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CR SUBMANIFOLDS OF A COMPLEX PROJECTIVE SPACE

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Dedicated to Professor Shoshichi Kobayashi on his 50th birthday

Introduction

CR submanifolds of a Kaehlerian manifold have been defined by A. Bejancu [1] in 1978 and are now being studied by various authors. See [1], [2], [4], [5], [9], [10], [11] and [12]. The purpose of the present paper is to study CR submanifolds of a complex projective space.

In §1 we first state generalities on submanifolds of Kaehlerian manifolds. We then define CR submanifolds and prove Theorem 1.1 which gives a necessary and sufficient condition for a submanifold of a Kaehlerian manifold to be a CR submanifold.

§2 is devoted to the study of a CR submanifold of a complex projective space with semi-flat normal connection.

In §3 we prove an integral formula which has been essentially given in [7], and in §4 we treat with the cases of CR submanifolds with parallel mean curvature vector.

Finally in §5 we consider CR submanifolds of a complex projective space with flat normal connection and parallel mean curvature vector.

1. Submanifolds of Kaehlerian manifolds

Let $\bar{M}$ be a complex m-dimensional (real $2m$-dimensional) Kaehlerian manifold with almost complex structure J. We denote by g the Hermitian metric tensor field of $\bar{M}$. Let M be a real n-dimensional Riemannian manifold isometrically immersed in $\bar{M}$. We denote by the same g the Riemannian metric tensor field induced on M from that of $\bar{M}$. The operator of covariant differentiation with respect to the Levi-Civita connection in $\bar{M}$ (resp. M) will

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be denoted by $\bar{\nabla}$ (resp. ∇). Then the Gauss and Weingarten formulas are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y), \quad \bar{\nabla}_X V = -A_V X + D_X V$$

for any vector fields X, Y tangent to M and any vector field V normal to M, where D denotes the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T(M)^{\perp}$ of M from that of $\bar{M}$. A and B are both called the second fundamental tensors of M and they are related by

$$g(B(X, Y), V) = g(A_V X, Y).$$

For the second fundamental tensor A we define its covariant derivative $\nabla_X A$ along X by

$$(\nabla_X A)_V Y = \nabla_X (A_V Y) - A_{D_X V} Y - A_V \nabla_X Y.$$

The *mean curvature vector* μ of M is defined to be $\mu = (\operatorname{Tr} B)/n$, where $\operatorname{Tr} B$ is the trace of B. If $B = 0$ (or $A = 0$) identically, then M is said to be *totally geodesic*, and if $\mu = 0$, then M is said to be *minimal*. A normal vector field V on M is said to be *parallel* if $D_X V = 0$ for any vector field X tangent to M.

For any vector field X tangent to M we put

$$JX = PX + FX, \tag{1.1}$$

where PX is the tangential part, and FX the normal part of JX. Then P is an endomorphism on the tangent bundle $T(M)$, and F is a normal bundle valued 1-form on the tangent bundle $T(M)$. Similarly, for any vector field V normal to M, we put

$$JV = tV + fV, \tag{1.2}$$

where tV is the tangential part, and fV the normal part of JV. For any vector field Y tangent to M, from (1.1) we have $g(JX, Y) = g(PX, Y)$ which shows that $g(PX, Y)$ is skew symmetric. Similarly, for any vector field U normal to M, from (1.2) we have $g(JV, U) = g(fV, U)$ which shows that $g(fV, U)$ is skew symmetric. We also have, from (1.1) and (1.2),

$$g(FX, V) + g(X, tV) = 0, \tag{1.3}$$

which gives the relation between F and t.

Now applying J to (1.1) and using (1.1), (1.2), we find

$$P^2 = -I - tF, \quad FP + fF = 0. \tag{1.4}$$

Applying J to (1.2) and using (1.1), (1.2) give

$$Pt + tf = 0, \quad f^2 = -I - Ft. \tag{1.5}$$

Definition. A submanifold M of a Kaehlerian manifold $\bar{M}$ is called a CR submanifold of $\bar{M}$ if there exists a differentiable distribution $\mathfrak{D}: x \to \mathfrak{D}_x \subset T_x(M)$ on M satisfying the following conditions:

(i) $\mathfrak{D}$ is holomorphic, i.e., $J\mathfrak{D}_x = \mathfrak{D}_x$ for each $x \in M$, and

(ii) the complementary orthogonal distribution $\mathfrak{D}^\perp : x \to \mathfrak{D}_x^\perp \subset T_x(M)$ is anti-invariant, i.e., $J\mathfrak{D}_x^\perp \subset T_x(M)^\perp$ for each $x \in M$.

