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ALMOST FLAT MANIFOLDS

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1. Introduction

A compact riemannian manifold M is said to be ε-flat if the riemannian sectional curvature K and the diameter d of M satisfy the inequality $|K|\, d^2 \leqslant \varepsilon$. In [3] Gromov proved that any sufficiently flat riemannian manifold possesses a finite cover which is diffeomorphic to a nil-manifold. In addition, Gromov demonstrated that every nil-manifold carries an ε-flat metric for any $\varepsilon > 0$. Recently, following Gromov's original ideas and improving the estimate for the pinching constant $\varepsilon = \varepsilon(n)$, Buser and Karcher [2] gave a detailed proof of this result.

In the present paper we prove a stronger version of this theorem by showing that M itself, and not only a finite cover, possesses a locally homogeneous structure. In fact, we prove that under suitable curvature assumptions, M is diffeomorphic to the quotient of a simply connected nilpotent Lie group N by an affine finite extension Γ of a lattice in N. In this form the theorem generalizes the well known Bieberbach theorem on euclidean space-forms.

The main idea in this proof is the same as in Min-Oo and Ruh [9], [10], where we solved a certain partial differential equation on M. Here the additional problem stems from the fact that the elliptic operator in question is not strictly positive. The cokernel of this operator is responsible for the fact that the model N is not known a priori, but has to be constructed in the proof.

In addition to the deformation techniques employed in [9], [10], we utilize one of the main ideas in Gromov's proof, the imitation of the proof of Bieberbach's theorem as presented in Chapter 3 of Buser-Karcher [2]. However, dependence on this chapter makes estimates for the pinching constant $\varepsilon = \varepsilon(n)$ better than $\exp(-\exp n^2)$ impossible. Nevertheless, we feel that a realistic constant would be $\exp(-n^2)$; accordingly no estimate of $\varepsilon(n)$, $n = \dim M$, is given in this paper.

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2. The result

The main result of this paper is the following generalization of Bieberbach's theorem on compact euclidean space forms and Gromov's theorem on almost flat manifolds. To formulate the theorem we recall that Γ is called an extension of L by H if $1 \to L \to \Gamma \to H \to 1$ is an exact sequence of group homomorphisms.

Theorem. *Let M denote a compact riemannian manifold, d the diameter, K the sectional curvature, and n the dimension of M. There exists a constant $\varepsilon = \varepsilon(n) > 0$ such that $|K| d^2 < \varepsilon$ implies that M is diffeomorphic to $\Gamma \setminus N$, where N is a simply connected nilpotent Lie group, and Γ is an extension of a lattice $L \subset N$ by a finite group H.*

Remark. As in the case of the Bieberbach theorem it is possible to specify the manner in which the fundamental group Γ of M acts on N. The constructions of the proof provide a riemannian metric as well as a connection D which is compatible with this metric. Left translation in the Lie group N coincides with parallel translation with respect to D. The fundamental group Γ acts as a group of affine isometries of N, i.e., the elements of Γ can be viewed as diffeomorphisms of N preserving the connection D as well as the metric.

In case M is flat, $N = \mathbf{R}^n$, and the above result is the Bieberbach theorem. The finite cover M' of M with covering group $H = \Gamma/L$ is a nil-manifold $L \setminus N$, and we recover Gromov's theorem. The estimate $|H| \leqslant 2 \cdot 14^{\dim O(n)}$ of Buser-Karcher [2] holds here as well.

To put the main result of this paper into proper perspective, we wish to introduce a qualitative version of the concept of Lie group. To motivate this concept, we recall that the structure of a Lie group on a simply connected manifold with basepoint e is determined by a connection D on the tangent bundle with curvature $R = 0$ and parallel torsion, i.e., $DT = 0$. The connection is provided by the left invariant vector fields and, vice versa, provides left invariant vector fields. With this in mind the following definition is natural.

Definition. A compact manifold P with metric connection D is called an ε-almost Lie group, if $(\|DT\| + \|R\|)d^2 \leqslant \varepsilon$, where d is the diameter of P.

