\underline{\tilde{m}}_1$ and $\underline{\tilde{m}}_2$ be the unit vector fields tangent to $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respectively. The forms $\tilde{B}_1$, $\tilde{B}_2$ are of the form $\tilde{B}_1 = \mu I$, $\tilde{B}_2 = \lambda\mathfrak{F}$. From $R(\underline{\tilde{m}}_\alpha, X)\underline{\tilde{m}}_\beta = 0$, $X(p) \in V_p^1$, we get

$$X\cdot\mu = \underline{\tilde{m}}_2\cdot\mu = 0, \quad \lambda^2 = \mu^2$$

by a simple computation, which proves the first part of the lemma completely.

Finally let us prove for the case $\nu(p) = 1$ that the unique second fundamental form B is of the form $B = \mu I$. Let $\underline{m}$ be the unit vector field pointing into $V^{(0)}$. Then the Ricci tensor is

$$R_{jl} = \kappa(g_{jl} - \underline{m}_j\underline{m}_l),$$

and we get, as in (4.20),

$$(4.25)\qquad (n-3)\nabla_l\kappa + 2(\underline{m}^s\nabla_s\kappa + \kappa\nabla_s\underline{m}^s)\underline{m}_l = 0.$$

First we prove that $\nabla_l\kappa \neq 0$. Indeed, if $B = \lambda\mathfrak{F} + \mu I$, then $\nabla_s\underline{m}^s = (n-1)\mu$. Let $c(s)$ be an integral geodesic of $\underline{m}$. The covector field $\nabla_l\kappa$ does not vanish on an open subset of $c(s)$, because otherwise on an open interval, $\kappa\nabla_s\underline{m}^s = 0$ which implies that $\mu = 0$ and $B = \lambda\mathfrak{F}$ since $\kappa \neq 0$. In this case $\lambda \neq 0$ on the interval, because $\nu(p) = 1$, and so on the interval we have

$$\nabla_{\underline{m}}B = -\lambda^2 I.$$

But this is impossible as the left side is a skew-symmetric, and the right is a symmetric endomorphism. Thus $\nabla_l\kappa \neq 0$ on an everywhere dense open subset of $c(s)$ and therefore on an everywhere dense open subset U of M. Let $p \in U$ be such a point, and H the hypersurface through p, on which κ has constant value. Then by (4.25) we get that $X^l\nabla_l\kappa = 0$ for every vector $X \in V^{(1)}$, i.e., the tangent space of the submanifold is just $V^{(1)}$. This means that $V^{(1)}$ is integrable, and thus B is symmetric. As $\mathfrak{F}$ is skew-symmetric, B must be of the form $B = \mu I$ on U. As U is everywhere dense in M, B is of the form $B = \mu I$ everywhere which proves the last statement.

Definition 4.1. The infinitesimally irreducible simple semi-symmetric leafs with $u(p) > 2$, $\nu(p) = 2$ are called Kaehlerian cones.

All these spaces are not complete, since by $\underline{\tilde{m}}_1(\mu) = -\mu^2$, $\underline{\tilde{m}}_1(\kappa) = -2\mu\kappa$, the curvature scalar κ has infinite value at a finite point of the integral curve $c(s)$ of $\underline{\tilde{m}}_1$.

We construct and describe all these cones in the continuation II of this paper.

Now we prove the first main theorem:

Theorem 4.3 (*First main theorem*). *Let (M^n, g) be an infinitesimally irreducible simple semi-symmetric leaf with $\nu(p) = 0$ and $u(p) > 2$. Then (M^n, g) is locally symmetric. If (M^n, g) is maximal and simple connected, then it is globally symmetric.*

Proof. First we prove a lemma proved also by A. Lichnerowicz [13].

