J. DIFFERENTIAL GEOMETRY
17 (1982) 307–335

A REGULARITY THEORY FOR HARMONIC MAPS

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0. Introduction

In this paper we develop a regularity theory for energy minimizing harmonic maps into Riemannian manifolds. Let $u\colon M^n \to N^k$ be a map between Riemannian manifolds of dimension n and k. It was shown by C. B. Morrey [17] in 1948 that if $n = 2$, then an energy minimizing harmonic map is Hölder continuous (and smooth if M and N are smooth). Since that time results have been found under special assumptions on N. Eells and Sampson [5] proved in 1963 that if N is compact and has nonpositive curvature, then every homotopy class of maps from a closed manifold M into N has a smooth harmonic representative. In the case where the image of the map is contained in a convex ball of N, there is a complete existence and regularity theory due to Hildebrandt and Widman [15] as well as Hildebrandt, Kaul and Widman [13]. Recently Giaquinta and Giusti obtained results for the case in which the image lies in a coordinate chart [9], [10].

In this paper we show that a bounded, energy minimizing map $u\colon M^n \to N^k$ is regular (in the interior) except for a closed set $\mathcal{S}$ of Hausdorff dimension at most $n - 3$. We also show $\mathcal{S}$ is discrete for $n = 3$. Moreover, we derive techniques (see Theorem IV) for lowering the dimension of $\mathcal{S}$ under the condition that certain smooth harmonic maps of spheres into N are trivial. This can be checked in some interesting cases, for example if N has nonpositive curvature or if the image of the map lies in a convex ball of N, we show $\mathcal{S} = \varnothing$ and any minimizing harmonic map into such a manifold is smooth. Using our methods, it is possible to reduce the dimension of $\mathcal{S}$ if N is a sphere or Lie group by studying harmonic spheres in N. Our methods work for functionals which are the energy plus lower order terms, and thus have direct bearing on the question of the existence of global Coulomb gauges in nonabelian gauge theories.

We point out that there is a strong historical precedent for partial regularity results in problems involving elliptic systems (see Almgren [1], De Giorgi [3],

Received January 26, 1982. Research of the first author was supported by the Sloan Foundation.

Giusti and Miranda [11] and Morrey [19]). Moreover, the types of singularities which arise in this problem have been observed in connection with other elliptic systems by De Giorgi [4] and others [11], and explicitly for harmonic maps by Hildebrandt, Kaul and Widman [13]. We also observe that the method we use for reducing the dimension of $\mathcal{S}$ is taken from H. Federer [7].

The problem which we deal with in this paper is to prove the regularity of vector-valued functions which minimize the ordinary Dirichlet integral subject to a family of smooth nonlinear (manifold) constraints. We use comparison maps as was done by Morrey for $n = 2$. The major difference is that for $n > 2$ one cannot localize the problem in the image. Finding comparison maps which satisfy the constraints is a major technical difficulty of this paper. If one assumes that the image is a bounded domain in Euclidean space (without constraints), the comparison construction is straightforward. Our methods work for a restricted class of functionals because we need a scaling inequality (see Proposition 2.4) to construct maps satisfying the constraints. In a second paper we carry out a similar program to obtain complete boundary regularity for solutions to the Dirichlet problem. The reason is that in this case the obstruction to regularity is a harmonic map of a hemisphere which is constant on the boundary, and one can show that such a map is trivial. This generalizes previous results by Hildebrandt and Widman [15] and R. Hamilton [12] on the boundary value problem.

1. Statement of results

Let M^n and N^k be Riemannian manifolds of dimension n and k. For technical reasons we assume that $N \subset R^k$ is isometrically embedded in Euclidean space. We assume throughout that M is compact, possibly with boundary, and that N is an open manifold. For $r \geqslant 0$ let $C^r(M, N)$ be the space of maps u: $M \to N$ which have continuous derivatives through order r, so that $C^r(M, N) \subseteq C^r(M, R^k)$ is a Banach submanifold. Likewise let $C^{r,\alpha}(M, N)$ for $\alpha \in (0, 1]$ denote the subset of $C^r(M, N)$ whose rth derivatives are Hölder continuous with exponent α.

In order to discuss harmonic maps from M to N, we work in the separable Hilbert space $L_1^2(M, R^k)$, the set of maps u: $M \to R^k$ whose component functions have first derivatives in L^2. By $L_{1,0}^2(M, R^k)$ we mean those L_1^2 maps which are zero on ∂M. Define

$$L_1^2(M, N) = \{u \in L_1^2(M, R^k) \colon u(x) \in N \text{ a.e. } x \in M\}.$$

Note that if $\dim M = 1$, then $L_1^2(M, N)$ is a Hilbert submanifold of $L_1^2(M, R^k)$, but this is not so for $\dim M > 1$. The set $L_1^2(M, N)$ inherits strong and weak topologies from $L_1^2(M, R^k)$. Moreover, the space $L_1^2(M, N)$ is a strongly closed subset with the additional property that the set $\{u \in L_1^2(M, N): \|u\|_{1,2} \leq C\}$ is weakly compact in $L_1^2(M, R^k)$.

For $u \in L_1^2(M, R^k)$, the energy functional is given by

$$E(u) = \int_M \langle du(x), du(x) \rangle \, dV = \int_M e(u).$$

Here the Lagrangian $e(u)$ is given in local coordinates by

$$e(u) = \sum_{\alpha,\beta} \sum_i g^{\alpha\beta} \frac{\partial u^i}{\partial x^\beta} \frac{\partial u^i}{\partial x^\alpha} (\det g_{\gamma\delta})^{1/2} \, dx,$$

where $g_{\alpha\beta}$ is the metric tensor of M. The norm on $L_1^2(M, R^k)$ is then given by

$$\|u\|_{1,2}^2 = E(u) + \int_M \sum_i (u^i(x))^2 \, dV,$$

where dV is the volume element of M. A harmonic map $u: M \to N$ is a weak solution of the Euler-Lagrange equation for E in $L_1^2(M, N)$ (see (2.1)).

For certain applications of our results we will need to consider critical points of functionals with additional lower order terms. Let $\tilde{E}(u) = E(u) + V(u)$ where $V(u) = \int_M v(u)$. In local coordinates, $v(u)$ will have the form

$$v(u) = \left[\sum_i \sum_\alpha \gamma_i^\alpha(x, u(x)) \frac{\partial u^i}{\partial x^\alpha}(x) + \Gamma(x, u(x)) \right] dV.$$

More invariantly, if $\mathcal{O}$ is an open neighborhood of N in R^k, then $\gamma \in C^r(M \times \mathcal{O}, T^*M \otimes R^k)$ and $\Gamma \in C^r(M \times \mathcal{O}, R)$. By an *$\tilde{E}$-minimizing map* we mean a map $u \in L_1^2(M, N)$ such that $\tilde{E}(u) \leq \tilde{E}(w)$ for any map $w \in L_1^2(M, N)$ with $(u - w) \in L_{1,0}^2(M, R^k)$. Throughout the paper we will assume that the metric on M is C^2 and $\gamma, \Gamma \in C^r$ for $r \geq 2$. Our first result is a "sufficiently small" type result. The explicit dependence on parameters is derived in detail in Theorem 3.1. Here $B_\sigma(a)$ is the geodesic ball about a in M.

Theorem I. *Let $u \in L_1^2(B_\sigma(a), N)$ be an $\tilde{E}$-minimizing map such that $u(x) \in N_0$ a.e. for some compact subset $N_0 \subseteq N$. If σ and $\sigma^{2-n}\tilde{E}(u)$ are sufficiently small, then u is Hölder continuous on $B_{\sigma/2}(a)$.*

It is well known, but difficult to find in the literature (see [2]), that u is smooth in the interior of $B_{\sigma/2}(a)$ once we have that u is continuous there.

For an $\tilde{E}$-minimizing map u, a point $x \in M$ is a *regular point* if u is continuous in a neighborhood of x. Let $\mathcal{R} = \mathcal{R}(u)$ be the set of all regular points and $\mathcal{S} = \mathcal{S}(u)$ be the complement of $\mathcal{R}$ in the interior of M. The *singular set* $\mathcal{S}$ is then obviously a closed subset of $\operatorname{int}(M)$. The reader should refer to [6,2.10.2] for a discussion of Hausdorff dimension.

Theorem II. *Let $u \in L_1^2(M, N)$ be $\tilde{E}$-minimizing with $u(x) \in N_0$ a.e. for a compact set $N_0 \subseteq N$. It then follows that $\dim(\mathcal{S} \cap \operatorname{int} M) \leqslant n - 3$ where $n = \dim M$ and $\dim A$ is the Hausdorff dimension of a set A. If $n = 3$, then $\mathcal{S}$ is a discrete set of points.*

We state the next results as a theorem only to motivate our results. Recall that a harmonic map is a weak solution to the Euler-Lagrange equations for E on $L_1^2(M, N)$. Suppose $u \in L_{1,\mathrm{loc}}^2(R^n, N)$ is a map such that $\partial u/\partial r = 0$ a.e. Then there is a map $w\colon S^{n-1} \to N$ such that $u(x) = w(x/|x|)$, and it is easy to see that u is harmonic if and only if w is harmonic. In fact $E(u \mid B_\sigma(0)) = (n-2)^{-1}\sigma^{n-2}E(w)$. Moreover, the map u has a singularity at 0 if and only if w is not a constant map.

Theorem III. *Let $u \in L_1^2(M, N)$ be an $\tilde{E}$-minimizing map and let $z \in \mathcal{S} \cap \operatorname{int} M$. There exists a sequence $\sigma_i \in R^+$, $\sigma_i \to 0$ such that the maps $u_i \in L_1^2(B_1(0), N)$, $u_i(x) = u(\exp_z \sigma_i x)$ converge to $u \in L_1^2(B_1(0), N)$. The map u is a nonconstant harmonic map satisfying $u(x) = w(x/|x|)$, $w \in L_1^2(S^{n-1}, N)$ harmonic.*

A homogeneous harmonic map with an isolated singularity at 0 will be referred to as a *tangent map* (TM). A tangent map which is $\tilde{E}$-minimizing on compact subsets of R^n is a *minimizing tangent map* (MTM).

Theorem IV. *Suppose there is an integer $l \geqslant 3$ such that every* MTM *from $R^j \to N$ is trivial, $3 \leqslant j \leqslant l$. Then if $u \in L_1^2(M, N)$ is $\tilde{E}$-minimizing with $u(x) \in N_0$ a.e., then $\dim(\mathcal{S} \cap \operatorname{int} M) \leqslant n - l - 1$. If $n = l + 1$, then $\mathcal{S}$ is a discrete set of points, and if $n < l + 1$, $\mathcal{S} = \varnothing$.*

This theorem has the following corollary which is closely related to the work of Eells and Sampson [5] and Hildebrandt, Kaul and Widman [13].

Corollary. *If the sectional curvature of N is nonpositive or if $u(M)$ is contained in a strictly convex ball of N, then $\mathcal{S} = \varnothing$; that is, any $\tilde{E}$-minimizing map $u \in L_1^2(M, N)$ is smooth.*

Proof. To prove the corollary from Theorem IV, it suffices to show that any tangent map $R^j \to N$ for $j \geqslant 3$ is trivial; that is, any smooth harmonic $u\colon S^{j-1} \to N$ is trivial. But this is elementary because if N has nonpositive curvature, we can lift u to a map $\tilde{u}\colon S^{j-1} \to \tilde{N}$ where $\tilde{N}$ is the universal cover of N. Since the square of the distance function ρ to a point is strictly convex we have that $\rho^2 \circ u$ is a2.1 subharmonic function on S^{j-1} which is hence constant. Thus u is constant. The same argument works if $u(M)$ is contained in a convex ball of N. This proves the corollary.

2. The Euler-Lagrange equation and scaling inequalities

We compute the Euler-Lagrange equation for $\tilde{E}$ and show that $\tilde{E}$-minimizing maps are weak solutions of this equation. As in the previous section, N_0 will be compact subset of N.

Lemma 2.1. *If u is $\tilde{E}$-minimizing on M and $u(x) \in N_0$ a.e., then u satisfies the formal Euler-Lagrange equations for E. These equations have the form*

$$(2.1) \qquad \Delta_M u - A(du, du) + \sum_{i,\alpha} B_{i,\alpha}(x, u(x)) \frac{\partial u^i}{\partial x^\alpha} + C(x, u(x)) = 0,$$

where A, B, C are smooth in their arguments, and A is quadratic in du.

