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GENERALIZED ROTATIONAL HYPERSURFACES OF CONSTANT MEAN CURVATURE IN THE EUCLIDEAN SPACES. I

WU-YI HSIANG

Introduction

In 1841, Delaunay [3] discovered a beautiful way of constructing rotational hypersurfaces of constant mean curvature in the euclidean 3-space E^3, namely, its generating curve can be obtained as the trace of a focus by rolling a given conic section on the axis. The above theorem of Delaunay was generalized to higher dimensional euclidean spaces in [9], namely, the generating curves of those $O(n-1)$-invariant hypersurfaces of constant mean curvature in E^n can again be obtained by rolling construction. However, from the viewpoint of equivariant differential geometry, a natural generalization of the rotational surfaces of E^3 should, at least, include those hypersurfaces which are invariant under an *isometric* transformation group (G, E^n) with *codimension two principal orbit type*. For example, in the case of E^4, there is the transformatiion group of type $O(2) \times O(2)$ acting on $E^2 \times E^2 = E^4$ besides the "usual" $O(3)$-action on E^4. Of course, in the final analysis, it will all depend on what kind of results such a generalization will lead to. As a preliminary indication, the study to generalized rotational hypersurfaces of $O(k) \times O(k)$-type already leads (here and the comparison paper [8]) to the discovery of a family of important new examples of constant mean curvature immersions of $(2k-1)$-spheres into E^{2k}. This result strongly suggests that the geometry of generalized rotational hypersurfaces definitely deserves a systematic investigation.

In this paper, we shall begin a systematic study of generalized rotational hypersurfaces of *constant mean curvatures* in E^n. The analytical problem of such a geometrical object can be reduced to the global solutions of certain specific ordinary differential equations. In §1, we shall recall some known results of [6,10], which will enable us to write down the reduced, ordinary differential equation for each type of generalized rotational transformation groups. One may naturally divide such transformation groups (G, E^n) into five

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types according to the geometric shape of their orbit spaces E^n/G, where E^n/G are linear cones of angle π/d, $d = 1, 2, 3, 4, 6$, respectively. The case of $d = 1$ corresponding to the usual rotational transformation group of $(O(n-1), E^n)$ and generalized rotational hypersurfaces of constant mean curvature of this type has already been thoroughly studied in [9]. The next case of $d = 2$ corresponds to the action of $O(p) \times O(q)$, $p, q \geqslant 2$, $p + q = n$, on $E^n = E^p \times E^q$. In this paper, we shall mainly study generalized rotational hypersurfaces of $O(p) \times O(q)$-type. We state the main results concerning constant mean curvature hypersurfaces of this type as follows:

The generating curves of generalized rotational hypersurfaces of $O(p) \times O(q)$-type with constant mean curvature h are solutions of the equation

$$\text{(II)} \qquad \dot{\sigma} = (p + q - 1)h + (p - 1)\frac{\cos\sigma}{y} - (q - 1)\frac{\sin\sigma}{x}.$$

A *global* solution curve is a solution curve $\gamma = \{(x(s), y(s)), -\infty < s < +\infty\}$ which are infinitely extendable in both directions. The following are the main results on the geometry of global solution curves of the above equation (II), for the general case $h \neq 0$. By a simple transformation of homothety, one may assume without loss of generality that $h = 1$:

(1) There are two straight line solutions of (II), namely, $x = (q-1)/(p+q-1)$ and $y = (p-1)/(p+q-1)$ whose inverse images are cylinders of type $\mathbf{R}^p \times S^{q-1}$ and $S^{p-1} \times \mathbf{R}^p$ respectively.

(2) Let $\gamma(s)$, $-\infty < s < +\infty$, be any given global solution curve of (II). Then $x = (q-1)/(p+q-1)$ is the asymptotic line of γ as $s \to +\infty$ and $y = (p-1)/(p+q-1)$ is the asymptotic line of γ as $s \to -\infty$ [cf. Theorem 1 and Corollary 1 of §2].

(3) Each global solution curve, γ, of (II) can have *at most* one (cusp) point on each axis. Therefore, one may classify the global solution curves of (II) into the following types, namely,

Type A. With no cusp point.

Type B. With exactly one cusp point on the x-axis.

Type C. With exactly one cusp point on the y-axis.

Type D. With exactly two cusp points (which must be one on each axis).

Type E. With exactly one cusp point at the origin.

(4) It is natural to define the direction function $\sigma_\gamma(s)$ on a given global solution curve $\gamma(s)$ such that it has a *jump of* $+\pi$ at each *cusp point* and *continuous* elsewhere. It follows from (2) that $\lim_{s\to+\infty}\sigma_\gamma(s)$ and $\lim_{s\to-\infty}\sigma_\gamma(s)$ both exist and

$$\Delta\sigma(\gamma) = \lim_{s\to+\infty}\sigma_\gamma(s) - \lim_{s\to-\infty}\sigma_\gamma(s) = 2n(\gamma)\pi - \frac{\pi}{2},$$

where $n(\gamma)$ is a suitable integer (called the winding number of γ). One has the following fundamental existence theorem.

Existence Theorem. (i) *If γ is of type* A, *then $n(\gamma) \geqslant 0$. Conversely, to each integer $k \geqslant 0$ there exists a global solution curve of type* A *with k as its winding number.*

(ii) *If γ is of type* B *or* C, *then $n(\gamma) \geqslant 1$. Conversely, to each integer $k \geqslant 1$, there exists a global solution curve of type* B *and* C *respectively whose winding number equals k.*

(iii) *If γ is of type* D, *then $n(\gamma) \geqslant 2$. Conversely, to each integer $k \geqslant 2$, there exists a global solution curve of type* D *with k as its winding number.*

As a corollary of the above existence theorem, one obtains the following.

Theorem. *There exist infinitely many noncongruent immersions $S^n \to E^{n+1}$ with constant mean curvature* 1, *for each $n \geqslant 3$.*

In fact, one obtains $([\frac{n}{2}] - 1)$ families of new examples of constant mean curvature immersions $S^n \to E^{n+1}$, namely, one infinite family of $O(p) \times O(q)$-invariant immersions for each decomposition of n into $p + q$ [cf. Theorem 2 and Corollary 2 of §3]. The special case of $n = 4$, $p, q = 2$ was announced in [7]. A different proof of the existence of such immersions for the case $p = q \geqslant 2$ was given in [8].

The analysis and the geometry of rotational hypersurfaces of other types will be studied in succeeding papers.

1. Orbital geometry and reduced ordinary differential equations

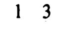
Orthogonal transformation groups $(G, \mathbf{R}^n)$ with codimension two principal orbit type were classified in [6]. They are exactly those *isotropy representations of symmetric spaces of rank* 2. Following É. Cartan, it is not difficult to compute the orbital geometry of such representations as follows.