Lemma 4.7 (*A. Lichnerowicz*). *For the curvature $R_{i\,kl}^{\,j}$ of a Riemannian manifold*

$$\begin{aligned}(4.26)\qquad &\nabla^m\nabla_m\left(R_{ijkl}R^{ijkl}\right) = 2\nabla_m R_{ijkl}\nabla^m R^{ijkl}\\ &+R^{ijkl}\left[4\left(\nabla_j\nabla_k R_{il} - \nabla_j\nabla_l R_{ik}\right) - 2g^{mn}\left(H_{jnkl;mi} + H_{nikl;mj}\right)\right],\end{aligned}$$

where

$$H_{jnkl;mi} := \nabla_m\nabla_i R_{jnkl} - \nabla_i\nabla_m R_{jnkl}.$$

Proof. In the following we use the Bianchi-identities:

$$\begin{aligned}&g^{mn}\nabla_m\nabla_n\left(R_{ijkl}R^{ijkl}\right)\\ &\quad= 2g^{mn}\left(\nabla_m\nabla_n R_{ijkl}\right)R^{ijkl} + 2\nabla_m R_{ijkl}\nabla^m R^{ijkl}\\ &\quad= 2\nabla_m R_{ijkl}\nabla^m R_{ijkl}\\ &\qquad -2g^{mn}R^{ijkl}\left[H_{jnkl;mi} + H_{nikl;mj} + \nabla_i\nabla_m R_{jnkl} + \nabla_j\nabla_m R_{nikl}\right].\end{aligned}$$

But also we get

$$\begin{aligned}&-2g^{mn}\left(\nabla_i\nabla_m R_{jnkl} + \nabla_j\nabla_m R_{nikl}\right)\\ &\qquad = 2g^{mn}\left(\nabla_i\nabla_k R_{jnlm} + \nabla_i\nabla_l R_{jnmk} + \nabla_j\nabla_k R_{nilm} + \nabla_j\nabla_l R_{nimk}\right),\end{aligned}$$

where the last 2 terms is just

$$4R^{ijkl}\left(\nabla_j\nabla_k R_{il} - \nabla_j\nabla_l R_{ik}\right),$$

which proves the lemma. q.e.d.

Now let us consider the space considered in the theorem. By the above lemma we get

$$(4.27)\qquad \nabla^m\nabla_m\left(R_{ijkl}R^{ijkl}\right) = 2\nabla_m R_{ijkl}\nabla^m R^{ijkl},$$

as $H_{jnkl;mi} = 0$, and because of the irreducibility the space is an Einstein space, i.e.,

$$(4.28)\qquad R_{ij} = \kappa g_{ij},$$

where κ = konstant, so that $\nabla_s R_{ij} = 0$.

If we consider the curvature operator $R_{k\,k}^{j\,l}$ as a symmetric linear endomorphism of the skew-symmetric linear endomorphisms, and denote its eigenvalues by $\lambda_1, \lambda_2, \cdots, \lambda_r$, then the function $R_{ijkl}R^{ijkl}$ satisfies

$$R_{ijkl}R^{ijkl} = 4 \sum_{i=1}^{r} \lambda_i^2, \quad r := \dim \mathcal{H}_p.$$

In the following we shall prove that the eigenvalue-functions λ_i are constant. Let us notice that showing this fact will complete our proof because in this case $R_{ijkl}R^{ijkl} =$ constant and, by (4.27), $\nabla_m R_{ijkl} \nabla^m R^{ijkl} = 0$. Since the metric is positive definite, we have $\nabla_m R_{ijkl} = 0$, i.e., the space is locally symmetric. Moreover, if the space is maximal and simply connected, then the space is a complete globally symmetric Riemannian space, which proves the theorem.

Now let us prove that λ_i's are constant indeed. The following lemma proves just this statement in a stronger form.

