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FINITE PROPAGATION SPEED, KERNEL ESTIMATES FOR FUNCTIONS OF THE LAPLACE OPERATOR, AND THE GEOMETRY OF COMPLETE RIEMANNIAN MANIFOLDS

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0. Introduction

Let M^n be a complete Riemannian manifold. Then the Laplacian $\Delta = -\delta d$ on functions is a nonpositive essentially self adjoint operator when restricted to functions of compact support. Thus functions $f(\sqrt{-\Delta})$ can be defined by the spectral theorem for unbounded self adjoint operators, according to the prescription

$$f\left(\sqrt{-\Delta}\right) = \int_0^\infty f(\lambda)\, dE_\lambda, \tag{0.1}$$

where dE_λ is the projection valued measure associated with $\sqrt{-\Delta}$.

A natural problem is to study the behavior of the explicit kernel $k_{f(\lambda)}(x_1, x_2)$ representing $f(\sqrt{-\Delta})$, in terms of the behavior of various geometric quantities on M^n. As a particularly important example we have the heat kernel $E(x_1, x_2, t) = k_{e^{-\lambda^2 t}}$. By use of the local parametrix and the standard elliptic estimates, one can show that for $t > 0$, $E(x_1, x_2, t)$ is a positive (symmetric) C^∞ function of x_1, x_2, t which for fixed t and (say) x_2, is in the domain of all positive powers of Δ as a function of x_1; see e.g. [9]. In works of Gårding [19] and Donnelly [16], upper estimates for $E(x_1, x_2, t)$ (and its derivatives) were given under the assumption that M^n has bounded geometry. They showed that as $x_2 \to \infty$, the behavior of $E(x_1, x_2, t)$ is roughly similar to that of the Euclidean heat kernel, $\dfrac{e^{-\rho^2(x_1, x_2)/4}}{(4\pi t)^{n/2}}$; ($\rho(x_1, x_2)$ denotes distance). Recall that M^n is said to have bounded geometry if the injectivity radius $i(x)$ of the

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exponential map at x stays uniformly bounded from below by some constant $\delta > 0$, and if (some or all of) the covariant derivatives $\nabla^i R$ of the curvature tensor R stay bounded in norm; see §3.

Recently, Cheng, Li and Yau [10] achieved a significant advance. They gave an estimate of the same general character as the above, but under a much weaker hypothesis. In particular, they showed that one need only assume a bound on the sectional curvature, $|K_M| < K$, and a lower bound on the injectivity radius of the exponential map $i(p)$ at some point p. Their argument makes use of three highly nontrivial facts which we now recall.

In what follows we will always let $|\ |_D$ denote the sup norm on $D \subset M^n$, and $\|\ \|_D$ the L^2 norm on D.

1. Moser iteration and the Sobolev constant. The Sobolev inequality states the existence of a constant $S_D > 0$, such that

$$S_D\left(\int_D |g|^{n/(n-1)}\right)^{(n-1)/n} \leqslant \int_D \|dg\|, \tag{0.2}$$

for any $g \in H_0^1(D)$. Let $B_R(x)$ be a metric ball, and let $u(x, t)$ be a solution of the heat equation $(\Delta - \partial/\partial t)u = 0$ on $[0, T] \times B_R(x)$. Then Moser's iteration argument [30, pp. 115–117] gives the powerful estimate

$$|u(x, T)| \leqslant c_n\left[\frac{1}{T^{(n+2)/2}} + \frac{1}{A^{n+2}}\right]^{1/2} S_D^{-n/2}\|u\|_{[0, T]\times B_A(x)}. \tag{0.3}$$

Similarly, if u is harmonic on $[-A/2, A/2] \times B_A(x)$, $((\partial^2/\partial t^2 + \Delta)u = 0)$, then

$$|u(x,0)| \leqslant c_n A^{-(n+1)/2} S_D^{-n/2}\|u\|_{[-A/2, A/2]\times B_A(x)}. \tag{0.3$'$}$$

2. The Sobolev constant equals the isoperimetric constant. In [4], [17], [28], it is shown that for any domain D in a Riemannian manifold M^n, the constant S_D is precisely equal to the optimal constant $\Phi(D)$ in the isoperimetric inequality

$$A(\partial U) \geqslant \Phi(D)(V(U))^{(n-1)/n}, \tag{0.4}$$

which relates the area $A(\partial U)$ and volume $V(U)$ of an arbitrary subdomain with smooth boundary $U \subset D \subset M^n$.

3. Croke's estimate of the isoperimetric constant. In [13] Croke defines a constant $\tilde{\omega}(D)$ as follows. Let $x \in D$, $v \in T_x M^n$, $\|v\| = 1$. Let γ_v be the geodesic with $\gamma'(0) = v$, and let t_v denote the first parameter value t, for which $\gamma_v(t) \in \partial D$. Finally, let $\mu()$ denote angular measure on the unit sphere in $T_x M^n$, normalized so that the total measure is 1. Set

$$\tilde{\omega}(D) = \inf_{x\in D} \mu(\{v \in T_x M^n \mid \|v\| = 1 \text{ and } \gamma_v \mid [0, t_v] \text{ is minimal}\}). \tag{0.5}$$

Croke proves the striking inequality

$$\Phi(D) \geqslant 2^{(n-1)/n} \frac{\alpha(n-1)}{(\alpha(n))^{(n-1)/n}} (\tilde{\omega}(D))^{(n+1)/n}, \tag{0.6}$$

where $\alpha(n)$ is the volume of the unit n-sphere; this result is sharp. The proof of (0.6) depends in an essential way on the fundamental work of Berger [2] and Kazdan [25].

It is apparent that (0.2), the equality $S_D = \phi(D)$, and (0.6) reduce the estimation of $E(x_1, x_2, t)$ to bounding the L^2-norm $\| E(x_1, x_2, t)\|_{B_{r_2}(x_2)}$ together with a lower estimate for $\tilde{\omega}(B_{r_2}(x_2))$. In particular, a considerable portion of the argument of [10] is devoted to obtaining the required L^2-estimate. We enlarge the above circle of ideas by noting that such L^2-estimates are an almost immediate consequence of the *finite propagation speed* of the *wave kernel* $\cos\sqrt{-\Delta}\, s\, k_{\cos \lambda s}$. In fact, the statement that $\cos\sqrt{-\Delta}\, s$ has unit propagation speed is itself a *sharp* estimate on $k_{\cos\lambda s}$ which holds *universally*. To get estimates on other kernels we need only to synthesize these kernels out of $k_{\cos\lambda s}$ by means of the Fourier transform; see (1.4). Thus we obtain a substantial simplification in the derivation of the required L^2-estimates, and extend their scope to a more general class of kernels k_f where $f \in \mathcal{F}_2^{\varphi, A}$; see §2 for the definition of $\mathcal{F}_2^{\varphi, A}$.

The pointwise estimates which follow from the previous discussion depend only on knowing a lower bound for the radius of a ball $B_r(x_0)$ on which we have a lower bound for $\tilde{\omega}(B_r(x_0))$. If we define the *injectivity angle* by

$$\tilde{\omega}(r, x) = \mu(\{\gamma'(0) \mid \gamma(0) = x, \gamma \mid [0, r] \text{ is minimal}\}), \tag{0.7}$$

then clearly

$$\tilde{\omega}(B_r(x_0)) \geqslant \inf_{x \in B_r(x_0)} \tilde{\omega}(2r, x). \tag{0.8}$$

If r is less than or equal to half the injectivity radius $i(x)$, for $x \in B_r(x_0)$ we have

$$\tilde{\omega}(B_r(x_0)) = \inf_{x \in B_r(x_0)} \tilde{\omega}(2r, x) = 1. \tag{0.9}$$

In [10] a lower bound for $i(x)$ is established, given $H \leqslant K_M \leqslant K$, a lower bound on $i(p)$ for some p, and the distance $\rho = \rho(p, x)$; they show in particular that even if $H < 0$, $i(x)$ decreases at most exponentially as a function of $\rho(p, x)$. The injectivity angle $\tilde{\omega}(2r, x)$ is then estimated using (0.9). However, a simple comparison argument (see Lemma 4.2) shows that a lower bound for $\tilde{\omega}(2r, x)$ follows directly from a lower bound on the Ricci curvature

Ric_M and a lower bound on the volume $V(B_{r_0}(p)) = V_{r_0}(p) > V > 0$; similar arguments are given in [13], [36] and [10], §2, Lemma 3, to prove related statements. From this we obtain (in §2) an upper estimate on the heat kernel, and, more generally, on k_f for a large class of functions f as described in §2, assuming $\mathrm{Ric}_M \geqslant (n-1)H$, $V_{r_0}(p) \geqslant V \geqslant 0$.

In fact, it turns out that if a lower bound on the injectivity radius $i(x)$ and a bound on $|K_M|$ is given, then the appeal to the deep machinery (1)–(3) above can be avoided. In this case, examination of a parametrix for Δ leads to an elliptic estimate, in which the constant can be bounded by a straightforward comparison argument. The resulting kernel estimates then apply to an essentially wider class of kernels k_f, $f \in \mathfrak{F}_1^N$, $N \geqslant n/4$; see §1. In this connection we also give an estimate on the decay of $i(x)$, which depends only on assuming $V(B_{r_0}(p)) \geqslant V > 0$, rather than $i(p) \geqslant i_0 > 0$; see Theorem 4.3. Our proof is based on ideas of [21]; unlike the argument of [10], it does not rely on Toponogov's theorem.

In dimensions 2 and 3, by means of a special argument we obtain estimates for the more general functions $f \in \mathfrak{F}_1^N$ assuming only $\mathrm{Ric}_M \geqslant (n-1)H$, $V_{r_0}(p) \geqslant V > 0$. We do not know if these estimates can ultimately be generalized to higher dimensions.

We now briefly summarize the contents of the remaining sections of the paper.

In §1 we give the estimates on kernels $k_f(x_1, x_2)$ for $f \in \mathfrak{F}_1^N$ assuming a bound on the sectional curvature, which generalize the results of [10]. In §2 we give estimates on kernels $k_f(x_1, x_2)$ for $f \in \mathfrak{F}_2^{\varphi, A}$, assuming only a lower bound on the Ricci tensor. In §3 we make a more detailed study of various situations involving bounded geometry. In particular we get bounds on $k_f(x_1, x_2)$ for certain classes of functions f holomorphic in a strip about the real axis, satisfying certain "symbol estimates." These bounds yield, in particular, some L^p-operator norm estimates of the sort investigated by Clerc and Stein [12] and by Stanton and Tomas [31] in the special case of certain classes of symmetric spaces.

§4 gives estimates on the volume, injectivity radius, and injectivity angle used in preceding sections. Here we give special emphasis to the role played by the *relative* volume estimates which follow from a lower bound on the Ricci curvature. As an application we consider manifolds for which the condition $\mathrm{Ric}_x \geqslant c/r^2$ holds outside some ball $B_{r_0}(p)$, where $r = \rho(p, x)$. We give sharp upper and *lower* bounds on the asymptotic growth of the volume $V_r(p)$ of $B_r(p)$, in terms of the constant c; the qualitative behavior of $V_r(p)$ is strongly influenced by the precise value of c. §4 can be read independently of the preceding sections, and some may wish to read it first.

1. Finite propagation speed, L^2 estimates, and bounded sectional curvature

In this section we derive estimates on the kernel $k_f(x_1, x_2)$ for $f(\sqrt{-\Delta})$ in an elementary fashion, under the assumption that M is a complete Riemannian manifold with bounded sectional curvature, $|K_M| \leq K$.

We begin by explaining how the finite propagation speed and L^2 operator norm boundedness of $\cos s\sqrt{-\Delta}$ lead to L^2 estimates on the kernels of functions of $\sqrt{-\Delta}$. These estimates, which are valid for arbitrary complete Riemannian manifolds, will also be applied in the following two sections.

Recall that $u = \cos s\sqrt{-\Delta}\ \delta_{x_2}$ satisfies the wave equation

$$\left(\frac{\partial^2}{\partial x^2} - \Delta\right) u = 0, \tag{1.1}$$

with initial data

$$u(0, x) = \delta_{x_2}, \quad \frac{\partial}{\partial s} u(0, x) = 0. \tag{1.2}$$

Thus by general results on hyperbolic equations (see, e.g., [32, Chapter IV]), it follows that

$$\operatorname{supp} \cos s\sqrt{-\Delta}\, \delta_{x_2} \subset \overline{B_{|s|}(x_2)}, \tag{1.3}$$

i.e., $\cos s\sqrt{-\Delta}$ has *unit* propagation speed. This fact can be brought to bear on the estimation of the kernels of other functions of $\sqrt{-\Delta}$, by employing the formula, valid for an even function $f(\lambda)$:

$$f\left(\sqrt{-\Delta}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(s) \cos s\sqrt{-\Delta}\, ds, \tag{1.4}$$

where

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(\lambda) \cos \lambda s\, d\lambda. \tag{1.5}$$

This is just the Fourier inversion formula for even functions, granted the spectral theorem for $\sqrt{-\Delta}$. In case $M = S_1^1$, the unit circle, (1.4) is essentially the Poisson summation formula. This approach to the study of $f(\sqrt{-\Delta})$ has been used by some of the authors before [32], [33], [8] in contexts more like those considered in §3 of this paper. The finite propagation speed property (1.3) has entered also into other qualitative studies of Δ on a complete Riemannian manifold. We mention particularly Chernoff's elegant proof that Δ^k is essentially self adjoint on $C_0^\infty(M)$ for all k, if M is complete; see [11]. Also, Kannai [37] has obtained estimates for the heat kernel on compact manifolds.

We now give a class of function $f(\lambda)$ for which we shall derive L^2 kernel estimates on $f(\sqrt{-\Delta})$ using (1.3) and (1.4).

Definition 1.1. We say

$$f \in \mathfrak{F}_1^k, \tag{1.6}$$

provided that f is even and, for each $I = (-\varepsilon, \varepsilon)$,

$$\left(\frac{d}{ds}\right)^l \hat{f}(s) \in L^1(\mathbf{R} \setminus I), \quad 0 \leqslant l \leqslant k. \tag{1.7}$$

We are now ready for our first general result.