If $\dim \mathfrak{D}_x^\perp = 0$ (resp. $\dim \mathfrak{D}_x = 0$), then the CR submanifold is a holomorphic submanifold (resp. anti-invariant submanifold) of $\bar{M}$. If in a CR submanifold $\dim \mathfrak{D}_x^\perp =$ codimension M, then the CR submanifold is what we call a generic submanifold of $\bar{M}$ (see [9], [10]).

Suppose that M is a CR submanifold of $\bar{M}$, and denote by l, $l^\perp$ the projection operators on $\mathfrak{D}_x$, $\mathfrak{D}_x^\perp$ respectively. Then we have

$$l + l^\perp = I, \quad l^2 = l, \quad l^{\perp 2} = l^\perp, \quad ll^\perp = l^\perp l = 0.$$

From (1.1) we have

$$l^\perp Pl = 0, \quad Fl = 0, \quad Pl = P,$$

which together with the second equation of (1.4) implies

$$FP = 0. \tag{1.6}$$

Thus we have

$$fF = 0. \tag{1.7}$$

From (1.3) and (1.7) we obtain

$$tf = 0, \tag{1.8}$$

and, in consequence of the first equation of (1.5),

$$Pt = 0. \tag{1.9}$$

Thus from the first equation of (1.4) it follows that

$$P^3 + P = 0, \tag{1.10}$$

which shows that P is an f-structure on M. Similarly, from the second equation of (1.5) we have

$$f^3 + f = 0, \tag{1.11}$$

which shows that f is an f-structure in the normal bundle $T(M)^\perp$ (see [8]).

Conversely, for a submanifold M of a Kaehlerian manifold $\bar{M}$, assume that we have (1.6), i.e., $FP = 0$, then we have (1.7), (1.8), (1.9), (1.10) and (1.11). We now put

$$l = -P^2, \quad l^\perp = I - l. \tag{1.12}$$

Then we can easily verify that

$$l + l^\perp = I, \quad l^2 = l, \quad l^{\perp 2} = l^\perp, \quad ll^\perp = l^\perp l = 0,$$

which mean that l and $l^{\perp}$ are complementary projection operators and consequently define complementry orthogonal distributions $\mathfrak{D}$ and $\mathfrak{D}^{\perp}$ respectively. From the first equation of (1.12) we have

$$Pl = P.$$

This equation can be written as

$$Pl^{\perp} = 0.$$

But $g(PX, Y)$ is skew-symmetric, and $g(l^{\perp} X, Y)$ is symmetric, and consequently the above equation gives

$$l^{\perp} P = 0,$$

and hence

$$l^{\perp} Pl = 0.$$

From the first equation of (1.12) we have

$$Fl = 0.$$

The above equations show that the distribution $\mathfrak{D}$ is invariant, and distribution $\mathfrak{D}^{\perp}$ is anti-invariant. Thus we have

Theorem 1.1. *In order for a submanifold M of a Kaehlerian manifold $\overline{M}$ to be a CR submanifold, it is necessary and sufficient that $FP = 0$.*

Theorem 1.2. *Let M be a CR submanifold of a Kaehlerian manifold $\overline{M}$. Then P is an f-structure in M, and f is an f-structure in the normal bundle of M.*

We next study the properties of the second fundamental tensor of a CR submanifold M of a Kaehlerian manifold $\overline{M}$.

From the Gauss and Weingarten formulas we have

$$tB(X, Y) + fB(X, Y) = (\nabla_X P)Y - A_{FY}X + B(X, PY) + (\nabla_X F)Y,$$

where we have put

$$(\nabla_X P)Y = \nabla_X(PY) - P\nabla_X Y, \quad (\nabla_X F)Y = D_X(FY) - F\nabla_X Y.$$

Comparing the tangential and normal parts of the both sides of this equation, we obtain

$$(1.13) \qquad (\nabla_X P)Y = A_{FY}X + tB(X, Y),$$

$$(1.14) \qquad (\nabla_X F)Y = -B(X, PY) + fB(X, Y).$$

Similarly, we have

$$(1.15) \qquad (\nabla_X t)V = A_{fV}X - PA_V X,$$

$$(1.16) \qquad (\nabla_X f)V = -FA_V X - B(X, tV),$$

where we have put

$$(\nabla_X t)V = \nabla_X(tV) - tD_X V, \quad (\nabla_X f)V = D_X(fV) - fD_X V.$$