Lemma 4.8. *Let $R(X, Y)Z$ be a continuous curvature operator field on an arcwise connected Riemannian space (M^n, g), $n > 2$, such that at every point p the operator $R_{|p}(X, Y)Z$ is irreducible and symmetric (i.e., the holonomy systems $[T_p(M), R_{|p}, \mathcal{H}_p]$, $p \in M$, are irreducible and symmetric), and that the Riemannian curvature scalar $\mathfrak{R} := R_{ij}^{\ \ ij}$ is constant on M^n. Then the eigenvalue functions λ_i of the curvature operator are constant.*

Proof. First let us assume that the curvature operator $R_{i\,k}^{j\,l}$ has distinct nonnull eigenvalues at a point $p \in M^n$. As we have seen, the system $[T_p(M), R_{|p}, \mathcal{H}_p]$ is an irreducible symmetric holonomy system. It is well known that the vector space $\mathcal{Q} = T_p(M) + h_p$ with the Lie bracket

$$[X, Y] = -R(X, Y), \quad X, Y \in T_p(M),$$
$$[A, B] = A \circ B - B \circ A, \quad A, B \in h_p,$$
$$[A, X] = A(X), \quad A \in h_p, X \in T_p(M)$$

is a semi-simple irreducible orthogonal symmetric Lie algebra [6]. Let G/H be the symmetric space as homogeneous space corresponding to the Lie algebra Q. Since $R_{|p}$ is $\mathcal{H}_p$-invariant, H is not simple; in fact, $R_{i\,k\,|p}^{j\,l}$ has distinct nonnull eigenvalues, and so the invariant subspaces corresponding to these several eigenvalues are ad H-invariant in h. Thus the symmetric space G/H is irreducible Riemannian globally symmetric space of type I or III, [6, p. 308, Theorems 5.3 and 5.4], because H is simple for the spaces of type II and IV.

The complete list of the spaces of type I and III can be found also in the Helgason's book [6, p. 354]. In the table below we collect from this list all the spaces for which H is nonsimple and thus the curvature operator has distinct nonnull eigenvalues. We list also these nonnull eigenvalues. We remark that these eigenvalues can be determined from formulas (4.5) where the eigenvalues λ_i^* of operator T come from (4.2), and also that as the Lie algebras e_6 and e_7 are simple, their Killing forms K_i are of the form $K_i = \gamma_i\langle\, ,\rangle$. The so defined constants γ_6 and γ_7 occur in the table below.

In the case of the lemma the curvature skalar κ and dim M are constant, since by the irreducibility and symmetry the Ricci tensor is of the form $R_{ij} = \kappa g_{ij}$, and so $\mathfrak{R} = n\kappa$.