Proposition 1.1. *Let M^n be any complete Riemannian manifold, and let $x_2 \in M^n$. Let $B_r(x_2)$ be the ball of radius R about x_2. If $f \in \mathfrak{F}_1$, then*

$$\| f\left(\sqrt{-\Delta}\right) u \|_{M \setminus B_R}(x_2) \leqslant \|u\| \frac{1}{\pi} \int_{R-r}^{\infty} |\hat{f}(s)|\, ds \tag{1.8}$$

provided

$$\operatorname{supp} u \subset B_r(x_2), \quad r < R. \tag{1.9}$$

Proof. Use the formula, valid for even f,

$$f\left(\sqrt{-\Delta}\right) u = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(s) \cos s\sqrt{-\Delta}\, u ds. \tag{1.10}$$

Now (1.3) and (1.9) imply

$$\cos s\sqrt{-\Delta}\, u = 0 \quad \text{on } M \setminus B_{r+|s|}(x_2). \tag{1.11}$$

Thus the left side of (1.8) is bounded by

$$\left\| \frac{1}{\pi} \int_{R-r}^{\infty} \hat{f}(s) \cos s\sqrt{-\Delta}\, u\, ds \right\| \leqslant \frac{1}{\pi} \int_{R-r}^{\infty} \|u\|\, |\hat{f}(s)|\, ds,$$

since

$$\|\cos s\sqrt{-\Delta}\,\| \leqslant 1. \tag{1.12}$$

This proves (1.8).

Corollary 1.2. *Under the hypotheses of Proposition* 1.1., *if $f \in \mathfrak{F}_1^{2k+2l}$, then*

$$\| \Delta^k f\left(\sqrt{-\Delta}\right) \Delta^l u \|_{M \setminus B_r(x_2)} \leqslant \|u\| \cdot \frac{1}{\pi} \int_{R-r}^{\infty} |\hat{f}^{(2k+2l)}(s)|\, ds. \tag{1.13}$$

Proof. The identity (1.4) shows that

$$\Delta^k f\left(\sqrt{-\Delta}\right) \Delta^l u = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}^{(2k+2l)}(s) \cos s\sqrt{-\Delta}\, u\, ds, \tag{1.14}$$

so (1.13) follows exactly as (1.8).

We now specialize to the case $|K_M| \leqslant K$. Choose

$$r_1 \leqslant \min\left(K^{-1/2}, i(x_1)\right), \quad r_2 \leqslant \min\left(K^{-1/2}, i(x_2)\right). \tag{1.15}$$

Then we claim that in a ball of radius r_j about x_j, the constant in the elliptic estimate for Δ can be controlled. Thus Corollary 1.2 leads to a pointwise estimate in this case. To see this, first assume that

$$r_j = 1, \quad |K_M| \leqslant 1, \tag{1.16}$$

and that (1.15) holds. Recall that if $g(r)$ is any function of r alone in a normal ball $B_{r_0}(x_0)$, we have

$$\Delta g = g'' + \frac{A'}{A} g'. \tag{1.17}$$

Here the prime denotes differentiation with respect to r, and, in general, $A(r, \theta)$ denotes the area element in polar coordinates, on the complement of the cut locus of x_0. Thus A/r^{n-1} is the Radon-Nikodym derivative of the pull-back by $\exp^*_{x_0}$ of the volume element of M with respect to Lebesgue measure on $T_{x_0}M$. A standard comparison (see [7, Chapter 1]) shows that if $|K_M| \leqslant 1$, then on $B_1(x_0)$

$$0 < \frac{c_1(n)}{r} < \frac{A'}{A} < \frac{c_2(n)}{r}. \tag{1.18}$$

Let ϕ be a smooth function supported on $[0, 1/2]$ with $\phi \mid [0, 1/4] \equiv 1$, $|\phi'| < 5$, and set $\phi_\varepsilon = \phi(r/\varepsilon)$. Let

$$P = \phi_\varepsilon(r) \frac{r^{2-n}}{\alpha(n-1)}, \quad \left(P - \phi_\varepsilon \frac{\log r}{2\pi}, n = 2\right).$$

Finally let $\Delta_x(P(x_0, x)) = Q$ denote Δ_x applied pointwise to P. Then on $B_1(x_0)$,

$$|Q(r)| \leqslant C_\varepsilon r^{2-n}. \tag{1.19}$$

If $g(x)$ is a smooth function on $B_1(x_0)$, then

$$\int_{B_1(x_0)} P(x_0, x) \Delta g(x)\, dx = g(x_0) - \int_{B_1(x_0)} Q(x) g(x)\, dx. \tag{1.20}$$

By the above mentioned comparison, the metric on $B_1(x_0)$ is bounded uniformly above and below by a constant times the Euclidean metric in normal coordinates. Fix ε sufficiently small so that the N^2 compositions below are defined and apply the standard iteration argument (see e.g. [15]) to (1.20). Thus if we write (1.20) as

$$P\Delta = I - Q, \tag{1.21}$$

and set

$$(1.22)\qquad \begin{aligned}&\mathcal{P} = (I + Q + \cdots Q^{N-1})P, \quad N = \left[\frac{n}{4}\right] + 1,\\ &\mathcal{Q} = Q^N,\end{aligned}$$

then it follows easily that

$$(1.23)\qquad \mathcal{P}^N(\Delta^N g) + \mathcal{P}^{N-1}\mathcal{Q}(\Delta^{N-1}g) + \cdots + \mathcal{P}\mathcal{Q}(\Delta g) + \mathcal{Q}g = g.$$

The kernels on the left hand are continuous and their sup norms are bounded, since the metric is boundedly related to the Euclidean metric in normal coordinates, and (1.19) holds. Thus we get

$$(1.24)\qquad \begin{aligned}&|g(x_0)| \leq c(n)\Big(\|g\|_{B_1(x_0)} + \|\Delta g\|_{B_1(x_0)} + \cdots \|\Delta^N g\|_{B_1(x_0)}\Big),\\ &\qquad\qquad N = \left[\frac{n}{4}\right] + 1.\end{aligned}$$

We can now obtain the corresponding estimate in the general case by multiplying the metric by a suitable constant. This gives

Proposition 1.3. *Let* $|K_M| \leq K$.

$$(1.25)\qquad r \leq \min\big(|K|^{-1/2}, i(x_0)\big), \quad N = \left[\frac{n}{4}\right] + 1,$$

Then

$$(1.26)\qquad |g(x_0)| \leq c(n)\Big(r^{-n/2}\|g\|_{B_r(x_0)} + \cdots + r^{2N-n/2}\|\Delta^N g\|_{B_r(x_0)}\Big).$$

In Theorem 4.2 we will give a lower estimate for $i(x)$ which will be incorporated into Theorem 1.4 below. First we introduce some notation.

Let $V_r(p)$ denote the volume of $B_r(p)$ in M^n, and V_r^H the volume of $B_r(q) \subset M_H^n$, where M_H^n is the simply connected n-dimensional space of curvature H. Thus

$$(1.27)\qquad V_r^H = \alpha(n-1)\int_0^r \mathcal{A}^H(s)\,ds,$$

where

$$(1.28)\qquad \mathcal{A}^H(s) = \begin{cases} \left(\sin\sqrt{H}s/\sqrt{H}\right)^{n-1}, & H > 0,\\ s^{n-1}, & H = 0,\\ \left(\sinh\sqrt{|H|}s/\sqrt{|H|}\right)^{n-1}, & H < 0,\end{cases}$$

and $\alpha(n-1) = V(S_1^{n-1})$.

Theorem 1.4. *Let M^n be complete and $H \leqslant K_M \leqslant K$. Fix $p, x_1, x_2 \in M^n$, $r_0 > 0$. Set $\rho_j = \rho(p, x_j)$, $j = 1, 2$, and $d = \rho(x_1, x_2)$. Let $4N > n$ and assume $\hat{f}^{(k)} \in L^1(R)$, $0 \leqslant k \leqslant 4N$. Fix r, r_0, s with $r_0 + 2s \leqslant \pi/\sqrt{k}$, $r_0 \leqslant \pi/4\sqrt{k}$.*

(i) *Then the kernel k_f of $f(\sqrt{-\Delta})$ is continuous on $\{(x_1, x_2) \in M \times M$: $x_1 \neq x_2\}$ and satisfies*

$$(1.29)\qquad |k_f(x_1, x_2)| \leqslant \frac{1}{\pi} c^2(n) \sum_{0 \leqslant i, j \leqslant N} \bar{r}_1^{2i-n/2} \bar{r}_2^{2j-n/2} \int_0^\infty |\hat{f}^{(2i+2j)}(s)|\, ds,$$

where $c(n)$ is the constant in (1.26) *and*

$$(1.30)\qquad \bar{r}_j = \min\left(|H|^{-1/2}, \frac{r_0}{2} \frac{1}{1 + \left(V^H_{r_0+s}/V_r(p)\right)\left(V^H_{\rho_j+r}/V^H_s\right)} \right).$$

(ii) *If only $f \in \mathfrak{F}_1^{4N}$, then for $r_j < \bar{r}_j$, $j = 1, 2$, and $r_1 + r_2 < d$,*

$$(1.31)\qquad \begin{aligned} &|k_f(x_1, x_2)| \\ &\leqslant \frac{2}{\pi} c^2(n) \sum_{0 \leqslant i, j \leqslant N} r_1^{2i-n/2} r_2^{2j-n/2} \int_{d-r_1-r_2}^\infty |\hat{f}^{(2i+2j)}(s)|\, ds. \end{aligned}$$

Proof of Theorem 1.4. Since the arguments for (1.29), (1.31) are essentially the same, we consider (1.31). Let $u \in L^2$ be supported on $B_{r_2}(x_2)$. We apply (1.26) of Proposition 1.3 with $g = \Delta_2^j k_f(x_1, x_2)(u(x_2))$, $x_0 = x_1$; by (1.30) and the estimate for $i(x_1)$ of Theorem 4.7, the hypothesis (1.25) is satisfied. If we combine the resulting estimate with (1.13) of Corollary 1.2, we get

$$(1.32)\qquad \begin{aligned} |\Delta_2^j k_f(x_1, x_2)(u)| &\leqslant \frac{c(n)}{\pi} \sum_{i=0}^N r_1^{2i-n/2} \|\Delta_1^i \Delta_2^j k_f(u)\|_{B_{r_1}(x_1)} \\ &\leqslant c(n) \|u\|_{B_{r_2}(x_2)} \sum_{i=0}^N r_1^{2i-n/2} \int_{d-r_1-r_2}^\infty |\hat{f}^{(2i+2j)}(s)|\, ds. \end{aligned}$$

Since u is arbitrary, for all x_1 we have

$$(1.33)\qquad \begin{aligned} &\|\Delta_2^j k_f(x_1, x_2)\|_{B_{r_2}(x_2)} \\ &\leqslant c(n) \sum_{i=0}^N r_1^{2i-n/2} \int_{d-r_1-r_2}^\infty |\hat{f}^{(2i+2j)}(s)|\, ds. \end{aligned}$$

If we now apply (1.26) on $B_{r_2}(x_2)$, (1.31) follows.

Remark. The estimate for $i(x_j)$ incorporated in Theorem 1.4 can be sharpened somewhat further, if ρ_j is sufficiently large; see Theorem 4.7(ii). Also, one can easily choose the constants r, r_0, s, so that the estimate is optimal. In particular cases in which M^n is compact, one can make use of known estimates on the injectivity radius; see e.g. [7].

We point out that Theorem 1.4 can be extended to the case $\partial M^n \neq \emptyset$, by use of a suitable generalization of Proposition 4.2.

2. Ricci curvature bounded below

We begin by assuming that M^n is an *arbitrary* complete Riemannian manifold with $\partial M^n = \emptyset$. Let $\mathcal{F}_2^{\varphi, A}$ be the class of even functions $\{f(\lambda)\}$, such that for some constants A, k_0, s_0 we have

$$(2.1) \qquad |\hat{f}^{(k)}(s)| \leq c_1 \left(\frac{k}{A}\right)^k \varphi(s), \quad k \geq k_0, s \geq s_0,$$

where $\varphi \geq 0$ belongs to $L^1(a, \infty)$ for all $a > 0$. If we set

$$(2.2) \qquad \psi(r) = \int_r^\infty \phi(s)\, ds,$$

then

$$(2.3) \qquad \int_r^\infty |\hat{f}^{(k)}(s)|\, ds \leq c_1 \left(\frac{k}{A}\right)^k \psi(r).$$

We are going to estimate k_f for $f \in \mathcal{F}_2^{\varphi, A}$. Our first step is to reduce the problem to estimating solutions to the homogeneous Laplace equation (harmonic functions) in dimension $(n+1)$. The device we employ was used by Taylor [34], and Lions and Magenes [27], in their independently discovered proof of the Kotake-Narasimhan Theorem. Observe that if $\rho(x_1, x_2) = d = 2A + r$, by Corollary 1.2 we have

$$(2.4) \qquad \| \Delta^k f\left(\sqrt{-\Delta}\right) \Delta^l u \|_{B_A(x_1)} \leq c \left(\frac{2k+2l}{A}\right)^{2k+2l} \psi(r) \| u \|_{B_A(x_2)},$$

for any $u \in L^2$ with $\operatorname{supp} u \subset B_r(x_2)$. Thus for such u, for $|y| \leq A/2$, the power series

$$(2.5) \qquad \begin{aligned} v(x, y) &= \sum_{k=0}^\infty \frac{y^{2k}}{(2k)!} (-\Delta)^k f\left(\sqrt{-\Delta}\right) u, \\ &= \left(\cosh \sqrt{-\Delta}\, y\right) f\left(\sqrt{-\Delta}\right) u(x), \text{ (formally)} \end{aligned}$$

converges to an element of $L^2([-A/2, A/2] \times B_r(x_1))$. Moreover,

$$(2.6) \qquad \left(\Delta_M + \frac{\partial^2}{\partial y^2}\right) v = 0,$$

and by (2.4),

$$(2.7) \qquad \| v \|_{[-A/2, A/2] \times B_r(x_1)} \leq c\psi(r) \| u \|_{B_r(x_2)} A^{1/2}.$$

As explained in §0, we can now combine the three facts emphasized in [10] ((0.3′), $S_D = \Phi(D)$, and (0.6)), to obtain the estimate

$$(2.8) \qquad |k_f(u)|_{B_{A/2}(x_1)} \leqslant k_n A^{-n/2}[\tilde{\omega}(B_A(x_1))]^{-(n+1)/2}\psi(r)\|u\|_{B_A(x_1)}.$$

This implies that for all x_1,

$$(2.9) \qquad \|k_f(x_1, x_2)\|_{B_A(x_2)} \leqslant k_n A^{-n/2}[\tilde{\omega}(B_A(x_1))]^{-(n+1)/2}\psi(r),$$

where k_n is a constant depending only on n. Similarly, we obtain

$$(2.10) \qquad \|\Delta_2^l k_f(x_1, x_2)\|_{B_A(x_2)} \leqslant k_n A^{-n/2}[\tilde{\omega}(B_A(x_1))]^{-(n+1)/2}\left(\frac{2l}{2A}\right)^{2l}\psi(r).$$

Then for each fixed x_1 we can set

$$(2.11) \qquad w(x_1, x_2, y) \sum_{l=0}^{\infty} \frac{y^{2l}}{(2l)!}\Delta_2 f\left(\sqrt{-\Delta}\right)\delta_{x_2},$$

to get a harmonic function on $[-A/2, A/2] \times B_A(x_2)$ whose L^2 norm satisfies

$$(2.12) \qquad \|w(x_1, x_2, y)\|_{[-A/2, A/2]\times B_A(x_2)} \leqslant c\|k_f(x_1, x_2)\|_{B_A(x_2)}.$$

Finishing the argument as above gives

$$(2.13) \qquad |k_f(x_1, x_2)|_{B_{A/2}(x_1)\times B_{A/2}(x_2)} \leqslant c_n^2 A^{-n}[\tilde{\omega}(B_A(x_1))\tilde{\omega}(B_A(x_2))]^{-(n+1)/2}\psi^2(r)$$

for $\rho(x_1, x_2) = d = 2A + r$. The same argument shows that for all x_1, x_2,

$$(2.14) \qquad \begin{aligned} &|k_f(x_1, x_2)|_{B_{A/2}(x_1)\times B_{A/2}(x_2)} \\ &\leqslant c_n^2 A^{-n}[\tilde{\omega}(B_A(x_1))\tilde{\omega}(B_A(x_2))]^{-(n+1)/2}\psi^2(0), \end{aligned}$$

provided $\varphi \in L^1(\mathbf{R}^+)$. Note that if ${}_g k_{f(\lambda)}$ denotes the kernel of $f(\sqrt{-\Delta})$ with respect to the metric g, then ${}_g k_{f(\lambda)} = c_{c^{-2}g}^{-n} k_{f(\lambda/c)}$. Thus using (2.13), (2.14) we can estimate ${}_g k_{f(\lambda)}$ in two different ways. It is straightforward to check that the estimates so obtained coincide; i.e., (2.13), (2.14) have the correct scaling property.