Moreover, the second fundamental tensor A of a CR submanifold M satisfies

$$A_{FX}Y = A_{FY}X \quad \text{for any } X, Y \in \mathfrak{D}_X^{\perp}. \tag{1.17}$$

In the sequel, we assume that M is a CR submanifold of a complex projective space CP^m of complex dimension m and with constant holomorphic sectional curvature 4. Then we have respectively the equations of Gauss and Codazzi as follows:

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(PY,Z)PX - g(PX,Z)PY \\ & + 2g(X,PY)PZ + A_{B(Y,Z)}X - A_{B(X,Z)}Y, \end{aligned} \tag{1.18}$$

$$\begin{aligned} & (\nabla_X A)_V Y - (\nabla_Y A)_V X \\ & \qquad = g(FX,V)PY - g(FY,V)PX - 2g(X,PY)tV, \end{aligned} \tag{1.19}$$

where R denotes the Riemannian curvature tensor of M.

We now define the curvature tensor $R^{\perp}$ of the normal bundle of M by

$$R^{\perp}(X,Y)V = D_X D_Y V - D_Y D_X V - D_{[X,Y]}V.$$

Then we have the equation of Ricci:

$$\begin{aligned} & g(R^{\perp}(X,Y)U,V) + g([A_V, A_U]X, Y) \\ & \quad = g(FY,U)g(FX,V) - g(FX,U)g(FY,V) + 2g(X,PY)g(fU,V), \end{aligned} \tag{1.20}$$

where $[A_V, A_U] = A_V A_U - A_U A_V$. If $R^{\perp}$ vanishes identically, then the normal connection of M is said to be *flat*.

For a CR submanifold M we have the following decomposition of the tangent space $T_x(M)$ at each point $x \in M$;

$$T_x(M) = \mathfrak{D}_x \oplus \mathfrak{D}_x^{\perp}.$$

Similarly, we have

$$T_x(M)^{\perp} = F\mathfrak{D}_x^{\perp} + \mathfrak{N}_x,$$

where $\mathfrak{N}_x$ is the orthogonal complement of $F\mathfrak{D}_x^{\perp}$ in $T_x(M)^{\perp}$. Then $J\mathfrak{N}_x = f\mathfrak{N}_x = \mathfrak{N}_x$.

We take an orthonormal frame $\{e_1, \cdots, e_{2m}\}$ of $\overline{M}$ such that, restricted to M, $e_1, \cdots, e_n$ are tangent to M. Then $e_1, \cdots, e_n$ form an orthonormal frame of M. We can take $e_1, \cdots, e_n$ in such a way that $e_1, \cdots, e_{n-p}$ form an orthonormal frame of $\mathfrak{D}_x$, and $e_{n-p+1}, \cdots, e_n$ form an orthonormal frame of $\mathfrak{D}_x^{\perp}$, where $p = \dim \mathfrak{D}_x^{\perp}$, and $n - p = \dim \mathfrak{D}_x$. Moreover, we take $\{e_{n+1}, \cdots, e_{2m}\}$ in such a way that $e_{n+1}, \cdots, e_{n+p}$ form an orthonormal frame of $F\mathfrak{D}_x^{\perp}$, and

$e_{n+p+1},\cdots,e_{2m}$ form an orthonormal frame of $\mathfrak{N}_x$. Unless otherwise stated, we shall use the conventions that the ranges of indices are respectively:

$$i, j, k = 1,\cdots,n; \quad r, s, t = 1,\cdots,n-p;$$
$$a, b, c = n-p+1,\cdots,n; \quad x, y, z = n+1,\cdots,n+p;$$
$$\lambda, \mu, \nu = n+p+1,\cdots,2m.$$

2. Semi-flat normal connection

Let M be a real n-dimensional CR submanifold of a complex projective space CP^m. If the curvature tensor R of the normal bundle of M satisfies

$$R^{\perp}(X, Y)V = 2g(X, PY)fV \tag{2.1}$$

for any vector fields X, Y tangent to M and any vector field V normal to M, then the normal connection of M is said to be *semi-flat* (see [11]). If $\nabla f = 0$, then the f-structure f is said to be *parallel*.