$M=G/H$	$M=G/H$	Eigenvalues	$\tilde{\lambda}$
$\dfrac{SU(p,q)}{S(U_p \times U_q)}$	$\dfrac{SU(p+q)}{S(U_p+U_q)}$	$\tilde{\lambda}, \dfrac{\tilde{\lambda}}{2p+1}, \dfrac{\tilde{\lambda}}{2q+1}$	$\tilde{\lambda} = \dfrac{-2pq\kappa}{2\left(1+\frac{p^2-1}{2p+1}+\frac{q^2-1}{2q+1}\right)}$
$\dfrac{SO_0(p,q)}{SO(p)\times SO(q)}$	$\dfrac{SO(p+q)}{SO(p)\times SO(q)}$	$\dfrac{\tilde{\lambda}}{2p+1}, \dfrac{\tilde{\lambda}}{2q+1}$	$\tilde{\lambda} = \dfrac{-pq\kappa}{\frac{p(p--1)}{2p+1}+\frac{q(q-1)}{2q+1}}$
$\dfrac{SO^*(2n)}{U(n)}$	$\dfrac{SO(2n)}{U(n)}$	$\dfrac{\tilde{\lambda}, \tilde{\lambda}}{1+2n}$	$\tilde{\lambda} = \dfrac{-n(n-1)\kappa}{2\left(1+\frac{n^2-1}{1+2n}\right)}$
$\dfrac{Sp(n,\mathbf{R})}{U(n)}$	$\dfrac{Sp(n)}{U(n)}$	$\dfrac{\tilde{\lambda}, \tilde{\lambda}}{1+2n}$	$\tilde{\lambda} = \dfrac{-(n+1)n\kappa}{2\left(1+\frac{n^2-1}{1+2n}\right)}$
$\dfrac{Sp(p,q)}{Sp(p)\times Sp(q)}$	$\dfrac{Sp(p+q)}{Sp(p)\times Sp(q)}$	$\dfrac{\tilde{\lambda}}{1+4p}, \dfrac{\tilde{\lambda}}{1+4q}$	$\tilde{\lambda} = \dfrac{-4pq\kappa}{2\left(\frac{p(2p+1)}{1+4p}+\frac{q(2q+1)}{1+4q}\right)}$
$e_6(-2)$, $su(6)+su(2)$	$e_{6(-78)}$, $su(6)+su(2)$	$\dfrac{\tilde{\lambda}}{13}, \dfrac{\tilde{\lambda}}{5}$	$\tilde{\lambda} = \dfrac{-40\kappa}{2\left(\frac{35}{13}+\frac{3}{5}\right)}$
$e_{6(-14)}$, $so(10)+\mathbf{R}$	$e_{6(-78)}$, $so(10)+\mathbf{R}$	$\tilde{\lambda}, \dfrac{\tilde{\lambda}}{21}$	$\tilde{\lambda} = \dfrac{-32\kappa}{2\left(1+\frac{45}{21}\right)}$
$e_{7(-5)}$, $so(12)+su(2)$	$e_{7(-133)}$, $so(12)+su(2)$	$\dfrac{\tilde{\lambda}}{25}, \dfrac{\tilde{\lambda}}{5}$	$\tilde{\lambda} = \dfrac{-64\kappa}{2\left(\frac{66}{25}+\frac{3}{5}\right)}$
$e_{7(-25)}$, $e_6+\mathbf{R}$	$e_{7(-133)}$, $e_6+\mathbf{R}$	$\tilde{\lambda}, \dfrac{\tilde{\lambda}}{1+\gamma_6}$	$\tilde{\lambda} = \dfrac{-54\kappa}{2\left(1+\frac{78}{1+\gamma_6}\right)}$
$e_{8(-24)}$, $e_7+su(2)$	$e_{8(-248)}$, $e_7+su(2)$	$\dfrac{\tilde{\lambda}}{1+\gamma_7}, \dfrac{\tilde{\lambda}}{5}$	$\tilde{\lambda} = \dfrac{-112\kappa}{2\left(\frac{3}{5}+\frac{54}{1+\gamma_7}\right)}$
$f_{4(4)}$, $sp(3)+su(2)$	$f_{4(-52)}$, $sp(3)+su(2)$	$\dfrac{\tilde{\lambda}}{13}, \dfrac{\tilde{\lambda}}{5}$	$\tilde{\lambda} = \dfrac{-28\kappa}{2\left(\frac{21}{13}+\frac{3}{5}\right)}$
$g_{2(2)}$, $su(2)+su(2)$	$g_{2(-14)}$, $su(2)+su(2)$	$\dfrac{\tilde{\lambda}}{5}, \dfrac{\tilde{\lambda}}{5}$	$\tilde{\lambda} = \dfrac{-8\kappa}{2\left(\frac{3}{5}+\frac{3}{5}\right)}$

From the table it can be seen that by such restrictions there are only finite many possible values for the eigenvalue of the curvature operator. But these eigenvalues are continuous on the arcwise connected manifold, thus they must be constant on M.

Now if at every point the curvature operator has only equal nonnull eigenvalues, then this eigenvalue is at a point $p \in M$ of the form

$$\lambda_{|p} = \frac{\dim M}{2 \dim \mathcal{H}_p} \kappa.$$

From this formula we get evidently that λ is constant on M because the values $\dim M$ and κ are constant, and $\lambda_{|p}$ is continuous.

Now we turn to the proof of the second main theorem.