As mentioned in §0, in Proposition 4.2 we give a lower estimate for $\tilde{\omega}(A, x_0)$, if A is sufficiently small. To simplify the statement of Theorem 2.1, we will use only Proposition 4.2(i) even though (ii) is sharper if $\rho(p, x_j) = \rho_j$ is sufficiently large. In §4 we will define a quantity $r(H, V)$ by

$$(2.15) \qquad V_{r(H,V)}^H = \frac{V}{2},$$

where V_r^H is given by (1.27).

Theorem 2.1. *Let M^n be a complete Riemannian manifold with $\partial M^n = \varnothing$, and* $\mathrm{Ric}_M \geqslant (n-1)H$. *Set* $V_{r_0} = V_{r_0}(p)$. *If* $f \in \mathcal{F}_2^{\varphi, A}$ *and* $A < \frac{1}{2}r(H, V)$, *then the kernel* $k_f(x_1, x_2)$ *is continuous for* $x_1 \neq x_2$ *and satisfies* (2.13), (2.14), *where*

$$\tilde{\omega}(B_A(x_j)) \geqslant \frac{1}{\left(2V^H_{r+r_0}/V_{r_0}\right) - 1}. \tag{2.16}$$

Remark. We point out that just as in [10, §4] (or more or less equivalently, by the method of "domination of semigroups"), we can in fact obtain pointwise bounds on the norm of the *gradient* of k_f under the assumptions of Theorem 2.1. We omit the details.

In the appendix to this section, we describe sufficient conditions for a function f to be in the class $\mathcal{F}_2^{\varphi, A}$. At this point we will illustrate our results and methods by applying them to the heat kernel.

Example 2.1 (*The heat kernel*). In this case (2.1)–(2.12) can by bypassed. We begin by recalling the obvious estimate

$$\int_r^\infty \frac{e^{-\frac{1}{4}\lambda^2/t}}{(4\pi t)^{\frac{1}{2}}}\, d\lambda \leqslant c e^{-\frac{1}{4}r^2/t}; \tag{2.17}$$

see [26] for a detailed result. Let $u \in L^2$, $\operatorname{supp} u \subset B_A(\underline{x}_2)$. Set $k_{e^{-\lambda^2 t}} = E(x_1, x_2, t)$ and

$$v(x_1, t) = \int E(x_1, x_2, t) u(x_2)\, dx_2. \tag{2.18}$$

Assume first that $\rho(\underline{x}_1, \underline{x}_2) = d > 2A$. By Proposition 1.1 we have for fixed s

$$\| v(x_1, s) \|_{B_A(x_1)} \leqslant c e^{-\frac{1}{4}(d-2A)^2/s} \| u \|. \tag{2.19}$$

Using the fact that $e^{-\frac{1}{4}(d-2A)^2/s}$ is an increasing function of s, we have an obvious estimate for $\| v(x_1, s) \|_{B_A(x_1) \times [\frac{z}{2}, \underline{z}]}$. Then using (0.3) we can apply steps 1–3 of §0 to get

$$\begin{aligned} | v(x_1, z) | \leqslant c_1 c_n \left[\frac{1}{z^{(n+2)/2}} + \frac{1}{A^{n+2}} \right]^{\frac{1}{2}} e^{-\frac{1}{4}(d-2A)^2/z} \\ \times \left(\tilde{\omega}(B_A(x_1)) \right)^{-\frac{1}{2}(n+1)} \| u \|. \end{aligned} \tag{2.20}$$

Thus for fixed $\underline{x}_1$, z,

$$\begin{aligned} \| E(\underline{x}_1, x_2, z) \| \leqslant c_1 c_n \left[\frac{1}{z^{n/2}} + \frac{z}{A^{n+2}} \right] e^{-\frac{1}{4}(d-2A)^2/z} \\ \times \left(\tilde{\omega}(B_A(x_1)) \right)^{-\frac{1}{2}(n+1)}, \end{aligned} \tag{2.21}$$

and applying steps 1–3 on $B_A(\underline{x}_2) \times [\frac{t}{2}, t]$, we have

$$|E(\underline{x}_1, \underline{x}_2, t)| \leq c_2 c_n^2 \left[\frac{1}{t^{n/2-1}} + \frac{2}{A^{n+2}}\right]\left[\frac{1}{t^{(n+2)/2}} + \frac{1}{A^{n+2}}\right] \times e^{-\frac{1}{4}(d-2A)^2/t}\left[\tilde{\omega}(B_A(x_1))\tilde{\omega}(B_A(x_2))\right]^{-\frac{1}{2}(n+1)}. \tag{2.22}$$

This gives, for $d > 2A$,

$$|E(\underline{x}_1, \underline{x}_2, t)| \leq c_3 c_n^2 \left[\frac{1}{t^{n/2}} + \frac{1}{A^n}\right] e^{-\frac{1}{4}(d-2A)^2/t} \times \left(\tilde{\omega}(B_A(x_1)) \cdot \tilde{\omega}(B_A(x_2))\right)^{-\frac{1}{2}(n+1)}. \tag{2.23}$$

Similarly, for all $\underline{x}_1$, $\underline{x}_2$, including those for which $d < 2A$,

$$|E(\underline{x}_1, \underline{x}_2, t)| \leq c_3 c_n^2 \left[\frac{1}{t^{n/2}} + \frac{1}{A^n}\right]\left[\tilde{\omega}(B_A(x_1))\tilde{\omega}(B_A(x_2))\right]^{-\frac{1}{2}(n+1)}. \tag{2.24}$$

If we combine (2.23), (2.24) with the estimate of $\tilde{\omega}(B_A(x))$ in Proposition 4.2, we get a generalization of the estimate of [10]. In this connection we wish to emphasize that for small time, say $0 < t \leq 1$, the uniform estimate provided by (2.24) together with Proposition 4.2 is *not* a consequence of the usual asymptotic expansion

$$E(\underline{x}, \underline{x}, t) \sim \frac{1}{(4\pi t)^{n/2}}\left[1 - \frac{1}{6}\tau(\underline{x})t + \cdots\right], \tag{2.25}$$

where τ denotes the scalar curvature. Although we have in fact assumed a lower bound on $\mathrm{Ric}_{B_A(x)}$, rather than on $\tau(\underline{x})$, in effect the upper bound on $\tau(\underline{x})$ has been replaced by a lower bound on $V(B_r(\underline{x}))$ for some r. Scaling the metric on a flat torus makes it obvious that a lower bound on $V(B_r(\underline{x}))$ is really necessary and that (2.25) need not hold uniformly.

As a second example, consider the family of metrics corresponding to a sharply rounded cone tip $\underline{x}$, as in Fig. 2.1 below.

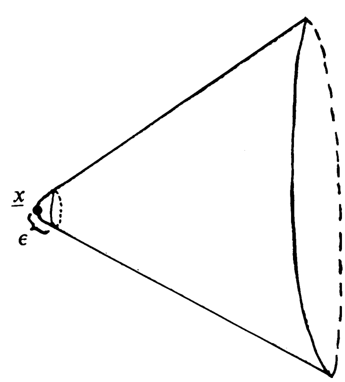

FIG. 2.1

The curvature is $\sim 1/\varepsilon^2$ at the tip and is *nonnegative* everywhere. The volume $V(B_1(\underline{x}))$ is bounded below independent of ε. Then according to (2.24), $E(\underline{x}, \underline{x}, t)$ is bounded independent of ε, for $0 < t < 1$. Again, (2.25) does not hold uniformly in ε.

We can also derive (2.22) as a consequence of Theorem 2.1. This requires estimating $\hat{f}_t(s)$ and its derivatives, where

$$f_t(\lambda) = e^{-t\lambda^2}. \tag{2.26}$$

Note that

$$\hat{f}_t(s) = t^{-\frac{1}{2}}\hat{f}_1(t^{-\frac{1}{2}}s), \tag{2.27}$$

which implies

$$\hat{f}_t^{(k)}(s) = t^{-\frac{1}{2}(k+1)}\hat{f}_1^{(k)}(t^{-\frac{1}{2}}s). \tag{2.28}$$

We will estimate $\hat{f}^{(k)}{}_{\frac{1}{4}}(s)$, since

$$\hat{f}_{\frac{1}{4}}(s) = \pi^{-\frac{1}{2}}e^{-s^2}. \tag{2.29}$$

Then

$$\hat{f}_{\frac{1}{4}}^{(k)}(s) = (-1)^k H_k(s) e^{-s^2}, \tag{2.30}$$

where $H_k(s)$ is the kth Hermite polynomial; see Lebedev [26, p. 60]. In order to get a good estimate on $H_k(s)$, it is convenient to use the generating function identity

$$\sum_{n=0}^{\infty} \frac{H_n(s)^2}{2^n n!}\eta^n = \left(1-\eta^2\right)^{-\frac{1}{2}} e^{2s^2\eta/(1+\eta)}, \quad |\eta| < 1; \tag{2.31}$$

see [26, p. 61]. Pick η fairly small, so that

$$\frac{|H_n(s)|^2}{2^n n!}|\eta|^n \leq c e^{\varepsilon s^2}, \tag{2.32}$$

with $\varepsilon > 0$ given. It follows that

$$|\hat{f}_{\frac{1}{4}}^{(k)}(s)| \leq c\left[\left(\frac{2}{|\eta|}\right)^k k!\right]^{\frac{1}{2}} e^{-(1-\varepsilon)s^2}. \tag{2.33}$$

Consequently,

$$\begin{aligned} |\hat{f}_t^{(k)}(s)| &\leq c\left[\left(\frac{2}{|\eta|}\right)^k k!\right]^{\frac{1}{2}} t^{-\frac{1}{2}(k+1)} e^{-\frac{1}{4}(1-\varepsilon)s^2/t} \\ &= c\left[\left(\frac{2}{|\eta| t}\right)^k k!\right]^{\frac{1}{2}} t^{-\frac{1}{2}} e^{-\frac{1}{4}(1-\varepsilon)s^2/t}. \end{aligned} \tag{2.34}$$

Thus we can take $f_t \in \mathfrak{F}_2^{\varphi, A}$ with $\varphi(r) = t^{-\frac{1}{2}} e^{-\frac{1}{4}(1-\varepsilon)r^2/t}$, and, by Theorem 2.1, recover essentially the same estimates (2.24), (2.25) on the heat kernel.

To close this section, we show how, for manifolds of dimension 2 or 3, the more general class of estimates of §1 can in fact be derived under the weaker assumption that Ricci curvature is bounded from below.

Let

$$Hg = e_i e_j(g) - \nabla_{\nabla_{e_i} e_j}(g) \tag{2.35}$$

denote the Hessian of g. We will show that the constant $c(r)$ in the coercive estimate

$$\| Hg \|^2_{B_{r/2}(x_0)} \leqslant c(r)\left(\| dg \|^2_{B_r(x_0)} + \| \Delta g \|^2_{B_r(x_0)}\right) \tag{2.36}$$

can be bounded from above in terms of r, H, if we assume $\mathrm{Ric}_M \geqslant H$; compare [10, §4].

However, the usual proof of the Sobolev inequality which estimates pointwise norm in terms of H^s, $s > n/2$, norm shows that the constant in that inequality can be bounded from above if a lower bound for the refined injectivity angle $\tilde{\omega}(H, \varepsilon, r_0, x_0)$, (as defined before Proposition 2.3), is known. If we combine this with (2.36), we find that if $\dim M = 2, 3$, we obtain directly a pointwise estimate on k_f, $f \in \mathfrak{F}_1^1$, only assuming $\mathrm{Ric}_M \geqslant (n-1)H$, $V_{r_0}(p) \geqslant V$. We turn to the proof of (2.36).