A similar generalization of Lie groups, called soft Lie groups, was proposed by Regge and Ne'eman [11]. In [7] Nomizu proposed to study a more general concept. A natural question to study is whether an almost Lie group is always diffeomorphic to $\Gamma \setminus G$ with Γ an extension of a lattice L of a Lie group G. The answer is yes if the model group is abelian, i.e., if the torsion is zero (close to zero is sufficient). This is the result of the present paper. Theorem 1 of [9] shows that the answer is yes if the model group is compact and semi-simple, i.e., if the "Killing form" constructed from the torsion is negative definite. We conjecture that, for a suitable $\varepsilon = \varepsilon(n)$, any ε-almost Lie group modelled on a compact Lie group is diffeomorphic to $\Gamma \setminus G$, with G a Lie group and Γ an extension of a lattice in G. Manifolds modelled on $U(2)$ should be of interest here because they occur in physics.

3. Outline of proof

The starting point of the present proof is Chapter 3 of Buser-Karcher [2], where the linear holonomy $h(\alpha)$ of closed loops α in M is studied. The main result is that if a loop α is not too long and if $h(\alpha)$ is close to the identity in $O(n)$, then, for a suitably small pinching constant $\varepsilon(n)$, $h(\alpha)$ is in fact extremely close to the identity. As a consequence, the points $h(\alpha) \in O(n)$ with α a closed loop in M occur in well defined clusters, and these clusters are the elements of a finite group H. Center of mass arguments show that H is isomorphic to a subgroup of the orthogonal group $O(n)$.

In the first step of the proof, we use the above information on the linear holonomy to construct a flat connection ∇' on TM with holonomy group H. In general, the torsion T' of ∇' is nonzero. We prove that $\|T'\|$ is bounded by a constant times the square root of $\varepsilon(n)$, and therefore is arbitarily small for a suitable choice of $\varepsilon(n)$.

In the second step, we construct a flat connection D near ∇' with parallel torsion $T = T^D$, i.e., $DT = 0$. To solve the partial differential equation $DT = 0$ we follow, as in [9], [10], the method of Newton-Kolmogorov-Moser and solve a linearized deformation equation well enough for the iteration to converge to a connection D with $DT = 0$. As in [9], [10], the construction of D requires the study of a certain elliptic operator Δ''. The additional difficulty here is that Δ'' is not strictly positive because no assumption is made on the sign of the curvature of the riemannian manifold.

We overcome this difficulty by comparing Δ'', lifted to a finite cover Q of M, to the Laplacian on functions. The kernels have the same dimension, and the L_2-norms on the orthogonal complements are nearly the same. Therefore an estimate of Li-Yau [8] on the first eigenvalue of the Laplacian yields an estimate from below for the norm of Δ'' restricted to the orthogonal complement of the kernel.

In the third and final step of this proof we observe that the connection D defines a Lie group structure on the universal cover $\tilde{M}$ of M. Now let N denote this Lie group with underlying manifold $\tilde{M}$. We identify the fundamental group of M with a subgroup Γ of the group A of affine transformations of M. N, via left translation, is a normal subgroup of A. We then define the groups L and H to be kernel and image respectively of the homomorphism $\Gamma \subset A \to A/N$. Finally, we show that, for suitably small $\varepsilon(n)$, the Lie group N is nilpotent.

The norms utilized in this paper are defined exactly as in [10]. In particular, $\| \ \|$ denotes the maximum norm, and $\| \ \|_{s,q}$ the Sobolev norm in L_q involving the first s derivatives.

4. The proof

We begin with a review of the results of [2, Ch. 3] which are relevant for the present proof. To simplify notation we normalize the diameter of M: $d = 1$, and work with curvature bounds $|K| < \Lambda^2$, i.e., $\Lambda^2 = \varepsilon(n)$. We pay no attention to estimates for $\varepsilon(n)$ and prove only the existence of $\varepsilon(n) > 0$ such that the theorem holds.

Let $p \in M$ denote an arbitrary point. The exponential map $\exp\colon T_pM \to M$ has maximal rank at least in a ball B_r with center at $0 \in T_pM$ and radius $r = \pi\Lambda^{-1}$. The curvature assumptions are such that r is large compared to the diameter $d = 1$ of M. On B_r we define the riemannian metric $\exp^* g$, where g is the metric on M, in order to turn exp into a local isometry. Let $u = (X_1, \cdots, X_n)$ denote the field of orthonormal frames on B_r defined by parallel translation, with respect to the Levi-Civita connection of $\exp^* g$, along geodesic rays of an orthonormal frame $u(0)$ in $0 \in B_r$. As usual we identify the frame $u(x)$ in T_xB_r with a linear map, an isometry in this case,

$$u(x)\colon \mathbf{R}^n \to T_xB_r.$$

Let $\pi_r = B_r \cap \exp^{-1}(\{p\})$. The elements of π_r are discrete in B_r. To each $\alpha \in \pi_r$ we associate a local isometry

$$S_\alpha\colon B_r \to B_r$$

defined as follows: Let $dS_\alpha(0)\colon T_0B_r \to T_pM \to T_\alpha B_r$ be the isometry provided by the exponential map mapping T_0B_r and $T_\alpha B_r$ respectively to T_pM, and extend this isometry of tangent spaces by mapping geodesic rays originating at $0 \in B_r$ to corresponding geodesic rays originating at $\alpha \in B_r$. The map S_α is a local isometry, because it projects to the identity on M and it can be extended as long as the geodesic rays remain in B_r.