Proof. We will say that a map is stationary for $\tilde{E}$ if $d\tilde{E}(u(t))/dt\,|_{t=0}$ for any differentiable curve of maps $u\colon (-\varepsilon, \varepsilon) \to L_1^2(M, R^k)$ where $u(t) \in L_1^2(M, N)$, $u(0) = u$, and $u(t) - u \in L_{1,0}^2(M, R^k)$. Of course it is true that an $\tilde{E}$-minimizing map is stationary. We will show that the space of admissible variations $\{\psi = u'(0)\colon u(t)$ satisfies the above condition$\}$ is large enough so that u satisfies the Euler-Lagrange equation for $\tilde{E}$. Note that it does not make sense to discuss differentiable curves in $L_1^2(M, N)$ per se because $L_1^2(M, N)$ does not have a local smooth structure.

Let $\mathcal{O}$ be an open neighborhood of N in R^k such that the map $\Pi\colon \mathcal{O} \to N$, given by $\Pi(y) =$ nearest point in N to y, is a smooth fibration. Since N_0 is a compact set, $\mathcal{O}$ contains a uniform neighborhood of N_0. Thus for t sufficiently small and any C^∞ R^k-valued function φ which is zero on ∂M, we can define

$$u(t)(x) = \pi(u(x) + t\varphi(x)).$$

Observe that for $x \in M$ the curve $t \mapsto f(t) = u(t)(x)$ is smooth for small t and $f'(t) = \varphi(X) d\Pi(u(x) + t\varphi(x))$ is uniformly bounded. Moreover, $du(t)(x) = (du(x) + td\varphi(x)) \cdot d\pi(u(x) + t\varphi(x))$ is smooth in t for almost all x. We find

$$\|u'(t)\|_{1,2} = \|\varphi(x) \cdot d\Pi(u(x) + t\varphi(x))\|_{1,2} \leq c(\varphi)[1 + \|du\|_{1,2}],$$

and $u(t)$ is differentiable in $L_1^2(M, R^k)$. Note that

$$u'(t)(x)\,|_{t=0} = \varphi(x) \cdot d\Pi_{u(x)},$$

where $d\Pi_y$ is the tangential projection $R^k \to T_y N$ for $y \in N$. Let

$$\mathcal{E}(u) = -2\Delta_M u - d_M^* \gamma \circ u + (d_N \gamma \cdot du) + d_N \Gamma$$

be the expression for the Euler-Lagrange equation of the unconstrained problem. Then we have

$$\frac{d}{dt}\tilde{E}(u(t))\,|_{t=0} = \int_M \langle \mathcal{E}(u), \Pi_{u(x)}(\varphi(x)) \rangle \, dV$$

$$= \int_M \langle \Pi_{u(x)}(\mathcal{E}(u)), \varphi(x) \rangle \, dV = 0$$

for all φ. We find that, in a distributional sense, $d\Pi_{u(x)}\mathcal{E}(u(x)) = 0$. Because

$$d\Pi_{u(x)}(\Delta_m u) = \Delta_M u - d^2\pi_{u(x)}(du, du) = \Delta_M u - A(du, du),$$

(A is the second fundamental form of N in R^k), the equations can be put in the required form. We are not really interested in the exact form of the lower order terms. This completes the proof of Lemma 2.1.

Our theorems will be proved by covering M with geodesic coordinate balls and proving regularity for $\tilde{E}$-minimizing maps on balls. Let $B_1 = B_1^n(0)$ be the unit ball in R^n. For $\Lambda > 0$, let $\mathcal{F}_\Lambda$ denote the class of functionals $\tilde{E}$ on B_1 with metric $g_{\alpha\beta}$ such that $g_{\alpha\beta}(0) = \delta_{\alpha\beta}$ and lower order terms satisfying, for $x \in B_1$, $u \in N_0$,

$$\sum_{\alpha,\beta,\tau}\left|\frac{\partial}{\partial x^\tau} g_{\alpha\beta}(x)\right| + |\gamma(x,u)| + |d_u\gamma(x,u)| + |\Gamma(x,u)|^{1/2} + |d_u\Gamma(x,u)|^{1/2} \leq \Lambda. \tag{2.2}$$

If u is $\tilde{E}$-minimizing for $\tilde{E} \in \mathcal{F}_\Lambda$ and $u(x) \in N_0$ a.e., we say $u \in \mathcal{H}_\Lambda$. The lower order terms are handled by showing that Λ is a dimensional constant which shrinks with the radius of a coordinate ball. In fact, if $\tilde{E}$ is a functional and $B_\sigma(p)$ is a geodesic ball in M of radius σ centered at $p \in M$, we define a functional $\tilde{E}^{p,\sigma}$ on B_1 by setting

$$\begin{aligned}\tilde{E}^{p,\sigma}(w) &= \int_{B_1}\left(|dw|^2_{g_\sigma} + \sigma(dw\cdot\gamma(y,w)) + \sigma^2\Gamma(y,w)\right)g_\sigma^{1/2}\,dy \\ &= \sigma^{2-n}\tilde{E}_{B_\sigma(p)}(u),\end{aligned} \tag{2.3}$$

where $w(y) = u(\sigma y)$ and $g_\sigma(y) = g(\sigma y)$. Since M and N_0 are compact, we can choose Λ so that $\tilde{E}^{p,\sigma} \in \mathcal{F}_\Lambda$ for all p and some $\sigma > 0$. It follows that if $\tilde{E}^{p,\sigma} \in \mathcal{F}_\Lambda$, then $\tilde{E}^{p,\lambda\sigma} \in \mathcal{F}_{\lambda\Lambda}$ for any $\lambda \in (0,1]$. We state the following.

Lemma 2.2. *Given $\Lambda > 0$, there exists $\sigma_0 > 0$ such that for $0 < \sigma \leq \sigma_0$ and $p \in M$, if u is $\tilde{E}$-minimizing, then $w(y) = u(\exp_p \sigma y)$ is $\tilde{E}^{p,\sigma}$-minimizing where $\tilde{E}^{p,\sigma} \in \mathcal{F}_\Lambda$.*

Thus we restrict our attention to $\tilde{E} \in \mathcal{F}_\Lambda$ where Λ is small. Let E be the energy functional in the Euclidean metric on B_1. Let E_σ, $\tilde{E}_\sigma$ denote energies taken over B_σ, $0 < \sigma \leq 1$. The inequalities

$$\begin{aligned}|E_\alpha(u) - \tilde{E}_\sigma(u)| &\leq c\Lambda\left(\sigma E_\alpha(u) + \sigma^{n/2}E_\alpha(u)^{1/2} + \Lambda\sigma^n\right) \\ &\leq \tfrac{3}{2}c\Lambda\left(\sigma E_\alpha(u) + \sigma^{n-1}\right)\end{aligned}$$

are straightforward provided $\Lambda\sigma \leqslant 1$. Consequently for $\sigma \in (0, 1]$ we have

$$\tilde{E}_\sigma(u) \leqslant (1 + \bar{c}\Lambda\sigma)E_\alpha(u) + \bar{c}\Lambda\sigma^{n-1},$$
$$E_\alpha(u) \leqslant (1 + \bar{c}\Lambda\sigma)\tilde{E}_\sigma(u) + \bar{c}\Lambda\sigma^{n-1},$$

provided $c\Lambda \leqslant \frac{1}{2}$. We have the following.

Lemma 2.3. *If Λ is sufficiently small, and u is $\tilde{E}$-minimizing for $\tilde{E} \in \mathcal{F}_\Lambda$ (i.e., $u \in \mathcal{H}_\Lambda$), then there exists a constant $c = c(n) > 0$ such that for $\sigma \in (0, 1]$*

$$E_\alpha(u) \leqslant (1 + c\Lambda\sigma)E_\alpha(w) + c\Lambda\sigma^{n-1}$$

for any $w \in L_1^2(B_1, N)$ with $w = u$ on $B_1 \sim B_\sigma$.

Proof. Since $\tilde{E}_\sigma(u) \leqslant \tilde{E}_\sigma(w)$, this follows directly from the above inequalities.

We can now prove the first basic inequality of the paper. We use the notation

$$E_\sigma^x(u) = \int_{B_\sigma(x)} |du|^2(y)\,dy.$$

Proposition 2.4. *Let $u \in \mathcal{H}_\Lambda$ for Λ sufficiently small. Then we have*

$$\sigma^{2-n}E_\sigma^x(u) \leqslant c\left[\rho^{2-n}E_\rho^x(u) + \Lambda\rho\right]$$

for $x \in B_{1/2}$, $0 < \sigma \leqslant \rho \leqslant \frac{1}{2}$.

Proof. By rescaling as discussed above, we can work on B_1 instead of $B_{1/2}(x)$. This will introduce at worst an extra multiplicative factor. For almost all $\sigma \in (0, 1]$ we have $\int_{|x|=\sigma} |du|^2\,d\xi < \infty$ where ξ is a variable on the sphere. Introduce the comparison map

$$v_\sigma(x) = u(x), \qquad |x| \geqslant \sigma,$$
$$v_\sigma(x) = u(\sigma x/|x|), \quad |x| \leqslant \sigma.$$

Since the result is trivial for $n = 2$, we assume $n > 2$. Denote by $|d_\xi u|^2$ the tangential energy along the spheres $|x| = r$, so that $|du|^2 = |d_\xi u|^2 + |\partial u/\partial r|^2$. We compute

$$E_\alpha(v_\sigma) = (n-2)^{-1}\sigma\int_{|x|=\sigma} |d_\xi u|^2\,d\xi$$
$$= (n-2)^{-1}\sigma\left(\frac{d}{d\sigma}E_\alpha(u) - \int_{|x|=\sigma}\left|\frac{\partial u}{\partial r}\right|^2 d\xi\right).$$

From Lemma 2.3 we get, with $\bar{c} = c\Lambda$,

$$E_\alpha(u) \leqslant (1 + \bar{c}\sigma)E_\alpha(v_\sigma) + \bar{c}\sigma^{n-1}$$
$$\leqslant (n-2)^{-1}\sigma(1 + \bar{c}\sigma)\left[\frac{d}{d\sigma}E_\alpha(u) - \int_{|x|=\sigma}\left|\frac{\partial u}{\partial r}\right|^2 d\xi\right] + \bar{c}\sigma^{n-1}.$$

This implies

$$(2.4)\qquad 0 \leqslant \sigma^{2-n}\int_{|x|=\sigma}\left|\frac{\partial u}{\partial r}\right|^2 d\xi \leqslant \frac{d}{d\sigma}\left[(1+\bar{c}\sigma)^{n-2}\sigma^{2-n}E_\alpha(u)\right] + \bar{\bar{c}}.$$

Since $E_\alpha(u)$ is a nondecreasing function, we can integrate this inequality from σ to ρ

$$(2.5)\quad (1+\bar{c}\sigma)^{n-2}\sigma^{2-n}E_\alpha(u) \leqslant (1+\bar{c}\rho)^{n-2}\rho^{2-n}E_\rho(u) + \bar{\bar{c}}(\rho-\sigma),$$

where we have discarded the radial derivative term. This implies directly the conclusion of Proposition 2.4. Note that $\bar{\bar{c}}$ is a constant times Λ.

If we take the radial derivative term into consideration in the above argument we can prove more. Note that if we set $u_\lambda(x) = u(\lambda x)$ for $\lambda \in (0,1]$, then as in (2.3) we have $u_\lambda \in \mathcal{H}_{\lambda\Lambda}$ for $u \in \mathcal{H}_\Lambda$, and

$$(2.6)\qquad E_1(u_\lambda) = \lambda^{2-n}E_\lambda(u).$$

Lemma 2.5. *There is a sequence* $\lambda(i) \to 0$, $\lambda(i) \in (0,1]$, *such that* $u_{\lambda(i)}$ *converge weakly in* $L_1^2(B_1, N)$ *to a limiting map* $u_0 \in L_1^2(B_1, N)$. *The map* u_0 *is a harmonic map satisfying* $\partial u_0/\partial r = 0$ *a.e. in* B_1.

Proof. From the previous result and (2.6), $E_1(u_\lambda)$ is bounded for $\lambda \in (0,1]$, and therefore we get a weakly convergent sequence $u_{\lambda(i)} \to u_0 \in L_1^2(B_1, N)$. Since $u_{\lambda(i)} \in \mathcal{H}_{\lambda(i)\Lambda}$, it satisfies Euler equations of the form (2.1).