Recall that for any 1-form ω we have the Bochner-Weitzenbock formula

$$(d\delta + \delta d)\omega = \sum_i \left(-\nabla_{e_i}\nabla_{e_i} + \nabla_{\nabla_{e_i} e_i}\right)\omega + \mathrm{Ric}(\omega^*), \tag{2.37}$$

where

$$\mathrm{Ric}(\omega^*) = (\mathrm{Ric}(\omega))^*, \tag{2.38}$$

and ω^* is the tangent vector dual to ω. Fix a metric ball $B_{\bar{r}}(x)$, and let $\psi(r)$ be defined by $\psi(r) \mid [0, \bar{r}/2] \equiv 1$, $\psi(r) \mid [\bar{r}/2, \bar{r}] = -2r/\bar{r} + 2$. By Stokes' theorem,

$$\int_{B_{\bar{r}}(x_0)} \psi^2 \, d\delta dg \wedge *dg = \int_{B_{\bar{r}}(x_0)} \psi^2 \delta \, dg \wedge *\delta dg - \int_{B_{\bar{r}}(x_0)} 2\psi \, d\psi \delta dg \wedge *dg. \tag{2.39}$$

Applying (2.37) to the left-hand side and integrating by parts gives

$$\begin{aligned}\int_{B_{\bar r}(x_0)} &\psi^2 d\delta dg \wedge {}^*dg \\ &= \int_{B_{\bar r}(x_0)} \Big\{\psi^2 \sum_i \big(-\nabla_{e_i}\nabla_{e_i} + \nabla_{\nabla_{e_i}e_i}\big) dg \wedge {}^*dg + \psi^2 \operatorname{Ric}(dg) \wedge {}^*dg\Big\} \\ &= \int_{B_{\bar r}(x_0)} \Big\{\psi^2 \sum_i \nabla_{e_i} dg \wedge {}^*\nabla_{e_i} dg + 2\psi \sum_i e_i(\psi) \nabla_{e_i} dg \wedge {}^*dg \\ &\qquad + \psi^2 \operatorname{Ric}(dg) \wedge {}^*dg\Big\}.\end{aligned} \tag{2.40}$$

If we substitute (2.40) into (2.39) and transpose the middle term we get

$$\begin{aligned}\int_{B_{\bar r}(x_0)} &\psi^2 \Big(\sum_i \nabla_{e_i} dg \wedge {}^*\nabla_{e_i} dg + \operatorname{Ric}(dg) \wedge {}^*dg\Big) \\ &= \int_{B_{\bar r}(x_0)} \psi^2 \delta dg \wedge {}^*\delta dg - \int_{B_{\bar r}(x_0)} 2\psi d\psi \delta dg \wedge {}^*dg \\ &\quad - \int_{B_{\bar r}(x_0)} 2\psi \sum_i e_i(\psi) \nabla_{e_i} dg \wedge {}^*dg.\end{aligned} \tag{2.41}$$

Note that $\nabla dg = Hg$. Applying the Schwarz inequality to the last two terms of the right-hand side of (2.41) gives

$$\begin{aligned}\int_{B_r} &\psi^2 \sum_i \nabla_{e_i} dg \wedge {}^*\nabla_{e_i} dg + (n-1)H \|\psi dg\|^2_{B_{\bar r}(x_0)} \\ &\leqslant \|\Delta g\|^2_{B_{\bar r}(x_0)} + \Big(\|\Delta g\|_{B_{\bar r}(x_0)} + \frac{4}{\bar r^2} \|dg\|^2_{B_{\bar r}(x_0)}\Big) \\ &\quad + \Big(\frac{1}{2}\int_{B_{\bar r}(x_0)} \psi^2 \sum_i \nabla_{e_i} dg \wedge {}^*\nabla_{e_i} dg + \frac{8}{\bar r^2} \|dg\|^2_{B_{\bar r}(x_0)}\Big).\end{aligned} \tag{2.42}$$

If we transpose the fourth term on the right-hand side, we obtain

Proposition 2.2. *Let M^n be complete, and $\operatorname{Ric}_M \geqslant (m-1)H$. Then for all x, r*

$$\begin{aligned}\|Hg\|^2_{B_{\bar r/2}(x)} &\leqslant 4\|\Delta g\|^2_{B_{\bar r}(x)} + \frac{24}{\bar r^2}\|dg\|^2_{B_{\bar r}(x)} \\ &\quad -2(n-1)H\|\psi dg\|^2_{B_{\bar r}(x)}.\end{aligned} \tag{2.43}$$

Let $A(r, \theta)$, $\mathscr{A}^H(r)$ be as in §1; see (1.28). Given $\varepsilon > 0$ and H, we define a refinement of the injectivity angle by letting $\tilde\omega(H, \varepsilon, r, x)$ be the angular measure of the set $\Omega^H(\varepsilon)$ of initial tangent vectors to geodesics γ from x such that $\gamma \mid [0, r]$ is minimal and

$$\frac{A(s,\theta)}{\mathcal{A}^H(s)} \geq \varepsilon, \quad 0 \leq s \leq r, \tag{2.44}$$

at $\gamma(s)$. Note that for all H, $\tilde{\omega}(H, 0, r, x) = \tilde{\omega}(r, x)$. Let $\nabla^i g$ denote the i-th covariant derivative of g.

Proposition 2.3. *Let g be a smooth function on $B_{\bar{r}}(x_0) \subset M^n$. Then for $N > n/2$,*

$$g^2(x_0) \leq \frac{c_3(N)}{\varepsilon\tilde{\omega}(H, \varepsilon, \bar{r}, x_0)} \left(\int_0^{\bar{r}} \frac{r^{2(N-1)}}{\mathcal{A}^H(r)} dr \right) \sum_{i=0}^{N} \bar{r}^{2(i-N)} \| \nabla^i g \|^2_{B_{\bar{r}}(x_0)}. \tag{2.45}$$

Proof. Let ψ be essentially as above, but made C^∞. Then

$$\begin{aligned} g^2(x_0) &\leq \frac{1}{\tilde{\omega}(H, \varepsilon, \bar{r}, x_0)} \int_{\Omega^H(\varepsilon)} \left| \int_0^{\bar{r}} \frac{d}{dr} (\psi g(r,\theta)) dr \right|^2 d\theta \\ &= \frac{1}{\tilde{\omega}(H, \varepsilon, \bar{r}, x_0)} \\ &\quad \cdot \int_{\Omega^H(\varepsilon)} \left[\int_0^{\bar{r}} \frac{r^{N-1}}{A^{\frac{1}{2}}(r,\theta)} \frac{d^N}{dr^N} (\psi g(r,\theta)) A^{\frac{1}{2}}(r,\theta) dr \right]^2 d\theta \\ &\leq \frac{c_3(N)}{\varepsilon\tilde{\omega}(H, \varepsilon, \bar{r}, x_0)} \left(\int_0^{\bar{r}} \frac{r^{2(N-1)} dr}{\mathcal{A}^H(r)} \right) \sum_{i=0}^{N} \bar{r}^{2(i-N)} \int_{B_{\bar{r}}(x_0)} \| \nabla^i g \|^2, \end{aligned} \tag{2.46}$$

and the claim follows.

In Proposition 4.2 we will give a lower estimate for $\tilde{\omega}(H, \varepsilon, \bar{r}, x_0)$ for ε, $\bar{r}$ sufficiently small. We will now combine this with Propositions 2.2 and 2.3 to get the following result. As in previous instances, the statement can be sharpened somewhat; see Proposition 4.2(iv).

Theorem 2.4. *Let M^n be complete, and* $\mathrm{Ric}_M \geq (n-1)H$. *Fix $p, x \in M^n$ and $r_0 > 0$. Set $V_{r_0} = V_{r_0}(p)$, $\rho = \rho(p, x)$, $V^H_{r(H, V_{r_0})} = \frac{1}{2} V_{r_0}$.*

(i) *Then for $N > n/2$,*

$$\begin{aligned} g^2(x) \leq & \frac{c_3(N)}{\left[\left(4V^H_{\rho + r_0} / V_{r_0} \right) - 3 \right]^2} \left(\int_0^{r(H, V_{r_0})} \frac{r^{2(N-1)} dr}{\mathcal{A}^H(r)} \right) \\ & \times \sum_{i=0}^{N} \bar{r}^{2(i-N)} \| \nabla^i g \|^2_{B_{\bar{r}}(x)}. \end{aligned} \tag{2.47}$$

(ii) *In particular, for* $n = 2, 3$,

$$g^2(x) \leqslant \frac{c_3(2)}{\left[\left(4V_\rho^H + r_0/V_{r_0}\right) - 3\right]^2}\left(\int_0^{r(H,V_{r_0})} \frac{r^2}{\mathfrak{C}^H(r)}\,dr\right)$$

$$\times\left[r_0^{-4}\|g\|^2_{B_{r_0}(x)} + 25r_0^{-2}\|dg\|^2_{B_{r_0}(x_0)} - 2(n-1)H\|\psi dg\|^2_{B_{r_0}(x_0)} + 4\|\Delta g\|^2_{B_{r_0}(x_0)}\right]. \tag{2.48}$$

In view of (2.48) the kernels k_f, $f \in \mathfrak{F}_1^1$, can now be treated by the method of §1.

Appendix to §2. The class $\mathfrak{F}_2^{\varphi,A}$, an example

As we saw above, the functions

$$f_t(\lambda) = e^{-t\lambda^2} \tag{A.1}$$

are elements of $\mathfrak{F}_2^{\varphi,A}$ with

$$\varphi(r) = t^{-\frac{1}{2}}e^{-\frac{1}{4}r^2/t'}, \quad t' > t. \tag{A.2}$$

Here we want to give another example of a large class of functions which belong to $\mathfrak{F}_2^{\varphi,A}$ with

$$\varphi(r) = e^{-Wr}. \tag{A.3}$$

Consider a region in the complex domain of the form

$$\Omega = \{z \in \mathbf{C} : |\operatorname{Im} z| \leqslant W\} \cup \{z \in \mathbf{C} : |\operatorname{Im} z| \leqslant B|\operatorname{Re} z|\}. \tag{A.4}$$

Suppose f is holomorphic in Ω and satisfies the estimate

$$|f(z)| \leqslant c(1 + |z|)^m, \quad z \in \Omega. \tag{A.5}$$

If W and B are decreased slightly, it follows from Cauchy's integral formula that

$$|f^{(k)}(z)| \leqslant c(ck)^k(1 + |z|)^{m-k}, \tag{A.6}$$

and in particular,

$$|z^k f^{(k+m)}(z)| \leqslant c(ck)^k. \tag{A.7}$$

Now if $g(z)$ is holomorphic on $|\operatorname{Im} z| \leqslant W$. then

$$\int_{-\infty}^{\infty} |g(x + iy)|\,dx \leqslant G, \quad |y| \leqslant W, \tag{A.8}$$

implies

$$|\hat{g}(s)| \leqslant Ge^{-W|s|}, \quad s \in \mathbf{R}. \tag{A.9}$$

Thus from (A.7) it follows that

$$|s^{k+m}\hat{f}^{(k)}(s)| \leqslant c_1(ck)^k e^{-W|s|}, \tag{A.10}$$

so

$$|\hat{f}^{(k)}(s)| \leqslant c_1(ck)^k e^{-W|s|}, \quad |s| \geqslant 1. \tag{A.11}$$

Thus $f \in \mathfrak{F}_2^{\varphi, A}$ with $\varphi(r)$ given by (A.3).

3. Manifolds with bounded geometry

We say a Riemannian manifold M has C^k-bounded geometry provided that about each $x \in M$ there is a geodesic ball $B_a(x)$ of radius a (independent of x) such that

$$\exp_x : T_x M \to M \tag{3.1}$$

is a diffeomorphism of $B_a(0) \subset T_x M$ onto $B_a(x)$ with the following property: the metric tensor g_{ij} on $B_a(x)$, pulled back to $B_a(0)$ by (3.1), is bounded in the C^k topology for $T_x M$, and the inverse matrix g^{ij} is bounded in sup norm. We include the case $k = \infty$ and also $k = \omega$; $C^\omega(B_a(0))$ is the space of real analytic functions on $B_a(0)$. Note that $\{u_k\}$ is bounded in $C^\omega(B_a(0))$ if, for any compact $K \subset B_a(0)$, there is a complex neighborhood $\tilde{K}$ of K in $\mathbf{C}T_x(M)$, on which $\{u_k\}$ extends as a uniformly bounded set of holomorphic functions.

Another common notion of "bounded geometry" involves bounds on (say the first k) covariant derivatives of the curvature tensor, plus a lower bound on the injectivity radius. For $k = \infty$ or ω, this concept coincides with C^k-bounded geometry defined above, but not for finite k. For example, a lower bound on $i(x)$ and bounded curvature together form a condition which is stronger than C^0-bounded geometry as defined above. Of course, if M has C^∞-bounded, or even C^2-bounded, geometry, then it has bounded curvature, so the results of §1 apply. But we will obtain further results on the operators $f(\sqrt{-\Delta})$ in this case.

First we consider weak boundedness assumptions on the geometry of M. An assumption even weaker than C^0-bounded geometry is the hypothesis that the injectivity radius is bounded from below:

$$i(x) \geqslant a > 0, \tag{3.2}$$

with no further assumptions. In this case, as remarked in the introduction, we have

$$\tilde{\omega}(B_{a/2}(x)) = 1. \tag{3.3}$$

The estimate (2.13) on $k_f(x_1, x_2)$ for $f \in \mathfrak{F}_2^{\varphi,A}$ holds with $A = a/2$, and we have

$$|k_f(x_1, x_2)| \leq c_n^2\left(\frac{2}{a}\right)^n \psi(r)^2, \tag{3.4}$$

if

$$\rho(x_1, x_2) = a + r. \tag{3.5}$$

If we do not assume a lower bound on the Ricci curvature, methods of estimating the gradient of $k_f(x_1, x_2)$ do not apply. We note that the assumption of C^0-bounded geometry implies a Hölder bound on $k_f(x_1, x_2)$ which improves the pointwise bound (3.4) for $f \in \mathfrak{F}_2^{\varphi,A}$.