Theorem 4.4 (*The second main theorem*). *Let (M^n, g) be an infinitesimally irreducible simple semi-symmetric leaf with $u(p) > 2$ and $\nu(p) = 1$ at a point $p \in M$. Then the space is locally isometric to an elliptic or a hyperbolic or a euclidean cone. If the space is simply connected and maximal, then it is a maximal elliptic or hyperbolic or euclidean cone.*

Proof. By Lemmas 4.6 and 4.8, $\nu = 1$ in a whole neighborhood U of p, and also $B = \mu I$, $\mu \neq 0$ on this open set. Later we shall see that $\nu = 1$ on the whole arcwise connected manifold M^n, but at this moment we examine the space only on U.

Since B is symmetric, $V^{(1)}$ is integrable distribution and, as we have seen in the proof of Lemma 4.6, the Ricci curvature R_{jl} is of the form

$$R_{jl} = \kappa(g_{jl} - \underline{m}_j \underline{m}_l),$$

where κ is constant on the integral manifolds of $V^{(1)}$. But also the function μ is constant on these integral manifolds of $V^{(1)}$. Indeed from (2.8) we get $(\tilde{\nabla}_X B)(Y) = (\tilde{\nabla}_Y B)(X)$, $X, Y \in V^{(1)}$. Thus for all $X, Y \in V_p^{(1)}$

$$\{\nabla_X \mu\} Y = \{\nabla_Y \mu\} X,$$

i.e., $\nabla_X \mu = 0$ which proves that μ is constant on the integral manifolds of $V^{(1)}$.

Let us consider an integral hypersurface $\tilde{M}$ of $V^{(1)}$, and let $\tilde{\nabla}$ be its Levi-Civita connection. Furthermore let $R^*(X, Y)Z$ be the restriction of the curvature R of (M^n, g) onto $\tilde{M}$, and let $\tilde{R}(X, Y)Z$ be the Riemannian curvature of $\tilde{M}$ in the induced metric.

Lemma 4.9. *On an integral manifold $\tilde{M}$ of $V^{(1)}$ the following equation holds*:

$$4(n-1)\kappa^2\mu^2 - 2(n-2)\mu^2 R^*_{ijkl} R^{*ijkl} = \tilde{\nabla}_s R^*_{ijkl} \tilde{\nabla}^s R^{*ijkl}. \tag{4.29}$$

Proof. From Lemma 4.7 we get

$$\nabla^s \nabla_s \left(R_{ijkl} R^{ijkl}\right) = 2\nabla_s R_{ijkl} \nabla^s R^{ijkl} + 4R^{ijkl}\left(\nabla_j \nabla_k R_{il} - \nabla_j \nabla_l R_{ik}\right). \tag{4.30}$$

First we compute the last part of the right side. In the computation we use the formulas

$$\nabla_j\kappa = -2\kappa\mu \underline{m}_j, \quad \nabla_j\mu = -\mu^2\underline{m}_j, \tag{4.31}$$

which come evidently from (4.25), (2.10) and the fact that μ is constant on the integral manifolds of $V^{(1)}$. As

$$R_{il} = \kappa(g_{il} - \underline{m}_i\underline{m}_l),$$
$$\nabla_i m_j = \mu(g_{ij} - \underline{m}_i\underline{m}_j),$$

we have

$$\nabla_k R_{il} - \nabla_l R_{ik} = \kappa\mu(\underline{m}_l g_{ik} - \underline{m}_k g_{il}),$$

and so

$$\begin{aligned} 4R^{ijkl}(\nabla_j\nabla_k R_{il} - \nabla_j\nabla_l R_{ik}) &= 4\kappa\mu R^{ijkl}\{(\nabla_j\underline{m}_l)g_{ik} - (\nabla_j\underline{m}_k)g_{il}\} \\ &= 4\kappa\mu^2 R^{ijkl}(g_{lj}g_{ik} - g_{kj}g_{il}) \\ &= 8\kappa\mu^2 R_{ij}{}^{ij} = -8(n-1)\kappa^2\mu^2. \end{aligned} \tag{4.32}$$