(i) It follows from the maximal tori theorem of É. Cartan (for the case of symmetric spaces) that there exists a 2-dimensional linear subspace, $\mathbf{R}^2$, which is the fixed point set of a chosen principal isotropy subgroup H of $(G, \mathbf{R}^n)$ and intersects every G-orbit *perpendicularly*.

(ii) The Weyl group, $W = N(H, G)/H$, acts on $\mathbf{R}^2$ as a group generated by reflections and $\mathbf{R}^n/G \simeq \mathbf{R}^2/W$. Therefore, the orbit space $\mathbf{R}^n/G$ can be identified with the Weyl chamber of $(W, \mathbf{R}^2)$ and the orbital distance metric is flat, namely, a linear cone of angle π/d, $d = 1, 2, 3, 4$ or 6.

Let ρ_m, μ_m, ν_m be the standard representation of $SO(m)$ (or $O(m)$), $SU(m)$ (or $U(m)$), $\mathrm{Sp}(m)$ on $\mathbf{R}^n$, $\mathbf{C}^n$, $\mathbf{H}^n$ respectively. Let the Weyl chamber be the linear cone given by $y \geqslant 0$ and $x \cdot \sin \pi/d - y \cdot \cos \pi/d \geqslant 0$ and $w(d, i)$ be the linear function $(x \cdot \sin i\pi/d - y \cdot \cos i\pi/d)$. Then the orbital distance metric of $\mathbf{R}^n/G$ is simply given by $ds^2 = dx^2 + dy^2$ and the needed geometric invariants of the orbit structure of those orthogonal transformation groups can be listed as follows (cf. [6]).

TABLE I

G	Φ	$n = \dim \Phi$	$d(W)$	$f(\xi) =$ [volume of $\xi]^2$	Asso. Sym. Space L/G
$SO(n-1)$	$1 + \rho_{n-1}$	n	1	$c \cdot y^{2n-4}$	$\mathbf{R}^1 \times S^{n-1}$
$SO(l) \times SO(m)$	$\rho_l + \rho_m$	$l + m$	2	$c \cdot x^{2m-2} \cdot y^{2l-2}$	$S^l \times S^m$
$SO(3)$	$S^2\rho_3 - 1$	5	3	$c \cdot \prod_{i=0}^{2} w(3, i)^2$	$SU(3)/SO(3)$
$SU(3)$	Ad	8	3	$c \cdot \prod_{i=0}^{2} w(3, i)^4$	$\frac{SU(3) \times SU(3)}{SU(3)}$
Sp(3)	$\Lambda^2\nu_3 - 1$	14	3	$c \cdot \prod_{i=0}^{2} w(3, e)^8$	$\frac{SU(6)}{\mathrm{Sp}(3)}$
F_4	1 	26	3	$c \cdot \prod_{i=0}^{2} w(3, i)^{16}$	$\frac{E_6}{F_4}$
$SO(5)$	Ad	10	4	$c \cdot x^4y^4 \cdot (x^2 - y^2)^4$	$\frac{SO(5) \times SO(5)}{SO(5)}$
$SO(2) \times SO(m)$	$\rho_2 \otimes \rho_m$	$2m$	4	$c \cdot (x \cdot y)^{2m-4} \cdot (x^2 - y^2)^2$	$\frac{SO(2+m)}{SO(2) \times SO(m)}$
$S(U(2) \times U(m))$	$[\mu_2 \otimes_{\mathbf{C}} \mu_m]_{\mathbf{R}}$	$4m$	4	$c \cdot (x \cdot y)^{4m-6} \cdot (x^2 - y^2)^4$	$\frac{SU(2+m)}{S(U(2) \times U(m))}$
$\mathrm{Sp}(2) \times \mathrm{Sp}(m)$	$\nu_2 \otimes_{\mathbf{H}} \nu_m^*$	$8m$	4	$c \cdot (x \cdot y)^{8m-10} \cdot (x^2 - y^2)^8$	$\frac{\mathrm{Sp}(2+m)}{\mathrm{Sp}(2) \times \mathrm{Sp}(m)}$
$U(5)$	$[\Lambda^2\mu_5]_{\mathbf{R}}$	20	4	$c \cdot (xy)^{10} \cdot (x^2 - y^2)^8$	$\frac{SO(10)}{U(5)}$
$U(1) \times \mathrm{Spin}(10)$	$[\mu_1 \otimes_{\mathbf{C}} \Delta_1^+]_{\mathbf{R}}$	32	4	$c \cdot (xy)^{18} \cdot (x^2 - y^2)^{12}$	$\frac{E_6}{U(1) \times \mathrm{Spin}(10)}$
G_2	Ad	14	6	$c \cdot \prod_{i=0}^{5} w(6, i)^4$	$\frac{G_2 \times G_2}{G_2}$
$SO(4)$	1 3 	8	6	$c \cdot \prod_{i=0}^{5} w(6, i)^2$	$G_1/SO(4)$

Next, let us recall the following proposition of [10] which reduces the computation of the mean curvature of a G-invariant submanifold $(G, N) \subset (G, M)$ to that of its image, $N/G \subset M/G$, at the level of orbit space.

Proposition 1. *Let (G, N) be a G-submanifold of (G, M) with the same principal orbit type as that of M, and let x be a point on a principal orbit $\xi \in N/G$. Let ν_x be an arbitrary unit normal to N at x and ν_ξ be its image at ξ. Then*

$$H(\nu_x) = H'(\nu_\xi) - \frac{1}{2}\frac{d}{d\nu_\xi}\ln f(\xi),$$

where $H(\nu_\xi)$ and $H'(\nu_\xi)$ are the mean curvatures of N and N/G in the directions ν_x and ν_ξ respectively, $f(\xi) = [$volume of the principal orbit $\xi]^2$ and $d/d\nu_\xi$ is the directional differentiation.

Combine the above reduction with the orbital geometric invariants of Table I, one has the following list of reduced, ordinary differential equations for generalized rotational hypersurfaces of constant mean curvature h. We let σ denote the angle of the tangent vector with the x-axis.

Proposition 2. *The differential equation of the "generating curve" of a generalized rotational hypersurface of constant mean curvature h in E^n is listed as follows according to its type.*

Type I, $d = 1$, $(O(n-1), \mathbf{R}^n)$:

$$\dot{\sigma} = (n-1)h + (n-2)\frac{\cos\sigma}{y}.$$

Type II, $d = 2$, $(O(p) \times O(q), \mathbf{R}^{p+q})$:

$$\dot{\sigma} = (p+q-1)h + (p-1)\frac{\cos\sigma}{y} - (q-1)\frac{\sin\sigma}{x}.$$

Type III, $d = 3$, $(SO(3), \mathbf{R}^5)$, $(SU(3), \mathbf{R}^8)$, $(\mathrm{Sp}(3), \mathbf{R}^{14})$ *or* $(F_4, \mathbf{R}^{26})$:

$$\dot{\sigma} = (3k+1)h + k\left\{\frac{\cos\sigma}{y} - \frac{\sin\left(\sigma + \frac{\pi}{6}\right)}{x\cos\frac{\pi}{6} - y\sin\frac{\pi}{6}} - \frac{\sin\left(\sigma - \frac{\pi}{6}\right)}{x\cos\frac{\pi}{6} + y\sin\frac{\pi}{6}}\right\},$$

where $k = 1, 2, 3, 4$ or 8 respectively.