Indeed, according to Corollary 1.2 if u is supported on $B_a(x_2)$, then

$$\|\Delta^k f\left(\sqrt{-\Delta}\,\right)\Delta^l u\|_{M\setminus B_{r+a}(x_2)} \leq \frac{1}{\pi}C(C(2k+2l))^{2k+2l}\psi(r)\|u\|. \tag{3.6}$$

Thus if $\rho(x_1, x_2) = r + 2a$, then

$$\|\Delta^k f\left(\sqrt{-\Delta}\,\right)\Delta^l u\|_{B_a(x_1)} \leq \frac{C}{\pi}(C(2k+2l))^{2k+2l}\psi(r)\|u\|. \tag{3.7}$$

We now argue similarly to the proof of Theorem 2.1. Given u supported in $B_a(x_2)$, if $|y| \leq B = \frac{1}{2}c^{-1}$, the power series

$$v(y, x) = \sum_{k=0}^{\infty} \frac{y^{2k}}{(2k)!}\Delta^k f\left(\sqrt{-\Delta}\,\right)u \tag{3.8}$$

converges to an element of $L^2([-B, B] \times B_a(x_1))$, and we have

$$\left(\frac{\partial^2}{\partial y^2} + \Delta\right)v = 0, \tag{3.9}$$

$$\|v\|_{[-B,B]\times B_a(x_1)} \leq C_1\psi(r)\|u\|_{L^2}. \tag{3.10}$$

Now in normal coordinates on $B_a(x_1)$ we have

$$\Delta w(x) = \sum_{j,k} g^{-\frac{1}{2}}\frac{\partial}{\partial x_j}\left(g^{\frac{1}{2}}g^{ij}\frac{\partial w}{\partial x_k}\right), \tag{3.11}$$

and our hypothesis that M has C^0-bounded geometry precisely gives uniform ellipticity of (3.11), and hence of (3.9), with C^0 bounds on the coefficients. Thus the di Georgi-Nash-Moser theorems on such elliptic equations in divergence form (see Gilbarg and Trudinger [20]) imply the pointwise estimate

$$|v(x_1)| \leq C_2\psi(r)\|u\| \tag{3.12}$$

for u supported in $B_a(x_2)$. Now (3.12) is equivalent to the assertion that $f(\sqrt{-\Delta})\delta_{x_1}$ is L^2 on $B_a(x_2)$ and

$$\| f\left(\sqrt{-\Delta}\right)\delta_{x_1} \|_{B_a(x_2)} \leq c_2\psi(r). \tag{3.13}$$

Similarly, we obtain

$$\| \Delta^l f\left(\sqrt{-\Delta}\right)\delta_{x_1} \|_{B_a(x_2)} \leq c_2(2Cl)^{2l}\psi(r). \tag{3.14}$$

So we can set, for $|y| \leq B$,

$$v_{x_1}(y, x) = \sum_{l=0}^{\infty} \frac{y^{2l}}{(2l)!}\Delta^l f\left(\sqrt{-\Delta}\right)\delta_{x_1} \tag{3.15}$$

to obtain

$$\left(\frac{\partial^2}{\partial y^2} + \Delta\right)v_{x_1} = 0, \tag{3.16}$$

$$\| v_{x_1} \|_{[-B, B]\times B_a(x_2)} \leq c_3\psi(r). \tag{3.17}$$

Again the di Georgi-Nash-Moser results yield a bound on $v_{x_1}(x_2)$, i.e.,

$$k_f(x_1, x_2) \leq c_4\psi(r). \tag{3.18}$$

But in fact the theory yields a Hölder bound

$$\| k_f(x_1, \cdot) \|_{C^\alpha(B_{a/2}(x_2))} \leq C_5\psi(r), \tag{3.19}$$

for a certain $\alpha > 0$ which depends only on the C^0-bound and ellipticity constant in (3.11). To summarize, we have proved the following.

Theorem 3.1. *If M has C^0-bounded geometry and $f \in \mathfrak{F}_2^{\varphi, A}$, then the kernel $k_f(x_1, x_2)$ satisfies the estimate* (3.19) *and also*

$$\| k_f(\cdot, x_2) \|_{C^\alpha(B_{a/2}(x_1))} \leq C_5\psi(r). \tag{3.20}$$

The fact that (3.20) holds follows from the symmetry of $k_f(x_1, x_2)$.

In particular, recall from the appendix to §2 that if $f(\lambda)$ is the restriction to $\lambda \in \mathbf{R}$ of a function $f(z)$ holomorphic on a region

$$\Omega = \{z \in \mathbf{C}: |\operatorname{Im} z| \leq W + \varepsilon\} \cup \{z \in \mathbf{C}: |\operatorname{Im} z| \leq B|\operatorname{Re} z|\}, \tag{3.21}$$

then $f \in \mathfrak{F}_2^{\varphi, A}$ with $\varphi(s) = Ce^{-W|s|}$, provided

$$|f(z)| \leq C(1 + |z|)^m, \quad z \in \Omega. \tag{3.22}$$

This gives

Corollary 3.2. *If M has C^0-bounded geometry, and $f(z)$ is holomorphic on* (3.21), *satisfying* (3.22), *then*

$$|k_f(x_1, x_2)| \leq Ce^{-Wr}, \tag{3.23}$$

provided

$$\rho(x_1, x_2) = r + 2a. \tag{3.24}$$

We will not pursue the implication of C^k-bounded geometry for finite positive k, but will devote the rest of this section to obtaining refined global results for manifolds with C^∞ or C^ω bounded geometry.

We now consider the case when M has C^∞-bounded geometry. Examples include all homogeneous spaces, i.e., manifolds with a transitive group of isometries, as well as more general sorts of manifolds such as leaves of C^∞ foliations of compact manifolds. Since such M has bounded curvature, the estimates of §1 apply to the estimate of the kernel $k_f(x_1, x_2)$ for $\rho(x_1, x_2) \geqslant a$. We complement this with a precise analysis of the behavior of the kernel of $f(\sqrt{-\Delta}$ for $\rho(x_1, x_2) \leqslant a$. Assume $f(\lambda) \in S^m_{\rho,0}(\mathbf{R})$, i.e.,

$$|f^{(j)}(\lambda)| \leqslant C_j(1 + |\lambda|)^{m-j\rho}. \tag{3.25}$$

We use here and below the notation for symbol classes and pseudodifferential operators of Hörmander; see [32, Chapter II]. From the representation

$$f\left(\sqrt{-\Delta}\right)u = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(s)\cos s\sqrt{-\Delta}\, u\, ds, \tag{3.26}$$

use a partition of unity $1 = \varphi_1(s) + \varphi_2(s)$, $\varphi_1(s) \in C_0^\infty(-a, a)$, even, $\varphi_1(s) = 1$ for $|s| \leqslant a/2$, to get

$$\begin{aligned} f\left(\sqrt{-\Delta}\right)u &= \frac{1}{2\pi}\int_{-a}^{a} \varphi_1(s)\hat{f}(s)\cos s\sqrt{-\Delta}\, u\, ds \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_2(s)\hat{f}(s)\cos sf\left(\sqrt{-\Delta}\right)u\, ds \\ &= F_1 u + F_2 u. \end{aligned} \tag{3.27}$$

Note that

$$\begin{aligned} F_2 u &= (1-\Delta)^{-k}\int_{-\infty}^{\infty}\left(1 - \frac{d^2}{ds^2}\right)^k \left(\varphi_2(s)\hat{f}(s)\right)\cos s\sqrt{-\Delta}\, u\, ds \\ &= (1-\Delta)^{-k} G_k u, \end{aligned} \tag{3.28}$$

with $\|G_k u\| \leqslant C_k \|u\|$. Thus the standard elliptic estimates and the Sobolev embedding theorem give

$$\|F_2 u\|_{C^l(B_a(x_1))} \leqslant C_l \|u\|_{L^2}. \tag{3.29}$$

Furthermore, if u is supported in $B_a(x_1)$,

$$\|F_2 u\|_{C^l(B_a(x_1))} \leqslant C_{lk} \|u\|_{H^{-k}(B_a(x_1))}, \tag{3.30}$$

where the norm on the right is a Sobolev norm.

Now consider

$$F_1 u = \frac{1}{2\pi}\int_{-a}^{a} \varphi_1(s)\hat{f}(s)\cos s\sqrt{-\Delta}\, u\, ds, \tag{3.31}$$

with u supported in $B_a(x_1)$. We will analyze (3.31) by replacing the operator $\cos s\sqrt{-\Delta}$, for $|s| \leq a$, by its geometrical optics approximation. This approach to the study of $f(\sqrt{-\Delta})$ on compact manifolds is emphasized in Chapter XII of [32], and in [33], and also plays a role in the work of [8] on the wave equation on a cone.

By shrinking a if necessary, we can assume the parametrix for $\cos s\sqrt{-\Delta}$ is of the following form, in a normal coordinate system centered at x_1:

$$\cos s\sqrt{-\Delta}\, u(x) = \sum_{j=1}^{2} \int a_j(s, x, \xi) e^{i\varphi_j(s,x,\xi)}\hat{u}(\xi)\, d\xi. \tag{3.32}$$

Here φ_j solves the eikonal equation

$$\frac{\partial}{\partial s}\varphi_j = (-1)^j \lambda_1(x, d\varphi_j) = (-1)^j (d\varphi_j, d\varphi_j)^{\frac{1}{2}}, \tag{3.33}$$

$$\varphi_j(0, x, \xi) = x \cdot \xi \tag{3.34}$$

for $|s| \leq a$, $x \in B_{2a}(x_1)$; $a_j(s, x, \xi)$ is determined by the usual transport equations of geometrical optics (see for example, [32, Chapter VIII]), with $a_j(0, x, \xi) = \frac{1}{2}$. Now if u is supported in $B_a(x_1)$, then, for $|s| \leq a$, $\cos s\sqrt{-\Delta}\, u$ is supported in $B_{2a}(x_1)$. If we set $\hat{g}(s) = \varphi_2(s)\hat{f}(s)$, then $g(\lambda) \in S^m_{\rho,0}$ differs from $f(\lambda)$ by an element of the Schwartz space $\mathcal{S}(\mathbf{R})$ of rapidly decreasing functions. Thus we have

$$\begin{aligned} F_1 u &= \sum_{j=1}^{2} \int_{-a}^{a} \hat{g}(s) a_j(s, x, \xi) e^{i\varphi_j(s,x,\xi)}\hat{u}(\xi)\, d\xi\, ds + S_1 u \\ &= \sum_j \int g(D_s)\big(a_j(s, x, \xi)e^{i\varphi_j(s,x,\xi)}\big)\Big|_{s=0}\hat{u}(\xi)\, d\xi + S_1 u \\ &= \int b(x, \xi) e^{ix\cdot\xi}\hat{u}(\xi) + S_1 u, \end{aligned} \tag{3.35}$$

where $S_1\colon \mathcal{E}'(B_a(x_1)) \to C^\infty(B_{3a}(x_1))$, and $b(x, \xi)$ is given by the fundamental asymptotic expansion lemma for pseudodifferential operators. Thus

$$b(x, \xi) = \sum_{j=1}^{2} b_j(0, x, \xi), \tag{3.36}$$

where

$$b_j(s, x, \xi) = e^{-i\varphi_j(s,x,\xi)} g(D_s)\big(a_j(s, x, \xi)e^{i\varphi_j(s,x,\xi)}\big) \tag{3.37}$$

belongs to $S^m_{\rho,1-\rho}$ if $\rho \geqslant \frac{1}{2}$ with

$$b_j(s, x, \xi) \sim a_j(s, x, \xi) g\left(\frac{\partial}{\partial s}\varphi_j\right) + \cdots. \tag{3.38}$$

Hence using (3.33) and (3.34) we obtain

$$b(x, \xi) \sim g(\lambda_1(x, \xi)) + \cdots. \tag{3.39}$$

The remainder terms in (3.38) and (3.39) sum to elements of $S^{m-(2\rho-1)}_{\rho,1-\rho}$ for $\rho \geqslant \frac{1}{2}$ and we have a complete asymptotic expansion for $\rho > \frac{1}{2}$.

Denoting by $OPS^m_{\rho,\delta}$ the set of pseudodifferential operators with symbols in the class $S^m_{\rho,\delta}$, we have the following conclusion.

Theorem 3.3. *Assume M has C^∞-bounded geometry. Let $f(\lambda) \in S^m_{\rho,0}(\mathbf{R})$, $\rho \geqslant \frac{1}{2}$ be even. If u is supported in $B_a(p)$ in normal coordinates centered at p, then we have*

$$f\left(\sqrt{-\Delta}\right)u = B_p(x, D)u;\ x \in B_a(0), \tag{3.40}$$

where $B_p(x, D) \in OPS^m_{\rho,1-\rho}$, and in fact

$$\{B_p(x, D) : p \in M\} \text{ is bounded in } OPS^m_{\rho,1-\rho}. \tag{3.41}$$

If $\rho > \frac{1}{2}$, the symbol of $B_p(x, D)$ has a complete asymptotic expansion of the form (3.39), *valid uniformly for $p \in M$.*

We remark that it would be quite natural to use a Hadamard type parametrix written invariantly in terms of the Riemannian distance instead of the Fourier integral representation (3.32), but we will not pursue this approach.

We can use Theorem 3.3 together with the estimates on the kernel $k_f(x_1, x_2)$ for $\rho(x_1, x_2) \geqslant a$ which follow as a special case of the results of §1, to obtain L^p-norm estimates on $f(\sqrt{-\Delta})$ for the following class of function f.

Definition 3.1. We say $f \in \mathfrak{S}^m_W$ if $f(z)$ is holomorphic on the strip

$$\Omega_W = \{z \in \mathbf{C} : |\operatorname{Im} z| < W\}, \tag{3.42}$$

and satisfies the $S^m_{1,0}$ symbol estimates on Ω_W:

$$|f^{(j)}(z)| \leqslant C_j(1 + |z|)^{m-j}, \quad z \in \Omega_W. \tag{3.43}$$

We have, according to (3.27),

$$f\left(\sqrt{-\Delta}\right) = \Phi_1 + \Phi_2, \tag{3.44}$$

where the kernel of Φ_1 is supported in the neighborhood $\{(x_1, x_2) : \rho(x_1, x_2) \leqslant 2a\}$ of the diagonal in $M \times M$ and is given in local normal coordinates as a bounded family of pseudodifferential operators. The kernel $\Phi_2(x_1, x_2)$ is smooth on $M \times M$ and satisfies the estimate

$$|\Phi_2(x_1, x_2)| \leqslant Ce^{-Wr}, \quad r = \rho(x_1, x_2). \tag{3.45}$$

From this we can deduce L^p continuity results for $f(\sqrt{-\Delta})$, as follows. First we show $\Phi_1: L^p(M) \to L^p(M)$, $1 < p < \infty$. Let $u \in L^p(M)$. We can write

$$u = \sum_{j=0}^{\infty} u_j, \tag{3.46}$$

where each u_j is supported in a ball $B_a(p_j)$ of radius a in M, and, because we are assuming M has bounded geometry, we can suppose there is a constant K such that, for any j, $B_{2a}(p_j)$ intersects no more than K other balls $B_{2a}(p_k)$, $p_k \neq p_j$. Thus we can arrange that

$$C^{-1} \sum_j \| u_j \|_{L^p} \leqslant \| u \|_{L^p} \leqslant C \sum_j \| u_j \|_{L^p}. \tag{3.47}$$

Now $\Phi_1 u = \Sigma_j \Phi_1 u_j$, and operators $b(x, D)$ with symbols in $S^0_{1,0}$ are continuous on L^p, $1 < p < \infty$, and bounded families of such operators have uniformly bounded operator norm on L^p, $1 < p < \infty$; see [32, Chapter XI]. Thus from (3.40) and (3.41) it follows that

$$\| \Phi_1 u_j \|_{L^p} \leqslant C_1 \| u_j \|_{L^p}, \tag{3.48}$$

provided $f \in S^0_{1,0}(\mathbf{R})$. Furthermore, $\Phi_1 u_j$ is supported in $B_{2a}(p_j)$. It follows that

$$\| \Phi_1 u \|_{L^p} \leqslant C_2(p) \| u \|_{L^p}, \quad \text{if } f \in S^0_{1,0}(\mathbf{R}). \tag{3.49}$$

As for the L^p continuity of Φ_2, we use the estimate (3.45) together with the elementary fact that the L^p operator norm of Φ_2 is bounded by

$$\sup_q \int_M | \Phi_2(p, q) | \, d\mathrm{vol}(q) + \sup_q \int_M | \Phi_2(p, q) | \, d\mathrm{vol}\, p. \tag{3.50}$$

See [32, Chapter XIII]. Now if M has C^∞-bounded geometry, or more generally if M has Ricci curvature bounded below, we have

$$\mathrm{vol}\{x \in M : \rho(x, p) \leqslant r\} \leqslant c e^{Kr} \tag{3.51}$$

for some K independent of p. Consequently, if (3.45) holds with $W > K$, we have a uniform bound on (3.50) and hence an L^p-operator bound on Φ_2, $1 \leqslant p \leqslant \infty$. Thus we have proved the following result.

