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HOLOMORPHIC CURVATURES OF ALMOST KÄHLER MANIFOLDS

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1. Introduction and notation

An almost Kähler manifold is an almost Hermitian manifold (M, g, F) such that:

$$(D_X F')(Y, Z) + (D_Y F')(Z, X) + (D_Z F')(X, Y) = 0, \tag{1}$$

where $F'(X, Y) = g(FX, Y)$, D is the natural metric connection, and X, Y, Z are arbitrary vector fields. In this paper we study holomorphic sectional and bisectional curvatures on an almost Kähler manifold. In particular, we obtain results similar to those given by Bishop, Goldberg and Kobayashi [1], [4] on Kähler manifolds and by Gray [5] on nearly Kähler manifolds.

Throughout the paper, we rely heavily on the following identity [2], [3]:

$$\begin{aligned} &K'(X, Y, \bar{Z}, W) + K'(X, Y, Z, \bar{W}) \\ &\quad = \tfrac{1}{2}F'((D_X F)Y - (D_Y F)X, (D_Y F)X, (D_Z F)W - (D_W F)Z), \end{aligned} \tag{2}$$

from which we obtain

$$\begin{aligned} &K'(X, Y, \bar{Z}, \bar{W}) - K'(X, Y, Z, W) \\ &\quad = -\tfrac{1}{2}g((D_X F)Y - (D_Y F)X, (D_Z F)W - (D_W F)Z), \end{aligned} \tag{3}$$

$$K'(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = K'(X, Y, Z, W), \tag{4}$$

$$\overline{K(X, Y, \bar{Z})} + K(X, Y, Z) = -\tfrac{1}{2}\left(D_{(D_X F)Y} - F_{(D_Y F)X}\right)Z, \tag{5}$$

where $K(X, Y, Z)$ is the Riemannian curvature tensor, $\bar{X} = FX$ and $K'(X, Y, Z, W) = g(K(X, Y, Z), W)$. The author would like to thank Professor Dr. R. S. Mishra for suggesting the problem and the useful discussions.

2. Holomorphic curvatures

The holomorphic bisectional curvature $h(X, Y)$ is given by

$$h(X, Y) = \frac{-K'(X, \bar{X}, Y, \bar{Y})}{g(X, X)g(Y, Y)}.$$

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Proposotion 2.1. *In an almost Kähler manifold we have*

$$h(X,Y)\|X\|^2\|Y\|^2 = K(X,Y)\|X\wedge Y\|^2 + K(\overline{X},Y)\|\overline{X}\wedge Y\|^2 + \|(D_XF)Y-(D_YF)X\|^2$$

where $K(X,Y)$ is the sectional curvature of plane fields determined by X, Y, and

$$\|X\wedge Y\|^2 = \|X\|^2\|Y\| - [g(X,Y)]^2.$$

Proof. Since

$$h(X,Y)\|X\|^2\|Y\|^2 = -K'(X,\overline{X},Y,\overline{Y}),$$

using Bianchi's first identity we get

$$h(X,Y)\|X\|^2\|Y\|^2 = K'(\overline{X},Y,X,\overline{Y}) - K'(X,Y,\overline{X},\overline{Y}).$$

In consequence of (2) and (3) the above equation takes the form

$$h(X,Y)\|X\|^2\|Y\|^2 = -K'(X,Y,X,Y) - K'(\overline{X},Y,\overline{X},Y) + \|(D_XF)Y-(D_YF)X\|^2.$$

Now since

$$K(X,Y) = \frac{-K'(X,Y,X,Y)}{g(X,X)g(Y,Y)-g(X,Y)g(X,Y)},$$

the result follows.