The parallelization u of B_r gives rise to a holonomy map

$$h\colon \pi_r \to O(n),$$

defined by $\alpha \mapsto u(\alpha)^{-1} \circ dS_\alpha \circ u(0)\colon \mathbf{R}^n \to T_0 B_r \to T_\alpha B_r \to \mathbf{R}^n$. In the following estimates the distance $d(a, b)$, $a, b \in SO(n)$ is defined to be the absolute value of the maximal rotation angle of ab^{-1}. The following proposition summarizes the main results of [2, Ch. 3] relevant to the present proof. The slight modification in notation is due to the fact that we wish to identify the geodesic loops α through $p \in M$ with the corresponding endpoints of geodesic segments starting in $0 \in B_r$, and view them as local diffeomorphisms S_α of B_r. The advantage of this is that the abstractly defined Gromov product of [2] turns into composition of the corresponding local diffeomorphisms.

Proposition. *Let $w = 2 \cdot 14^{\dim SO(n)}$, $\rho \geqslant 10^4 w$. For suitably small $\varepsilon(n) > 0$ the following assertions hold*:

(i) *Let $\alpha, \beta \in \pi_\rho$ satisfy $d(h(\alpha), h(\beta)) < 0.47$. Then $d(h(\alpha), h(\beta)) < 0.01$, and $\alpha \sim \beta$ if $d(h(\alpha), h(\beta)) < 0.47$ is an equivalence relation.*

(ii) *The set H of equivalence classes is a group with multiplication defined by composition of representative local isometries S_α.*

(iii) *The order of H is at most equal to w.*

(iv) *Each element of H can be represented by an element $\alpha \in B_{2w}$.*

In the sequel we wish to view H as a subgroup of $O(n)$. To achieve this we define a map $\omega_0\colon H \to O(n)$, where the image of an equivalence class is its center of mass in $O(n)$. This map is an almost homomorphism in the sense of [4]. Theorem (3.8) of [4] shows that there is a homomorphism $\omega\colon H \to O(n)$ near ω_0. Because of the proposition, assertion (i), ω is injective and we identify H with its image in $O(n)$. Without change in notation, we modify the definition of the holonomy map $\pi_r \to O(n)$ slightly to

$$h\colon \pi_r \to H \subset O(n),$$

where the arrow maps a loop α to the corresponding equivalence class.

After these preparations everything is ready for the first step of the proof, the construction of a flat connection ∇' on TM. To this effect we construct a reduction Q of the structure group $O(n)$ of the principal bundle P of orthonormal frames to H.

Let P_q, $q \in M$ denote the set of orthonormal frames in T_qM, and η a positive function with support in the interior of B_ρ. Let u denote the frame field on B_ρ defined above, and let $\exp^{-1}\{q\}$ denote the inverse image of q, where exp is the exponential map $T_p \to M$ considered earlier. As a consequence of the proposition and the small curvature, the orthonormal frames $\{(\exp_* u(y))g;\ y \in \exp^{-1}\{q\},\ g \in H\} \subset P_q \cong O(n)$ fall into $|H| \leqslant w$ equivalence classes. For every $y \in \exp^{-1}\{q\}$ and $g \in H$ chosen suitably, $(\exp_* u(y))g$

is contained in every equivalence class exactly once, and H permutes the classes. We define the center for each class with respect to the weight $\frac{1}{\nu}\eta(y)$, where $\nu = \Sigma\,\eta(y)$; compare [6] for details. Because the center of mass construction is equivariant with respect to isometries, (right action of H in P_q) H permutes the centers. Let $Q_q \subset P_q$ denote the set of these centers. Now $Q = \bigcup_{q\in M} Q_q$ is a principal bundle with fiber H, and is a reduction of P to the structure group H.