Theorem 3.4. *Assume M has C^∞ bounded geometry. If*

$$f \in \mathbb{S}^0_W, \quad W > K, \tag{3.52}$$

where (3.51) *holds for K, then*

$$f\left(\sqrt{-\Delta}\right) : L^p(M) \to L^p(M), \quad 1 < p < \infty. \tag{3.53}$$

At this point we make the following remark. Sometimes one knows that the spectrum of $-\Delta$ on M is contained in $[\alpha_0, \infty)$ for some $\alpha_0 > 0$. In such a case, the same considerations as above, or indeed elsewhere in this paper, apply to

$f(\sqrt{-\Delta - \alpha_0})$. In particular we have, in the context of Theorem 3.4,

$$(3.54) \qquad f\left(\sqrt{-\Delta - \alpha_0}\right): L^p(M) \to L^p(M), \quad 1 < p < \infty,$$

for $f \in \mathbb{S}_w^0$. Some results a bit weaker than Theorem 3.4 were announced in [33].

We pass now to a still more rigid class of manifolds, those with C^ω-bounded geometry. This class still includes homogeneous spaces. In this case, if $f \in \mathfrak{F}_2^{\varphi, A}$, the construction (3.8) yields a harmonic function on $(-B, B) \times B_a(x_1)$, and also (3.15) yields a harmonic function on $(-B, B) \times B_a(x_2)$, i.e., these functions are annihilated by the operator $\partial^2/\partial y^2 + \Delta$, and in normal coordinates we have Δ of the form (3.11). This time the coeffcients of $\partial^2/\partial y^2 + \Delta$ form a bounded family of analytic functions on $B_a(0)$ with a uniform ellipticity constant, so the analyticity estimates for solutions to $(\partial^2/\partial y^2 + \Delta)w = 0$ imply the estimates on the kernel $k_f(x_1, x_2)$:

$$(3.55) \qquad \sup_{x \in B_a(x_1)} |D_2^\alpha k_f(x_1, x_2)| \leq C(C|\alpha|)^{|\alpha|}\psi(r),$$

if $\rho(x_1, x_2) = r + 2a$, where one uses normal coordinates centered at x_1 to express D_x^α. In particular, if $f(z)$ is holomorphic in the region

$$(3.56) \quad \Omega = \{z \in \mathbf{C}: |\operatorname{Im} z| \leq W + \varepsilon\} \cup \{z \in \mathbf{C}: |\operatorname{Im} z| \leq B|\operatorname{Re} z|\}$$

and satisfies

$$(3.57) \qquad |f(z)| \leq C(1 + |z|)^m, \quad z \in \Omega,$$

then, as discussed in the appendix to §2, the estimate (3.55) becomes

$$(3.58) \qquad \sup_{x \in B_a(x_1)} |D_2^\alpha k_f(x_1, x_2)| \leq C(C|\alpha|)^{|\alpha|} e^{-rW}.$$

Since the function $f_t(\lambda) = e^{-t\lambda^2}$ belongs to $\mathfrak{F}_2^{\varphi, A}$ with $\varphi(r) = e^{-(1-\varepsilon)r^2/4t}$, we also obtain from (3.55) estimates on the heat kernel proved by Gårding [19], for the case where M is a Lie group with left invariant metric.

Finally, we note a dampening of the kernel $k_f(x_1, x_2)$ when M is a rank 1 symmetric space, i.e., when the isotropy group of M fixing a point x acts transitively on the unit sphere in T_xM. We suppose M is of noncompact type with negative curvature. In such a case, the estimate (3.51) can be refined to the following sharper result on the volume of a spherical shell

$$(3.59) \qquad A_r(q) = \{x \in M: a + r \leq \rho(q, x) \leq 3a + r\}.$$

Namely,

$$(3.60) \qquad \operatorname{vol} A_r(q) \sim C(a)e^{Kr},$$

which implies

$$\operatorname{vol}\{x \in M\colon \rho(x,q) \leqslant r + 2a\} \sim Ce^{Kr}. \tag{3.61}$$

Now $f(\sqrt{-\Delta})$ is invariant under rotations of M about q. This fact together with (3.60) and the usual elliptic estimates yields

$$|k_f(x,q)| \leqslant Ce^{-Kr/2} \sup_{0 \leqslant k \leqslant [\frac{n}{2}]+1} \int_r^\infty |\hat{f}^{(2k)}(s)|\, ds, \tag{3.62}$$

if

$$\rho(x,q) = r + 2a. \tag{3.63}$$

In particular, by (3.60) the function $u_q(x) = k_f(x,q)$ is integrable on $\{x \in M\colon \rho(x,q) \geqslant 1\}$ with L^1 norm independent of q, provided

$$\int_r^\infty |\hat{f}^{(2k)}(s)|\, ds \leqslant Cr^{-1-\varepsilon}e^{-Kr/2}, \quad 0 \leqslant k \leqslant \left[\frac{n}{2}\right] + 1. \tag{3.64}$$

It follows that, for a rank-one symmetric space M, if (3.66) holds,

$$f\left(\sqrt{-\Delta}\right)\colon L^p(M) \to L^p(M), \quad 1 < p < \infty. \tag{3.65}$$

More generally, if the spectrum of $-\Delta$ is contained in $[\alpha_0, \infty)$, and

$$f \in \mathbb{S}^0_w, \quad W > \frac{1}{2}K, \tag{3.66}$$

then

$$f\left(\sqrt{-\Delta - \alpha_0}\right)\colon L^p(M) \to L^p(M), \quad 1 < p < \infty. \tag{3.67}$$

This result was proved by Stanton and Tomas [31] by a different method, following the work of Clerc and Stein [12] on complex symmetric spaces. As these authors point out, one can make use of the Kunze-Stein phenomenon which implies that the operator Φ_2 of (3.44) is continuous on $L^p(M)$, $1 < p < \infty$, provided that, for some $b > 1$,

$$\Phi_2(x_1, \cdot) \in L^q(M), \quad q \in (1, b). \tag{3.68}$$

which is slightly weaker than the requirement

$$\Phi_2(x_1, \cdot) \in L^1(M), \tag{3.69}$$

we sought above. Indeed, this allows us to replace $W > \frac{1}{2}K$ by

$$W = \frac{1}{2}K \tag{3.70}$$

in (3.66).

4. Volume, injectivity radius and injectivity angle

In this section we prove the geometric estimates used in the previous sections.

We begin by discussing the volume. Let $x \in M^n$, and $A(r, \theta)$, $\mathcal{A}^H(r)$, V_r^H be as defined after (1.7) and in (1.16), (1.17). Suppose $\mathrm{Ric}_M \geqslant (n-1)H$ and regard $A(r, \theta)$ as a function on the tangent space $T_x M$. If $H > 0$, we restrict attention to $B_{\pi/\sqrt{H}}(0) \subset T_x M$. Let $U \subset T_x M$ denote the interior of the cut locus in the tangent space. The proof of the standard volume comparison (see [3, pp. 253–257]) shows that on U,

$$\text{(4.1)} \qquad \frac{A(r,\theta)}{\mathcal{A}^H(r)} \downarrow,$$

where $\downarrow$ indicates a decreasing function. If $S_r(0) \subset T_x M$ denotes the sphere of radius r about the origin, (4.1) implies

$$\text{(4.2)} \qquad \frac{\displaystyle\int_{S_r(0)\cap U} A(r,\theta)}{\displaystyle\int_{S_r(0)} \mathcal{A}^H(r)} \downarrow;$$

the possible existence of the cut points only helps matters. In general, if $f(r)$, $g(r)$ are positive functions such that $\frac{f}{g}\downarrow$, then it follows that

$$\text{(4.3)} \qquad \frac{\displaystyle\int_0^r f(s)}{\displaystyle\int_0^r g(s)} \downarrow .$$

To see this, set $f/g = h$. Then $h\downarrow$ and

$$\text{(4.4)} \quad \int_0^r f \int_r^R g = \int_0^r gh \int_r^R g \geqslant \left[\int_0^r g\right] h(r) \int_r^R g \geqslant \left[\int_0^r g\right] \int_r^R gh = \int_0^r g \int_r^R f.$$

Thus, equivalently,

$$\text{(4.5)} \quad \int_0^r f \int_r^R g = \int_0^r f \int_0^r g + \int_0^r f \int_r^R g \geqslant \int_0^r f \int_0^r g + \int_0^r g \int_r^R f = \int_0^r g \int_0^R f,$$

which gives (4.3). From (4.2) and (4.3) we obtain the following *relative* volume estimates which were emphasized in [21], [23].

Proposition 4.1. *Let M^n be complete and $\mathrm{Ric}_M \geqslant (n-1)H$. Let $p, x \in M^n$ and $\rho(p,x) = \rho$. Then*

$$\text{(i)} \qquad \frac{V_r(p)}{V_r^H} \downarrow,$$

or equivalently, for $r_1 \leqslant r_2 \leqslant r_3$,

$$\text{(ii)} \qquad \frac{V_{r_3}(p) - V_{r_2}(p)}{V^H_{r_3} - V^H_{r_2}} \leqslant \frac{V_{r_1}(p)}{V_{r_1}H}.$$

Moreover,

$$\text{(iii)} \qquad V_{r_2}(p)\frac{V_{r_1}H}{V^H_{\rho+r_2}} \leqslant V_{r_1}(x).$$

If $r_2 + r < \rho$, *then*

$$\text{(iv)} \qquad V_{r_2}(p)\frac{V^H_r}{V^H_{\rho+r_2} - V^H_{\rho-r_2}} \leqslant V_r(x).$$

Proof. (i) and (ii). Let

$$(4.10) \qquad f = \int_{S_r^{n-1}(0)\cap U} A(r,\theta), \quad g = \int_{S_r^{n-1}(0)} \mathcal{Q}^H(r).$$

Then (i) and (ii) follow from (4.2), (4.3).

(iii) We have

$$(4.11) \qquad V_{r_2}(p) \leqslant V_{\rho+r_2}(x) \leqslant V^H_{\rho+r_2}\frac{V_{r_1}(x)}{V_{r_1}H}.$$

(iv) Using (ii) we obtain

$$(4.12) \quad V_{r_2}(p) \leqslant V_{\rho+r_2}(x) - V_{\rho-r_2}(x) \leqslant \left(V^H_{\rho+r_2} - V_{\rho-r_2}H\right)\frac{V_r(x)}{V^H_r}.$$

Relative volume estimates are implicit in the proof of the more familiar volume comparison $V_r(p) \leqslant V_r^H$, and have, in fact, been used previously (compare e.g. [5]). However, recent applications in [22], [23] indicate that their significance was not fully appreciated.

Note that it follows immediately from Proposition 4.1(iv) that if $\mathrm{Ric}_M \geqslant 0$, $d > 2r$, then

$$(4.13) \qquad \frac{(d-2r)^n}{d^n - (d-2r)^n} V_r(p) \leqslant V_d(p).$$

Thus we recover the result of Calabi and Yau [35] that $V_d(p)$ grows linearly as $d \to \infty$. At the end of this section we will give a sharp generalization of (4.13) to manifolds whose Ricci curvature decays no slower than $c/[\rho(p,x)^2]$ outside some $B_{r_0}(p)$.

Remark 4.1. Let $S \subset M^n$ be any set with the property that if $q \in S$ is not on the cut locus of p, then the unique minimal geodesic from p to q is contained in S. Clearly, Proposition 4.1(i) and (ii) continue to hold with $V_r(p)$ replaced by $V(S \cap B_r(p))$, as do the corresponding generalizations of (iii) and (iv).

Remark 4.2. Let M^n be a *Ricci model* in the sense of [9]. Thus $\underline{M}^n$ is a metric ball (possibly of infinite radius), with metric $dr^2 + f^2(r)g$, where $f(r) \sim r - Kr^3/6 + \cdots$ for some K. If at distance r from p, then we have $\mathrm{Ric}_{M^n} \geqslant -(n-1)f''(r)/f(r)$, Remark 4.1 still applies with V_r^H replaced by $f^{n-1}(r)$.

Remarks 4.1 and 4.2 will be used without further comment below.

We now consider the refined injectivity angle $\tilde{\omega}(H, \varepsilon, r, x)$ introduced in §2 just before Proposition 2.3. We retain the assumptions of Proposition 4.1.