It follows easily that u_0 satisfies the Euler-Lagrange equation for E and hence u_0 is harmonic. To see that $\partial u_0/\partial r = 0$ a.e., first note that (2.5) implies the existence of a number L_0 with

$$(2.7)\qquad L_0 = \lim_{\sigma\to 0}\sigma^{2-n}E_\alpha(u) = \lim_{\sigma\to 0}E_1(u_\sigma).$$

If we integrate (2.4) from 0 to λ keeping the radial derivative term, we have

$$\int_{B_\lambda} r^{2-n}\left|\frac{\partial u}{\partial r}\right|^2 dx \leqslant \left[(1+\bar{c}\lambda)^{n-2}\lambda^{2-n}E_\lambda(u) - L_0\right] + \bar{\bar{c}}\lambda.$$

By a change of variables

$$\int_{B_1} r^{2-n}\left|\frac{\partial u_\lambda}{\partial r}\right|^2 dx = \int_{B_\lambda} r^{2-n}\left|\frac{\partial u}{\partial r}\right|^2 dx.$$

Therefore we have

$$\lim_{\lambda\to 0}\int_{B_1} r^{2-n}\left|\frac{\partial u_\lambda}{\partial r}\right|^2 dx = 0,$$

which implies that $\partial u_0/\partial r = 0$ a.e., for any weak limit u_0. This proves Lemma 2.5.

In §4 we will show that this convergence is actually norm convergence. To do that will require some preliminary regularity results. The first is (see Theorem 3.1)

Regularity Estimate 2.6. *There exists $\bar{\varepsilon} > 0$ depending only on n and $N_0 \subseteq N$ such that if $u \in \mathcal{H}_\Lambda$, $\Lambda \leq \varepsilon$, and $E_1(u) \leq \bar{\varepsilon}$, then u is Hölder continuous on $B_{1/2}$ and satisfies $|u(x) - u(y)| \leq c|x - y|^\alpha$ for $x, y \in B_{1/2}$ where $c, \alpha > 0$ depend only on n, N_0.*

We first note that Theorem I follows immediately from this result by rescaling, see Lemma 2.2. Secondly, we can immediately prove the corollary.

Corollary 2.7. *If $u \in L_1^2(B_1, N)$ is in $\mathcal{H}_\Lambda$ and $\mathcal{S}$ is the singular set of u, then $\mathcal{H}^{n-2}(\mathcal{S} \cap B_{1/2}) = 0$. More generally, if $u \in L_1^2(M, N)$ is $\tilde{E}$-minimizing, then $\mathcal{H}^{n-2}(\mathcal{S} \cap \operatorname{int} M) = 0$.*

Proof. By rescaling, the second statement follows from the first. For $x \in \mathcal{S} \cap B_{1/2}$, choose normal coordinates y centered at x. Let $u_{x,\lambda}(y) = u(\exp_x \lambda y)$. Then the maps $u_{x,\lambda}$ are in $\mathcal{H}_{\lambda\Lambda}$ (see Lemma 2.2). By the regularity estimate we have

$$\bar{\varepsilon} \leq E_1(u_{x,\lambda}) = \lambda^{2-n}\int_{B_\lambda(x)} |du|^2\, dx \tag{2.8}$$

for all $x \in \mathcal{S} \cap B_{1/2}$, $\lambda\Lambda \leq \bar{\varepsilon}$. For $\delta \in (0, \bar{\varepsilon}/\Lambda)$, let $\{B_\delta(x_1), \ldots, B_\delta(x_l)\}$ be a maximal family of $l = l(\delta)$ disjoint balls of radius δ with center $x_i \in \mathcal{S} \cap B_{1/2}$. By maximality, $\mathcal{S} \cap B_{1/2} \subseteq \bigcup_{j=1}^{l} B_{2\delta}(x_i)$. Applying (2.8) on each ball and summing, we get

$$l\delta^{n-2} \leq \bar{\varepsilon}^{-1}\int_{\bigcup_i B_\delta(x_i)} |du|^2\, dx \leq \bar{\varepsilon}^{-1}E(u). \tag{2.9}$$

Since $\mathcal{S} \cap B_{1/2} \subseteq \bigcup_i B_{2\delta}(x_i)$, we see that $\mathcal{H}^{n-2}(\mathcal{S} \cap B_{1/2}) \leq cE(u)$. In particular $\mathcal{H}^n(\bigcup_i B_\delta(x_i)) \leq \bar{c}\delta^2 E(u)$, and by the dominated convergence theorem

$$\lim_{\delta \to 0} \int_{\bigcup_i B_\delta(x_i)} |du|^2\, dx = 0.$$

Using this in (2.9) then shows $\mathcal{H}^{n-2}(\mathcal{S} \cap B_{1/2}) = 0$.

3. The ϵ-regularity theorem

In this section we prove regularity of minimizing maps under the assumption that the energy is small. Precisely we prove

Theorem 3.1. *There exists a constant $\bar{\varepsilon} = \bar{\varepsilon}(n, N_0)$ such that if $u \in \mathcal{H}_\Lambda$, $\Lambda \leq \bar{\varepsilon}$, and $E_1(u) \leq \bar{\varepsilon}$, then u is Hölder continuous on $B_{1/2}$ and $|u(x) - u(y)| \leq c|x - y|^\alpha$ for $x, y \in B_{1/2}$ where $\alpha = \alpha(n) > 0$ and $c = c(n, N_0)$.*

We will prove this theorem by establishing energy decay estimates on small balls. We first show that if $\bar{\varepsilon}$ is chosen small, it is possible to approximate u by smooth maps into N. To see this we let $\varphi: R^n \to R^+$ be a smooth radial mollifying function so that support $(\varphi) \subseteq B_1$ and $\int_{R^n} \varphi(x)\, dx = 1$. We then note that if $u^* = \int_{B_1} \varphi(x) u(x)\, dx$, we can apply a version of the Poincaré inequality to assert

$$\int_{B_1} | u - u^* |^2\, dx \leq c_1 E_1(u) \leq c_1 \bar{\varepsilon}.$$

(Throughout this section $c_1, c_2, \cdots$ will denote constants depending only on n, N_0.) This inequality implies in particular that u^* lies near many values of $u(x)$ for $x \in B_1^n$. Hence in particular we see that $\operatorname{dist}(u^*, N) \leq c_2(\bar{\varepsilon})^{1/2}$. This inequality gains in power when it is combined with the scaling inequality Proposition 2.4, for we can apply it on the ball $B_h^n(x)$ for any $x \in B_{1/2}^n$, $0 < h \leq \frac{1}{4}$. That is, we apply it to the scaled map $u_{x,h}: B_1^n \to N$ given by

$$u_{x,h}(y) = u(x - hy).$$

By Proposition 2.4 we have

$$E_1(u_{x,h}) = h^{2-n} E_{B_h(x)}(u) \leq c_3 E_1(u) + c_3 \bar{\varepsilon} \leq c_4 \bar{\varepsilon},$$

provided $x \in B_{1/2}$, $h \in (0, \frac{1}{4}]$. Thus if we set

$$u^{(h)}(x) = \int_{B_1^n} \varphi(y) u(x - hy)\, dy = \int_{B_1^n} \varphi^{(h)}(x - z) u(z)\, dz$$

where $\varphi^{(h)}(x) = h^{-n} \varphi(x/h)$, we have

$$\operatorname{dist}\left(u^{(h)}(x), N_0\right) \leq c_5 \bar{\varepsilon}^{1/2} \tag{3.1}$$

for any $x \in B_{1/2}$, $h \in (0, \frac{1}{4}]$. Let $\mathcal{O}$ be a normal neighborhood of N in R^k, and let $\Pi: \mathcal{O} \to N$ denote the smooth nearest point projection map. Since N_0 is compact, $\mathcal{O}$ contains a uniform neighborhood of N_0. By (3.1), if $\bar{\varepsilon}$ is chosen small we will have $u^{(h)}(x) \in \mathcal{O}$ for all $x \in B_{1/2}$, and we can define a smooth map $u_h: B_{1/2} \to N$ by $u_h(x) = \Pi \circ u^{(h)}(x)$. We note the following result.

Lemma 3.2. *Let $\bar{h} = \bar{\varepsilon}^{1/4}$, and suppose $h \in (0, \bar{h}]$. Then we have*

$$\int_{B_{1/2}} | du^{(h)} |^2\, dx \leq c_6 E_1(u),$$

$$\sup_{x \in B_{1/2}} | u^{(\bar{h})}(x) - u^{(\bar{h})}(0) |^2 \leq c_6 \bar{\varepsilon}^{1/2}.$$

Proof. The first inequality is standard. To prove the second, observe

$$|du^{(\bar h)}|^2(x) = \left| \int_{B_1} \varphi^{(\bar h)}(x-y)du(y)\,|\,dy \right|^2$$
$$\leqslant \int_{B_1} \varphi^{(\bar h)}(x-y)\,|du|^2(y)\,dy.$$

By Proposition 2.4 this implies

$$|du^{(\bar h)}|^2(x) \leqslant c_7 \bar h^{-n} E_{B_{\bar h}(x)}(u) \leqslant c_8 \bar h^{-2}\bar\varepsilon = c_8 \bar\varepsilon^{1/2}$$

for any $x \in B_{1/2}$. This implies the second inequality and completes the proof of Lemma 3.2.

In order to compare our approximating maps to u, we must force them to agree with u on the boundary of some small ball. To achieve this we observe that inequality (3.1) is a pointwise inequality, that is, x is fixed and h arbitrarily small. Thus we can choose $h = h(x)$ and the inequality still holds for each x. Let $\tau = \bar\varepsilon^{1/8}$, and suppose $\theta \in (\tau, \frac{1}{4})$. We choose $h = h(r)$, $r = |x|$ to be a nonincreasing smooth function of r satisfying

(3.2) $\qquad h(r) = \bar h \quad \text{for } r \leqslant \theta, h(\theta+\tau) = 0, |h'(r)| \leqslant 2\bar\varepsilon^{1/8}.$

We can then set

$$u^{(h(x))}(x) = \int_{B_1} \varphi^{(h(x))}(x-y)u(y)\,dy,$$

and by (3.1), $u_{h(x)}(x) = \Pi \circ u^{(h(x))}(x)$. We can prove the following result.

Lemma 3.3. *For $\theta \in (\tau, \frac{1}{4}]$, the map u_h is in $L_1^2(B_{1/2}, N)$ and satisfies $u_h = u$ on $B_{1/2} \sim B_{\theta+\tau}$, and*

$$\int_{B_{\theta+\tau}\sim B_\theta} |du_h|^2\,dx \leqslant c_9 \int_{B_{\theta+2\tau}\sim B_{\theta-\tau}} |du|^2\,dx.$$

Proof. Since Π is a smooth map, it suffices to prove the lemma for $u^{(h)}$ instead of u_h. We first note that by a change of variable, $u^{(h)}(x)$ can be written

$$u^{(h)}(x) = \int_{B_1} \varphi(y)u(x-h(x)y)\,dy.$$

From this expression it is clear that $u^{(h)}$ is smooth if u is smooth. We first consider a smooth $u: B_{1/2} \to R^k$. If Ω is a domain compactly contained in $B_{1-\tau}$, then we compute

$$\frac{\partial u^{(h)}}{\partial x^\alpha} = \int_{B_1} \varphi(y)\left[\frac{\partial u}{\partial x^\alpha}(x-hy) - \frac{\partial h}{\partial x^\alpha} y \cdot \nabla u(x-hy)\right] dy.$$

Thus it follows that

$$\int_\Omega |du^{(h)}|^2\,dx \leqslant c_{10}\int_\Omega\int_{B_1} \varphi(y)\,|du|^2\,(x-hy)\,dydx.$$

For $x \in \Omega$, we let $z = x - h(x)y$. This defines a map $F_y\colon \Omega \to \Omega_\tau$, where we let $\Omega_\tau = \{x\colon \operatorname{dist}(x, \Omega) < \tau\}$. By (3.2) we see that F_y is a diffeomorphism onto $F_y(\Omega)$ with Jacobian approximately one. Thus by change of variables we can estimate

$$\int_\Omega |du|^2\,(x - h(x)y)\,dx \leqslant 2\int_{\Omega_\tau} |du|^2\,dx.$$

Therefore we have

$$\int_\Omega |du^{(h)}|^2\,dx \leqslant c_{11}\int_{\Omega_\tau} |du|^2\,dx. \tag{3.3}$$

Now if u_i is a sequence of smooth maps $B_{3/4} \to R^k$ converging strongly to u in $L_1^2(B_{3/4}, N)$, then (3.3) implies

$$\int_{B_{1/2}} |du_i^{(h)} - du_j^{(h)}|^2\,dx \leqslant c_{11}\int_{B_{3/4}} |du_i - du_j|^2\,dx,$$

and hence $\{u_i^{(h)}\}$ is Cauchy in $L_1^2(B_{1/2}, R^k)$. From this it follows that $\lim_{i\to\infty} u_i^{(h)} = u^{(h)}$, and $u^{(h)} = u$ on $B_{1/2} \sim B_{\theta+\tau}$. We can then apply (3.3) with $\Omega = B_{\theta+\tau} \sim B_\theta$ to get the conclusion of Lemma 3.3.