On the other hand by

$$\begin{aligned} &(\nabla_{\underline{m}}R)(X,Y)Z = -2\mu R(X,Y)Z, \\ &X^r\underline{m}^iY^jZ^kV^l\nabla_r R_{ijkl} = -\mu X^iY^jZ^kV^lR_{ijkl}, \quad X,Y,V,Z \in V_p^{(1)}, \end{aligned} \tag{4.33}$$

we get

$$\nabla_s R_{ijkl}\nabla^s R^{ijkl} = \tilde{\nabla}_s R^*_{ijkl}\tilde{\nabla}^s R^{*ijkl} + 8\mu^2 R^*_{ijkl}R^{*ijkl}. \tag{4.34}$$

Finally let us consider the left side of (4.30). By Lemma 4.8 the function $R_{ijkl}R^{ijkl}$ is constant on $\tilde{M}$. Thus by (4.31) and (4.33) we have

$$\begin{aligned} \nabla_s\nabla^s(R_{ijkl}R^{ijkl}) &= (n-1)\mu\nabla_{\underline{m}}(R_{ijkl}R^{ijkl}) + \nabla_{\underline{m}}\nabla_{\underline{m}}(R_{ijkl}R^{ijkl}) \\ &= -4(n-1)\mu^2 R^*_{ijkl}R^{*ijkl} + 20\mu^2 R^*_{ijkl}R^{*ijkl}. \end{aligned} \tag{4.35}$$

By formulas (4.30), (4.32), (4.34) and (4.35) we get (4.29). q.e.d.

Let us also compute the left side of (4.29) with the help of the eigenvalues of the curvature operator. If $\lambda_1, \lambda_2, \cdots, \lambda_r$ denote the distinct nonnull eigenvalues of the curvature operator $R_{i\ k}^{\ j\ l}$ with multiplicity $k_1, k_2, \cdots, k_r$, then the following equations hold trivially:

$$\begin{aligned} &-(n-1)\kappa = 2(k_1\lambda_1 + k_2\lambda_2 + \cdots + k_r\lambda_r), \\ &R_{ijkl}R^{ijkl} = R^*_{ijkl}R^{*ijkl} = 4(k_1\lambda_1^2 + \cdots + k_r\lambda_r^2), \\ &k_1 + \cdots + k_r = \dim \mathcal{H}_p. \end{aligned}$$

Thus for the left side of (4.29) we get

$$4(n-1)\kappa^2\mu^2 - 2(n-2)\mu^2 R^*_{ijkl}R^{*ijkl}$$
$$= \frac{8\mu^2}{n-1}\left[2(k_1\lambda_1 + \cdots + k_r\lambda_r)^2 - (n-1)(n-2)\left(k_1\lambda_1^2 + \cdots + k_r\lambda_r^2\right)\right].$$

On the other hand for the theorem in the brackets we get

$$\sum_{i=1}^{r}\left(2k_i^i - (n-1)(n-2)k_i\right)\lambda_i^2 + 2\sum_{\substack{i,j=1\\ i\neq j}}^{r} k_i k_j \lambda_i \lambda_j$$
$$\leqslant \sum_{i=1}^{r}\left(2k_i^2 - (n-1)(n-2)k_i\right)\lambda_i^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{r} k_i k_j\left(\lambda_i^2 + \lambda_j^2\right)$$
$$= \sum_{i=1}^{r} k_i\left(2\sum_{i=1}^{r} k_i - (n-1)(n-2)\right)\lambda_i^2$$
$$= \sum_{i=1}^{r} k_i\left(2\dim \mathcal{H}_p - (n-1)(n-2)\right)\lambda_i^2.$$