The fiber bundle Q on M defines a flat connection ∇' on the bundle P of orthonormal frames. We proceed to estimate the torsion T' of ∇'. Essentially, ∇' is obtained by averaging the connections defined by the images under the exponential map of the frame field u defined on B_ρ. A first contribution to T' is therefore the torsion of the flat connection ∇^u defined by the frame field u. To estimate this contribution we estimate the difference $\nabla - \nabla^u$, where ∇ is the Levi-Civita connection on B_ρ with respect to the metric $\exp^* g$ defined earlier. Such an estimate was obtained in [5]. The result is $\|\nabla - \nabla^u\| \leq c\,|K| \cdot\rho$, where K is the sectional curvature, c is a constant depending on the definition of $\|\ \|$ only, and the assumption on the pinching constant $\varepsilon(n)$ is such that B_ρ is roughly isometric to a euclidean ball of radius ρ. Since $\|T^u\| \leq \|\nabla - \nabla^u\|$, the above estimate implies that, for sufficiently small $\varepsilon(n)$, $\|T^u\|$ is arbitrarily small. A second contribution to T' is caused by the nonzero derivative with respect to $y \in B_\rho$ of the weight function η utilized in the definition of Q_q. Let η be equal to 1 on $B_{\rho-1}$ and decreasing to zero on $B_\rho - B_{\rho-1}$, approximately with gradient 1. The contribution of $d\eta$ is then proportional to $1/\rho$. A third contribution to T' is caused by the deviation of the center of mass construction from the corresponding linear average of connections. This contribution is negligeable since the equivalence classes utilized in the construction are contained in small balls in $P_q \cong O(n)$. The final estimate is

$$\|T'\| < c\Lambda, \tag{1}$$

where c a constant depending on the dimension, and Λ is the square root of $\varepsilon(n)$.

In the second and main step of the proof we construct a flat connection D near ∇' with parallel torsion. D will be the limit of a sequence of flat connections on M. We start with a few definitions. Let β denote a p-form on M with values in TM provided with a flat metric connection D'. We define

$$\begin{aligned} d'\beta(X_0,\cdots,X_p) &= \sum_{i=0}^{p} (-1)^i (D'_{X_i}\beta)(X_0,\cdots,\hat{X}_i,\cdots,X_p), \\ \delta'\beta(X_2,\cdots,X_p) &= \sum_{k=1}^{n} (D'_{e_k}\beta)(e_k, X_2,\cdots,X_p), \end{aligned} \tag{2}$$

where $(e_1,\cdots,e_n)$ is an orthonormal basis in TM, and (X_i) are vector fields on M.

For further reference we list the expressions for $d' \circ d'$ and $\delta' \circ \delta'$ respectively:

$$\begin{aligned} &d' \circ d'\beta(X_0,\cdots,X_{p+1}) \\ (3) \qquad &= \sum_{i<j} (-1)^{i+j}\big(T'(X_i, X_j)\beta\big)\big(X_0,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{p+1}\big), \end{aligned}$$

where T' is the torsion of D', and $T'(X_i, X_j)\beta$ is an abbreviation for the covariant derivative of β with respect to D' in direction $T'(X_i, X_j)$,

$$(4) \qquad \delta' \circ \delta'\beta(X_3,\cdots,X_p) = \sum_{i,j=1}^{n} \big(T'(e_i, e_j)\beta\big)\big(e_i, e_j, X_3,\cdots,X_p\big).$$

The sequence of flat connections D^i with limit D will be constructed by means of gauge transformations. We continue with listing properties of gauge transformations; compare [1]. Let σ denote a section of $TM \otimes T^*M$, and let $s = (I + \sigma)^{-1}$, where I is the identity in $\mathrm{Hom}(TM, TM) \cong TM \otimes T^*M$. For σ small enough, $s = (I + \sigma)^{-1}$ exists and is a section of $\mathrm{Aut}\, TM$ and thus an element of the gauge group. The gauge group acts on the affine space of connections on TM. The image $s^*D = D^s$ of D under s is

$$D^s = s \circ D \circ s^{-1}.$$

If D is flat, then D^s is flat as well. Let T and T^s denote the torsion of D and D^s respectively. Then

$$\begin{aligned} T^s(X, Y) &= D^s_X Y - D^s_Y X - [X, Y] \\ &= sD_X(s^{-1}Y) - sD_Y(s^{-1}X) - [X, Y] \\ (5) \qquad &= T(X, Y) + s\big(D_X s^{-1}\big)Y - s\big(D_Y s^{-1}\big)X \\ &= T(X, Y) + (I + \sigma)^{-1} d'\sigma(X, Y), \end{aligned}$$

where d' is the "exterior derivative" defined above.