In order to prove Theorem 3.1, it suffices by Morrey's Lemma [16,2.4.1] to prove that

$$r^{2-n}E_{B_r(x)}(u) \leqslant c_{12}r^{2\alpha}$$

for any $x \in B_{1/2}$, $r \in (0, \frac{1}{4}]$. We will prove that if $\bar{\varepsilon}$ is sufficiently small, then we have

$$r^{2-n}E_r(u) \leqslant c_{12}r^{2\alpha} \tag{3.4}$$

for $r \in (0, \frac{1}{2}]$. The previous estimate can then be gotten by reapplying (3.4) with varying center point. We now state a result which is a discrete version of (3.4).

Proposition 3.4. *There exists $\bar{\varepsilon} = \bar{\varepsilon}(n, N_0) > 0$ such that if $u \in \mathcal{H}_\Lambda$, $\Lambda \leqslant \bar{\varepsilon}$, and $E_1(u) \leqslant \bar{\varepsilon}$, then we have*

$$\bar{\theta}^{2-n}E_{\bar{\theta}}(u) + \bar{\theta}\Lambda \leqslant \tfrac{1}{2}(E_1(u) + \Lambda)$$

for some $\bar{\theta} = \bar{\theta}(n, N_0) \in (0, 1)$.

Proof of Theorem 3.1. We show how Theorem 3.1 follows from Proposition 3.4. We will prove (3.4) by an iterative procedure. Observe that the scaled map

$u_{\bar\theta}(x) = u(\bar\theta x)$ lies in $\mathcal{K}_{\bar\theta\Lambda}$ and from Proposition 3.4 we have

$$E_1(u_{\bar\theta}) = \bar\theta^{2-n}E_{\bar\theta}(u) \leqslant \bar\varepsilon.$$

Thus Proposition 3.4 is applicable to $u_{\bar\theta}$, and we get

$$\bar\theta^{2-n}E_{\bar\theta}(u_{\bar\theta}) + \bar\theta^2\Lambda \leqslant \tfrac{1}{2}\big(E_1(u_{\bar\theta}) + \bar\theta\Lambda\big).$$

But this is the same as

$$(\bar\theta^2)^{2-n}E_{\bar\theta^2}(u) + \bar\theta^2\Lambda \leqslant \tfrac{1}{2}\big(\bar\theta^{2-n}E_{\bar\theta}(u) + \bar\theta\Lambda\big).$$

Applying Proposition 3.4 on the right we get

$$(\bar\theta^2)^{2-n}E_{\bar\theta^2}(u) + \bar\theta^2\Gamma L \leqslant \left(\tfrac{1}{2}\right)^2\big(E_1(u) + \Lambda\big).$$

Repeating this argument i times we get

$$(\bar\theta^i)^{2-n}E_{\bar\theta^i}(u) + \bar\theta^i\Lambda \leqslant 2^{-i}\big(E_1(u) + \Lambda\big)$$

for any nonnegative integer i. Given any $r \in (0, 1)$, there is an integer i so that $r \in [\bar\theta^{i+1}, \bar\theta^i]$. Setting $\alpha = (\log 2)/(2\log\bar\theta^{-1})$, we have

$$(\bar\theta^i)^{2-n}E_r(u) + r\Lambda \leqslant (\bar\theta^i)^{2\alpha}\big(E_1(u) + \Lambda\big)$$

for any $r \in (0, 1)$. This implies

$$r^{2-n}E_r(u) \leqslant \bar\theta^{2-n-2\alpha}r^{2\alpha}\big(E_1(u) + \Lambda\big),$$

which verifies (3.4). As we have observed, this proves Theorem 3.1.

Proof of Proposition 3.4. Let v be the solution of the linear Dirichlet problem

$$\Delta v = 0, \quad \text{in } B_{1/2},$$
$$v = u^{(\bar h)} \quad \text{on } \partial B_{1/2}.$$

Thus $v\colon B_{1/2} \to R^k$ is a smooth harmonic map. We observe the following properties of v. First we note that by Lemma 3.2, $u^{(\bar h)}(B_{1/2}) \subseteq B^k_{c\bar\varepsilon^{1/4}}(u^{(\bar h)}(0))$. Thus it follows that the image of v is also contained in this ball. In particular we have

$$\sup_{B_{1/2}} |v - u^{(\bar h)}|^2 \leqslant c_{13}\bar\varepsilon^{1/2}. \tag{3.5}$$

The mean value inequality for subharmonic functions implies the result

$$\sup_{B_{1/2}} |dv|^2 \leqslant c_{14}\int_{B_{1/2}} |dv|^2.$$

Since v minimizes energy on $B_{1/2}$ for its boundary, we have

$$\int_{B_{1/2}} |dv|^2 \leqslant \int_{B_{1/2}} |du^{(\bar h)}|^2 \leqslant E_1(u)$$

by Lemma 3.2. Therefore, we have

$$\sup_{B_{1/4}} |dv|^2 \leq c_{15} E_1(u). \tag{3.6}$$

For any $\theta \in (0, \frac{1}{4}]$ we can estimate

$$\begin{aligned} \theta^{2-n} E_\theta(u_{\bar{h}}) &\leq c_{16} \theta^{2-n} E_\theta(u^{(\bar{h})}) \\ &\leq 2c_{16}\theta^{2-n} \int_{B_\theta} \left\{ |d(u^{(\bar{h})} - v)|^2 + |dv|^2 \right\} dx, \end{aligned} \tag{3.7}$$

where we have used the inequality (3.1) and the smoothness of Π. By (3.6) we see that

$$\theta^{2-n} \int_{B_\theta} |dv|^2 \leq c_{17} \theta^2 E_1(u) \tag{3.8}$$

for any $\theta \in (0, \frac{1}{4})$. Integrating by parts we have

$$\int_{B_{1/2}} |d(u^{(\bar{h})} - v)|^2 = -\int_{B_{1/2}} (u^{(\bar{h})} - v) \cdot \Delta(u^{(\bar{h})} - v).$$

Using (3.5) and the harmonic property of v we get

$$\int_{B_{1/2}} |d(u^{(\bar{h})} - v)|^2 \leq c_{18} \bar{\varepsilon}^{1/4} \int_{B_{1/2}} |\Delta u^{(\bar{h})}|. \tag{3.9}$$

The Euler equation (2.1) for u tells us

$$\begin{aligned} \Delta u^{(\bar{h})}(x) &= \int_{R^n} \left[\Delta_x \varphi^{(\bar{h})}(x-y) \right] u(y)\, dy \\ &= \int_{R^n} \left[\Delta_y \varphi^{(\bar{h})}(x-y) \right] u(y)\, dy \\ &= \int_{R^n} \varphi^{(\bar{h})}(x-y) [A(du, du) - B\, du - C]\, dy. \end{aligned}$$

From the form of B, C we conclude

$$|\Delta u^{(\bar{h})}(x)| \leq c_{19} \int_{R^n} \varphi^{(\bar{h})}(x-y) \left[|du|^2 + \Lambda \right] dy.$$

Integrating over $x \in B_{1/2}$ we finally have

$$\int_{B_{1/2}} |\Delta u^{(\bar{h})}| \leq c_{19}(E_1(u) + \Lambda).$$

Combining this with (3.7), (3.8), and (3.9) gives

$$\theta^{2-n} E_\theta(u_{\bar{h}}) \leq c_{20} \theta^{2-n} \bar{\varepsilon}^{1/4} (E_1(u) + \Lambda) + c_{20} \theta^2 E_1(u) \tag{3.10}$$

for any $\theta \in (0, \frac{1}{4})$.

Let $\gamma_n \in (0, 1/16]$ be a number to be chosen depending only on n, and let $\bar{\theta} = \bar{\varepsilon}^{\gamma_n}$. Let p be the greatest integer less than or equal to $\bar{\theta}/(3\tau)$ where $\tau = \bar{\varepsilon}^{1/8}$ and write

$$[\bar{\theta}, \bar{\theta} + 3p\tau] = \bigcup_{i=1}^{p} I_i, \qquad |I_i| = 3\tau,$$

where each I_i is a closed interval of length 3τ. Since $\gamma_n \leq \frac{1}{16}$, we have $p \geq \frac{1}{3}\bar{\varepsilon}^{-1/16} - 1$. We have

$$\int_{r \in [\theta, \theta + 3p\tau]} |du|^2 \, dx = \sum_{i=1}^{p} \int_{r \in I_i} |du|^2 \, dx \leq E_1(u).$$

Thus we can choose an interval I_j for some j with $1 \leq j \leq p$ such that

$$\int_{r \in I_j} |du|^2 \, dx \leq p^{-1} E_1(u) \leq c_{21} \bar{\varepsilon}^{1/16} E_1(u). \tag{3.11}$$

Let θ be the number such that $I_j = [\theta - \tau, \theta + 2\tau]$, and let $h(x)$ be as in Lemma 3.3. Thus $u_{h(x)}(x) \in L_1^2(B_{1/2}, N)$ and satisfies $u_h = u$ for $r \geq \theta + \tau$, and

$$\int_{r \in [\theta, \theta + \tau]} |du_h|^2 \leq c_{22} \int_{r \in I_j} |du|^2.$$

Thus by (3.11) we have

$$\int_{r \in [\theta, \theta + \tau]} |du_h|^2 \leq c_{23} \bar{\varepsilon}^{1/16} E_1(u). \tag{3.12}$$

By Lemma 2.3 we have

$$E_{\theta+\tau}(u) \leq c_{24} E_{\theta+\tau}(u_h) + c_{24} \Lambda \bar{\theta}^{n-1},$$

since $\theta + \tau \leq 2\bar{\theta}$. By (3.12) this implies

$$E_\theta(u) \leq c_{25} E_\theta(u_{\bar{h}}) + c_{25} \bar{\varepsilon}^{1/16} E_1(u) + c_{25} \Lambda \bar{\theta}^{n-1}.$$

Combining this with (3.10) and using $\theta \in [\bar{\theta}, 2\bar{\theta}]$ we have

$$\bar{\theta}^{2-n} E_\theta(u) \leq c_{26} (\bar{\theta}^{2-n} \bar{\varepsilon}^{1/16} + \bar{\theta}^2)(E_1(u) + \Lambda_1).$$

Since $\bar{\theta} = \bar{\varepsilon}^{\gamma_n}$ this gives

$$\bar{\theta}^{2-n} E_{\bar{\theta}}(u) \leq c_{27} (\bar{\varepsilon}^{1/16 - \gamma_n(n-2)} + \bar{\varepsilon}^{2\gamma_n})(E_1(u) + \Lambda).$$

We choose $\gamma_n = \min\{[32(n-2)]^{-1}, 64^{-1}\}$ and hence

$$\bar{\theta}^{2-n} E_{\bar{\theta}}(u) \leq c_{27} \bar{\varepsilon}^{2\gamma_n}(E_1(u) + \Lambda).$$

This implies

$$\bar{\theta}^{2-n}(E_{\bar{\theta}}(u) + \theta\Lambda) \leq (c_{27} + 1)\bar{\varepsilon}^{2\gamma_n}(E_1(u) + \bar{\theta}\Lambda).$$

Choosing $\bar{\varepsilon}$ so small that $(c_{27}+1)\bar{\varepsilon}^{2\gamma_n} \leqslant \frac{1}{2}$ finishes the proof of Proposition 3.4.

4. Extension and compactness theorems

In this section we study convergence of $\tilde{E}$-minimizing maps. This study involves the construction of lots of comparison maps. Our basic tool is the cone-type comparison already used in the proof of Proposition 2.4. For a fixed point $u^* \in R^k$, we introduce the notation

$$W_\sigma(u) = \int_{B_\sigma} |u - u^*|^2\, dx.$$

This notation will be used throughout this section. Recall that $E_\Omega(u)$ denotes the energy taken over a region $\Omega \subseteq R^n$; likewise $W_\Omega(u) = \int_\Omega |u - u^*|^2\, dx$. Let $C_\sigma^n = B_\sigma^{n-1} \times [-\sigma, \sigma]$ be the cylinder of height and diameter 2σ.