Lemma 2.1. *Let M be a CR submanifold of CP^m with semi-flat normal connection. If the f-structure f is parallel, then*

$$A_{fV} = 0 \tag{2.2}$$

for any vector field V normal to M, that is, $A_\lambda = 0$ where $A_\lambda = A_{e_\lambda}$.

Proof. Since the f-structure f is parallel, (1.16) gives

$$A_V tU = A_U tV \tag{2.3}$$

for any vector fields U, V normal to M. On the other hand, the Ricci equation and (2.1) imply that

$$g([A_V, A_U]X, Y) = g(FY, U)g(FX, V) - g(FX, U)g(FY, U). \tag{2.4}$$

Using (1.14) we obtain

$$\begin{aligned} 0 = g((\nabla_X f)fV, FY) &= -g\big(f^2V, (\nabla_X F)Y\big) \\ &= g\big(A_{f^2V}X, PY\big) + g\big(A_{fV}X, Y\big), \end{aligned}$$

from which it follows that

$$g\big(A_{fV}X, A_{fV}X\big) = -g\big(A_{f^2V}X, PA_{fV}X\big). \tag{2.5}$$

Moreover, from (2.4) we have

$$A_{fV}A_{f^2V} = A_{f^2V}A_{fV}. \tag{2.6}$$

From (2.5) and (2.6) we see that $\operatorname{Tr} A^2_{fV} = 0$ and hence $A_{fV} = 0$ for any vector field V normal to M.

Lemma 2.2. *Let M be a CR submanifold of CP^m with semi-flat normal connection and parallel f-structure f. If $PA_V = A_V P$ for any vecor field V normal to M, then*

$$g(A_U X, A_V Y) = g(X, Y)g(tU, tV) - g(FX, U)g(FY, V) \tag{2.7}$$
$$- \sum_i g(A_U tV, e_i)g(A_{Fe_i} X, Y).$$

Proof. From the assumption we have $g(A_U PX, tV) = 0$, which implies

$$g((\nabla_Y A)_U PX, tV) + g(A_U(\nabla_Y P)X, tV) + g(A_U PX, (\nabla_Y t)V) = 0.$$

Thus from (1.13), (1.15) and (2.2) we find

$$g((\nabla_{PX} A)_U PX, tV) + g(A_U tB(PY, X), tV) - g(PA_U X, P^2 A_V Y) = 0,$$

from which it follows that

$$g((\nabla_{PY} A)_U PX, tV) - \sum_i g(A_U tV, e_i)g(A_{Fe_i} X, PY) + g(PA_U X, A_V Y) = 0.$$

From this and the Codazzi equation we have

$$g(PX, PY)g(tU, tV) - \sum_i g(A_U tV, e_i)g(A_{Fe_i} PX, PY)$$
$$+g(P^2 A_U X, A_V Y) = 0.$$

On the other hand, we obtain

$$g(PX, PY)g(tU, tV) = g(X, Y)g(tU, tV) - g(FX, FY)g(tU, tV),$$
$$- \sum_i g(A_U tV, e_i)g(A_{Fe_i} PX, PY)$$
$$= - \sum_i g(A_U tV, e_i)g(A_{Fe_i} X, Y) + g(A_U tV, A_{FY} X),$$
$$g(P^2 A_U X, A_V Y) = -g(A_U X, A_V Y) - g(A_U X, A_{FY} tV).$$

Moreover, from (2.4) we see that

$$g(A_U tV, A_{FY} X) - g(A_U X, A_{FY} tV) = g(tU, tV)g(FX, FY)$$
$$-g(FX, U)g(FY, V).$$

From these equations we have

$$g(X, Y)g(tU, tV) - g(FX, U)g(FY, V)$$
$$- \sum_i g(A_U tV, e_i)g(A_{Fe_i} X, Y) - g(A_U X, A_V Y) = 0,$$

which proves (2.7).

A parallel section U of the normal bundle of M is called an *isoperimetric section* if $\operatorname{Tr} A_U =$ constant $\neq 0$, and a *minimal section* if $\operatorname{Tr} A_U = 0$.

Lemma 2.3. *Let M be a CR submanifold of CP^m. For any isoperimetric or minimal section U of the normal bundle of M, we have*

$$\sum_j \left(\nabla_{e_j} A\right)_U e_j = 0. \tag{2.8}$$

Proof. For any vector field X tangent to M, we have

$$\begin{aligned}\sum_j g\left(\left(\nabla_{e_j} A\right)_U e_j, X\right) &= \sum_j \left[g\left((\nabla_X A)_U e_j, e_j\right) + g\left(Fe_j, U\right)g\left(PX, e_j\right)\right.\\ &\qquad \left.-g\left(FX, U\right)g\left(Pe_j, e_j\right) - 2g\left(e_j, PX\right)g\left(tU, e_j\right)\right]\\ &= 0,\end{aligned}$$

because of the Codazzi equation.