Type IV, $d = 4$:

$$\dot{\sigma} = (2k+2l+1)h + k\left\{\frac{\cos\sigma}{y} - \frac{\sin\sigma}{x}\right\}$$
$$-\sqrt{2}\,l\left\{\frac{\sin\left(\sigma - \frac{\pi}{4}\right)}{x-y} + \frac{\sin\left(\sigma - \frac{\pi}{4}\right)}{x+y}\right\},$$

where

$$(k, l) = \begin{Bmatrix} (2,2) \\ (5,4) \\ (9,6) \\ (m-2,1) \\ (2m-3,2) \\ (4m-5,4) \end{Bmatrix}$$

for

$$\begin{cases} (SO(5), \mathbf{R}^{10}) \\ (U(5), \mathbf{R}^{20}) \\ (U(1) \times \mathrm{Spin}(10), \mathbf{R}^{32}) \\ (SO(2) \times SO(m), \mathbf{R}^{2m}) \\ (S(U(2) \times U(m)), \mathbf{R}^{4m}) \\ (\mathrm{Sp}(2) \times \mathrm{Sp}(m), \mathbf{R}^{8m}). \end{cases}$$

Type V, $d = 6$, $(SO(4), \mathbf{R}^8)$ *or* $(G_2, \mathbf{R}^{14})$:

$$\dot{\sigma} = (6k+1)h + k\left\{ \frac{\cos\sigma}{y} - \frac{\sin\sigma}{x} - \frac{2\sin\left(\sigma + \frac{\pi}{6}\right)}{\sqrt{3}\,x - y} - \frac{2\sin\left(\sigma - + \frac{\pi}{6}\right)}{\sqrt{3}\,x + y} - \frac{2\sin\left(\sigma + \frac{\pi}{3}\right)}{x - \sqrt{3}\,y} - \frac{2\sin\left(\sigma - \frac{\pi}{3}\right)}{x + \sqrt{3}\,y} \right\},$$

where $k = 1, 2$ *for* $(SO(4), \mathbf{R}^8)$, $(G_2, \mathbf{R}^{14})$ *respectively.*

Proof. It follows from Propositioin 1 and the list of orbital geometric data by straightforward computations. q.e.d.

In the special case of generalized rotational *minimal* hypersurfaces, i.e., $h = 0$, it is not difficult to see that all the above equations are invariant under homotheties. Hence, it is advantageous to use polar coordinates (r, θ) to transform them into the following first order equations in terms of θ and σ.

Proposition 2′. *The differential equations of the generating curve of a generalized rotational minimal hypersurface can be reduced to the following first order equations according to its type.*

Type I. $\sin\theta \cdot \sin(\sigma - \theta) \cdot d\sigma - (n-2)\cos\sigma\, d\theta = 0.$

Type II.

$$\sin\theta\cos\theta\sin(\sigma-\theta)\cdot d\sigma+[(q-1)\sin\sigma\sin\theta \\ -(p-1)\cos\sigma\cos\theta]\,d\theta=0.$$

Type III.

$$\sin(\sigma-\theta)\,d\sigma+k\left\{\frac{\sin\left(\sigma+\frac{\pi}{6}\right)}{\cos\left(\theta+\frac{\pi}{6}\right)}+\frac{\sin\left(\sigma-\frac{\pi}{6}\right)}{\cos\left(\theta-\frac{\pi}{6}\right)}-\frac{\cos\sigma}{\sin\theta}\right\}d\theta=0$$

where $k=1,2,4,8$.

Type IV.

$$\sin(\sigma-\theta)\,d\sigma+\left\{k\left[\frac{\sin\sigma}{\cos\theta}-\frac{\cos\sigma}{\sin\theta}\right]\right. \\ \left.+l\left[\frac{\sin\left(\sigma+\frac{\pi}{4}\right)}{\cos\left(\theta+\frac{\pi}{4}\right)}+\frac{\sin\left(\sigma-\frac{\pi}{4}\right)}{\cos\left(\theta-\frac{\pi}{4}\right)}\right]\right\}d\theta=0,$$

where $(k,l)=(2,2),(5,4),(9,6),(m-2,1),(2m-3,2)$ *or* $(4m-5)$.

Type V.

$$\sin(\sigma-\theta)\,d\sigma+k\left\{\frac{\sin\left(\sigma+\frac{\pi}{6}\right)}{\cos\left(\theta+\frac{\pi}{6}\right)}+\frac{\sin\left(\sigma-\frac{\pi}{6}\right)}{\cos\left(\theta-\frac{\pi}{6}\right)}+\frac{\sin\left(\sigma+\frac{\pi}{3}\right)}{\cos\left(\theta+\frac{\pi}{3}\right)}\right. \\ \left.+\frac{r\sin\left(\sigma-\frac{\pi}{3}\right)}{\cos\left(\theta-\frac{\pi}{3}\right)}+\frac{\sin\sigma}{\cos\theta}-\frac{\cos\sigma}{\sin\theta}\right\}\cdot d\theta=0,$$

$k=1,2$.

Proof. One substitutes $h=0$, $x=r\cos\theta$ and $y=r\sin\theta$ into the equations of Proposition 2. Notice that $d\sigma/d\theta=\dot{\sigma}\cdot ds/d\theta$ and $ds/(rd\theta)=1/\sin(\sigma-\theta)$. It is straightforward to reduce those equations of Proposition 2 into the above first order equations respectively.

Remarks. (i) The basic reason for the existence of the above reduction to a first order equation in terms of (θ,σ) is exactly the *homothetic invariance* of the special case of *minimal* hypersurfaces. Indeed, each integral curve in the (θ,σ)-space corresponds exactly to a family of integral curves in the (x,y)-space which are equivalent under homotheties.

(ii) It is not difficult to see that the singularities of the above first order equations are quite regular. Therefore, the usual method of Poincaré-Bendixson is readily applicable to analyze the geometry of the solution curves of the above equations. One of the special cases (i.e. type II with $p = q$) was treated in [2] which played an imporatnt role in the study of Bernstein's problem and minimal cones. We refer to [12] for a systematic treatment of all the above five types of equations via Poincaré-Bendixson theory.

In the general case of nonzero constant mean curvature $h \neq 0$, homotheties will change the value of h. However, the equation of type I (with nonzero h) is still invariant under the translations in the x-direction. Corresponding to this invariance property of type I equation, one has the following "first integral".

Proposition 3 [11]. *Up to an equivalence of translations in the x-direction, the global solutions of the equation*

$$\text{(I)} \qquad \dot{\sigma} = (n-1)h + (n-2)\frac{\cos\sigma}{y}$$

is uniquely characterized by $J = y^{n-2}\cos\sigma + hy^{n-1} = c$.