If $K'(X,\overline{X},X,\overline{X})$ is denoted by $Q(X)$, then by direct computations we get

Proposition 2.2. *In an almost Kähler manifold we have*

$$K'(X,Y,X,Y) = \frac{1}{32}\{3Q(X+\overline{Y}) + 3Q(X-\overline{Y}) - Q(X+Y) - Q(X-Y) \times 4Q(X) - 4Q(Y)\} + \frac{3}{8}\|(D_XF)Y-(D_YF)X\|^2,$$

$$K'(X,\overline{X},Y,\overline{Y}) = \frac{1}{16}\{Q(X+\overline{Y}) + Q(X-\overline{Y}) + Q(X+Y) + Q(X-Y) \times 4Q(X) - 4Q(Y)\} - \tfrac{1}{4}\|(D_XF)Y-(D_YF)X\|^2.$$

The holomorphic sectional curvature $H(X)$ is defined by

$$H(X) = \frac{-K'(X,\overline{X},X,\overline{X})}{\|X\|^4} = \frac{-Q(X)}{\|X\|^4}.$$

Thus $H(X) = h(X,X)$.

The relation between sectional, holomorphic sectional and holomorphic bisectional curvatures is given by the following.

Proposition 2.3. *The sectional curvature $K(X, Y)$ and the holomorphic bisectional curvature $h(X, Y)$ with respect to the fields of planes determined by X, Y with $\|X\| = \|Y\| = 1$, $g(X, Y) = \operatorname{Cos}\phi \geqslant 0$ and $g(X, \bar{Y}) = \operatorname{Cos}\theta \geqslant 0$ are respectively given by*

$$K(X, Y) = \frac{1}{8}\{3(1 + \operatorname{Cos}\theta)^2 H(X + \bar{Y}) + 3(1 - \operatorname{Cos}\theta)^2 H(X - \bar{Y}) \\ -H(X + Y) - H(X - Y) - H(X) - H(Y)\} \\ -\frac{3}{8}\|(D_X F)Y - (D_Y F)X\|^2,$$

if $g(X, Y) = \operatorname{Cos}\phi = 0$ *and*

$$h(X, Y) = \tfrac{1}{4}\{(1 + \operatorname{Cos}\theta)^2 H(X + \bar{Y}) + (1 - \operatorname{Cos}\theta)^2 H(X - \bar{Y}) \\ + (1 + \operatorname{Cos}\phi)^2 H(X + Y) + (1 + \operatorname{Cos}\phi)^2 H(X - Y) \\ -H(X) - H(Y)\} + \tfrac{1}{4}\|(D_X F)Y - (D_Y F)X\|^2.$$

Proof. The proof follows from Proposition 2.2.

Proposition 2.4. *Let M be an almost Kähler manifold, and X a unit vector field for which the holomorphic sectional curvature $H(X)$ assumes its minimum at $m \in M$. Then for all $Y \in T_m M$, (the tangent space at m) with $g(X, Y) = F'(X, Y) = 0$ and $\|Y\| = 1$ we have*

$$H(X) < \tfrac{3}{2}\|(D_X F)Y - (D_Y F)X\|^2 - 3K'(X, Y, X, Y) - K'(X, \bar{Y}, X, \bar{Y}).$$

If $H(X)$ assumes its maximum, then the inequality is reversed.

Proof. If a, b are real numbers such that $a^2 + b^2 = 1$, then under the conditions of the proposition we have

$$H(ax + b\bar{y}) = -Q(ax + b\bar{y}),$$
$$H(ax + b\bar{y}) + H(ax - b\bar{y}) - 2a^4 H(x) - 2b^4 H(y) \\ = -4a^2b^2\{K'(X, \bar{X}, Y, \bar{Y}) + K'(X, Y, X, Y) + K'(X, Y, \bar{X}, \bar{Y})\} \\ = -4a^2b^2\{K'(X, Y, X, Y) + 2K'(X, Y, \bar{X}, \bar{Y}) - K'(X, \bar{Y}, \bar{X}, Y)\} \\ = -4a^2b^2\{3K'(X, Y, X, Y) + K'(X, \bar{Y}, X, \bar{Y}) \\ -\tfrac{3}{2}\|(D_X F)Y - (D_Y F)X\|^2\}.$$

If $H(X)$ assumes its minimum at X, then

$$(1 - a^4)H(X) < b^4 H(Y) - 2a^2b^2\{3K'(X, Y, X, Y) + K'(X, \bar{Y}, X, \bar{Y}) \\ -\tfrac{3}{2}\|(D_X F)Y - (D_Y F)X\|^2\}.$$

The result follows when we take limit of both sides of the inequality as $a \to 1$.