Lemma 2.4. *Let M be a CR submanifold of CP^m with semi-flat normal connection and parallel f-structure f. If the mean curvature vector of M is parallel, and $PA_V = A_V P$ for any vector field V normal to M, then the square of the length of the second fundamental tensor is constant.*

Proof. Due to (2.2) and (2.7) the square of the length of the second fundamental tensor is given by

$$\sum_x \operatorname{Tr} A_x^2 = (n-1)p + \sum_{x,y} g\left(A_x te_x, te_y\right)\operatorname{Tr} A_y,$$

where $A_x = A_{e_x}$. On the other hand, for any vector field $V \in F\mathfrak{D}^\perp$, we have $D_X V \in F\mathfrak{D}^\perp$ because of $\nabla f = 0$. From (1.14) we also have, for any $V \in \mathfrak{N}$, $D_X V \in \mathfrak{N}$. Therefore, since $R^\perp(X, Y)V = 0$ for any $V \in F\mathfrak{D}^\perp$, we can take an orthonormal frame $\{e_x\}$ of $F\mathfrak{D}^\perp$ such that $De_x = 0$ for each x (see [3, p. 99]). Then we see that $\nabla_X(te_x) = -PA_x X$. Since the mean curvature vector of M is parallel and $PA_V = A_V P$, from the Codazzi equation and (2.3) we find

$$\nabla_X\left(\sum_x \operatorname{Tr} A_x^2\right) = \sum_{x,y} g\left(\left(\nabla_{te_x} A\right)_y te_x, X\right)\operatorname{Tr} A_y.$$

On the other hand, using $PA_V = A_V P$ and (1.13) we have

$$\sum_i g\left(\left(\nabla_{Pe_i} A\right)_x Pe_i, tV\right) = 0, \quad \sum_i g\left(\left(\nabla_{Pe_i} A\right)_x Pe_i, PX\right) = 0$$

for any vector field V normal to M and any vector field X tangent to M. Consequently, we obtain

$$\sum_s \left(\nabla_{e_s} A\right)_x e_s = \sum_i \left(\nabla_{Pe_i} A\right)_x Pe_i = 0$$

for each x. Since the mean curvature vector of M is parallel, (2.8) implies

$$0 = \sum_i (\nabla_{e_i} A)_x e_i = \sum_s (\nabla_{e_s} A)_x e_s + \sum_a (\nabla_{e_a} A)_x e_a$$
$$= \sum_a (\nabla_{e_a} A)_x e_a = \sum_y (\nabla_{te_y} A)_x te_y$$

for each x, and hence $\sum_x \operatorname{Tr} A_x^2 = \text{constant}$.

3. Integral formulas

For any vector field X of a Riemannian manifold M, we generally have (see [7])

$$\operatorname{div}(\nabla_X X) - \operatorname{div}((\operatorname{div} X)X)$$
$$= S(X, X) + \tfrac{1}{2}|L(X)g|^2 - |\nabla X|^2 - (\operatorname{div} X)^2, \tag{3.1}$$

where S denotes the Ricci tensor of M, $L(X)g$ the Lie derivative of g with respect to X, and $|Y|$ the length with respect to g of Y on M.

Let M be an n-dimensional CR submanifold of CP^m with semi-flat normal connection and parallel f-structure f. Suppose that U is a parallel section of the normal bundle of M. Then from equation of Ricci and (2.2) we have $fU = 0$ and hence $U \in F\mathfrak{D}^\perp$. We also have $\nabla_X tU = -PA_U X$, and hence

$$\operatorname{div} tU = \sum_i g(\nabla_{e_i} tU, e_i) = -\operatorname{Tr} PA_U = 0,$$

since P is skew-symmetric, and A_U is symmetric. Then we have, from (3.1),

$$\operatorname{div}(\nabla_{tU} tU) = S(tU, tU) + \tfrac{1}{2}|L(tU)g|^2 - |\nabla tU|^2. \tag{3.2}$$