Proof.

$$\frac{dJ}{ds} = y^{n-2}\dot{y}\left\{-\dot{\sigma} + (n-1)h + (n-2)\frac{\cos\sigma}{y}\right\} = 0.$$

Therefore, $J = c$ along each integral curve of the above equation. Moreover, J is obviously invariant under translation and it is not difficult to show that two solution curves with the same values of J are translationally equivalent. q.e.d.

We refer to [9] for further discussion of the geometry of the above solution curves, e.g., a generalization of Delaunay's theorem.

2. The analysis of global solutions of the equation of type II

In this section, we shall study the properties of global solutions of the following equation of type II:

$$\text{(II)} \qquad \dot{\sigma} = (p+q-1)h + (p-1)\frac{\cos\sigma}{y} - (q-1)\frac{\sin\sigma}{x}.$$

By a proper choice of orientation and a suitable homothetic transformation, we may assume that $h = 1$ in the above equation. In analyzing the behavior of solutions of (II), it is quite natural to compare them with those solutions of either

$$\text{(III)} \qquad \dot{\sigma} = (p+q-1) - (q-1)\frac{\sin\sigma}{x},$$

or

$$\text{(III}')\qquad \dot{\sigma} = (p+q-1) + (p-1)\frac{\cos\sigma}{y},$$

which are essentially equations of type (I). By Proposition 3, one has the following "*first integrals*" of (III) and (III′) respectively,

$$I = x^{q-1}\sin\sigma - \frac{p+q-1}{q}x^q \quad\text{and}\quad J = y^{p-1}\cos\sigma + \frac{p+q-1}{p}y^p.$$

Following [8], one also has the following important fact of monotonicity of I and J along any solution curve.

Proposition 4 [8]. *Let $\gamma = \{x(s), y(s)\}$ be a solution curve of* (II) *and s be the arc length. Then*

$$\frac{dI}{ds} = (p-1)\frac{x^{q-1}\cos^2\sigma}{y} \geqslant 0 \quad\textit{and}\quad \frac{dJ}{ds} = (q-1)\frac{y^{p-1}\sin^2\sigma}{x} \geqslant 0$$

and hence both I and J are monotonically increasing along γ.

Proof. Straightforward differentiation will show that

$$\begin{aligned}\frac{dI}{ds} &= x^{q-1}\cos\sigma\left\{\dot{\sigma} + (q-1)\frac{\sin\sigma}{x} - (p+q-1)\right\}\\ &= (p-1)\frac{x^{q-1}\cos^2\sigma}{y} \geqslant 0,\\ \frac{dJ}{ds} &= y^{p-1}\sin\sigma\left\{-\dot{\sigma} + (p-1)\frac{\cos\sigma}{y} + (p+q-1)\right\}\\ &= (q-1)\frac{y^{p-1}\sin^2\sigma}{x} \geqslant 0. \quad \text{q.e.d.}\end{aligned}$$

There are two simple-minded solutions of (II) which are characterized by the condition $\dot{\sigma} \equiv 0$, namely,

Lemma 1. *If $\dot{\sigma} \equiv 0$, then*

$$\begin{cases} \textit{either} \quad \sigma = \pi, \quad y = \dfrac{p-1}{p+q-1}, \\ \textit{or} \quad \sigma = \dfrac{\pi}{2}, \quad x = \dfrac{q-1}{p+q-1}. \end{cases}$$

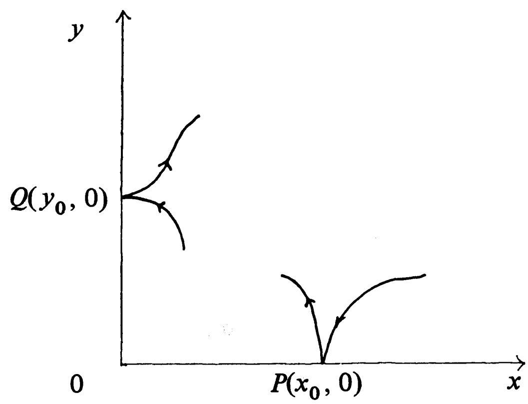
Proof. It is easy to see that the above two curves are solutions of (II). Let us prove that they are the only solutions with $\dot{\sigma} \equiv 0$. Suppose γ is such a solution curve. Then $\sigma \equiv \sigma_0$ and γ satisfies the following algebraic equation, namely

$$(p-1)\cos\sigma_0\frac{1}{y} - (q-1)\sin\sigma_0\frac{1}{x} + (p+q-1) = 0.$$

Hence, either $\sin\sigma_0 = 0$ or $\cos\sigma_0 = 0$, for otherwise, the above equation defines a nondegenerate hyperbola which obviously contradicts the assumption $\dot{\sigma} \equiv 0$.

Proposition 5. *To each point $P(x_0, 0)$ (resp. $Q(0, y_0)$) on the x-axis (resp. y-axis), there exists a unique solution curve of* (II) *passing through P (resp. Q) which is automatically analytic and forms a perpendicular cusp point. Conversely, a global solution curve of* (II) *can have at most one point on the x-axis (resp. y-axis).*

Proof. Suppose γ is a solution curve of (II) passing through $P(x_0, 0)$. Then, it is easy to show that γ forms a perpendicular cusp point at P. If we consider the incoming and the outgoing branches of γ separately, then their inverse images in E^{p+q} are *regular hypersurfaces of constant mean curvature*. Hence, they are automatically analytic and this implies that each branch of γ must also be analytic.



In view of the above fact, it is natural to use the usual method of power series substitution and majoration to establish the uniqueness and existence of such a solution curve γ. We refer to [5] for a detailed proof of such a majoration. Finally, it follows from the monotonicity of J along γ that γ cannot have any other point of the x-axis.

Remarks. (i) It follows from the above power series substitution that the unique solution curve γ_{x_0} (which has a cusp point at $(x_0, 0)$) *depends on x_0 analytically*.

(ii) In the special case of $(0,0)$, namely, $x_0 = 0 = y_0$, the uniqueness and the existence of an *analytic* solution passing through $(0,0)$ are still valid. However, the *automatic analyticity* of such a solution needs a new proof.