After these preparations we define a sequence of gauge transformations s^i transforming the flat connection ∇' into a flat connection D with parallel torsion. Starting with $D^0 = \nabla'$ we define inductively

$$D^{i+1} = s^* D^i$$

where $s = (I + \sigma^i)^{-1}$, and σ^i is a section in $TM \otimes T^*M$ to be defined below. To facilitate notation we denote σ^i and T^i, the torsion of D^i, by σ and T respectively. Let d'' and δ'' denote the adjoint operators of δ' and d' respectively with respect to a metric g' on TM with $D'g' = 0$. We define

$$\begin{aligned} \Delta' &= d'\delta' + \delta'd', \\ \Delta'' &= d''\delta'' + \delta''d'', \end{aligned} \tag{6}$$

and let α'_0 and α'_1 denote the projections of a differential form α on M with values in TM to $\ker \Delta'$ and $(\ker \Delta')^{\perp}$ respectively. (Same definition for Δ''.)

In the following main lemma we define the differential form σ utilized in the inductive definition of D^i, and state the estimates sufficient to prove convergence.

Main lemma. *Let D' be a flat metric connection on TM with holonomy group H of order $|H| \leqslant w$, and T the torsion of D'. There exists a constant $A > 0$ depending on w, the dimension n, and the curvature K of M such that $\|T\| < A$ implies that $\sigma = \delta''\beta$, where β is the unique solution of $\Delta''\beta = -T'_1$ perpendicular to* $\ker \Delta''$, *satisfies*

(i) $\|d'\sigma + T'_1\| < c\|T\| \cdot \|T'_1\|_{0,q}$

(ii) $\|\sigma\|_{1,q} < c\|T'_1\|_{0,q}$, *where c is a constant depending on w, n and $d =$* diam M, $\| \;\|$ *is the maximum norm, and* $\| \;\|_{i,q}$ *is the Sobolev norm defined in* [10, (5.1)].

The proof of the main lemma is similar to the proof of the main lemma of [10]. The following modifications are required. The L_2-estimate for β in [10] follows from a Bochner formula. The corresponding formula in the present case does not furnish the required information, because no assumption is made on the sign of the curvature. The Bochner formula will be replaced by a comparison of Δ'' to the Laplace operator on functions. The improvement of the L_2-bound to a bound for higher Sobolev norms is the same as in [10] and does not require further discussions. The second modification of the proof is necessary, because the present main lemma is sharper than its counterpart in [10]. Here we estimate $\|d'\sigma + T'_1\|$ in terms of $\|T\| \cdot \|T'_1\|_{0,q}$ and not just $\|T\|^2$.

We first estimate $\|\beta\|_{0,2}$. To do this we work on the holonomy bundle Q over M. The reason is that TQ is trivial. Let $u = (e_1, \cdots, e_n)$ denote an orthonormal frame field, parallel with respect to D' lifted to Q. To estimate the norm of Δ'' on $(\ker \Delta'')^{\perp}$ we compare Δ'' to the Laplace operator Δ (for functions on Q) operating on the component functions β^i_{jk}, with respect to the frame field u of β.

The adjoints of d' and δ' respectively with respect to the metric defined by u are

$$\begin{aligned}\delta''\alpha(X_2,\cdots,X_p) &= -\sum_{k=1}^{n}\left(D_{e_k}\alpha\right)(e_k, X_2,\cdots,X_p)\\ &\quad-\sum_{k=1}^{n}(\operatorname{div} e_k)\alpha(e_k, X_2,\cdots,X_p),\end{aligned}\tag{7}$$

$$\begin{aligned}d''\alpha(X_0,\cdots,X_p) &= \sum_{i=0}^{p}(-1)^i\left(D_{X_i}\alpha\right)(X_0,\cdots,\hat{X}_i,\cdots,X_p)\\ &\quad+\sum_{i=0}^{p}(-1)^i(\operatorname{div} X_i)\alpha(X_0,\cdots,\hat{X}_i,\cdots,X_p),\end{aligned}\tag{8}$$

where $\operatorname{div} e_i$ is the divergence of e_i with respect to the volume form provided by u, and the vector fields X_i are parallel.

The Laplace operator applied to the component functions of a 2-form α with values in TQ is given by the formula

$$\Delta\alpha^i_{jk} = -\alpha^i_{jk;pp} - (\operatorname{div} e_p)\alpha^i_{jk;p},\tag{9}$$

where the index p after the semi-colon indicates the derivative in direction e_p, and the Einstein summation convention is in effect.