On the other hand, due to (1.18) and (2.2), the Ricci tensor S of M is given by

$$S(tU, tU) = (n-1)g(tU, tU) + \sum_x \operatorname{Tr} A_x g(A_x tU, tU)$$
$$- \sum_x g(A_x^2 tU, tU). \tag{3.3}$$

Moreover, we have

$$|\nabla tU|^2 = \operatorname{Tr} A_U^2 - \sum_x g(A_x^2 tU, tU). \tag{3.4}$$

From (3.2), (3.3) and (3.4) it follows that

$$\operatorname{div}(\nabla_{tU} tU) = (n-1)g(tU, tU) + \sum_x \operatorname{Tr} A_x g(A_x tU, tU)$$
$$- \operatorname{Tr} A_U^2 + \tfrac{1}{2}|L(tU)g|^2. \tag{3.5}$$

We now take an orthonormal frame $\{e_x\}$ such that $De_x = 0$ for each x. Then (3.5) implies

$$\operatorname{div}\Big(\sum_x \nabla_{te_x} te_x\Big) = (n-1)p + \sum_{x,y} \operatorname{Tr} A_x g(A_x te_y, te_y) \tag{3.6}$$
$$- \sum_x \operatorname{Tr} A_x^2 + \tfrac{1}{2}\sum_x |L(te_x)g|^2.$$

By (2.2) it is easy to show that the right-hand side of (3.6) is independent of the choice of an orthonormal frame of $T_x(M)^{\perp}$. We notice that $|L(tU)g|^2 = |[P, A_U]|^2$.

Theorem 3.1. *Let M be a compact orientable n-dimensional CR submanifold of CP^m with semi-flat normal connection and parallel f-structure f. Then*

$$\int_M \Big[(n-1)p - \sum_x \operatorname{Tr} A_x^2 + \sum_{x,y} \operatorname{Tr} A_x g(A_x te_y, te_y)\Big] *1 \tag{3.7}$$
$$= -\frac{1}{2}\int_M \sum_x |[P, A_x]|^2 *1.$$

Theorem 3.2. *Let M be a compact orientable n-dimensional minimal CR submanifold of CP^m with semi-flat normal connection and parallel f-structure f. Then*

$$\int_M \Big[(n-1)p - \sum_x \operatorname{Tr} A_x^2\Big] *1 = -\frac{1}{2}\int_M \sum_x |[P, A_x]|^2 *1. \tag{3.8}$$

4. Parallel mean curvature vector

Let M be an n-dimensional CR submanifold of CP^m with semi-flat normal connection and parallel f-structure f, and suppose that the mean curvature vector of M is parallel. In the following we compute the Laplacian of the second fundamental tensor of M (see [6], [11]).

Using (2.2) and (2.8), by a straightforward computation we obtain

$$g(\nabla^2 A, A) = \sum_{x,i,j} g\big(\nabla_{e_i}\nabla_{e_i} A\big)_x e_j, A_x e_j$$
$$= (n-3)\sum_x \operatorname{Tr} A_x^2 - \sum_x (\operatorname{Tr} A_x)^2 + 6\sum_x \big[\operatorname{Tr}(A_x P)^2 - \operatorname{Tr} A_x^2 P^2\big]$$
$$+ 3\sum_{x,y} \big[g(A_x te_y, A_x te_y) - g(A_x te_x, A_y te_y)\big] \tag{4.1}$$
$$-\frac{1}{2}\sum_{x,y,i} g\big([A_x, A_y]e_i, [A_x, A_y]e_i\big)$$
$$+ \sum_{x,y} \big[3g(A_x te_x, te_y)\operatorname{Tr} A_y - (\operatorname{Tr} A_x A_y)^2 + (\operatorname{Tr} A_y)(\operatorname{Tr} A_x^2 A_y)\big],$$

where we have taken $\{e_x\}$ such that $De_x = 0$ for each x, and used the fact that $(\nabla_X A)_V Y = 0$ for any $V \in \mathfrak{N}_x$. On the other hand, from (2.4) we have

$$\sum_{x,y,i} g([A_x, A_y]e_i, [A_x, A_y]e_i) = 2p(p-1), \tag{4.2}$$

$$\sum_{x,y} [g(A_x te_y, A_x te_y) - g(A_x te_x, A_y te_y)] = p(p-1). \tag{4.3}$$