Lemma 2 [8]. *Along any given solution curve γ, one has the following upper bound (resp. lower bound) for I (resp. J), namely,*

$$I(s) \leqslant \frac{1}{q}\left\{\frac{q-1}{p+q-1}\right\}^{q-1} \quad \left(\text{resp. } J(s) \geqslant -\frac{1}{p}\left\{\frac{p-1}{p+q-1}\right\}^{p-1}\right).$$

Moreover, if

$$I(s_0) = \frac{1}{q}\left\{\frac{q-1}{p+q-1}\right\}^{q-1} \quad \left(\text{resp. } J(s_0) = -\frac{1}{p}\left\{\frac{p-1}{p+q-1}\right\}^{p-1}\right),$$

then $x(s) = \dfrac{q-1}{p+q-1}$ *for* $s \geqslant s_0$ *(resp.* $y(s) = \dfrac{p-1}{p+q-1}$ *for* $s \leqslant s_0$*).*

Proof.

$$I = x^{q-1}\sin\sigma - \frac{p+q-1}{q}x^q \leqslant x^{q-1} - \frac{p+q-1}{q}x^q, \qquad x \geqslant 0,$$

(resp. $J = y^{p-1}\cos\sigma + (p+q-1)y^p/q \geqslant -y^{p-1} + (p+q-1)y^p/p, y \geqslant 0$) which reaches a maximal value of $\frac{1}{q}\{\frac{q-1}{p+q-1}\}^{q-1}$ at $x = \frac{q-1}{p+q+1}$ (resp. a minimal value of $-\frac{1}{p}\{\frac{p-1}{p+q-1}\}^{p-1}$ at $y = \frac{p-1}{p+q+1}$). Hence, Lemma 2 follows.

Theorem 1. *Let* $\gamma = \{(x(s), y(s)), \| -\infty < s < +\infty\}$ *be a global solution curve of* (II). *Then*

$$\operatorname*{Lim}_{s\to+\infty} I(s) = \frac{1}{q}\left\{\frac{q-1}{p+q-1}\right\}^{q-1} \quad \textit{and} \quad \operatorname*{Lim}_{s\to-\infty} J(s) = -\frac{1}{p}\left\{\frac{p-1}{p+q-1}\right\}^{p-1}.$$

Corollary 1. $x = \dfrac{q-1}{p+q-1}$ *is the asymptotic line of* γ *as* $s \to +\infty$ *and* $y = \dfrac{p-1}{p+q-1}$*s the asymptotic line of* γ *as* $s \to -\infty$.

Notation. From now on, we shall fix our notation to denote the upper bound of I by U and the lower bound of J by L, namely,

$$U = \frac{1}{q}\left\{\frac{q-1}{p+q-1}\right\}^{q-1}, \quad L = -\frac{1}{p}\left\{\frac{p-1}{p+q-1}\right\}^{p-1}.$$

Proof of Corollary 1. Purely algebraically, it is easy to show that to any given $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$x \geqslant 0$ and $x^{q-1} - \dfrac{p+q-1}{q}x^q > U - \delta_1$ implies $\left| x - \dfrac{q-1}{p+q-1} \right| < \varepsilon$,

$y \geqslant 0$ and $-y^{p-1} + \dfrac{p+q-1}{p}y^p < L + \delta_2$ implies $\left| y - \dfrac{p-1}{p+q-1} \right| < \varepsilon$.

Hence, Corollary 1 follows readily from Theorem 1.

Proof of Theorem 1. Since the proof of $\mathrm{Lim}_{s\to+\infty} = U$ and that of $\mathrm{Lim}_{s\to-\infty} J(s) = L$ are essentially the same, we shall only give the proof of the first limit in the following.

By Proposition 4 and Lemma 2, $I(s)$ is an increasing function with U as an upper bound. Hence $\mathrm{Lim}_{s\to+\infty} I(s)$ *exists* and $\leqslant U$. We need only to show that it is *equal* to U. Suppose to the contrary that $\mathrm{Lim}_{s\to+\infty} I(s) = a < U$. Let $\delta > 0$ be a positive real number which is *much smaller* than $(U - a)$. Then, there exists a sufficiently large s_0 such that

$$a - \delta < I(s) \leqslant a \quad \text{for all } s \geqslant s_0.$$

Observe that, for any given constant, $c < U$, the differential equation

$$I = x^{q-1}\dot{y} - \frac{p+q-1}{q}x^q = c$$

determines a family of solution curves which are periodic and translationally invariant in the direction of y. Moreover, there exists a positive constant depending only on c, say $k(c)$, such that the above curve contains two intervals satisfying

$$|\dot{x}| = |\cos\sigma| \geqslant k(c)$$

within each period. Let $T(C)$ be the period (in y) and $l(c)$ be the arc length of a single period. Since the given global solution curve γ is closely approximated by a *suitable* solution curve of $I = a$ for any stretch of length $l(a)$ after s_0, it is not difficult to estimate the increment of $I(s)$ along such a stretch of γ as follows.

Let $\gamma[s_1, s_2] = \{(x(s), y(s)); s_1 \leqslant s \leqslant s_2\}$ be an interval of γ such that

(i) $|\dot{x}(s)| \geqslant k_1$ for all $s_1 \leqslant s \leqslant s_2$,

(ii) $|x(s_1) - x(s_2)| \geqslant k_2$ where k_1, k_2 are two positive constants only depending on a. Let Y be the maximal of $\{y(s), s_1 \leqslant s \leqslant s_2\}$. Then, it follows from Proposition 4 that

$$\begin{aligned}\Delta I = I(s_2) - I(s_1) &= \int_{s_1}^{s_2} \frac{dI}{ds}\,ds = \int_{s_1}^{s_2} (p-1)\frac{x^{q-1}}{y}\dot{x}^2\,ds \\ &\geqslant \int_{s_1}^{s_2} \frac{(p-1)k_1}{qY}\left|\frac{d}{ds}x^q\right| ds = \frac{(p-1)k_1}{qY}\,|x(s_1)^q - x(s_2)^q|\,.\end{aligned}$$

It follows from the above simple estimate of ΔI that

$$\mathop{\mathrm{Lim}}_{s\to+\infty} I(s) - I(s_0) \geqslant c' \sum_{j=0}^{\infty} \frac{1}{Y_0 + jT},$$

where c', Y_0 and T are positive constants. This is clearly a contradiction to the boundedness of $I(s)$ because the series of the right-hand side diverges. Hence $\operatorname{Lim}_{s\to+\infty} I(s)$ must be equal to U. Almost the same proof will show that $\operatorname{Lim}_{s\to-\infty} J(s) = L$.

3. The geometry of global solutions of (II)

In this section, we shall study some basic geometric properties of the global solution curves of (II). It was proved in §2 that every global solution curve γ approaches $x = (q-1)/(p+q-1)$ (resp. $y = (p-1)/(p+q-1)$) asymptotically as $s \to +\infty$ (resp. $s \to -\infty$). Moreover, a global solution curve γ has at most one cusp point on the x-axis (resp. the y-axis). Suppose s_1 is such a cusp point of γ. We shall define the direction function, $\sigma(s)$, of γ in such a way that it has a jump of $+\pi$ at s_1. Then $\operatorname{Lim}_{s\to+\infty}\sigma(s)$ and $\operatorname{Lim}_{s\to-\infty}\sigma(s)$ both exist. We shall simply denote them by $\sigma_\gamma(+\infty)$ and $\sigma_\gamma(-\infty)$ respectively.