Lemma 4.1. *Let $u \in L_1^2(\partial C_\sigma^n, N)$ be given such that $u(x, -\sigma) = u_1(x)$, $u(x, \sigma) = u_2(x)$ for $x \in B_\sigma^{n-1}$ with $u_1, u_2 \in L_1^2(B_\sigma^{n-1}, N)$. Suppose also that $u(x,t) = {}^\circ u(x)$ for $(x,t) \in S_\sigma^{n-2} \times [-\sigma, \sigma]$. In particular we have $u^1 = u^2 = {}^\circ u$ on $\partial B_\sigma^{n-1} = S_\sigma^{n-2}$ with ${}^\circ u \in L_1^2(S_\sigma^{n-2}, N)$. Then there exists an extension $\bar{u} \in L_1^2(C_\sigma^n, N)$, $\bar{u} = u$ on ∂C_σ^n, satisfying the inequalities*

$$E(\bar{u}) \leqslant c\sigma\big(E_\sigma(u_1) + E_\sigma(u_2) + \sigma E({}^\circ u)\big),$$
$$W(\bar{u}) \leqslant c\sigma\big(W_\sigma(u_1) + W_\sigma(u_2) + \sigma W({}^\circ u)\big).$$

Proof. By scaling the domain we assume $\sigma = 1$. The easiest proof of this lemma is to observe that there exists a bi-Lipschitz homeomorphism $f\colon \partial B_1^n \to \partial C_1^n$ which extends to a bi-Lipschitz homeomorphism $\bar{f}\colon B_1^n \to C_1^n$ where $\bar{f}(x) = |x| f(x/|x|)$. Let $\Pi\colon B_1^n \sim \{0\} \to \partial B_1^n$ be the radial projection, i.e., $\Pi(x) = x/|x|$. Define a projection map $\hat{\Pi}\colon C_1^n \sim \{(0,0)\} \to \partial C_1^n$ by $\hat{\Pi} = f \circ \Pi \circ \bar{f}^{-1}$. Then define $\bar{u} = u \circ \hat{\Pi}$. As in Proposition 2.4

$$E(\bar{u} \circ \bar{f}) \leqslant (n-2)^{-1} E(u \circ f).$$

Due to Lipschitz equivalence

$$E(\bar{u}) \leqslant K E(\bar{u} \circ \bar{f}), \qquad E(u \circ f) \leqslant K E(u)$$

with constant K depending on the Lipschitz constants of $\bar{f}^{-1}$ and f. These inequalities and a similar argument for W yields the results of Lemma 4.1.

Since our main extension lemma, Lemma 4.3, will be proved by induction on dimension, we first give the result for $n = 2$ where the proof is quite different and the result much simpler.

Lemma 4.2. *If* $u \in L_1^2(S_\sigma^1, N)$ *and* $E(u)W(u) \leqslant \delta_1^2$ *for a number* $\delta_1 = \delta_1(N_0)$, *then there exists* $\bar{u} \in L_1^2(B_\sigma^2, N)$ *with* $\bar{u}|_{\partial B_\sigma^2} = u$ *and*

$$E_\sigma(\bar{u}) \leqslant c_1(E(u)W(u))^{1/2}, \qquad W_\sigma(\bar{u}) \leqslant c_1\sigma W(u).$$

Proof. As usual we take $\sigma = 1$. Let $\delta^2 = E(u)W(u)$ so that $\delta^2 \leqslant \delta_1^2$. If S^1 is parametrized by $\theta \in [0, 2\pi)$, then we have

$$|u(\theta) - u^*|^2 \leqslant 2\int_0^{2\pi} |u(\theta) - u^*||u'(\theta)|\,d\theta \leqslant 2\delta.$$

Thus if δ_1 is small, then $\min|u - u^*|$ is small. Let $\bar{u}$ be the (Morrey [18,5.4]) harmonic map minimizing E for boundary values given by u. Since δ_1 is small, the boundary values of $\bar{u}$ lie in a convex ball (see Hildebrandt, Kaul and Widman [13]), so it follows that $\|u - u^*\|_\infty \leqslant c_2\delta^{1/2}$. The Euler-Lagrange equation for $\bar{u}$ is

$$\Delta\bar{u} = A_{\bar{u}}(d\bar{u}, d\bar{u}).$$

This implies the inequality (in weak form)

$$\begin{aligned}\tfrac{1}{2}\Delta|\bar{u} - u^*|^2 - |d\bar{u}|^2 &= \langle \bar{u} - u^*, A_{\bar{u}}(d\bar{u}, d\bar{u})\rangle \\ &\geqslant -\|\bar{u} - u^*\|_\infty \|A\|_\infty |d\bar{u}|^2.\end{aligned}$$

Thus if δ_1 is small, we have $\Delta|\bar{u} - u^*|^2 \geqslant 0$, so by the mean value inequality $W(\bar{u}) \leqslant \frac{1}{2}W(u)$.

To get the estimate on $E(\bar{u})$, we compare $\bar{u}$ to $\Pi \circ v$, where $v\colon B_1^2 \to R^k$ is the solution of the linear Dirichlet problem with boundary values u and Π is projection from a normal neighborhood of N_0 onto N. By standard $H_{1/2}$ norm estimates, $E(v) \leqslant c\delta^{1/2}$. Since $\bar{u}$ is minimizing, we have

$$E(\bar{u}) \leqslant E(\Pi \circ v) \leqslant c_3E(v) \leqslant c_4\delta^{1/2}.$$

This proves Lemma 4.2.

We now prove a higher dimensional version of this.

Lemma 4.3. *For* $n \geqslant 2$ *there exists* $\delta = \delta(n, N_0)$ *and a constant* $q = q(n)$ *such that if* $\varepsilon \in (0,1)$ *is given and* $u \in L_1^2(\partial B_\sigma^n, N_0)$ *satisfies* $\sigma^{4-2n}E(u)W(u) \leqslant \delta^2\varepsilon^q$ (*note that* W *depends also on a fixed vector* $u^* \in R^k$), *then there exists* $\bar{u} \in L_1^2(B_\sigma^n, N)$, $\bar{u}|_{\partial B_\sigma^n} = u$ *such that*

$$E(\bar{u}) \leqslant c_5(\varepsilon\sigma E(u) + \varepsilon^{-q}\sigma^{-1}W(u)),$$
$$W(\bar{u}) \leqslant c_5\varepsilon^{-q}\sigma W(u).$$

Proof. First note that Lemma 4.2 implies Lemma 4.3 for $n = 2$ with $q(n) = 1$ which proves the first step of our induction. Also, by rescaling we take $\sigma = 1$. The following lemma is part of our construction. We leave its proof until later.

Lemma 4.4. *Let $\sigma \in (0, \frac{1}{2})$, $A_\sigma = S^{n-1} \times [-\sigma, \sigma]$. Assume Lemma 4.3 is true for $n-1$, and let $v \in L_1^2(S^{n-1}, N)$ satisfy $E(v)W(v) \leqslant \sigma^{2n-4}(\delta')^2$ where $\delta' = \delta'(n-1, N_0)$ depends on the constants arising from Lemma 4.3 for $n-1$. Then there exists a combinatorial constant $\alpha = \alpha(n) < 1$, a constant $K = K(n)$, and a map $\bar{v} \in L_1^2(A_\sigma, N)$, $\bar{v}|_{S^{n-1}\times\{\sigma\}} = v$, $\bar{v}|_{S^{n-1}\times\{-\sigma\}} = v'$ where $v' \in L_1^2(S^{n-1}, N)$ such that*

$$\begin{aligned} E(\bar{v}) &\leqslant K\sigma E(v) + K\sigma^{-1}W(v),\\ W(\bar{v}) &\leqslant K\sigma W(v),\\ E(v') &\leqslant \sigma E(v) + K\sigma^{-2}W(v),\\ W(v') &\leqslant KW(v). \end{aligned}$$

We assume for now that Lemma 4.4 is true and proceed with the proof of Lemma 4.3. Let $\varepsilon \in (0,1)$ be given and choose an integer s with $\alpha^s \approx \varepsilon$ ($\alpha = \alpha(n)$ given in Lemma 4.4) and a cylinder of height $2\sigma = \varepsilon$. Consider the s disjoint annuli $A_{i,\sigma}$ given by

$$A_{i,\sigma} = \{x \in B_1 \colon 1 - 2i\sigma \leqslant |x| \leqslant 1 - 2(i-1)\sigma\}$$

for $i = 1, \cdots, s$. Apply Lemma 4.4 on each of the $A_{i,\sigma}$ by taking $v = v_1 = u$ on the outer boundary of $A_{1,\sigma}$ and at the ith step $v = v_i = v'_{i-1}$ where v'_{i-1} is the value on the inner boundary obtained via Lemma 4.4 at the previous step. Note that as long as $2s\sigma = \varepsilon s < \frac{1}{2}$, each annulus $A_{i,\sigma}$ is uniformly equivalent (Lipschitz) to $S^{n-1} \times [-\sigma, \sigma]$. In order to apply Lemma 4.4 we must have $E(v_i)W(v_i) \leqslant \sigma^{2n-4}(\delta')^2$, but from the $(i-1)$-st application we get

$$\begin{aligned} E(v_i) &\leqslant \alpha E(v_{i-1}) + K\sigma^{-2}W(v_{i-1}),\\ W(v_i) &\leqslant KW(v_{i-1}). \end{aligned}$$

By iteration this gives (provided $k \geqslant 2$)

$$\begin{aligned} E(v_i) &\leqslant \alpha^{i-1}E(u) + 2\sigma^{-2}K^iW(u),\\ W(v_i) &\leqslant K^{i-1}W(u). \end{aligned}$$

Thus we may continue s times provided

$$\begin{aligned} K^sE(u)W(u) &\leqslant 2^{-1}\sigma^{2n-4}(\delta')^2,\\ 2\sigma^{-2}K^{2s-1}W^2(u) &\leqslant \sigma^{2n-4}(\delta')^2 2^{-1}. \end{aligned} \tag{4.1}$$

We check these inequalities by noting

$$K^s = \alpha^{s\ln K/\ln\alpha} \approx \varepsilon^{\ln K/\ln\alpha}, \qquad \sigma = \varepsilon/2,$$

so that the first inequality is equivalent to

$$E(u)W(u) \leqslant c_6\varepsilon^{2n-4-\ln K/\ln\alpha}(\delta')^2,$$

that is, taking $q = 2n - 4 - \ln K/\ln \alpha$, $\delta(n, N_0) = c_6^{1/2}\delta'$ in the hypothesis of Lemma 4.3 for n. Now if the second inequality of (4.1) fails and the first holds, we have $K^s E(u)W(u) \leq 2\sigma^{-2}K^{2s-1}W^2(u)$ which implies $E(u) \leq 2\sigma^{-2}K^{s-1}W(u)$. In this case we do not need to apply Lemma 4.4 at all since we can extend u homogeneously into B_1 by $\bar{u}(x) = u(x/|x|)$, and we have

$$\begin{aligned} E(\bar{u}) &\leq \varepsilon E(u) + 2\sigma^{-2}K^{s-1}W(u) \\ &\leq \varepsilon E(u) + c_7\varepsilon^{-2+\ln K/\ln \alpha}W(u) \end{aligned}$$

as required.

We can thus assume that (4.1) is true and that we have applied Lemma 4.4 s times. We then let $\bar{u}|_{A_{i,\sigma}} = \bar{v}_i$, the extension obtained at the ith state from Lemma 4.4. We extend $\bar{u}$ into $B_{1-s\varepsilon}$ by setting $\bar{u}(x) = v_s'((1 - s\varepsilon)x/|x|)$. This gives a map $\bar{u} \in L_1^2(B_1, N)$ with $\bar{u}|_{\partial B_1^n} = u$. By Lemma 4.4 we have

$$\begin{aligned} E(\bar{u}\,|\,A_{i,\sigma}) &= E(\bar{v}_i) \leq K\big(\sigma E(v_i) + \sigma^{-1}W(v_i)\big) \\ &\leq K\big(\sigma\alpha^{i-1}E(u) + 3\sigma^{-1}K^iW(u)\big), \\ W(\bar{u}\,|\,A_{i,\sigma}) &= W(\bar{u}_i) \leq K\sigma W(v_i) \leq \sigma K^iW(u). \end{aligned}$$

In the ball $B_{1-s\varepsilon}^n$

$$\begin{aligned} E(\bar{u}\,|\,B_{1-s\varepsilon}^n) &\leq E(v_s) \leq \alpha^{s-1}E(u) + 2\sigma^{-2}K^sW(u), \\ W(\bar{u}\,|\,B_{1-s\varepsilon}^n) &\leq W(v_s) \leq K^{s-1}W(u). \end{aligned}$$

Adding these up we get

$$\begin{aligned} E(\bar{u}) &\leq \left(\alpha^{s-1} + K\sigma \sum_{i=1}^{s} \alpha^{i-1}\right)E(u) + c_8\sigma^{-2}K^sW(u), \\ W(\bar{u}) &\leq c_8K^sW(u). \end{aligned}$$

Since $\alpha^s + K\sigma\sum_{i=1}^s \alpha^{i-1} \leq (1 + 2K(1 - \alpha)^{-1})\varepsilon = c_9\varepsilon$ and $K^s = \varepsilon^{\ln K/\ln \alpha}$, we get the conclusion of Lemma 4.3.