Remarks. (i) The above proof is similar (with a difference in sign) to that given by Goldberg and Kobayashi [4] and Gray [5].

(ii) We take limit as $a \to 1$, rather than taking $a = 1$, $b = 0$ as was done in the above mentioned references, because in the latter case both sides of the inequality reduce to 0.

3. Almost Kähler manifolds with constant holomorphic sectional curvature

In this part, we assume that the holomorphic sectional curvature $H(X)$ has a constant value k.

Theorem 3.1. *A necessary and sufficient condition for an almost Kähler manifold M to admit a constant holomorphic sectional curvature k is that for any vector fields X, Y, Z, W on M the following identity hold:*

$$\begin{aligned}8K'(X,Y,Z,W) &- 2g((D_XF)Y-(D_YF)X,(D_ZF)W-(D_WF)Z\\&-g((D_XF)W-(D_WF)X,(D_ZF)Y-(D_YF)Z)-g((D_YF)W\\&-(D_WF)Y,(D_XF)Z-(D_ZF)X))\\&=-2k\{2F'(X,Y)F'(Z,W)+g(X,Z)g(Y,W)-g(Y,Z)g(X,W)\\&\qquad+F'(Y,Z)F'(W,X)+F'(X,Z)F'(Y,W)\},\end{aligned}$$

or equivalently the following identity holds:

$$\begin{aligned}8K(X,Y,Z) &= -2\big(D_{(D_XF)Y}-F_{(D_YF)X}\big)Z-\big(D_{(D_ZF)Y}-F_{(D_YF)Z}\big)X\\&\quad-\big(D_{(D_XF)Z}-F_{(D_ZF)X}\big)Y-2k\{2F'(X,Y)\overline{Z}+g(X,Z)Y\\&\quad-g(Y,Z)X-F'(Y,Z)\overline{X}+F'(X,Z)\overline{Y}\}.\end{aligned}$$

Proof. A necessary and sufficient condition for an almost Hermitian manifold to be of constant holomorphic sectional curvature, given by Mishra [6], is

$$\begin{aligned}&3K'(X,Y,Z,W)+3K'(\overline{X},\overline{Y},Z,W)+3K'(X,Y,\overline{Z},\overline{W})\\&\quad+3K'(\overline{X},\overline{Y},\overline{Z},\overline{W})+K'(X,W,\overline{Z},\overline{Y})+K'(Y,W,\overline{X},\overline{Z})\\&\quad+K'(Y,Z,\overline{W},\overline{X})+K'(Z,X,\overline{W},\overline{Y})+K'(Y,\overline{W},Z,\overline{X})\\&\quad+K'(\overline{W},X,Z,\overline{Y})+K'(X,\overline{Z},W,\overline{Y})+K'(\overline{Z},Y,W,\overline{X})\\&=-4k\{2F'(X,Y)F'(Z\,W)+g(X,Z)g(Y,W)-g(Y,Z)g(X,W)\\&\qquad+F'(Y,Z)F'(W,X)+F'(X,Z)F'(Y,W)\}.\end{aligned}$$

Our result follows from the above and identities (1)–(5).

In fact, the above result can be improved as follows.

Theorem 3.2. *A necessary and sufficient condition for an almost Kähler manifold to be of constant holomorphic sectional curvature is that $K(X, Y, Z)$ be a linear combination of* $(D_{(D_XF)Y} - F_{(D_YF)X})Z$, $(D_{(D_ZF)Y} - F_{(D_YF)Z})X$, $(D_{(D_XF)Z} - F_{(D_ZF)X})Y$, X, Y, $\overline{X}$, $\overline{Y}$ *and* $\overline{Z}$.