Definition. The total change of direction of γ is defined to be

$$\Delta\sigma(\gamma) = \sigma_\gamma(+\infty) - \sigma_\gamma(-\infty).$$

Remark. It follows from Corollary 1 of §2 that $\Delta\sigma(\gamma)$ is equal to $(2n\pi - \pi/2)$ for a suitable integer n. We shall call it the *winding number* of γ and denote it by $n(\gamma)$.

Lemma 3. *Let γ be a given global solution curve. If $I(s_1) > 0$ (resp. $J(s_2) < 0$) then*

$$|\sigma_\gamma(+\infty) - \sigma_\gamma(s_1)| < \frac{\pi}{2} \quad \left(\text{resp. } |\sigma_\gamma(-\infty) - \sigma_\gamma(s_2)| < \frac{\pi}{2}\right).$$

Proof. By Proposition 4, $I(s) \geqslant I(s_1) > 0$ for all $s > s_1$, (resp. $J(s) \leqslant J(s_2) < 0$ for $s < s_2$). Hence $\sin\sigma(s) > 0$ for all $s > s_1$, (resp. $\cos\sigma(s) < 0$ for all $s < s_2$) and Lemma 3 immediately follows.

Remark. Suppose $\gamma[s_2, s_1]$ is an interval of γ with $I(s_1) > 0$ and $J(s_2) < 0$. Then $n(\gamma)$ is uniquely determined by $\sigma(s_1) - \sigma(s_2)$, namely,

$$\begin{aligned}\left|(2n(\gamma)\cdot\pi - \frac{\pi}{2} - [\sigma(s_1) - \sigma(s_2)]\right| &\leqslant |\sigma_\gamma(+\infty) - \sigma(s_1)| \\ &\quad + |\sigma_\gamma(-\infty) - \sigma(s_2)| < \pi.\end{aligned}$$

Examples. (1) Let γ_i, $i = 1, 2, 3$, be the unique global solution curve with a cusp point at $((q-1)/(p+q-1), 0)$, $(0, (p-1)/(p+q-1))$ and $(0,0)$ respectively. Then $n(\gamma_i) = 1$.

(2) Let γ be the unique global solution curve with $x(0) = g(0) = \delta$ and $\sigma(0) = 3\pi/4$, $\delta < \operatorname{Min}\{p/((p+q-1)\sqrt{2}), q/((p+q-1)\sqrt{2})\}$. Then $I(0) > 0$ and $J(0) < 0$ and hence $n(\gamma) = 0$.

Lemma 4. *To any positive integer N, there exists a global solution curve γ with $n(\gamma) \geqslant N$.*

Proof. We may assume that $N > 1$. Let (x_0, y_0) be a point with $x_0, y_0 > 2(N+1)$ and γ be the unique global solution curve with $\sigma_0 = 3\pi/4$. Let $s_2 < 0$ be the first point of γ with $y(s_2) = 1\frac{1}{2}$ and $s_1 > 0$ be the first point with $x(s_1) = 1\frac{1}{2}$, namely $\gamma[s_2, s_1]$ is the interval of γ which lies completely within the region $x \geqslant 1\frac{1}{2}$ and $y \geqslant 1\frac{1}{2}$. It is not difficult to show that

$$\dot{\sigma}(s) = (p+q-1) + (p-1)\frac{\cos\sigma}{y} - (q-1)\frac{\sin\sigma}{x} > \begin{cases} q & \text{for } s_2 < s \leqslant 0, \\ p & \text{for } 0 \leqslant s < s_1. \end{cases}$$

Therefore, it is easy to estimate that

$$\Delta\sigma[s_2, s_1] = \sigma(s_1) - \sigma(s_2) = \int_{s_2}^{s_1} \dot{\sigma}\, ds \geqslant 2N(p+q) \geqslant 8N.$$

Moreover, it is easy to see that

$$\dot{\sigma} = \begin{cases} -(q-1)\cdot\dfrac{I}{x^q} + (p-1)\dfrac{\dot{x}}{y} + \dfrac{p+q-1}{q}, \\[2ex] (p-1)\cdot\dfrac{J}{y^p} - (q-1)\dfrac{\dot{y}}{x} + \dfrac{p+q-1}{p}. \end{cases}$$

Hence, if $I \leqslant 0$, $J \geqslant 0$ and $\dot{\sigma} \leqslant 0$, then $\dot{x} < 0$ and $\dot{y} > 0$. Combining the above fact with Lemma 3, it is quite straightforward to show that

$$\Delta\sigma(\gamma) \geqslant \Delta\sigma[s_2, s_1] - 2\pi \geqslant 8N - 2\pi,$$

which clearly implies that $n(\gamma) \geqslant N$. q.e.d.

Next let us study various deformations among solution curves.

Deformation of type 1. Let $\beta(t)$, $0 \leqslant t \leqslant 1$, be a C^2-curve in the interior of the orbit space, i.e., consists of no points of the x or y-axis, and $\mathbf{V}(t)$ be a C^2-vector field of unit length along $\beta(t)$. By the existence and uniqueness theorem of ordinary differential equations, there exists a unique family of global solution curves $\gamma_t(s)$ such that

$$\gamma_t(0) = \beta(t) \quad \text{and} \quad \dot{\gamma}_t(0) = \mathbf{V}(t), \qquad 0 \leqslant t \leqslant 1.$$

We shall call the above continuous family of solution curves a deformation of type 1.

Deformation of type 2. By Proposition 5, there exists a unique global solution curve $\gamma_u(s)$ (resp. $\beta_u(s)$) with $\gamma_u(0) = (u, 0)$ (resp. $\beta_u(0) = (0, u)$). Now, let u vary between an interval $[a, b]$, $a > 0$. Then one obtains a continuous family of global solution curves $\{\gamma_u(s);\ a \leqslant u \leqslant b\}$ (resp. $\{\beta_u(s);\ a \leqslant u \leqslant b\}$). We shall call such a family a deformation of type 2.

It is obvious that two global solution curves with cusp point on the x-axis (resp. the y-axis) are linked by a deformation of type 2. It is not difficult to show that two arbitrarily given global solution curves γ_0 and γ_1 can always be linked by a deformation of type 1.

A deformation is called a C^1-deformation if $\sigma(\gamma_t(s))$ is continuous in t for each fixed value of s_0 if $\gamma_t(s_0)$ are regular points and $\gamma_t(s_0)$ are all *cusp points* of γ_t on the x-axis (resp. the y-axis) if it is the case for $\gamma_0(s_0)$.

Lemma 5. *Let $\gamma_t(s)$ be a deformation of type* 1 *and $[s_2, s_1]$ be a given finite interval containing* 0. *If the point set*

$$U\gamma_t[s_2, s_1] = \{\gamma_t(s);\ s \in [s_2, s_1],\ t \in [0,1]\}$$

contains no boundary point, then $\gamma_t[s_2, s_1]$ is a C^1-deformation and hence $\Delta\sigma\{\gamma_t[s_2, s_1]\}$ is a continuous function of t.