Proof of Lemma 4.4. It remains to prove that Lemma 4.4 follows from Lemma 4.3 for $n - 1$. Let $x_1, \cdots, x_q$ be a maximal array of points on S^{n-1} satisfying $|x_i - x_j| \geq \sigma$ for $i \neq j$. By maximality, $S^{n-1} \subseteq \bigcup_{i=1}^q B_\sigma^{n-1}(x_i)$. For each $i \leq q$, let $\sigma_i \in [\sigma, 2\sigma]$ be chosen so that the restriction $v_i = v|_{\partial B_{\sigma_i}^{n-1}(x_i)}$ lies in $L_1^2(\partial B_{\sigma i}^{n-1}(x_i), N)$ and

$$E(v_i) \leq c_{10}\sigma^{-1}\int_{B_{2\sigma}^{n-1}(x_i)} |dv|^2\, d\mu,$$

$$W(v_i) \leq c_{10}\sigma^{-1}\int_{B_{2\sigma}^{n-1}(x_i)} |v - u^*|^2\, d\mu.$$

Let $B_i = B^{n-1}_{\sigma_i}(x_i) \subseteq S^{n-1}$. The existence of such σ_i follows from Fubini's theorem. We have

$$\sigma_i^{6-2n}E(v_i)W(v_i) \leq c_{11}\sigma^{4-2n}E(v)W(v).$$

Thus we can apply Lemma 4.3, for ε to be chosen, on B_i provided

$$c_{11}\sigma^{4-2n}E(v)W(v) \leq (\bar{\delta}\,)^2\varepsilon^{\bar{q}}, \tag{4.2}$$

where $\bar{\delta} = \delta(n-1, N_0)$, $\bar{q} = q(n-1)$. Let $v_i' \in L_1^2(B_i, N)$ be given by Lemma 4.3 satisfying

$$E(v_i') \leq c_{12}\big(\varepsilon\sigma E(v_i) + \varepsilon^{-\bar{q}}\sigma^{-1}W(v_i)\big),$$
$$W(v_i') \leq c_{12}\varepsilon^{-\bar{q}}\sigma W(v_i).$$

From the choice of σ_i this implies

$$(4.3)\qquad \begin{aligned} E(v_i') &\leq c_{13}\left(\varepsilon\int_{B^{n-1}_{2\sigma}(x_i)} |dv|^2\,d\mu + \varepsilon^{-\bar{q}}\sigma^{-2}\int_{B^{n-1}_{2\sigma}(x_i)} |v-u^*|^2\,d\mu\right),\\ W(v_i') &\leq c_{13}\varepsilon^{-\bar{q}}\int_{B^{n-1}_{2\sigma}(x_i)} |v-u^*|^2\,d\mu. \end{aligned}$$

We also observe that since any point $x \in S^{n-1}$ is contained in a bounded number (depending only on n) of balls $B^{n-1}_{2\sigma}(x_i)$, we have

$$(4.4)\qquad \begin{aligned} \sum_{i=1}^{q}\int_{B^{n-1}_{2\sigma}(x_i)} |dv|^2\,d\mu &\leq c_{14}E(v),\\ \sum_{i=1}^{q}\int_{B^{n-1}_{2\sigma}(x_i)} |v-u^*|^2\,d\mu &\leq c_{14}W(v). \end{aligned}$$

From the choice of $x_1,\cdots,x_q$ we see that there is a fixed integer I_n and families $\mathfrak{B}_1,\cdots,\mathfrak{B}_{I_n}$ of balls such that

$$\bigcup_{j=1}^{I_n}\mathfrak{B}_j = \{B_i : i = 1,\cdots,q\},$$

and each $\mathfrak{B}_j$ is comprised of a family of disjoint balls. Since $S^{n-1} \subseteq \bigcup_{i=1}^q B_i$, we have $\sum_{i=1}^q E(v \mid B_i) \geq E(v)$. Thus for some $\mathfrak{B}_j$, say $\mathfrak{B}_1$, we have

$$\sum_{B_i\in\mathfrak{B}_j} E(v \mid B_i) \geq I_n^{-1}E(v). \tag{4.5}$$

Let $\mathcal{O} = \bigcup_{B_i \in \mathfrak{B}_1} B_i$, and define the extension $\bar{v}$ on $(S^{n-1} \sim \mathcal{O} \times [-\sigma, \sigma])$ by $\bar{v}(x, t) = v(x)$. On each cylinder $B_i \times [-\sigma, \sigma]$, apply Lemma 4.1 to get $\bar{v} \in L_1^2(B_i \times [-\sigma, \sigma], N)$ satisfying $\bar{v}(x, \sigma) = v(x)$, $\bar{v}(x, -\sigma) = v_i'(x)$, and $\bar{v}(x, t) = v(x)$ for $(x, t) \in (\partial B_i) \times [-\sigma, \sigma]$. We have

$$E(\bar{v} \mid B_i \times [-\sigma, \sigma]) \leqslant c_{15}\sigma(E(v \mid B_i) + E(v_i') + \sigma E(v_i)),$$
$$W(\bar{v} \mid B_i \times [-\sigma, \sigma]) \leqslant c_{15}\sigma(W(v \mid B_i) + W(v_i') + \sigma W(v_i)),$$
$$E(\bar{v} \mid (S^{n-1} \sim \mathcal{O}) \times [-\sigma, \sigma]) = 2\sigma E(v \mid S^{n-1} \sim \mathcal{O}),$$
$$W(\bar{v} \mid (S^{n-1} \sim \mathcal{O}) \times [-\sigma, \sigma]) = 2\sigma W(v \mid S^{n-1} \sim \mathcal{O}).$$

Therefore we have by (4.3) and (4.4)

$$\begin{aligned} E(\bar{v}) &\leqslant c_{16}\sigma E(v) + c_{16}\varepsilon^{-\bar{q}}\sigma^{-1}W(v), \\ W(\bar{v}) &\leqslant c_{16}\varepsilon^{-\bar{q}}W(v). \end{aligned} \tag{4.6}$$

Let $v' = \bar{v} \mid S^{n-1} \times \{-\sigma\}$, so that by (4.3), (4.4), and (4.5)

$$\begin{aligned} E(v') &\leqslant E(v \mid S^{n-1} \sim \mathcal{O}) + \sum_{i=1}^{q} E(v_i') \\ &\leqslant (1 - I_n^{-1})E(v) + c_{17}\varepsilon E(v) + c_{17}\varepsilon^{-\bar{q}}\sigma^{-2}W(v), \\ W(v') &\leqslant W(v) + \sum_{i=1}^{q} W(v_i') \leqslant c_{17}\varepsilon^{-\bar{q}}W(v). \end{aligned}$$

We now fix ε so small that $\alpha(n) = (1 - I_n^{-1} + c_{17}\varepsilon) < 1$. This fixes all constants $E(v)W(V) \leqslant \sigma^{2n-4}(\delta')^2$ with $(\delta')^2 = c_{11}^{-1}(\bar{\delta})^2\varepsilon^{\bar{q}}$. This completes the proof of Lemma 4.4.

The main application of Lemma 4.4 is to give a significant improvement of Theorem 3.1. We now prove this.

Proposition 4.5. *Given $B > 0$ there exists a constant $\varepsilon_0 = \varepsilon_0(n, N_0, B)$ such that if $u \in \mathcal{H}_\Lambda$, $\Lambda \leqslant \varepsilon_0$, $E_1(u) \leqslant B$, and $W_1(u) \leqslant \varepsilon_0$, then u is Hölder continuous on $B_{1/2}$ and $|u(x) - u(y)| \leqslant c|x - y|^\alpha$ for $x, y \in B_{1/2}$ where $\alpha = \alpha(n) \geqslant 0$ and $c = c(n, N_0)$.*

Proof. By Fubini's theorem, there exists $\sigma \in [\frac{3}{4}, 1]$ such that

$$W(u \mid \partial B_\sigma) = \int_{\partial B_\sigma} |u - u^*|^2\, d\xi \leqslant 8W_1(u) \leqslant 8\varepsilon_0,$$
$$E(u \mid \partial B_\sigma) \leqslant 8E_1(u) \leqslant 8B.$$

Applying Lemma 4.3 on B_σ, there exists $\bar{u} \in L_1^2(B_\sigma, N_0)$ with $\bar{u} \mid \partial B_\sigma = u$ such that if $W(u \mid \partial B_\sigma) \leqslant 8^{-1}\sigma^{2n-4}\delta^2\varepsilon^q B^{-1}$

$$E_\sigma(\bar{u}) \leqslant 8c_5(\varepsilon\sigma B + \varepsilon^{-q}\sigma^{-1}\varepsilon_0).$$

Since $u \in \mathcal{H}_\lambda$, we can apply Lemma 2.3 to get

$$E_\sigma(u) \leq (1 + c\Lambda\sigma)E_\sigma(\bar{u}) + c\Lambda\sigma^{n-1} \leq c_{18}(\varepsilon B + \varepsilon^{-q}\varepsilon_0).$$

Fix ε so small that $c_{18}\varepsilon B \leq 2^{-1}\bar{\varepsilon}\sigma^{n-2}$ where $\bar{\varepsilon}$ is given in Theorem 3.1. If ε_0 is so small that $c_{18}\varepsilon^{-q}\varepsilon_0 \leq 2^{-1}\bar{\varepsilon}\sigma^{n-2}$ and $8\varepsilon_0 B \leq 8^{-1}\sigma^{2n-4}\delta^2\varepsilon^q$, then we have $\sigma^{2-n}E_\sigma(u) \leq \bar{\varepsilon}$, $\sigma\Lambda \leq \bar{\varepsilon}$ so we can apply Theorem 3.1 to assert that u is Hölder continuous on $B_{\sigma/2}$. Proposition 3.5 now follows.

We now look at weak limits of minimizing maps. We cannot show that these are again minimizing. Our first result is a compactness theorem which says that weak convergence is actually strong. The key ingredient in its proof is Proposition 4.5.

Proposition 4.6. *Let $\{u_i\} \subseteq \mathcal{H}_\Lambda$ be a weakly convergent (in L_1^2) sequence with limit u_0 such that $E_1(u_i) \leq c_{19}$. Then u_0 is locally Hölder continuous outside a closed set $\mathcal{S}_0$ with $\mathcal{H}^{n-2}(\mathcal{S}_0) = 0$. Moreover, u_i converges to u_0 in L_1^2-norm on $B_{1/2}$ and uniformly on compact subsets of $\overline{B}_{1/2} \sim \mathcal{S}_0$.*

Proof. Since the u_i have uniformly bounded energy, we may assume they converge in L^2-norm to u_0. By lower semi-continuity $E_1(u_0) \leq c_{19}$. Let $\mathcal{S}_0$ be the singular set of u_0. We prove that $\mathcal{S}_0$ is small by noting that if $W_\sigma^x(u_0) < \varepsilon_0\sigma^n$ for some $u^* \in R^k$ and $\varepsilon_0 > 0$ (ε_0 given by Proposition 4.5), by L^2-convergence we have $W_\sigma^x(u_i) < \varepsilon_0\sigma^n$ for large i. By Proposition 2.4, there exists $B > 0$ such that $\sigma^{2-n}E_\sigma^x(u_i) \leq B$ for all i. Moreover, if Proposition 4.5 is applied on a ball of radius σ, then the assumption on Λ is $\Lambda \leq \varepsilon_0\sigma^{-1}$ which is automatically satisfied for σ small. Thus we can apply Proposition 4.5 to u_i giving us a uniform Hölder estimate on u_i in $B_{\sigma/2}(x)$, and hence u_i converges uniformly on $B_{\sigma/2}(x)$ to u_0 and u_0 is Hölder continuous there. In particular, by the Poincaré inequality, if $\sigma^{2-n}E_\sigma^x(u_0)$ is small, then u_0 is Hölder continuous on $B_{\sigma/2}(x)$. Thus by the argument given in Corollary 2.7 we have $\mathcal{H}^{n-2}(\mathcal{S}_0) = 0$. We have also shown that u_i converges uniformly to u_0 on compact subsets of $\overline{B}_{1/2} \sim \mathcal{S}_0$.