Since $\mathcal{H}_p$ is a subgroup of $SO(n-1)$, $\dim \mathcal{H}_p \leqslant (n-1)(n-2)/2$. So the left side of (4.29) is always negative except the case where $\mathcal{H}_p \cong SO(n-1)$ and there is only one simple eigenvalue λ of the curvature operator. In this case the left side of (4.29) vanishes, but the right side is nonnegative, as it is $\tilde{\nabla}_s R^*_{ijkl}\nabla^s R^{*ijkl}$. Thus (4.29) holds only in the case $\mathcal{H}_p \cong SO(n-1)$, and so $\tilde{\nabla}_s R^*_{ijkl} = 0$, i.e., the tensor field R^*_{ijkl} is parallel on $\tilde{M}$. From $\mathcal{H}_p \cong SO(n-1)$ it follows that the curvature form R is of constant curvature on the $(n-1)$-dimensional subspace $V_p^{(1)}$, i.e., it is of the form

$$R(X,Y)Z = \frac{\kappa}{n-2}\{g(Y,Z)X - g(X,Z)Y\}, \quad X, Y, Z \in V_p^{(1)}.$$

From (2.13) we get

$$\tilde{R}(X,Y)Z = \left(\frac{\kappa}{n-2} + \mu^2\right)\{g(Y,Z)X - g(X,Z)Y\},$$

and thus also the induced metric on $\tilde{M}$ is of constant curvature.

We prove that a simply connected neighborhood of U is isometric to an open subset of the hyperbolic or elliptic cone. Let $(x^2, x^3, \cdots, x^r)$ be a coordinate neighborhood of $\tilde{M}$, and $\partial_2, \partial_3, \cdots, \partial_n$ such vector fields on U such that $[\underline{m}, \partial_i] = 0$ and $\partial_{i/\tilde{M}} = \partial/\partial x^i$. If $X_2, X_3, \cdots, X_n$ are such vector fields on U, which are parallel on the integral curves of $\underline{m}$ and satisfy $X_{i/\tilde{M}} = \partial/\partial x^i$, then it follows evidently from $[\underline{m}, \partial_i] = 0$ that

$$\partial_i = \left(x^1 + \frac{1}{\mu_0}\right) X_i,$$

where μ_0 is the constant value of μ on $\tilde{M}$, and x^1 the signed distance of the point from $\tilde{M}$ along the integral curves of $\underline{m}$. Thus the metrical tensor field g_{ij} of the space in the coordinate system is

$$g_{\alpha\beta} = \left(x^1 + \frac{1}{\mu_0}\right)^2 g_{\alpha\beta}(x^2, \cdots, x^n), \quad \alpha, \beta > 1,$$
$$g_{1\alpha} = 0, g_{11} = 1,$$

where $g_{\alpha\beta}(x^2, \cdots, x^n)$ is the metrical tensor field of a Riemannian space of constant curvature. It can be seen obviously that the space is locally isometric to the elliptic or hyperbolic cone or to the *euclidean cone* (defined by $g_{\alpha\beta} = \delta_{\alpha\beta}$ in the above formulas).

Finally we prove that μ never vanishes on M. Indeed, μ is constant on the integral manifold of $V^{(1)}$, and furthermore it is of the form $1/(s + 1/\mu_0)$ on an integral geodesic $c(s)$ of the vector field $\underline{m}$, where $\mu_0 = \mu(c(0))$, as it satisfies $d\mu/ds = -\mu^2$. Thus μ does not vanish at a point which is at finite distance from U. This proves the statement. We get also obviously that a simply connected and maximal space corresponding to the theorem is globally isometric to an elliptic or a hyperbolic cone. q.e.d.

Considering the main theorems and Theorem 1.3 we have the following local structure theorem for semi-symmetric spaces.

Theorem 4.5 (*The local structure theorem*). *For every semi-symmetric Riemannian space there exists an everywhere dense open subset U such that around every point of U the space is locally isometric to a space which is the direct product of symmetric spaces, two-dimensional Riemannian spaces, spaces foliated with $(n-2)$-dimensional Euclidean spaces, elliptic cones, hyperbolic cones, eucliden cones, Kaehlerian cones.*

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