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THE CHARACTERISTIC NUMBERS OF 4-DIMENSIONAL KÄHLER MANIFOLDS

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Dedicated to the author's teacher Professor Buchin Su

1. Introduction

There have been many results about the relation between the curvature of a Riemannian manifold M and its characteristic numbers. S. S. Chern and J. Milnor [3] proved that a 4-dimensional manifold with sectional curvature everywhere of the same sign has nonnegative Euler number. M. Berger [1] and N. Hitchin [6] considered the case of an Einstein manifold. H. Donnelly [4] obtained inequalities involving the Euler number and the Pontrjagin number of Einstein Kähler manifolds. S. T. Yau [11] and A. Polombo [8] generalized Gray-Hitchin-Thorpe [5], [6], [9] inequality to k-Ricci pinched manifolds and considered the k-sectionally pinched case.

In the present paper the similar problem for k-Ricci pinched Kähler manifold is considered, and a generalization of Donnelly's inequalities is obtained (Theorem 1).

On the other hand R. Bishop and S. I. Goldberg [2] proved that a 4-dimensional Kähler manifold with holomorphic sectional curvature everywhere of the same sign has nonnegative Euler number. This result is improved in Theorem 2 of this paper.

Thus the main results are the following two theorems.

Theorem 1. *Let M be a compact oriented 4-dimensional Kähler manifold with Euler number χ and Pontrjagin number p. If M is k-Ricci pinched with $k \geqslant \sqrt{2}/2$, then the inequalities*

$$\chi + \frac{3 - 5k^2}{2k^2} p \geqslant 0, \tag{1}$$

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$$\chi + \frac{1}{2}p \geqslant 0 \tag{2}$$

are valid. Furthermore, if the equality in (1) *occurs, then M must be in one of the following three cases*:

(i) *M has constant holomorphic curvature,*

(ii) *the universal covering manifold of M is a K_3 surface,*

(iii) *M is flat.*

If the equality in (2) *occurs, then M must be in one of cases* (*ii*) *and* (*iii*) *above.*

Theorem 2. *Let M be a compact oriented* 4-*dimensional Kähler manifold with Euler number χ and Pontrjagin number p. If M is λ-holomorphically pinched with $\lambda \geqslant 0$, then*

$$\chi + \frac{1}{2}p \geqslant 0, \quad \chi + \min\left(\frac{1 - 2\lambda - 5\lambda^2}{6\lambda^2}, \frac{\lambda^2}{\lambda^2 - 4}\right)p \geqslant 0 \quad \textit{for } \frac{1}{4} \leqslant \lambda \leqslant 1,$$

and, otherwise

$$\chi + \frac{\lambda^2}{1 - 4\lambda^2}p \geqslant 0, \quad \chi + \frac{\lambda^2}{\lambda^2 - 4}p \geqslant 0. \tag{3}$$

We should point out that A. Polombo [7] has obtained similar results, which however do not cover the above theorems.

2. Preliminary notation

First of all, we construct a special Hermitian basis at any point p in a 4-dimensional Kähler manifold M. Let e_1 and e_2 be unit eigenvectors of the Ricci curvature such that it reaches its maximum and minimum respectively. It is clear that e_1 and e_2 are mutually perpendicular. Therefore using the canonical almost complex structure J we obtain a Hermitian basis $\{e_1, Je_1, e_2, Je_2\}$ which diagonalizes the Ricci curvature tensor. In this case $R_{11} = R_{22}$ and $R_{33} = R_{44}$.

From the author's previous paper [10], we have the Euler number χ and the Pontrjagin number p for any 4-dimensional Kähler manifold:

$$\chi = \frac{1}{8\pi^2}\int_M\left(|W^-|^2 + \frac{S^2}{12} - 2|R^{+-}|^2\right)\Omega, \tag{4}$$

$$p = \frac{1}{4\pi^2}\int_M\left(\frac{S^2}{24} - |W^-|^2\right)\Omega, \tag{5}$$

where W^- is the antiself dual part of the conformal curvature tensor, R^{+-} is the part of the Riemannian curvature tensor which is self dual on the first two indices as well as antiself dual on the last two [10], S is the scalar curvature, and Ω is the volume form of the manifold.

Equivalently, (4) and (5) can be expressed in another form:

$$\chi = \frac{1}{8\pi^2}\int_M \left(|R^{--}|^2 + \frac{1}{16}S^2 - 2|R^{+-}|^2\right)\Omega, \tag{4'}$$

$$p = \frac{1}{4\pi^2}\int_M \left(\frac{S^2}{16} - |R^{--}|^2\right)\Omega, \tag{5'}$$

where R^{--} is the part of the Riemannian curvature tensor and is antiself dual on both pairs of indices.

By directly computing, we have

$$|R^{+-}|^2 = \frac{1}{4}(R_{1212} - R_{3434})^2, \tag{6}$$

$$|R^{+-}|^2 = \frac{1}{16}(R_{11} + R_{22} - R_{33} - R_{44})^2 = \frac{1}{4}(R_{11} - R_{33})^2 \tag{6'}$$

under the special Hermitian basis $\{e_1, Je_1, e_2, Je_2\}$.