To prove the L_1^2-convergence of u_i to u_0, we now observe that we can cover $\mathcal{S}_0 \cap B_{1/2}$ by a family of balls $\{B_{r_i}(x_i)\}$ such that $\sum_i r_i^{n-2} < \varepsilon$ for any $\varepsilon > 0$. If $\mathcal{O} = \bigcup_i B_{r_i}(x_i)$, we can estimate by Proposition 2.4

$$E_{\mathcal{O}}(u_j) \leq \sum_i E_{r_i}^{x_i}(u_j) \leq c_{20}\sum_i r_i^{n-2} < c_{20}\varepsilon \tag{4.7}$$

for any j. On the other hand, we have shown uniform convergence of u_j to u_0 on $B_{1/2} \sim \mathcal{O}$ so subtracting the Euler equations (2.1) for u_j and u_k, multiplying by $u_j - u_k$ and putting in a cutoff function we easily get

$$\int_{B_{1/2}\sim\mathcal{O}} |d(u_j - u_k)|^2\, dx \leq c(\mathcal{O}, \Lambda) \sup_{\overline{B}_{1/2}\sim\mathcal{O}} |u_j - u_k|\,.$$

Therefore we have from 4.7

$$\int_{B_{1/2}} |d(u_j - u_k)|^2 \, dx \leqslant c_{21}\varepsilon + c(\mathcal{O}, \Lambda) \sup_{\overline{B}_{1/2} \sim \mathcal{O}} |u_j - u_k| \,.$$

Thus $\{u_j\}$ is a Cauchy sequence in $L_1^2(B_{1/2}, N_0)$ which therefore converges in L_1^2-norm to u_0. This completes the proof of Proposition 4.6.

Let $\mathcal{H}_{\Lambda,B}$ denote the set of maps $u \in \mathcal{H}_\Lambda$ with $E_1(u) \leqslant B$. Let $\overline{\mathcal{H}}_{\Lambda,B}$ denote the closure of $\mathcal{H}_{\Lambda,B}$ taken in $L_1^2(B_1, N_0)$. We now prove a strong version of Lemma 2.5.

Proposition 4.7. *Given $u \in \overline{\mathcal{H}}_{\Lambda,B}$ and $x_0 \in B_{1/2}$, there is a sequence $\lambda(i) \to 0$, $\lambda(i) \in (0, \frac{1}{2}]$, such that the maps $u_{x_0,\lambda(i)} \in L_1^2(B_1, N_0)$ defined by $u_{x_0,\lambda(i)}(x) = u(\lambda(i)(x - x_0))$ converge in L_1^2-norm on B_1^n to a harmonic map $u_0 \in \mathcal{H}_{0,B'}$ satisfying $\partial u_0/\partial r = 0$ a.e. on B_1. Moreover, the convergence is uniform on compact subsets of $\overline{B}_1 \sim \mathcal{S}_0$.*

Proof. Since u is a strong limit of minimizing maps, we get inequality (2.4) satisfied for u and hence by Lemma 2.5 there is a sequence $\lambda(i) \to 0$ such that $u_{x_0,\lambda(i)}$ converges weakly to a harmonic map u_0 satisfying $\partial u_0/\partial r = 0$ a.e. Since $u_{x_0,\lambda(i)} \in \overline{\mathcal{H}}_{\lambda(i)\Lambda,B}$, it follows that there is $\tilde{u}_i \in \mathcal{H}_{\lambda(i)\Lambda,B}$ with $\|\tilde{u}_i - u_{x_0,\lambda(i)}\|_{1,2} < i^{-1}$. From this we see that $\tilde{u}_i$ converges weakly to u_0 and hence strongly by Proposition 4.6. Therefore, $\{u_{x_0,\lambda(i)}\}$ converges strongly to u_0 and we have proven Proposition 4.7.

5. Dimension reduction of $\mathcal{S}$

We prove Theorems II and IV simultaneously in this section by adapting to our setting a dimension reducing argument of H. Federer [7]. We first need some preliminary results.

Lemma 5.1. *Suppose $l \geqslant 3$ and $u \in L^2_{1,\mathrm{loc}}(R^l, N)$ satisfies $g,u/\partial x^l = 0$ a.e. Then there exists $u_0 \in L^2_{1,\mathrm{loc}}(R^{l-1}, N)$ such that $u(x', x^l) = u_0(x')$ a.e. $x' \in R^{l-1}$. If u is E-minimizing on each compact subset of R^l, then u_0 is E-minimizing on each compact subset of R^{l-1}.*

Proof. Suppose on the contrary that $v \colon B_\sigma^{l-1} \to N_0$ satisfies $v = u_0$ on ∂B_σ^{l-1} and $E_\sigma(v) \leqslant E_\sigma(u_0) - \eta$ for some $\eta > 0$. Let $\lambda \gg 0$ be a large number and consider a map $\bar{v} \colon B_\sigma^{l-1} \times [-\lambda - 2\sigma, \lambda + 2\sigma] \to N_0$ satisfying $\bar{v}(x', x^l) = v(x')$ for $|x^l| \leqslant \lambda$ and constructed by Lemma 4.1 on $B_\sigma^{l-1} \times [-\lambda - 2\sigma, -\lambda]$ and $B_\sigma^{l-1} \times [\lambda, \lambda + 2\sigma]$ so that $\bar{v} = u$ on $\partial(B_\sigma^{l-1} \times [-\lambda - 2\sigma, \lambda + 2\sigma])$ and $E(\bar{v}) \leqslant 2\lambda E_\sigma(v) + c$ where c depends on σ, u, v. By the minimizing property of u in $R^l \sim \mathcal{S}$ we have

$$(2\lambda + 4\sigma)E_\sigma(u) \leqslant 2\lambda E_\sigma(v) + c.$$

Choosing λ large we contradict the inequality $E_\sigma(v) \leqslant E_\sigma(u) - \eta$. This proves Lemma 5.1.

The next result guarantees that, in a very simple situation, limits of minimizing maps are minimizing.

Lemma 5.2. *Let $u_0 \in L^2_{1,\text{loc}}(R^l, N_0)$ with $l \geqslant 3$ be a harmonic map with an isolated singularity at 0 such that u_0 satisfies $\partial u_0/\partial r = 0$ a.e. Suppose $u \in L^2_{1,\text{loc}}(R^n, N_0)$, $n \geqslant l$, is given by $u(x', x'') = u_0(x')$, $x' \in R^l$, $x'' \in R^{n-l}$. Suppose there is a sequence $u_i \in \mathcal{K}_{\Lambda_i, B}$ such that u_i converges to u in $L^2_1(B_1, N_0)$ and $\Lambda_i \to 0$. Then both u, u_0 are E-minimizing on compact subsets of R^n, R^l. In particular, u_0 is a minimizing tangent map* (MTM).

Proof. We first show that u is minimizing. Note that because of the homogeneity of u it suffices to prove the u is E-minimizing on $B^l_1 \times B^{n-l}_1$. We first prove a preliminary inequality. Suppose $v_0 \in L^2_1(B^l_1, N_0)$ such that $v_0 = u_0$ on ∂B^l_1. We will modify v_0 to make it agree with u near the origin. For any $\delta > 0$, define a map $v_\delta \in L^2_1(B^l_1, N)$ by

$$v_\delta(x) = \begin{cases} v(x) & \text{for } |x| \geqslant \delta, \\ v\left(\delta \dfrac{x}{|x|}\right) & \text{for } |x| \leqslant \delta. \end{cases}$$

We then have

$$E_\delta(v_\delta) \leqslant \delta \int_{\partial B^l_\delta} |dv_0|^2\, dx. \tag{5.1}$$

For any $\varepsilon \in (0, 2^{-1}\delta)$, define $v_{\delta,\varepsilon}\colon B^l_1 \to N$ by

$$v_{\delta,\varepsilon}(r, \xi) = \begin{cases} v_\delta(r, \xi) & \text{for } r \geqslant 2\varepsilon, \\ u_0(r, \xi) & \text{for } r \leqslant \varepsilon, \\ v_\delta(\rho(r), \xi) & \text{for } \varepsilon < r < 2\varepsilon \end{cases}$$

where r, ξ are polar coordinates in R^l and $\rho(r)$ is the linear function

$$\rho(r) = (2 - 2\varepsilon) - \varepsilon^{-1}(1 - 2\varepsilon)r.$$

Since $u_0(r, \xi)$ is independent of r and $v_0 = u_0$ on ∂B^l_1, we see that $v_{\delta,\varepsilon} \in L^2_1(B^l_1, N_0)$ and satisfies $v_{\delta,\varepsilon} = u_0$ on ∂B^l_1, $v_{\delta,\varepsilon} = u_0$ in B_ε. We now estimate

$$E_1(v_{\delta,\varepsilon}) \leqslant E_1(v_\delta) + E_\varepsilon(u_0) + \int_{B^l_{2\varepsilon} \sim B_\varepsilon} |dv_{\delta,\varepsilon}|^2\, dx.$$

By (5.1) and the homogeneity of u this implies

$$\begin{aligned} E_1(v_{\delta,\varepsilon}) \leqslant{}& E_1(v_0) + \delta \int_{\partial B^l_\delta} |dv_0|^2 + \varepsilon^{l-2} E_1(u_0) \\ &+ \int_{B^l_{2\varepsilon} \sim B^l_\varepsilon} |dv_{\delta,\varepsilon}|^2\, dx. \end{aligned} \tag{5.2}$$

We estimate the last term in two parts

$$\int_{B^l_{2\varepsilon}\sim B^l_\varepsilon}\left|\frac{\partial v_{\delta,\varepsilon}}{\partial r}\right|^2 dx \leq \varepsilon^{l-2}\int_\varepsilon^{2\varepsilon}\int_{S^{l-1}}\left|\frac{\partial v_\delta}{\partial \rho}\right|^2(\rho,\xi)r^{-1}\,d\xi dr.$$

By change of variable this gives

$$\int_{B^l_{2\varepsilon}\sim B^l_\varepsilon}\left|\frac{\partial v_{\delta,\varepsilon}}{\partial r}\right|^2 dx \leq \varepsilon^{l-2}\int_{2\varepsilon}^{1}\int_{S^{l-1}}\left|\frac{\partial v_\delta}{\partial \rho}\right|^2(\rho,\xi)\,d\xi d\rho.$$

From the definition of v_δ this implies (since $\partial v_\delta/\partial r = 0$ on $|x|\leq\delta$)

$$\int_{B^l_{2\varepsilon}\sim B^l_\varepsilon}\left|\frac{\partial v_{\delta,\varepsilon}}{\partial r}\right|^2 dx \leq \varepsilon^{l-2}\delta^{1-l}\int_{B_1\sim B_\delta}\left|\frac{\partial v_0}{\partial r}\right|^2 dx. \tag{5.3}$$

We also can estimate

$$\int_{B_{2\varepsilon}\sim B_\varepsilon} r^{-2}|d_\xi v_{\delta,\varepsilon}|^2\,dx \leq c_1\varepsilon^{l-3}\int_\varepsilon^{2\varepsilon}\int_{S^{l-1}}|d_\xi v_\delta|^2(\rho,\xi)\,d\xi dr.$$

Changing variables we get

$$\int_{B_{2\varepsilon}\sim B_\varepsilon} r^{-2}|d_\xi v_{\delta,\varepsilon}|^2\,dx \leq c_2\varepsilon\int_{B_1\sim B_{2\varepsilon}} r^{-2}|d_\xi v_\delta|^2\,dx. \tag{5.4}$$

Combining (5.2), (5.3) and (5.4) we get

$$E_1(v_{\delta,\varepsilon}) \leq \left(1+c_2\varepsilon+\varepsilon^{l-2}\delta^{1-l}\right)E_1(v_0)+c_3\delta\int_{\partial B^l_\delta}|dv_0|^2+\varepsilon^{l-2}E_1(u_0). \tag{5.5}$$