The corresponding formula for Δ' is

$$(\Delta'\alpha)^i_{jk} = -\alpha^i_{jk;pp} + T^q_{pj}\alpha^i_{kp;q} - T^q_{pk}\alpha^i_{pj;q},\tag{10}$$

where T^q_{pj} are the components of the torsion T of D'. From (9) and (10) it follows that

$$\Delta\alpha^i_{jk} - (\Delta'\alpha)^i_{jk} = -(\operatorname{div} e_p)\alpha^i_{jk;p} - T^q_{pj}\alpha^i_{kp;q} + T^q_{pk}\alpha^i_{jp;q}.\tag{11}$$

We observe that the difference between the two operators is a first order differential operator of norm $\|T\|$. In addition, the kernel of Δ, which is equal to the space of forms with constant components, is contained in the kernel of Δ'. Let λ denote the first eigenvalue of Δ on the manifold Q, and λ' a lower bound for the L_2-norm of Δ' applied to the elements in $(\ker\Delta)^\perp$. (11) yields

$$\lambda' > \lambda - c\|T\|.\tag{12}$$

Thus, for $\|T\|$ small enough, λ' is essentially bounded from below by λ. To obtain a lower bound for λ' we utilize the estimate

$$\lambda \geqslant \frac{\pi^2}{4d_Q^2} + \min\{(n-1)\delta, 0\}\tag{13}$$

of Li-Yau [8], where δ is the Ricci curvature of the riemannian manifold Q, and d_Q is its diameter. In our case, for suitably small pinching constant $\varepsilon(n)$, the metric on Q is a small perturbation, in the C^1-topology, of the original metric on M, and the above estimate applies with δ close to zero. In addition, $d_Q < dw$. The conclusion is that $\ker \Delta = \ker \Delta'$, and that $\Delta'^{-1}\colon (\ker \Delta'')^{\perp} \to (\ker \Delta')^{\perp}$ is bounded in L_2 by a constant depending only on the dimension of M. (The diameter d of M was normalized to 1.) The analogous estimate holds for Δ''^{-1}, and we obtain the following estimate for the differential form β of the main lemma:

$$\|\beta\|_{0,2} < c\|T_1'\|_{0,q}. \tag{14}$$

As in [10, (5.7)] interior regularity estimates for the elliptic operator Δ'' yield

$$\|\beta\|_{1,q} < c\|T_1'\|_{0,q}. \tag{15}$$

We continue with an estimate for $\operatorname{div} e_p$. The Levi-Civita connection for the metric defined by u is equal to

$$D' + A^{-1}T,$$

where $A\colon V \wedge V \otimes V \to V \otimes V \wedge V$ is the vector space isomorphism defined by skew symmetrization in the last two variables, $V \cong V^*$ is the tangent space of Q at the point under consideration, and T is the torsion of D'. Let U^i_{jk} denote the components of $U = A^{-1}T$ with respect to the frame field $u = (e_1, \cdots, e_n)$. Because $D'e_i = 0$, the divergence of e_k is

$$\operatorname{div} e_k = -U^p_{pk}.$$

We define $U_0' = A^{-1}T_0'$, $U_1' = A^{-1}T_1'$, where T_0' and T_1' are, as defined earlier, the projections of T to $\ker \Delta'$ and $(\ker \Delta')^{\perp}$ respectively. Now T_0', and therefore U_0', have constant coefficients with respect to u. Since the integral over Q of $\operatorname{div} e_k$ is zero, and $\|T\| < 1$, we obtain

$$\|\operatorname{div} e_k\|_{0,q} < c\|T_1'\|_{0,q}. \tag{16}$$

As a consequence, the norms of the following differential operators of order zero are

$$\|d' - d''\|_{0,q} < c\|T_1'\|_{0,q}, \tag{17}$$

$$\|\delta' - \delta''\|_{0,q} < c\|T_1'\|_{0,q}. \tag{18}$$

We continue with an estimate for $d'T_1'$. The first Bianchi equation, because the curvature of D' is zero, reads

$$d'T'(e_i, e_j, e_k) = d'T_1'(e_i, e_j, e_k) = -ST'\big(T'(e_i, e_j), e_k\big),$$

where S denote the cyclic sum with respect to i, j, k. We abbreviate $T'(T'(e_i, e_j), e_k) = T' \circ T'(e_i, e_j, e_k)$ and have

$$d'T_1' = S(T_0' \circ T_0') + S(T_1' \circ T_0') + S(T_0' \circ T_1') + S(T_1' \circ T_1').$$