Let X and Y be perpendicular unit tangent vectors of M at any point p, such that $\langle X, JY\rangle = 0$. Then we have the formula [2]

$$\begin{aligned} K(X, Y) + K(X, JY) = \frac{1}{4}[&H(X + JY) + H(X - JY) + H(X + Y) \\ &+ H(X - Y) - H(X) - H(Y)], \end{aligned} \tag{7}$$

where $K(X, Y)$ is the sectional curvature of the plane spanned by X, Y, and $H(X) = K(X, JX)$. By (7) we obtain the components of the Ricci curvature tensor:

$$\begin{aligned} R_{11} &= K(e_1, Je_1) + K(e_1, e_2) + K(e_1, Je_2) \\ &= H(e_1) + \frac{1}{4}[H(e_1 + Je_2) + H(e_1 - Je_2) + H(e_1 + e_2) \\ &\qquad + H(e_1 - e_2) - H(e_1) - H(e_2)], \\ R_{33} &= H(e_2) + \frac{1}{4}[H(e_1 + Je_2) + H(e_1 - Je_2) + H(e_1 + e_2) \\ &\qquad + H(e_1 - e_2) - H(e_1) + H(e_2)], \end{aligned}$$

from which it follows that

$$\begin{aligned} S = {} & H(e_1 + Je_2) + H(e_1 - Je_2) + H(e_1 + e_2) \\ & + H(e_1 - e_2) + H(e_1) + H(e_2), \end{aligned} \tag{8}$$

and (6) can be written in the following form:

$$|R^{+-}|^2 = \frac{1}{4}[H(e_1) - H(e_2)]^2. \tag{9}$$

By definition a k-Ricci pinched manifold is one in which there is a number $k > 0$ such that

$$\frac{1}{4}|S| \geqslant k|R_{ii}| \tag{10}$$

for all i. It is easy to see $k \leqslant 1$. If the equality in (10) occurs, then either $k = 1$ or $S = 0$. Both conditions imply that the manifold is an Einstein manifold; furthermore in the second case it must be Ricci flat.

3. Proof of Theorem 1

From the pinching condition (10), we have

$$R_{11}^2 + R_{33}^2 \leqslant \frac{S^2}{8k^2}.$$

Substituting the above inequality into (6′) yields the following:

$$\begin{aligned} |R^{+-}|^2 &= \frac{1}{4}(R_{11} - R_{33})^2 = \tfrac{1}{2}(R_{11}^2 + R_{33}^2) - \frac{1}{4}(R_{11} + R_{33})^2 \\ &\leqslant \frac{S^2}{16K^2} - \frac{S^2}{16} = \frac{1 - K^2}{16K^2}S^2. \end{aligned} \tag{11}$$

If the equality holds above, then the equality also holds in (10) for k-Ricci pinched manifolds. Thus the equality in (11) occurs iff $k = 1$ or $S = 0$.

From (4), (5) and (11), we have

$$\chi + bp \geqslant \frac{1}{8\pi^2}\int_M \left[(1 - 2b)|W^-|^2 + \frac{5k^2 - 3 + 2bk^2}{24k^2}S^2\right]\Omega \tag{12}$$

for any real b. Taking $b = \frac{1}{2}(3 - 5k^2)/k^2$, we reduce (12) to

$$\chi + \frac{3 - 5k^2}{2k^2}p \geqslant \frac{3}{8\pi^2}\int_M \frac{2k^2 - 1}{k^2}|W^-|^2\Omega, \tag{12′}$$

which gives (1) when $K \geqslant \sqrt{2}/2$. The equality in (1) occurs only if one of the following conditions holds:

(i) $K = 1$, $|W^-| = 0$ and $S \neq 0$;
(ii) $S = 0$, $k^2 = \frac{1}{2}$ and $|W^-| \neq 0$;
(iii) $S = 0$, $|W^-| = 0$.

Under the first condition M has constant holomorphic curvature [10]. The second condition means that the universal covering of M is a K_3 surface [6]. When $S = 0$ and $|W^-|^2 = 0$, then $\chi = 0$, which forces M to be flat [1]. Taking $b = \frac{1}{2}$, from (12) we have (2) provided $k \geqslant \sqrt{2}/2$. If the equality holds in (2), then M satisfies either (ii) or (iii) above. The same discussion as above would not be repeated.

4. Proof of Theorem 2.

If M is a λ-holomorphically pinched Kähler manifold with $\lambda > 0$, then there is a constant $A > 0$ such that

$$\lambda A \leqslant H(X) \leqslant A, \tag{13}$$

for any $X \in T_p(M)$.