Now, given a map $v\colon B^l_1\times B^{n-1}_1\to N_0$ which agrees with u on the boundary, we apply (5.5) on each R^l slice (with x'' fixed) and integrate over B^{n-l}_1. We use the notation $|dW|^2=|d'W|^2+|d''W|^2$ for a map $W(x',x'')$. We have from (5.5)

$$\int_{B^l_1\times B^{n-l}_1}|d'v_{\delta,\varepsilon}|^2\,dx'dx''\leq\left(1+c_2\varepsilon+\varepsilon^{l-2}\delta^{1-l}\right)\int_{B^l_1\times B^{n-l}_1}|d'v|^2$$

$$\leq c_3\delta\int_{\partial B^l_\delta\times B^{n-l}_1}|d'v|^2+\varepsilon^{l-2}\int_{B^l_1\times B^{n-l}_1}|du|^2. \tag{5.6}$$

From the definition of $v_{\delta,\varepsilon}$ we can estimate

$$\int_{B^l_{2\varepsilon}}|d''v_{\delta,\varepsilon}|^2\,dx'\leq c_4\varepsilon^{l-1}\int_\varepsilon^{2\varepsilon}\int_{S^{l-1}}|d''v_\delta|^2(\rho,\xi,x'')\,d\xi dr.$$

Changing variables we get

$$\int_{B^l_{2\varepsilon}}|d''v_{\delta,\varepsilon}|^2\,dx'\leq c_4\varepsilon\left(\int_{B^l_1}|d''v|^2\,dx'+\int_{B^l_\delta}|d''v_\delta|^2\,dx'\right).$$

We also see that $\int_{B_\delta} |d''v_\delta|^2\, dx' \leq \delta \int_{\partial B_\delta} |d''v|^2$. Combining with (5.6) we then have

$$\int_{B_1^l \times B_1^{n-l}} |dv_{\delta,\varepsilon}|^2\, dx'dx'' \leq \left(1 + c_5\varepsilon + \varepsilon^{l-2}\delta^{1-l}\right) \int_{B_1^l \times B_1^{n-l}} |dv|^2\, dx'dx''$$

$$+ c_5\delta \int_{\partial B_\delta^l \times B_1^{n-l}} |dv|^2 + \varepsilon^{l-2} \int_{B_1^l \times B_1^{n-l}} |du|^2\, dx'dx''.$$

We can choose δ so that

$$\delta \int_{\partial B_\delta^l \times B_1^{n-l}} |dv|^2 \leq 2 \int_{B_{2\delta}^l \times B_1^{n-l}} |dv|^2\, dx'dx''.$$

Thus we can choose ε, δ small so that

$$(5.7) \qquad \int_{B_1^l \times B_1^{n-l}} |dv_{\delta,\varepsilon}|^2\, dx \leq \int_{B_1^l \times B_1^{n-l}} |dv|^2\, dx + \eta$$

for any given $\eta > 0$. The point is we have from the above construction $v_{\delta,\varepsilon} = u$ on $\partial(B_1^l \times B_1^{n-l})$ as well as $v_{\delta,\varepsilon} = u$ on $B_\delta^l \times B_1^{n-l}$, a neighborhood of $\mathbb{S} = \{0\} \times B_1^{n-l}$. Since by Proposition 4.6, u_i converges uniformly to u away from $\mathbb{S}$, it is obvious that u minimizes on each compact subset of $R^n \sim \mathbb{S}$. Therefore we have

$$\int_{B_1^l \times B_1^{n-l}} |du|^2 \leq \int_{B_1^l \times B_1^{n-l}} |dv_{\delta,\varepsilon}|^2.$$

By (5.7), since η is arbitrarily small we have

$$\int_{B_1^l \times B_1^{n-l}} |du|^2 \leq \int_{B_1^l \times B_1^{n-l}} |dv|^2,$$

and u is minimizing. Applying Lemma 5.1 successively, we get that u_0 is minimizing, and this finishes Lemma 5.2.

We now return to Hausdorff measure, and define for $E \subseteq R^n$, $s \geq 0$,

$$\varphi^s(E) = \inf\left\{ \sum r_i^s \colon E \subseteq \bigcup_i B_{r_i}(x_i) \right\}.$$

Following [7], we observe that for any E

$$(5.8) \qquad \varphi^s(E) = 0 \Leftrightarrow \mathcal{H}^s(E) = 0.$$

We also need the following density result (see [6,2.10.19(2)])

$$(5.9) \qquad \overline{\lim_{\lambda \to 0}}\, \lambda^{-s} \varphi^s(E \cap B_\lambda^n(x)) \geq c_6 > 0$$

for φ^s a.e. $x \in E$. We need the following result on the behaviour of φ^s under weak convergence.

Lemma 5.3. *Suppose u_i is a sequence in $\mathcal{H}_\Lambda$ which converges weakly to u in $L_1^2(B_1^n, N)$. If $\mathbb{S}_i$, $\mathbb{S}$ denote the singular sets of u_i, u respectively, then we have*

$$\varphi^s(\mathbb{S} \cap B_{1/2}^n) \geq \overline{\lim_{i \to 0}}\, \varphi^s(\mathbb{S}_i \cap B_{1/2}^n)$$

for any $s \geq 0$.

Proof. For any $\varepsilon > 0$, let $\{B_{r_i}(x_i)\}$ be a covering of $\mathbb{S} \cap B_{1/2}^n$ by balls satisfying

$$\sum_i r_i^s \leq \varphi^s(\mathbb{S} \cap B_{1/2}^n) + \varepsilon.$$

Now the set $K = \bar{B}_{1/2}^n \sim \bigcup_i B_{r_i}^n(x_i)$ is compact subset of $\bar{B}_{1/2}^n \sim \mathbb{S}$, so by Proposition 4.6 it follows that for j sufficiently large, the map u_j is smooth on K. Thus we have

$$\mathbb{S}_j \cap B_{1/2}^n \subseteq \bigcup_i B_{r_i}^n(x_i)$$

for j large. In particular we have

$$\varphi^s(\mathbb{S}_j \cap B_{1/2}^n) \leq \varphi^s(\mathbb{S} \cap B_{1/2}^n) + \varepsilon$$

for any $\varepsilon > 0$, j large. This gives the conclusion of Lemma 5.3.

Proof of Theorems II *and* IV. Suppose $u \in L_1^2(M, N)$ is $\tilde{E}$-minimizing with singular set $\mathbb{S} \subset \operatorname{int} M$. Let $0 \leq s < n-2$ be such that $\varphi^s(\mathbb{S}) > 0$. Then by (5.9) we can choose $p_0 \in \mathbb{S}$ such that

$$\lim_{\lambda_i \to 0} \lambda_i^{-s} \varphi^s(\mathbb{S} \cap B_{\lambda_i/2}^n) > 0 \tag{5.10}$$

for a sequence $\lambda_i \to 0$, where B_λ is taken in normal coordinates x centered at p_0. We look at the scaled maps $u_\lambda(x) = u(\lambda x)$. By Proposition 4.7 we can choose a subsequence of λ_i, call it λ_i, so that u_{λ_i} converges weakly in $L_1^2(B_1^n, N)$ to a harmonic map u_0, strongly in $L_1^2(B_{1/2}^n, N)$, where u_0 satisfies $\partial u_0/\partial r = 0$ a.e. If $\mathbb{S}_\lambda$ denotes the singular set of u_λ in B_1^n, we clearly have $\mathbb{S}_\lambda \cap B_{1/2}^n = \{x/\lambda : x \in \mathbb{S} \cap B_{\lambda/2}^n\}$ and hence $\varphi^s(\mathbb{S}_\lambda \cap B_{1/2}^n) = \lambda^{-s}\varphi^s(\mathbb{S} \cap B_{\lambda/2}^n)$. Thus (5.10) implies

$$\lim_{\lambda_i \to 0} \varphi^s(\mathbb{S}_{\lambda_i} \cap B_{1/2}^n) > 0.$$

Thus by Lemma 5.3 we have

$$\varphi^s(\mathbb{S}_0 \cap B_{1/2}^n) > 0.$$

Since $\partial u_0/\partial r = 0$ a.e., we have $\lambda \mathbb{S}_0 \subseteq \mathbb{S}_0$ for any $\lambda \geq 0$, and there are two possibilities: either we have $s \leq 0$, or we can choose a point $x_1 \in \mathbb{S}_0 \cap \partial B_1^n$ by (5.9) such that

$$\overline{\lim_{\lambda \to 0}}\, \lambda^{-s} \varphi^s(\mathbb{S}_0 \cap B_\lambda^n(x_1)) > 0.$$

We choose Euclidean coordinates centered at x_1 so that x^1 is radial at x_1. Repeating the above argument at x_1 we get a radially independent harmonic map $u_1 \in L^2_{1,\mathrm{loc}}(R^n, N_0)$ with $\varphi^s(\mathbb{S}_1 \cap B^n_1) > 0$. Since u_0 satisfied $\partial u_0/\partial r = 0$, it follows that $\partial u_1/\partial x^1 = 0$ a.e. If $s - 1 \leqslant 0$, we stop. Otherwise, there is a point $x_2 \in \mathbb{S}_1 \cap \partial B^{n-1}_1$, $R^{n-1} = \{(0, x^2, \cdots, x^n)\}$ and we repeat the argument at x_2. If we repeat this procedure m times, we get harmonic maps $u_j \in L^2_{1,\mathrm{loc}}(R^n, N_0)$ for $j = 1, \cdots, m$ such that $u_j \,|\, B^n_1 \in \overline{\mathcal{K}}_{\Lambda,B}$ for suitable B (see Proposition 4.7) and $\partial u_j/\partial r = \partial u^j/\partial x^\alpha = 0$ a.e. $\alpha = 1, \cdots, j$. Also we would have $\varphi^s(\mathbb{S}_j \cap B^n_1) > 0$ for $j = 1, \cdots, m$. We can repeat the argument until we have $s - m \leqslant 0$. In order to have constructed u_m, we must have had $s - m + 1 > 0$. Since $s < n - 2$, and m is an integer we then have $m \leqslant n - 2$. If $m = n - 2$, then we would have $\mathbb{S}_m \supseteq R^{n-2} = \{(x^1, \cdots, x^{n-2}, 0, 0)\}$ contradicting the fact that $\mathcal{H}^{n-2}(\mathbb{S}_m) = 0$. Therefore we have $m \leqslant n - 3$, and hence $\varphi^t(\mathbb{S}_m \cap B^n_1) = 0$ for $t > n - 3$. Since $\varphi^s(\mathbb{S}_m \cap B^n_1) > 0$, we have $s \leqslant n - 3$, and since s can be any number smaller than $\dim \mathbb{S}$ we have shown $\dim \mathbb{S} \leqslant n - 3$.

If we make the additional assumption that for $j = 1, \cdots, l$ we have no nontrivial MTM from $R^j \to N_0$, then we can say more. If $m = n - 3$, then we have $u_m \in L^2_{1,\mathrm{loc}}(R^n, N_0)$ such that $u_m \,|\, B^n_1 \in \overline{\mathcal{K}}_{\Lambda,B}$, and $u_m(x', x'') = \tilde{u}_m(x'')$ for $x' \in R^{n-3}$, $x'' \in R^3$ where $\tilde{u}_m$ has an isolated singularity at $x'' = 0$. Therefore, by Lemma 5.2, $\tilde{u}_m \in L^2_{1,\mathrm{loc}}(R^3, N_0)$ is an MTM and hence trivial by assumption. Thus we had $m \leqslant n - 4$. We can repeat the same reasoning for $m = n - 4, \cdots, n - l$ and we conclude that $m \leqslant n - l - 1$ which then implies $s \leqslant n - l - 1$ for any $s < \dim \mathbb{S}$ and hence $\dim \mathbb{S} \leqslant n - l - 1$.

Finally, suppose we had $n = l + 1$ and $\mathbb{S} \neq \varnothing$. If $p_0 \in \mathbb{S}$ and $u_0 \in L^2_{1,\mathrm{loc}}(R^n, N_0)$ is a blown-up harmonic map at p_0, then the above argument shows that u_0 has singular set $\mathbb{S}_0 = \{0\}$ and u_0 is an MTM. If there were a sequence $p_i \in \mathbb{S}$ with $p_i \to p_0$, then we could choose $\lambda(i) = 4\,\mathrm{dist}(p_i, p_0)$ and consider the scaled maps $u_{\lambda(i)} \in L^2_1(B^n_1, N_0)$. By the choice of $\lambda(i)$, we have $\mathbb{S}_{\lambda(i)} \cap \partial B^n_{1/4} \neq \varnothing$ for each i. Since the limit u_0 has an isolated singularity at 0, this contradicts Proposition 4.7. Therefore $\mathbb{S}$ is discrete. The same argument shows that, in general, for $n = 3$ either $\mathbb{S} = \varnothing$ or $\mathbb{S}$ is discrete. This completes the proofs of both Theorems II and IV.

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