Proposition 4.2. (i) *Given* ε, r_2, r, *we have*

$$\tilde{\omega}(H, \varepsilon, r, x) \geqslant \frac{1}{1-\varepsilon}\left[\frac{V_{r_2}(p) - V_r^H}{V_{\rho+r_2}^H - V_r^H} - \varepsilon\right], \tag{4.14}$$

where the right-hand side is > 0, *if* ε, r *are sufficient small.*

(ii) *If moreover* $r_2 < \rho$, *then*

$$\tilde{\omega}(H, \varepsilon, \rho - r_2, x) \geqslant \frac{1}{1-\varepsilon}\left[\frac{V_{r_2}(p)}{V_{\rho+r_2}^H - V_{\rho-r_2}^H} - \varepsilon\right]. \tag{4.15}$$

(iii) *Thus if* $r(H, V)$ *is defined by* $\frac{V}{2} = V_{r(H,V)}^H$, *then*

$$\begin{aligned}\tilde{\omega}&\left(H, \frac{V_{r_2}(p)}{2\left(2V_{\rho+r_2}^H - V_{r_2}(p)\right)}, r\left(H, V_{r_2}(p)\right), x\right)\\ &\geqslant \frac{V_{r_2}(p)}{4V_{\rho+r_2}^H - 3V_{r_2}(p)}.\end{aligned} \tag{4.16}$$

(iv) *If moreover* $r_2 < \rho$, *then*

$$\tilde{\omega}\left(H, \frac{V_{r_2}(p)}{4\left(V_{\rho+r_2}^H - V_{\rho-r_2}^H\right)}, \rho - r_2, x\right) \geqslant \frac{V_{r_2}(p)}{4\left(V_{\rho+r_2}^H - V_{\rho-r_2}H\right) - V_{r_2}(p)}.$$

Proof. (i) Since $V_r(x) \leqslant V_r^H$ and $B_{r_2}(p) \subset B_{\rho+r_2}(x)$, we have

$$V_{r_2}(p) - V_r^H \leqslant V_{\rho+r_2}(x) - V_r(x). \tag{4.17}$$

Using (4.2) and integrating in polar coordinates, we easily get

$$\begin{aligned}&V_{\rho+r_2}(x) - V_r(x)\\ &\quad\leqslant \left(V_{\rho+r_2}^H - V_r^H\right)\left[\tilde{\omega}(H, \varepsilon, r, x) + \varepsilon(1 - \tilde{\omega}(H, \varepsilon, r, x))\right],\end{aligned} \tag{4.18}$$

from which (i) follows.

(ii) This follows as in (i), by noting that

$$V_{r_2}(p) \leqslant V_{\rho+r_2}(x) - V_{\rho-r_2}(x). \tag{4.19}$$

(iii) and (iv) follow immediately from (i) and (ii).

We now discuss the injectivity radius. Let M^n be a complete Riemannian manifold and $p \in M^n$. Let γ be a geodesic loop on p of shortest length $L[\gamma] = 2l$. Let $K_M \leqslant K$ on $B_l(p)$. Then by a result of Klingenberg (see e.g. [7]), the injectivity radius of the exponential map at p is bounded below by $\min(l, \pi/\sqrt{K})$; if $K \leqslant 0$, we interpret $\pi/\sqrt{K}$ as ∞.

In [6] it was shown that there exists a constant $c_n(d, V, H) > 0$ with the following properties. Let M^n be *compact*. Suppose the diameter $d(M^n)$ and the volume $V(M^n)$ satisfy $d(M^n) \leqslant d$, $V(M^n) \geqslant V$, and that $K_M \geqslant H$. Then every *smooth* closed geodesic on M^n has length $\geqslant c_n(d, V, H)$. For compact manifolds Klingenberg showed in addition that the injectivity radius at all points is bounded below by the minimum of $\pi/\sqrt{K}$ (where $K_M \leqslant K$) and one half the length of a *smooth* closed geodesic. Thus it follows that for all $p \in M^n$,

$$i(p) \geqslant \min\left(\frac{1}{2}c_n(d, V, H), \pi/\sqrt{K}\right). \tag{4.20}$$

The method of [6] depends on Toponogov's theorem. It leads to a sharp constant in certain situations, and no part of the hypothesis can be removed unless something further is added; see e.g. [7] for further results of various authors in more special situations. However, in [24] Heintze and Karcher gave a new derivation. Their method was to estimate the volume of a tube around a smooth closed geodesic. This led to a better constant in most cases and did not require the use of Toponogov's Theorem. Whether the lower bound $K_M \geqslant H$ can be replaced by $\mathrm{Ric}_M \geqslant (n-1)H$ in the above estimates, is still an open question.

As mentioned in the introduction, in [10] a relative bound on the behavior of the injectivity radius $i(x)$ was derived, which did not require the global hypothesis of compactness. In particular, the authors worked with geodesic loops which were not necessarily smooth, but they assumed a lower bound on $i(p)$ for some p. The estimate which we now give replaces this assumption by a lower bound on the volume $V_r(p)$ for some r, p. Thus we get a purely local generalization of the results of [6], [24], in so far as they pertain to the injectivity radius. However, the method by its very nature depends on knowing a lower bound for the distance to the conjugate locus. Without this assumption, no information on the length of a closed geodesic γ is obtained, even if γ is smooth.

To emphasize the local nature of our result, we now let $B_r(p)$ be a metric ball in a Riemannian manifold such that for $r' < r$, $\overline{B_{r'}(p)}$ is compact. Assume that on $B_r(p)$, $K_M \leqslant K$ and $r \leqslant \pi/\sqrt{K}$, (r arbitrary if $K \leqslant 0$). We let $V_r^0(p)$ denote the volume of $B_r(0) \subset T_pM$ with respect to the *pulled back metric.*

Theorem 4.3. *Let $B_r(p)$ be as above. If γ is a geodesic loop on p of length $2l$, and $r_0 + 2s \leqslant r$, $r_0 \leqslant r/4$, then*

$$l \geqslant \frac{r_0}{2} \frac{1}{1 + V^0_{r_0+s}(p)/V_s(p)}. \tag{4.21}$$

If in addition $H < K_M \leqslant K$, then

$$l \geqslant \frac{r_0}{2} \frac{1}{1 + V^H_{r_0+s}(p)/V_s(p)}. \tag{4.22}$$

To prepare for the proof of Theorem 4.3, we begin with some elementary observations.

(i) By the Gauss Lemma, every curve c of length $L[c] < r$ with $c(0) = p$ has a unique lift to a curve $\tilde{c} \subset B_r(0) \subset M_p$ with $\tilde{c}(0) = 0$. In particular, a geodesic segment γ of length $< r$ with $\gamma(0) = p$ has a unique radial lift. Thus points of $B_r(0)$ can be identified uniquely with such γ. With this identification, the projection $\exp_p$: $B_r(0) \to B_r(p)$ is given by $\gamma \to e(\gamma)$, where $e(\gamma)$ denotes the endpoint of γ.

(ii) Let c_1, c_2 be piecewise smooth curves, and write $c_1 \underset{A}{\sim} c_2$ if c_1 and c_2 are homotopic over curves of length $\leqslant A$ keeping endpoints fixed. If $L[c] = l < r$, clearly $\tilde{c} \underset{l}{\sim} \tilde{\gamma}$ with $\tilde{\gamma}$ the unique radial geodesic such that $e(\tilde{c}) = e(\tilde{\gamma})$. Hence $c \underset{l}{\sim} \gamma$.

(iii) Since $\exp_p \mid B_r(0)$ is nonsingular, if γ_1, γ_2 are *distinct* radial geodesics with $L[\gamma_i] < r$, then by a standard argument on lifting homotopies, γ_1, γ_2 do *not* satisfy $\gamma_1 \underset{r}{\sim} \gamma_2$. In particular, the geodesic segment γ in (ii) is the unique geodesic segment with $c \underset{r}{\sim} \gamma$. We write $[c]$ for γ, and $[p]$ for the constant loop on p.

Let θ_1, θ_2 be closed curves on p, and let c be an arc from p with $L[\theta_i] \leqslant 2l$, $L[c] = s$ and $2(l + s) < r$. Let $\theta_i \cup c$ denote the arc from p which is equal to θ_i followed by c.

Lemma 4.4. *If $2(l + s) < r$ and $[\theta_1 \cup c] = [\theta_2 \cup c]$, then $[\theta_1] = [\theta_2]$.*

Proof. If $[\theta_1 \cup c] = [\theta_2 \cup c]$, then $\theta_1 \cup c \underset{2l+s}{\sim} \theta_2 \cup c$. Thus

$$\theta_1 \underset{2(l+s)}{\sim} \theta_1 \cup c \cup -c \underset{2(l+s)}{\sim} \theta_1 \cup c \cup -c \underset{2(l+s)}{\sim} \theta_2.$$

By (iii) above $[\theta_1] = [\theta_2]$.

Let $\#(q, a)$ denote the number of distinct inverse images of $q \in B_r(p)$ in $B_a(0)$, $a < r$.

Lemma 4.5 (*Even covering lemma*). *If $r_0 + 2s < r/4$ and $q \in B_s(p)$, then* $\#(q, 3r_0) \geq \#(p, r_0)$.

Proof. By (i) above, $\#(p, r_0) = m$, where $\gamma_1 \cdots \gamma_m$ are the distinct geodesic loops on p of length $< r_0$. Let σ be a minimal geodesic from p to q. By Lemma 4.4, $[\gamma_i \cup \sigma]$, $i = 1, \cdots, m$, represent distinct inverse images of q in $B_{r_0+s}(p)$.

Let θ be a loop on p, and σ a geodesic arc from p with $L(\theta) = 2l$, $L(\sigma) = s$ and $2(l + s) < r$. The correspondence $\sigma \to [\theta \cup \sigma]$ defines a map $\pi_{[\theta]}$: $B_s(0) \to B_{2l+s}(0)$ which clearly depends only on $[\theta]$. It is also obvious that $\exp_p \pi_{[\theta]}(x) = \exp_p x$ for $x \in B_s(0)$. Hence $\pi_{[\theta]}$ is an isometry from $B_s(0)$ to $\pi_{[\theta]}(B_s(0))$ with respect to the pulled back metric on $B_{2l+s}(0)$. It follows immediately from Lemma 4.4 that $\pi_{[\theta]}$ *has no fixed points* unless $[\theta] = p$.

Set $i\gamma = \overbrace{\gamma \cup \cdots \gamma}^{i}$ and $L(\gamma) = 2l$.

Lemma 4.6. *Let γ be a geodesic loop on p with $2l \cdot N < \frac{1}{4}r$. Then $[\gamma]$, $[2\gamma], \cdots, [N\gamma]$ are all distinct.*

Proof. Since $(i+j)\gamma = i\gamma \cup j\gamma$, if $[(i+j)\gamma] = [i\gamma \cup j\gamma] = [j\gamma] = [j\gamma \cup p]$, then $[i\gamma] = p$ by Lemma 4.1. Regard $[p], [\gamma], \cdots, [(i-1)\gamma]$ as points in the strictly convex ball $B_{2il}(0)$. Then $\pi_{[\gamma]}$: $B_{2il}(0) \to B_{2(i+1)l}(0)$ preserving distance. Hence $\pi_{[\gamma]}$ preserves the unique center of gravity of the set $\{[p], [\gamma], \cdots [(i-1)\gamma]\}$. By definition the center of gravity is the unique minimum of the function $y \to \Sigma_j \rho^2([j\gamma], y)$. It exists because $2il < \frac{1}{2}\pi/\sqrt{K}$ implies that $B_{2il}(0)$ is convex. This contradicts the fact that π_γ has no fixed points.

Proof of Theorem 4.3. Consider a geodesic loop γ with $L(\gamma) = 2l < r_0$. Let $N = [r_0/2l]$. By Lemma 4.6, p has at least N inverse images in $B_{r_0}(0)$. By Lemma 4.5, every point in $B_s(p)$ has at least N inverse images in $B_{r_0+s}(0)$. Since $\exp_p$ is nonsingular and orientation preserving, $N \cdot V_s(p) \leq V^0_{r_0+s}(p)$. Hence $1/[r_0/2l] \leq V(B_s(p))/V(B_{r_0+s}(0))$, and (4.20) follows. Then (4.21) follows from the standard volume comparison.

If we combine Theorem 4.3 with Proposition 4.1, we immediately obtain the following lower bound for the injectivity radius.

Theorem 4.7. *Let M^n be complete with $H \leq K_M \leq K$. Let $\rho = \rho(p, x)$, and fix r, r_0, s, with $r_0 + 2s < \pi/\sqrt{k}$, $r_0 \leq \pi/4\sqrt{k}$. Then*

(i)

$$i(x) \geq \frac{r_0}{2} \frac{1}{1 + \left(V^H_{r_0+s}/V_r(p)\right)\left(V^H_{\rho+r}/V^H_s\right)}. \tag{4.23}$$

(ii) *Moreover, if* $r + s < \rho$, *then*

$$i(x) \geqslant \frac{r_0}{2} \frac{1}{1 + \left(V^H_{r_0+s}/V_r(p)\right)\left(V^H_{\rho+r} - V^H_{\rho-r}\right)/V^H_s}. \tag{4.24}$$

As an application of the relative volume estimates proved at the beginning of this section, we now consider complete manifolds whose Ricci curvature at infinity is "*almost nonnegative*" in the following sense. Fix $p \in M^n$ and set $r = \rho(p, x)$. We assume that for some $r_0 > 0$ and all $r \geqslant r_0$,

$$\mathrm{Ric}_M(x) \geqslant (n-1)\frac{\frac{1}{4} \pm \nu^2}{r^2}. \tag{4.25}$$

(For Theorem 4.8 and Theorem 4.9 (iii) it suffices to assume that (4.25) holds only for *radial* directions from p; however for Theorem 4.9 (i) and (ii) this does not suffice.) By convention we take $\nu \geqslant 0$. Observe that the Jacobi equation

$$g'' = \frac{-\left(\frac{1}{4} - \nu^2\right)g}{r^2} \tag{4.26}$$

corresponding to sectional curvature $(\frac{1}{4} - \nu^2)r^2$ admits the pair of solutions

$$g_{\pm\nu}(r) = r^{\frac{1}{2}\pm\nu}, \tag{4.27}$$

and the particular solution

$$J_{\nu,R} = \frac{1}{2\nu}\left(-r^{\frac{1}{2}+\nu} + R^{2\nu}r^{\frac{1}{2}-\nu}\right), \tag{4.28}$$

satisfying $J_{\nu,R}(R) = 0$, $J'_{\nu,R}(R) = -1$. For the case $\frac{1}{4} + \nu^2$, we replace ν by $i\nu$ in (4.27), and obtain the solutions

$$r^{\frac{1}{2}}\cos(\nu \log r), \quad r^{\frac{1}{2}}\sin(\nu\log r). \tag{4.29}$$

Corresponding to (4.28) we have

$$\mathcal{J}_{i\nu,R}(r) = \frac{R}{\nu} r^{\frac{1}{2}}\sin(\nu\log r/R). \tag{4.30}$$

Note that

$$\mathcal{J}_{i\nu,R}(\mathrm{Re}^{\pi/\nu}) = \mathcal{J}_{i\nu,R}(R) = 0. \tag{4.31}$$

This gives the following extension of Myer's theorem.