Proof. If the holomorphic sectional curvature is constant, then the result follows from Theorem 3.1.

Now suppose that $K(X, Y, Z)$ is a linear combination of the terms mentioned, that is,

$$\begin{aligned} K(X, Y, Z) = {} & \alpha\big(D_{(D_XF)Y} - F_{(D_YF)X}\big)Z + \beta\big(D_{(D_ZF)Y} - F_{(D_YF)Z}\big)X \\ & +\gamma\big(D_{(D_XF)Z} - F_{(D_ZF)X}\big)Y + 2A(X, Y)\overline{Z} + B(X, Z)Y \\ & -B(Y, Z)X - C(Y, Z)\overline{X} + C(X, Z)\overline{Y}, \end{aligned}$$

where α, β, γ are real numbers, A, B, C are bilinear, and A is skewsymmetric. Applying Bianchi's first identity gives

$$\alpha = \beta + \gamma,\ B \text{ is symmetric}, \quad 2A(X, Y) = C(X, Y) - C(Y, X).$$

Since K is skewsymmetric in X, Y, we have

$$\beta = \gamma \to \beta = \gamma = \tfrac{\alpha}{2}.$$

Now

$$\begin{aligned} & K'(X, Y, Z, W) \\ & \quad = -\alpha g((D_XF)Y - (D_YF)X, (D_ZF)W - (D_WF)Z) \\ & \quad\quad -\tfrac{\alpha}{2} g((D_XF)W - (D_WF)X, (D_ZF)Y - (D_YF)Z) \\ & \quad\quad -\tfrac{\alpha}{2} g((D_YF)W - (D_WF)Y, (D_XF)Z - (D_ZF)X) \\ & \quad\quad +2A(X, Y)F'(Z, W) + B(X, Z)g(Y, W) - B(Y, Z)g(X, W) \\ & \quad\quad -C(Y, Z)F'(X, W) + C(X, Z)F'(Y, W). \end{aligned}$$

Using (3) we get $\alpha = -\frac{1}{4}$, $\beta = \gamma = -\frac{1}{8}$ and $C(Y, Z) = -B(Y, \overline{Z})$, $C(X, Z) = -B(X, \overline{Z})$. Similarly, for $\overline{X}$, $\overline{Y}$ we have $A(\overline{X}, \overline{Y}) = A(X, Y)$, $B(\overline{X}, \overline{Z}) = B(X, Z)$, C is skewsymmetric, and $A(X, Y) = C(X, Y) = B(\overline{X}, Y)$.

Using the symmetry of $K'(X, Y, Z, W)$ in the pairs X, Y and Z, W yields

$$\begin{aligned} & - B(Y, Z)g(X, W) - B(\overline{Y}, Z)F'(X, W) \\ & \qquad\qquad + B(W, X)g(Z, Y) + B(\overline{X}, W)F'(Y, Z) \\ & \qquad\qquad = 2B(\overline{Z}, W)F'(X, Y) - 2B(\overline{X}, Y)F'(Z, W). \end{aligned} \tag{6}$$

Also since $K'(X, Y, Z, W)$ is skewsymmetric in Z, W, we have

$$\tag{7} \begin{aligned} &-B(Y,Z)g(X,W) - B(\overline{Y},Z)F'(X,W) \\ &+B(W,X)g(Z,Y) + B(\overline{X},W)F'(Y,Z) \\ &\qquad = -B(X,Z)g(Y,W) - B(\overline{X},Z)F'(Y,W) \\ &\qquad\quad +B(Y,W)g(X,Z) + B(\overline{Y},W)F'(X,Z). \end{aligned}$$