From (4.1), (4.2) and (4.3) it follows that

$$\begin{aligned} g(\nabla^2 A, A) &= (n-3)\sum_x \operatorname{Tr} A_x^2 - \sum_x (\operatorname{Tr} A_x)^2 + 3\sum_x |[P, A_x]|^2 + 2p(p-1) \\ &\quad + \sum_{x,y} \left[3g(A_x te_x, te_y)\operatorname{Tr} A_y - (\operatorname{Tr} A_x A_y)^2 + (\operatorname{Tr} A_y)(\operatorname{Tr} A_x^2 A_y)\right]. \end{aligned} \tag{4.4}$$

Thus by (3.6) and (4.4) we obtain

$$\begin{aligned} -g(\nabla^2 A, A) - 2(n-p)p + \frac{3}{2}\sum_x |[P, A_x]|^2 + 3\operatorname{div}\Big(\sum_x \nabla_{te_x} te_x\Big) \\ = \sum_{x,y} (\operatorname{Tr} A_x A_y)^2 - n\sum_x \operatorname{Tr} A_x^2 + \sum_x (\operatorname{Tr} A_x)^2 \\ - \sum_{x,y} (\operatorname{Tr} A_x)(\operatorname{Tr} A_y^2 A_x) + (n-1)p. \end{aligned} \tag{4.5}$$

We now assume that $PA_V = A_V P$. Then $\nabla_{te_x} te_x = -PA_x te_x = 0$. Moreover, from Lemma 2.4 we see that $g(\nabla A, \nabla A) = -g(\nabla^2 A, A)$. Thus (4.5) reduces to

$$\begin{aligned} g(\nabla A, \nabla A) - 2(n-p)p &= \sum_{x,y} (\operatorname{Tr} A_x A_y)^2 - n\sum_x \operatorname{Tr} A_x^2 \\ &\quad + \sum_x (\operatorname{Tr} A_x)^2 - \sum_{x,y} (\operatorname{Tr} A_x)(\operatorname{Tr} A_y^2 A_x) + (n-1)p. \end{aligned} \tag{4.6}$$

Now using (2.7) we have

$$\sum_{x,y} (\operatorname{Tr} A_x A_y)^2 = (n-1)\sum_x \operatorname{Tr} A_x^2 + \sum_{x,y,z} (\operatorname{Tr} A_z)(\operatorname{Tr} A_x A_y) g(A_x te_y, te_z),$$

$$-\sum_{x,y} (\operatorname{Tr} A_x)(\operatorname{Tr} A_y^2 A_x) = -\sum_x (\operatorname{Tr} A_x)^2 + \sum_{x,y} \operatorname{Tr} A_x g(A_x te_y, te_y).$$

Substituting these equations into (4.6) we find

$$\begin{aligned} & g(\nabla A, \nabla A) - 2(n-p)p \\ &= -\sum_x \operatorname{Tr} A_x^2 + \sum_{x,y} \operatorname{Tr} A_x g(A_x te_y, te_y) + (n-1)p. \end{aligned} \tag{4.7}$$

Again from (2.7) we see that the right-hand side of (4.7) vanishes, and consequently we obtain

Lemma 4.1. *Let M be an n-dimensional CR submanifold of CP^m with semi-flat normal connection, parallel f-structure f and parallel mean curvature vector. If $PA_V = A_V P$ for any vector field V normal to M, then we have $g(\nabla A, \nabla A) = 2(n-p)p$.*

Example. Let $S^m(r)$ denote an m-dimensional sphere with radius r. We consider a Riemannian fibre bundle π: $S^{n+k}(1) \to CP^{(n+k-1)/2}$. Then we can see that $\pi(S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k))$ is a generic submanifold of $CP^{(n+k-1)/2}$ with parallel mean curvature vector, where $\sum_{i=1}^k r_i^2 = 1$, $\sum_{i=1}^k m_i = n+1$ and $m_1, \cdots, m_k$ are odd numbers. Moreover, if $r_i = (m_i/(n+1))^{1/2}$ $(i = 1, \cdots, k)$, then $\pi(S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k))$ is minimal (see [10]).

We need the following

Theorem A [11]. *Let M be a complete n-dimensional CR submanifold of CP^m with semi-flat normal connection and $n - p \geqslant 4$. If the f-structure f is parallel, and $g(\nabla A, \nabla A) = 2(n-p)p$, then M is a totally geodesic holomorphic submanifold $CP^{n/2}$ of CP^m, or M is a generic submanifold of $CP^{(n+p)/2}$ in CP^m and is*

$$\pi(S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k)), \quad \sum_{i=1}^k m_i = n+1,$$

$$\sum_{i=1}^k r_i^2 = 1, \quad 2 \leqslant k \leqslant n-3,$$

where $m_1, \cdots, m_k$ are odd numbers, and $p = k-1$.