Proof. Since the point set $U\gamma_t[s_2, s_1]$ is compact and assumed to be away from the boundary, it is of a finite distance away from the boundary, say $x, y \geqslant c > 0$. Within the region $\Omega_c = \{(x, y);\ x, y \geqslant c\}$, equation (II) satisfies the Lipschitz condition and hence Lemma 5 follows from the standard estimate for solutions of systems of ordinary differential equations satisfying the Lipschitz condition.

Lemma 5′. *Let $\gamma_u(s)$, $u \in [a, b]$, be a deformation of type* 2, *i.e.*, $\gamma_u(0) = (u, 0)$ (*resp.* $\gamma_u(0) = (0, u)$), *and* $[0, s_1]$ (*resp.* $[s_2, 0]$) *be a given finite interval. If the point set*

$$U\gamma_u[0, s_1] = \{\gamma_u(s),\ a \leqslant u \leqslant b,\ 0 < s \leqslant s_1\}$$

(*resp.* $U\gamma_u[s_2, 0) = \{\gamma_u(s),\ a \leqslant u \leqslant b,\ s_2 \leqslant s < 0\}$ *contains no boundary point, then* $\gamma_u[0, s_1]$ (*resp.* $\gamma_u[s_2, 0]$) *is a C^1-deformation. In particular,* $\Delta\sigma\{\gamma_u[0, s_1]\}$ (*resp.* $\Delta\sigma\{\gamma_u[s_2, 0]\}$) *is a continuous function of u.*

Proof. The two cases of the above lemma are essentially the same. We shall only show the case with initial points on the x-axis. it follows from the analytical dependence of $\gamma_u(s)$ on the parameter u (cf. Proposition 5) that there exists a sufficiently small $\delta > 0$ such that $\beta(u) = \gamma_u(\delta)$, $a \leqslant u \leqslant b$, is an analytic curve and $\mathbf{V}(u) = \dot{\gamma}_u(\delta)$ is an analytic vector field of unit length along $\beta(u)$. Hence, Lemma 5′ follows directly from Lemma 5.

Proposition 6. *Let $\gamma_t(s)$ be the family of global solution curves of Lemma* 5. *If no curve of the above family contains any boundary point, then $\Delta\sigma(\gamma_t)$ is a constant.*

Proof. It is not difficult to see that $I(\gamma_t(s)) = I(s, t)$ and $J(\gamma_t(s)) = J(s, t)$ are both continuous functions of (s, t). Moreover, there exists sufficiently large finite intervals $[s_2, s_1]$, $s_2 < 0 < s_1$, such that $I(s_1, t) \geqslant 0$ and $J(s_2, t) \leqslant 0$ for all $t \in [0, 1]$. Therefore, Proposition 6 follows from Lemma 3 and Lemma 5.

Proposition 6′. *Let $\gamma_u(s)$ be the family of global solution curves of Lemma 5′. If no curve of the above family contains any more boundary point other than the initial point, then $\Delta\sigma(\gamma_u)$ is a constant.*

Proof. It follows from Lemma 3 and Lemma 5′ in the same way as that of Proposition 6.

Proposition 7. *Let $\gamma_t(s)$ be the family of global solution curves of Lemma 5. Suppose that γ_0 is the only one which contains any boundary point and $\gamma_0(s_0) = (x_0, 0)$ (resp. $(0, y_0)$) is the only boundary point of γ_0. Then either $n(\gamma_t)$ is still a constant or $n(\gamma_t) = n(\gamma_0) - 1$ for $0 < t \leqslant 1$.*

Proposition 7′. *Let $\gamma_u(s)$ be the family of global solution curves of Lemma 5′. Suppose that γ_0 is the only one which contains another boundary point. Then either $n(\gamma_u)$ is still a constant or $n(\gamma_u) = n(\gamma_0) - 1$.*

Proof. The proofs of the above two propositions are quite similar. We shall only show that of Proposition 7 as follows: By Proposition 6, one needs only to show that

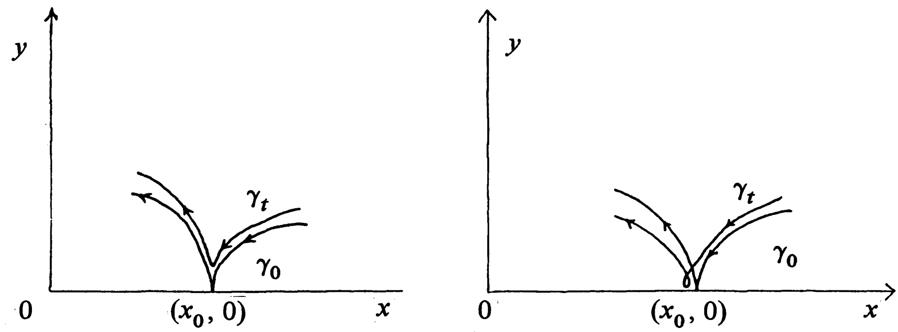
$$n(\gamma_t) = n(\gamma_0) \quad \text{or} \quad n(\gamma_0) - 1$$

for a sufficiently small $t > 0$. Let $(x_0, 0) = \gamma_0(s_0)$ be the cusp point of γ_0 and W be a sufficiently small neighborhood of $(x_0, 0)$ so that each curve γ_t, $t > 0$, can have *at most* one point with $\cos\sigma = 1$. Such a neighborhood W exists because it is not difficult to establish a *lower bound* (only depending on x_0) of the distance between two consecutive points on a given solution curve γ with $\cos\sigma = 1$ and are close to $(x_0, 0)$. [By estimating a lower bound of ΔJ between two such points on a given γ.] Since there exists a Lipschitz constant K_δ for the equation (II) over the region $\Omega_\delta = \{x, y \geqslant \delta\}$ such that $K_\delta \to \infty$ as $\delta \to 0$, it is not difficult to show that $\gamma_t(s) \to \gamma_0(s)$ as $t \to 0$ even for $s = s_0$. Therefore, there are only the following two possibilities.

(i) There exists a sufficiently small neighborhood, W, of $(x_0, 0)$ and a sufficiently small $\varepsilon > 0$ such that γ_t has *no point* with $\cos\sigma = 1$ in W for each $0 < t < \varepsilon$. In this case, it is easy to see that $n(\gamma_t) = n(\gamma_0) - 1$.

(ii) There exist a sufficiently small neighborhood, W, of $(x_0, 0)$ and a sufficiently small $\varepsilon > 0$ such that γ_t has *exactly one point* with $\cos\sigma = 1$ in W (for each $0 < t < \varepsilon$). Then, it is easy to show that each γ_t, t sufficiently small, has exactly one loop in W and the size of this loop tends to zero as $t \to 0$. Hence $n(\gamma_t) = n(\gamma_0)$ in this case.

Geometrically, the above two cases correspond to the following two kinds of deformations:



Summarizing the above discussion, we shall classify the global solution curves of (II) into the following five types, namely,

Type A. Global solution curves with no cusp point.