Theorem 4.8. *Let M^n be a complete Riemannian manifold such that for some $p \in M^n$, $r_0 > 0$, and $\nu > 0$ we have*

$$\mathrm{Ric}_M(x) \geqslant (n-1)\frac{\left(\frac{1}{4} + \nu^2\right)}{r^2} \tag{4.32}$$

for all $r \geqslant r_0$. Then M^n is compact and the diameter d_p from p satisfies

$$d_p < e^{\pi/\nu} r_0. \tag{4.33}$$

A weaker result of this type is derived in [1]. In [19] the compactness of M^n is derived from (4.32), but no explicit bound on the diameter is given.

Proof of Theorem 4.8. We use the field $\mathcal{J}_{i\nu, e^{\pi/\nu}r_0}$, and argue by index comparison as in the standard proof of Myers theorem. If $\gamma \mid [0, e^{\pi/\nu}r_0]$ is minimal, (4.31) and (4.32) lead to the conclusion that $\gamma \mid [r_0, e^{\pi/\nu}r_0]$ contains a pair of conjugate points.

Remark 4.3. By considering a metric on $\mathbf{R}^n$ which is of the form

$$dr^2 + rg(\theta) \tag{4.34}$$

outside some compact set, where $g(\theta)$ is the standard metric on S^{n-1}, we see that M^n need not be compact if $\nu = 0$. However, further refinements are possible. For example, a similar argument shows that if

$$\operatorname{Ric}_M(x) \geqslant (n-1)\left\{\frac{\frac{1}{4}}{r^2} + \frac{\frac{1}{4} + \nu^2}{r^2 \log r}\right\}, \tag{4.35}$$

then M^n is compact; see also [14] for a sharp result of this type.

We now consider the case $(\frac{1}{4} - \nu^2)/r^2$. Observe that for $\nu > 0$, the conditions

$$\nu \leqslant \frac{(n+1)}{2(n-1)}, \qquad -\frac{n}{(n-1)^2} \leqslant \frac{1}{4} - \nu^2, \quad 0 \leqslant (n-1)(\tfrac{1}{2} - \nu) + 1 \tag{4.36}$$

are equivalent to one another. Also recall that if M^n is not compact, a normal geodesic γ: $[0, \infty] \to M^n$ is called a *ray* if each segment of γ is minimal. Parts (i) and (ii) of the following theorem give a lower bound on the rate of growth of the volume of a ball, which generalizes the previously mentioned result of Yau and Calabi for the case $\operatorname{Ric}_M \geqslant 0$. A novel feature of the argument, as distinct from the case $\operatorname{Ric}_M \geqslant (n-1)H$, will be the use of a family of model spaces with metric

$$dr^2 + J^2_{\nu,R}(r)g, \quad 0 \leqslant r \leqslant R.$$

Here the metric is expressed in polar coordinates, but *the origin is at R*. For $\nu < \frac{1}{2}$, the metric is singular at the *boundary* $r = 0$, while for $\nu > \frac{1}{2}$ there is a conjugate point at the boundary. For $\nu = \frac{1}{2}$ we have the usual flat metric on a ball of radius R.

Theorem 4.9. *Let M^n be a complete noncompact Riemannian manifold, and let γ: $[0, \infty) \to M^n$ be a ray from some $p \in M^n$.*

(i) *Fix $0 \leqslant \nu \leqslant \frac{1}{2}(n+1)/(n-1)$. If $\nu \geqslant \frac{1}{2}$, assume that for some $r_0 > 0$ and all $r > r_0$ we have*

$$\operatorname{Ric}_M(x) \geqslant (n-1)\frac{(\frac{1}{4} - \nu^2)}{r^2}. \tag{4.37}$$

If $0 \leqslant \nu < \frac{1}{2}$, *assume that for some* r_1 *such that* $r_0 < r_1 < \frac{3}{2}r_0$ *and for all* $r > r_0$,

$$\mathrm{Ric}_M(x) \geqslant (n-1)\frac{\left(\frac{1}{4}-\nu^2\right)}{\left(r-2(r_1-r_0)\right)^2}. \tag{4.38}$$

If $0 \leqslant \nu < \frac{1}{2}(n+1)/(n-1)$, *then there exists* C_{r_1} *independent of* M^n *such that for* $r > 2r_1 - r_0$

$$c_{r_1}V_{(r_1-r_0)}(\gamma(r_1))r^{(n-1)(\frac{1}{2}-\nu)+1} \leqslant V_{r+(r_1-r_0)}(p). \tag{4.39}$$

(ii) *If* $\nu = \frac{1}{2}(n+1)/(n-1)$, *and* (4.37) *holds, then for* $r > 2r_1 - r_0$

$$c_{r_1}V_{(r_1-r_0)}(\gamma(r_1))\log r \leqslant V_{r+(r_1-r_0)}(p). \tag{4.40}$$

(iii) *For all* $\nu \geqslant 0$, *if* (4.37) *holds* (*in radial directions*), *then*

$$V_r(p) \leqslant c_M r^{(n-1)(\frac{1}{2}+\nu)+1}. \tag{4.41}$$

Proof. (i) We first consider the case $\nu \geqslant \frac{1}{2}$. Fix x with $\rho(p, x) = r$. Let $\sigma(t)$ be a normal geodesic emanating from p, and set $\rho(\gamma(r), \sigma(t)) = s$. We consider $t < r - r_0$ and note that $s > r - t$ by the triangle inequality. Since $\nu \geqslant \frac{1}{2}$,

$$\mathrm{Ric}_M\,\sigma(t) \geqslant (n-1)\frac{\left(\frac{1}{4}-\nu^2\right)}{s^2} \geqslant (n-1)\frac{\left(\frac{1}{4}-\nu^2\right)}{(r-t)^2}. \tag{4.42}$$

The argument can now be completed as in Proposition 4.1 (iv) except that we use $J_{\nu,r}^{n-1}$ in place of $\mathcal{Q}^H$. This gives

$$\frac{V_{(r_1-r_0)}(\gamma(r_1))}{V_{(r-r_1)}(\gamma(r))} \leqslant \frac{\int_{r_1}^{2r_1-r_0} J_{n,r}^{n-1}(t)\,dt}{\int_{2r_1-r_0}^{r} J_{n,r}^{n-1}(t)\,dt} \sim \frac{r^{(n-1)2\nu}}{r^{(n-1)(\frac{1}{2}+\nu)+1}}, \tag{4.43}$$

and (4.39) follows easily.

Now consider the case $0 \leqslant \nu < \frac{1}{2}$. It will suffice to obtain the analog of (4.42). Moreover, clearly we can restrict attention to those σ which are minimal from $\gamma(r)$ to some $\sigma(t) \in B_{r_1-r_0}(\gamma(r_1))$, and to $0 \leqslant t \leqslant \underline{t}$. Let $a = \rho(p, \sigma(\underline{t}))$. Then by the triangle inequality,

$$r_1 + (r_1 - r_0) \geqslant a, \tag{4.44}$$

$$(r - r_1) + (r_1 - r_0) \geqslant \underline{t} = (\underline{t} - t) + t, \tag{4.45}$$

$$a + (\underline{t} - t) \geqslant s. \tag{4.46}$$

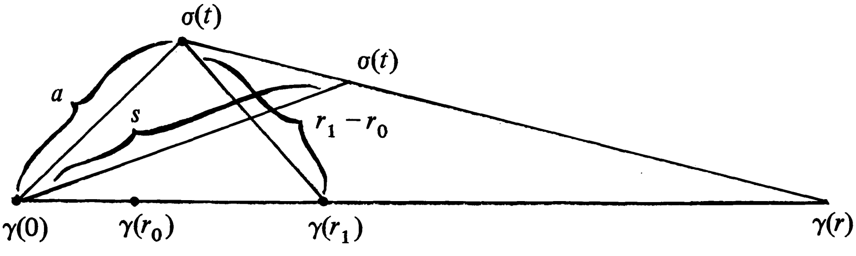

FIG. 4.1

Adding these and cancelling terms give

$$r + 2(r_1 - r_0) \geqslant t + s. \tag{4.47}$$

But $r - t \geqslant r_0$ and so, by the triangle inequality,

$$s \geqslant r_0. \tag{4.48}$$

Combining (4.47), (4.48) and the assumption $3r_0 > 2r_1$, we obtain

$$r - t \geqslant s - 2(r_1 - r_0) \geqslant s - r_0 > 0. \tag{4.49}$$

Thus, since $\nu < \frac{1}{2}$, we get

$$\mathrm{Ric}_M(x) \geqslant (n-1)\frac{\frac{1}{4} - \nu^2}{\left(s - 2(r_1 - r_0)\right)^2} \leqslant (n-1)\frac{\frac{1}{4} - \nu^2}{(r-t)^2}. \tag{4.50}$$

As above, this implies (4.39) for $\nu < \frac{1}{2}$.

(ii) The argument is as in (i).

(iii) The estimate (4.41) follows from the proof of the Heintze-Karcher comparison theorem [24] applied ;to $\partial B_{r_1} \setminus C_1$, where $r_1 > r_0$ and C is the cut locus of p.

Remark 4.4. By considering a metric on $\mathbf{R}^n$ which is of the form

$$dr^2 + r^{1 \pm 2\nu} g \tag{4.51}$$

outside some compact set, we see that the estimates of Theorem 4.9 are sharp. Also as in Proposition 4.2 we see that the injectivity angle $\tilde{\omega}$ decays at most like $r^{-(n-1)(\frac{1}{2}+\nu)}$.

References

[1] W. Ambrose, *A theorem of Myers*, Duke Math. J. **24** (1957) 345.

[2] M. Berger, *Blaschke's conjecture for spheres*, Appendix D in A. L. Besse, *Manifolds all of who geodesics are closed*, Ergebnisse Math. und ihrer Grenzgebiete, Vol. 93, Springer, Berlin, 1978, 236–242.

[3] R. Bishop & B. Crittenden, *Geometry of manifolds*, Academic Press, New York, 1964.

[4] E. Bombieri, *Theory of minimal surfaces, and a counter-example to the Bernstein conjecture in high dimension*, Lecture Notes, Courant Inst., New York, 1970.

[5] J. Cheeger, *The relation between the Laplacian and the diameter for manifolds of non-negative curvature*, Arch. Math. **19** (1968) 558–560.
[6] ______, *Finiteness theorems for riemannian manifolds*, Amer. J. Math. **92** (1970), 61–64.
[7] J. Cheeger & D. G. Ebin, *Comparison theorems in Riemannian geometry*, North Holland, Amsterdam, 1975.
[8] J. Cheeger & M. Taylor, *On the diffraction of waves by conical singularities*, Comm. Pure Appl. Math., to appear.
[9] J. Cheeger & S. T. Yau, *A lower bound for the heat kernel*, Comm. Pure Appl. Math. **34** (1981) 465–480.
[10] S. Y. Cheng, P. Li & S. T. Yau, *On the upper estimate of the heat kernel of a complete riemannian manifold*, Amer. J. Math. **103** (1981) 1021–1063.
[11] P. Chernoff, *Essential self-adjointness of powers of generators of hyperbolic equations*, J. Functional Analysis **12** (1973) 401–414.
[12] J. Clerc & E. Stein, *L^p multipliers for noncompact symmetric spaces*, Proc. Nat. Acad. Sci. U.S.A. **71** (1974) 3911–3912.
[13] C. Croke, *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Sci. École Norm. Sup. **13** (1980) 419–435.
[14] B. V. Dekster & I. Kupka, *Asymptotics of curvature in a space of positive curvature*, J. Differential Geometry **15** (1980) 553–568.
[15] G. de Rham, *Varieties differentiables*, Publ. Inst. Math. Univ. Nancago III, Hermann, Paris, 1960.
[16] H. Donnelly, *Spectral geometry for certain noncompact riemannian manifolds*, Math. Z. **169** (1979) 63–76.
[17] H. Federer & W. Flemming, *Normal integral currents* Ann. of Math. **72** (1960) 458–520.
[18] G. Galloway, *Compactness criteria for Riemannian manifolds*, Proc. Amer. Math. Soc., **84** (1982) 106–110.
[19] L. Gårding, *Vecteurs analytiques dans les representations des groupes de Lie*, Bull. Soc. Math. France **188** (1960) 73–93.
[20] D. Gilbarg & M. Trudinger, *Elliptic partial differential equations of second order*, Springer, Berlin, 1977.
[21] M. Gromov, *Almost flat manifolds*, J. Differential Geometry **13** (1978) 231–241.
[22] ______, *Curvature, diameter, and Betti numbers*, Comment. Math. Helv. **56** (1981) 179–197.
[23] ______, *Paul Levy's isoperimetric inequality*, Inst. Hautes Études Sci.l Publ. Math., to appear.
[24] E. Heintze & H. Karcher, *A general comparison theorem with applications to volume estimates for submanifolds*, Ann. Sci. École Norm. Sup. **11** (1978) 451–470.
[25] J. Kazdan, *An inequality arising in geometry, Appendix E*, in A. L. Besse, *Manifolds all of whose geodesics are closed*, Ergebnisse Math. und ihrer Grenzgebiete, Vol. 93, Springer, Berlin, 1978, 243–246.
[26] N. N. Lebedev, *Special functions and their applications*, Dover, New York, 1972.
[27] J. Lions & E. Magenes, *Non-homogeneous boundary value problems and appliations*, Vol. 3, Springer, New York, 1972.
[28] V. G. Maz'ya, *Classes of domains and imbedding theorems for function spaces*, Dokl. Akad. Nauk SSSR **133** (1960) 527–530; English Translation, Soviet Math. Dokl. **1** (1960) 882–885.
[29] J. Milnor, *A note on curvature and fundamental group*, J. Differential Geometry **2** (1968) 1–7.
[30] J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **17** (1964) 101–134.
[31] R. Stanton & P. Tomas, *Expansions for spherical functions on noncompact symmetric spaces*, Acta. Math. **140** (1978) 251–276.
[32] M. Taylor, *Pseudodifferential operators*, Princeton Univ. Press, Princeton, 1981.

[33] ______, *Fourier integral operators and harmonic analysis on compact manifolds*, Proc. Sympos. Pure Math. Vol. 35, Part 2, 1979, 115–136.

[34] ______, *Analytic properties of elliptic and conditionally elliptic operators*, Proc. Amer. Math. Soc. **28** (1971), 317–318.

[35] S. T. Yau, *Some function theoretic properties of complete Riemannian manifolds, and their applications to geometry*, Indiana Univ. Math. J. **25** (1976) 659–670.

[36] ______, *Survey on partial differential equations in differential geometry*, preprint.

[37] Y. Kannai, *Off diagonal short time asymptotics for fundamental solutions of diffusion equations*, Comm. Partial Differential Equations **2** (1977) 781–830.

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