Because of the estimate (18) and $D'T_0' = 0$, we obtain

$$\|d'T_1'\|_{0,2} < c\|T\| \cdot \|T_1'\|_{0,q}.$$

In addition, $\|d'T_1'\| < c\|T\| < 1$, and therefore

$$\begin{aligned} &\|d'T_1'\|_{0,q} < c\|T\| \cdot \|T_1'\|_{0,q}, \\ &\|d''T_1'\|_{0,q} < c\|T\| \cdot \|T_1'\|_{0,q}. \end{aligned} \tag{19}$$

The basic strategy for the proof of the main theorem is the same as in [10]. Let $W_{s,q}$ denote the Sobolev space of differential forms on Q with values in TQ involving the first s derivatives and $q > n$ with norm defined in [10, (5.1)]. The estimate (15) yields $W_{0,q}$ estimates for $\sigma = \delta''\beta$ and $d''\beta$. First, we improve these estimates to $W_{1,q}$ estimates. The basic tool is the following L_q-interior regularity estimate for the solution u of a linear elliptic system $Lu = f$ of order s.

$$\|u\|_{r+s,q} < c(\|u\|_{0,2} + \|f\|_{r,q}), \tag{20}$$

where the constant c depends on the coefficients of the operator L and the domain of definition of u. As in [9], [10], it is important to be sure that c depends only on the data provided in the main lemma. Since the proof of this is exactly the same as in the previous papers [9], [10] we omit the discussion of the constants here. As in [9], [10], we cannot use the full strength of (20) to estimate $\|\beta\|_{2,q}$ instead of the estimate (15), because the assumptions on the coefficients of Δ'' are not strong enough for this. Instead we estimate $\|\sigma\|_{1,q}$ of the main lemma in the following more complicated fashion.

We define

$$\gamma = -(d''\sigma + T_1'), \tag{21}$$

and obtain the elliptic systems

$$d''\sigma = -T_1' - \gamma, \quad \delta''\sigma = \delta'' \circ \delta''\beta. \tag{22}$$

$$d''(d''\beta) = d'' \circ d''\beta, \quad \delta''(d''\beta) = \gamma. \tag{23}$$

Because of (3), (4), and (15) the right-hand sides of both systems are bounded by $(\|T_1'\|_{0,q} + \|\gamma\|_{0,q})$, and (20) yields

$$\|\sigma\|_{1,q} + \|d'\beta\|_{1,q} \leqslant c(\|T_1'\|_{0,q} + \|\gamma\|_{0,q}). \tag{24}$$

The differential form γ satisfies the elliptic system

$$d''\gamma = -d''(d''\sigma + T_1'), \quad \delta''\gamma = \delta'' \circ \delta''(d''\beta). \tag{25}$$

In addition, (21), together with (17) and (18), shows that the norm $\|\gamma_0''\|_{0,2}$ of the projection of γ to $\ker\Delta''$ is bounded by $c\|T_1'\|_{0,q}^2$. On the other hand, the L_2-norm of Δ'' on $(\ker\Delta'')^\perp$ is bounded from below essentially by $\frac{1}{4}\pi^2/d_Q^2$. This follows from (13), (17) and (18). Now (19), (20), (24) and (25) yield

$$(26)\qquad \begin{gathered}\|\gamma\|_{1,q} < c\|T\| \cdot \left(\|T_1'\|_{0,q} + \|\gamma\|_{0,q}\right),\\ \|\gamma\|_{1,q} < c\|T\| \cdot \|T_1'\|_{0,q}.\end{gathered}$$

This estimate, by the Sobolev lemma, implies the first assertion of the main lemma. The second follows from (24) and (26).