From (6) and (7) it follows that

$$\tag{8} \begin{aligned} &2B(\overline{Z},W)F'(X,Y) - 2B(\overline{X},Y)F'(Z,W) \\ &\quad = -B(X,Z)g(Y,W) - B(\overline{X},Z)F'(Y,W) \\ &\quad\quad +B(Y,W)g(X,Z) + B(\overline{Y},W)F'(X,Z). \end{aligned}$$

Since the left-hand side of (8) is skewsymmetric in X, Y and in Z, W, the right-hand side has the same property. So interchanging X, Y with Z, W we should have the same quantity, and the right-side of (8) also equals

$$-B(Y,W)g(X,Z) - B(\overline{Y},W)F'(X,Z) + B(X,Z)g(Y,W) + B(\overline{X},Z)F'(Y,W).$$

Comparing this with what we have in (8) gives

$$2B(\overline{Z},W)F'(X,Y) - 2B(\overline{X},Y)F'(Z,W) = 0,$$

or

$$\frac{B(\overline{Z},W)}{g(\overline{Z},W)} = \frac{B(\overline{X},Y)}{g(\overline{X},Y)} = k,$$

from which follows the result.

Now if we take the trace of the curvature tensor in Theorem 3.1 we get

Theorem 3.3. *In an almost Kähler manifold of constant holomorphic sectional curvature k, the Ricci tensor is given by*

$$8\,\mathrm{Ric}(Y,Z) - 2k(n+2)g(Y,Z) - 2e^i\left(D_{(D_{e_i}F)Y} - F_{(D_YF)e_i}\right)Z$$

$$-e^i\left(D_{(D_{e_i}F)Z-(D_ZF)e_i}\right)Y,$$

where n is the dimension of the manifold, $\{e_i\}$ is a local basis of the tangent space, and $\{e^i\}$ its dual basis.

Theorem 3.4. *Let M be an almost Kähler manifold of constant holomorphic sectional curvature k. Then the sectional curvature $K(X, Y)$ with respect to vector fields X, Y with $\|X\| = \|Y\| = 1$ and $g(X, Y) = 0$ is given by*

$$K(X,Y) = \tfrac{k}{4}\left(3F'(X,Y)^2 + 1\right) - \tfrac{3}{8}\|(D_XF)Y - (D_YF)X\|^2,$$

and the holomorphic bisectional curvature $h(X, Y)$ *by*

$$h(X, Y) = \tfrac{k}{2}\left(F'(X, Y)^2 + 1\right) + \tfrac{1}{4}\|(D_X F)Y - (D_Y F)X\|^2.$$

The proof follows from Proposition 2.3.

Theorem 3.5. *Let M be an almost Kähler manifold of constant holomorphic sectional curvature k. If X, Y are vector fields such that* $\|X\| = \|Y\| = 1$ *and* $g(X, Y) = 0$, *then*

$$k \leqslant 4K(X, Y) + \tfrac{3}{2}\|(D_X F)Y - (D_Y F)X\|^2 \leqslant 4k, \quad k > 0.$$

$$k \leqslant 2h(X, Y) - \tfrac{1}{2}\|(D_X F)Y - (D_Y F)X\|^2 \leqslant 2k, \quad k > 0.$$

If $k < 0$, *the above inequalities are reversed.*

References

[1] R. L. Bishop & S. I. Goldberg, *Some implications of the generalised Gauss-Bonnet theorem*, Trans. Amer. Math. Soc. **112** (1964) 508–535.

[2] H. Farran, *On almost Kähler manifolds*, Progress of mathematics **13** (1979) 47–52.

[3] ______, *On almost Kähler manifolds. II*, Progress of mathematics **14** (1980) 21–28.

[4] S. I. Goldberg & S. Kobayashi, *Holomorphic bisectional curvature*, J. Differential Geometry **1** (1967) 225–233.

[5] A. Gray, *Nearly Kähler Manifolds*, J. Differential Geometry **4** (1970) 283–309.

[6] R. S. Mishra, *Structures on differentiable manifolds*, to be published.

[7] K. Yano, *Differential geometry on complex and almost complex spaces*, Pergamon Press, New York, 1965

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