Type B. Global solution curves with exactly one cup point on the x-axis.

Type C. Global solution curves with exactly one cusp point on the y-axis.

Type D. Global solution curves with two cusp points (which must be exactly one on each axis).

Type E. Global solution curves with one cusp point at the origin [conjecture: there is only one curve of type E].

Theorem 2. *To every integer $k > 0$, there exist global solution curves of* (II) *with winding number $n(\gamma) = k$ which are respectively of type* A, B, C *and* D. ($k > 1$ *in the case of* D.)

Corollary 2. *There exist infinitely many distinct immersions of S^{p+q-1} into E^{p+q} with constant mean curvature* 1 *which are $O(p) \times O(q)$-invariant.*

Proof of Corollary 2. Let γ be a global solution curve of D with $n(\gamma) = k$. Then the segment of γ between its two cusp points consists of $(k - 2)$ loops and the inverse image of such a segment is automatically an immersion of S^{p+q-1} into E^{p+q} with constant mean curvature 1.

Remark. The special case of $p = q = 2$ of the above corollary was announced in [7]. A different proof for the case of $p = q$ of the above corollary was given in [8].

Corollary 3. *The inverse images of curves* (*or segments of curves*) *of type* A, B *or* C *provide infinite varieties of interesting examples of complete immersions of*

$$\mathbf{R}^1 \times S^{p-1} \times S^{q-1}, \quad \mathbf{R}^p \times S^{q-1} \quad \text{or} \quad \mathbf{R}^q \times S^{p-1}$$

into E^{p+q} with constant mean curvature 1.

Proof of Theorem 2. (1) By Lemma 4, there exists a global solution curve γ with $n(\gamma) \geqslant k + 2$. On the other hand, those examples following Lemma 3 show that there exist curves of type A with winding number 0 and curves of type B or C with winding number 1. Let γ_1 be a global solution curve whose winding number $n(\gamma_1) \geqslant k + 2$ and γ_0 be a curve of type A with zero winding number. One may *link* γ_1 to γ_0 by a *deformation of type* 1, say $\{\gamma_t, 0 \leqslant t \leqslant 1\}$. By Propositions 6, 6′, 7 and 7′, the function of winding numbers $n(\gamma_t) = f(t)$ is an integral valued step function with jumps of absolute value 1. Moreover, a decrease (resp. increase) of 1 occurs only at the critical stage of the deformation when a cusp point suddenly pops up (resp. vanishes).

Suppose γ_1 is of type A. Then there must exist a curve γ_{t_0} in the above family which is of type B, C or D and $n(\gamma_{t_0}) \geqslant k + 2$. If γ_{t_0} is of type B or C, then it is easy to use a deformation of type 2 to obtain another curve of type D with winding number $\geqslant k + 2$. If γ_{t_0} is of type D, then one may again use a deformation of type 2 to obtain curves of type B (resp. C) whose winding numbers $\geqslant k + 1$. This proves the existence of curves of type B, C and D respectively whose winding numbers are at least $(k + 1)$, k can be arbitrarily large.

(2) Let β be a curve of type D and $n(\beta) = N$. Then, one may use deformation of type 2 to show the existence of a curve of type B, C or D respectively whose winding number is any given integer k less than N.

(3) Let β_k be a curve of type B and $n(\beta_k) = k$. Let $(x_0, 0)$ be the cusp point of β_k. Let (x_1, y_1) be a point on β_k sufficiently close to the cusp point $(x_0, 0)$ and $\mathbf{V}_1$ be a direction at (x_1, y_1) which is sufficiently close to the direction of β_k at (x_1, y_1). Then, it is not difficult to show that the unique solution curve γ_k of (II) with (x_1, y_1) as its initial point and $\mathbf{V}_1$ as its initial direction must be a curve of type A and $n(\gamma_k) = k$ or $k - 1$ (cf. Proposition 7). This completes the proof of Theorem 2.

Concluding Remarks. (1) *C^1-deformation.* If two global solution curves γ_0 and γ_1 can be linked by a C^1-deformation, then it is obvious that γ_0 and γ_1 must be of the same type and have the same winding number. We conjecture that the converse is also true, namely,

Conjecture. Two global solution curves of (II) can be linked by a C^1-deformation if and only if they are of the same type and have the same winding number.

(2) *Rigidity and uniqueness.* In this paper, we prove the *existence* of infinitely many $O(p) \times O(q)$-invariant, constant mean curvature immersions of S^{p+q-1} into E^{p+q} whose generating curves contain a different number of loops. Therefore, in the euclidean space of dimension $n \geqslant 4$, there are $[\frac{n}{2}] - 1$ different types of constant mean curvature immersions of S^{n-1} into E^n. It is

rather natural to ask the questions of *uniqueness* and *rigidity* about the above family of constant mean curvature immersions of S^{n-1} into E^n.

Conjecture. To each decomposition of n into $p + q = n$; $p, q \geqslant 2$, and to each integer $k \geqslant 0$, there exists a *unique* (up to congruences) *immersed* S^{n-1} *of constant mean curvature* 1 which is invariant under an isometry group of $O(p) \times O(q)$-type and whose generating curve contains exactly k loops.

Conjecture. All the above examples of constant mean curvature immersions of S^{n-1} in E^n are *rigid*.

(3) *Problem of symmetry.* Let γ be a global solution curve of type C and γ' be the segment of γ with $I(s) \leqslant 0$. For example, the straight line $y = (p-1)/(p+q-1)$, $x \geqslant 0$ is a special example of such a curve. The inverse image of the above straight line is a q-cylinder, $Z^q((p-1)/(p+q-1))$, consisting of points of distance $(p-1)/(p+q-1)$ to $(\mathbf{R}^q, 0)$ in $\mathbf{R}^q \oplus \mathbf{R}^p$. By Theorem 1, the inverse image of γ' is clearly an example of constant mean curvature immersion of $\mathbf{R}^q \times S^{p-1}$ which is $O(p)$-invariant and is asymptotic to the above $Z^q((p-1)/(p+q-1))$ at infinity. It is rather natural to ask the following converse problem.

Problem of symmetry. Let M be an $O(p)$-invariant immersion of $\mathbf{R}^q \times S^{p-1}$ in E^{p+q} which is of constant mean curvature 1 and is asymptotic to $Z^q((p-1)/(p+q-1))$ at infinity. Is it true that M is necessarily also $O(q)$-invariant?

(4) *Generalized rotational hypersurfaces in symmetric spaces.* Suppose M is a Riemannian manifold and G is an isometric transformation group (not necessarily compact) of M with codimension two principal orbit type. Then, it is quite natural to consider G-invariant hypersurfaces of M as a type of *generalized rotational hypersurfaces* in M. For example, if M is a global symmetric space, then usually there are various types of such transformation groups and the study of generalized rotational hypersurfaces in symmetric spaces will surely lead to interesting results and basic understandings of the geometry of symmetric spaces.

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UNIVERSITY OF CALIFORNIA, BERKELEY