The main lemma proves that the sequence (D^i), $i \in \mathbf{N}$, of flat connections converges, if $\varepsilon(n)$ is chosen suitably small, to a flat connection D with parallel torsion. To check this claim let $(T^i)_{i\in\mathbf{N}}$ denote the sequence of torsion tensors of the connections D^i, and let T_0^i and T_1^i denote the projections of T^i to the space of parallel tensors with respect to D^i, and its orthogonal complement respectively. We prove that $\sum_{i=0}^{\infty}\|T_1^i\|_{0,q}$ converges with the speed of a geometric series $\sum r^i$ with $0 < r < \theta < 1$, where a suitable choice of $\varepsilon(n)$ permits any $\theta > 0$. The computation (5) implies

$$(27)\qquad T^{i+1} = T_0^i + T_1^i + d'\sigma + O(\|\sigma\| \cdot \|d'\sigma\|),$$

where $O(\|\sigma\| \cdot \|d'\sigma\|)$ denotes a continuous term of maximum norm $c\|\sigma\| \cdot \|d'\sigma\|$. The main lemma implies

$$(28)\qquad T^{i+1} = T_0^i + O\left(\|T^i\| \cdot \|T_1^i\|_{0,q}\right).$$

While T_0^i is by definition parallel with respect to D^i, it may not be parallel with respect to $D^{i+1} = D^i + (I+\sigma)^{-1}D^i\sigma$. However, because of assertion (ii) of the main lemma, the Sobolev lemma and estimate (20), the projection of T_0^i to the orthogonal complement of the kernel of $\Delta^{i+1} = d^{i+1}\delta^{i+1} + \delta^{i+1}d^{i+1}$ satisfies the estimate

$$(29)\qquad \|T_1^{i+1}\|_{0,q} < c\|T^i\| \cdot \|T^i\|_{0,q}.$$

The iteration starts with the connection $D^0 = \nabla'$ whose torsion, because of (1), can be rendered as small as we please by a suitable choice of $\varepsilon(n)$. (28) shows that the sequence of continuous torsion tensors (T^i), for a suitable choice of $\varepsilon(n)$, is bounded. In fact we can achieve $c\|T^i\| < \theta$ for c as in (29) and any $\theta > 0$.

The main lemma, assertion (ii), and (29) now imply that $(D^i)i \in \mathbf{N}$ converges in $W_{0,q}$ to a connection D. The torsion tensors converge to the torsion tensor T of D in $W_{0,q}$ as well. The projection of T to $(\ker\Delta)^\perp$ is zero in $W_{0,q}$ and therefore vanishes identically, and T has constant coefficients in terms of a parallel section u. T satisfies the Jacobi identity and defines a Lie algebra $\mathfrak{g}$

with bracket $[\,,] = T$ on $\mathbf{R}^n = u^{-1}T_pQ$. The differential form $\omega\colon TM \to \mathfrak{g}$, defined by $X \mapsto u^{-1}X$, satisfies the system of elliptic differential equations

$$d\omega + [\omega, \omega] = 0, \quad \delta\omega = 0.$$

Therefore ω, and hence u and D, are smooth.

In the final step of the proof of the theorem we lift the covariant derivative D to the universal cover $\tilde{M}$ of M. Let $e \in \tilde{M}$ denote an arbitrary point, and $u(e)$ an orthonormal frame in $T_e\tilde{M}$. Let $\{X_1, \cdots, X_n\}$ denote the vector fields on $\tilde{M}$ obtained by parallel translation of $u(e)$ with respect to the flat connection D. Because $DT = 0$, where T is the torsion of D, the vector fields $\{X_i\}$ are a basis of some Lie algebra. Let N denote the Lie group with underlying manifold $\tilde{M}$ and neutral element $e \in \tilde{M}$ defined by $\{X_i\}$ viewed as a basis of the left invariant vector fields. N will turn out to be a nilpotent Lie group.

The Lie group of affine transformations A on $N = \tilde{M}$ is by definition the group of diffeomorphisms of $\tilde{M}$ leaving the connection D invariant. The fundamental group Γ of M acts by decktransformations on $\tilde{M}$ and is a subgroup of A. The Lie group N, via left translations, is also a subgroup of A. N is in fact a normal subgroup because the differential of a diffeomorphism $\zeta \in A$ rotates the vector fields by a constant orthogonal matrix. We define the lattice L and the finite group H of the theorem to be the kernel and image respectively of the homomorphism $\Gamma \subset A \to A/N$.

To complete the proof we refer to a theorem of Zassenhaus and Kazdan-Margulis, compare Raghunathan [12, Th. 8.16]: There exists a neighborhood $U \subset N$ such that if L is any discrete subgroup of N, then $L \cap U$ is contained in a connected nilpotent Lie subgroup of N. In addition, the size of the neighborhood U can be estimated in terms of the Lie bracket. Since the index of $L \subset \Gamma$ is bounded by the number w of the proposition, the pinching constant $\varepsilon(n)$ can be chosen so small that the nilpotent subgroup of the above theorem coincides with N and N is nilpotent.

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