Remark. In Theorem A, if $PA_V = A_V P$, then we can prove the result without the assumption $n - p \geqslant 4$ (see Lemma 2.2 of [11]).

From Lemma 4.1 and Theorem A we have

Theorem 4.1. *Let M be a complete n-dimensional CR submanifold of CP^m with semi-flat normal connection, parallel f-structure f and parallel mean curvature vector. If $PA_V = A_V P$ for any vector field V normal to M, then M is a totally geodesic holomorphic submanifold $CP^{n/2}$ of CP^m, or M is a generic submanifold of $CP^{(n+p)/2}$ in CP^m and is*

$$\pi(S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k)), \quad \sum_{i=1}^k m_i = n+1, \quad \sum_{i=1}^k r_i^2 = 1,$$

where $m_1, \cdots, m_k$ are odd numbers and $p = k-1$.

Theorem 4.2. *Let M be a compact orientable n-dimensional minimal CR submanifold of CP^m with semi-flat normal connection and parallel f-structure f. If the square of the length of the second fundamental tensor of M is $(n-1)p$, then*

M is a generic submanifold of $CP^{(n+p)/2}$ in CP^m and is

$$\pi(S^{m_1}(r_1)\times\cdots\times S^{m_k}(r_k)),$$

$$\sum_{i=1}^{k} m_i = n+1,\quad r_i = (m_i/(n+1))^{1/2}\quad (i=1,\cdots,k),$$

where $m_1,\cdots,m_k$ are odd numbers and $p = k-1$.

Proof. Since $\Sigma_x \operatorname{Tr} A_x^2 = (n-1)p$, (3.8) implies that $PA_V = A_V P$ for any vector field V normal to M. On the other hand, by the assumption, M is not totally geodesic. Thus our assertion follows from Theorem 4.1.

Since the scalar curvature r of M is given by

$$r = (n^2+2n-3p) - \sum_x \operatorname{Tr} A_x^2,$$

we have

Theorem 4.3. *Let M be a compact orientable n-dimensional minimal CR submanifold of CP^m with semi-flat normal connection and parallel f-structure f. If $r = (n+2)(n-p)$, then M is a generic submanifold of $CP^{(n+p)/2}$ in CP^m and is*

$$\pi(S^{m_1}(r_1)\times\cdots\times S^{m_k}(r_k)),\quad \sum_{i=1}^{k} m_i = n+1,$$

$$r_i = (m_i/(n+1))^{1/2}\quad (i=1,\cdots,k),$$

where $m_1,\cdots,m_k$ are odd numbers, and $p = k-1$.

5. Flat normal connection

In this section we assume that M is an n-dimensional CR submanifold of CP^m with flat normal connection. Then the Ricci equation implies

$$g([A_{fU}, A_U]PX, X) = 2g(PX, PX)g(fU, fU) \tag{5.1}$$

for any vector field X tangent to M and any vector field U normal to M. Thus we have

$$\operatorname{Tr} A_{fU}A_U P - \operatorname{Tr} A_U A_{fU} P = 2(n-p)g(fU, fU). \tag{5.2}$$

If $PA_V = A_V P$ for any vector field V normal to M, then we have $\operatorname{Tr} A_{fU}A_U P = \operatorname{Tr} A_U A_{fU} P$, and hence (5.2) implies that $n = p$, that is, $P = 0$, and M is an anti-invariant submanifold of CP^m, or $f = 0$, that is, M is a generic submanifold of CP^m. Therefore from Theorem 3 of [12] and Theorem 4.1 we have

Theorem 5.1. *Let M be a compact orientable n-dimensional CR submanifold of CP^m with flat normal connection and parallel mean curvature vector. If $PA_V = A_V P$ for any vector field V normal to M, then M is*

$$\pi(S^1(r_1) \times \cdots \times S^1(r_{n+1})), \quad \sum_{i=1}^{n+1} r_i^2 = 1$$

in CP^n in CP^m, or M is

$$\pi(S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k)), \quad \sum_{i=1}^{k} m_i = n + 1, \quad \sum_{i=1}^{k} r_i^2 = 1,$$

where $m_1, \cdots, m_k$ are odd numbers, and $p = k - 1$, $2m = n + p$.

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