The pinching condition (13) and (8) give the inequality

$$6\lambda A \leqslant S \leqslant 6A. \tag{14}$$

From (9) and (13) we have

$$|R^{+-}|^2 \leqslant \frac{1}{4}(1-\lambda)^2 A^2. \tag{15}$$

For any $b \geqslant -1$, (4), (5) and (15) give

$$\chi + bp \geqslant \frac{1}{8\pi^2}\int_M \left\{(1-2b)|W^-|^2 + \left[\left(\frac{5}{2}\lambda^2 + \lambda - \frac{1}{2}\right) + 3b\lambda^2\right]A^2\right\}\Omega. \tag{16}$$

Taking $b = \frac{1}{2}$ in (16), we have

$$\chi + \frac{1}{2}p \geqslant \frac{1}{8\pi^2}\int_M \left(4\lambda^2 + \lambda - \frac{1}{2}\right)A^2\Omega.$$

Thus

$$\chi + \frac{1}{2}p \geqslant 0, \quad \text{for } \frac{1}{4} \leqslant \lambda \leqslant 1. \tag{17}$$

Taking $b = \frac{1}{6}(1 - 2\lambda - 5\lambda^2)/\lambda^2$ in (16), we have

$$\chi + \frac{1 - 2\lambda - 5\lambda^2}{6\lambda^2}p \geqslant 0, \tag{18}$$

when $\frac{1}{4} \leqslant \lambda \leqslant 1$.

If we denote

$$\begin{aligned}
\varepsilon_1^- &= e_1 \wedge Je_1 - e_2 \wedge Je_2,\\
\varepsilon_2^- &= e_1 \wedge e_2 - Je_2 \wedge Je_1,\\
\varepsilon_3^- &= e_1 \wedge Je_2 - Je_1 \wedge e_2,
\end{aligned}$$

then

$$\begin{aligned}
\langle R_{\varepsilon_1^-}, \varepsilon_1^-\rangle &= \langle R_{\varepsilon_1^-}(e_1), Je_1\rangle - \langle R_{\varepsilon_1^-}(e_2), Je_2\rangle\\
&= -\frac{S}{2} + 2\langle R_{e_1Je_1}(e_1), Je_1\rangle + 2\langle R_{e_2Je_2}(e_2), Je_2\rangle\\
&= -\frac{S}{2} + 2H(e_1) + 2H(e_2),
\end{aligned}$$

from which it follows that

$$|R^{--}|^2 = \left(H(e_1) + H(e_2) - \frac{S}{4}\right)^2 + L^2, \tag{19}$$

where L^2 is the sum of the squares of all the other entries of the matrix (R^{--}).

For any b from (9) we have

$$\begin{aligned}
(1 - 2b)&\left(H(e_1) + H(e_2) - \frac{1}{4}S\right)^2 + \frac{1+2b}{16}S^2 - 2|R^{+-}|^2 \\
&= (1 - 2b)\left(H(e_1) + H(e_2) - \frac{1}{4}S\right)^2 + \frac{1+2b}{16}S^2 - \frac{1}{2}(H(e_1) - H(e_2))^2 \\
&= \left(\tfrac{1}{2} - 2b\right)\left[H(e_1) + H(e_2) - \frac{1}{2}S\right]^2 + \frac{b}{2}\left[S - H(e_1) - H(e_2)\right]^2 \\
&\quad - \frac{b}{2}\left[H(e_1) + H(e_2)\right]^2 + 2H(e_1)H(e_2).
\end{aligned} \tag{20}$$

For $0 \leqslant b \leqslant \frac{1}{4}$ it follows from (13), (14) and (20) that

$$\begin{aligned}
(1 - 2b)&\left(H(e_1) + H(e_2) - \frac{1}{4}S\right)^2 + \frac{1+2b}{16}S^2 - 2|R^{+-}|^2 \\
&\geqslant (8\lambda^2 b - 2b + 2\lambda^2)A^2 = 2((4\lambda^2 - 1)b + 2\lambda^2)A^2.
\end{aligned} \tag{21}$$

For $0 \leqslant b \leqslant \frac{1}{4}$, (4′), (5′), (19) and (21) give

$$\chi + bp \geqslant \frac{1}{8\pi^2}\int_M \{(1 - 2b)L^2 + 2[(4\lambda^2 - 1)b + \lambda^2]A^2\}\Omega. \tag{22}$$

Taking $b = \lambda^2/(1 - 4\lambda^2)$ in (22) yields

$$\chi + \frac{\lambda^2}{1 - 4\lambda^2}p \geqslant \frac{1}{8\pi^2}\int_M \frac{1 - 6\lambda^2}{1 - 4\lambda^2}L^2\Omega.$$

Note that $0 \leqslant b \leqslant 1/4$. Thus if $\lambda \leqslant \sqrt{2}/4$, then

$$\chi + \frac{\lambda^2}{1 - 4\lambda^2}p \geqslant 0. \tag{23}$$

We consider again the case $b \leqslant 0$ in (20). In this case

$$\begin{aligned}
(1 - 2b)&\left(H(e_1) + H(e_2) - \frac{1}{4}S\right)^2 + \frac{(1+2b)}{16}S^2 - 2|R^{+-}|^2 \\
&\geqslant (8b - 2b\lambda^2 + 2\lambda^2)A^2 = \left[(8 - 2\lambda^2)b + 2\lambda^2\right]A^2,
\end{aligned}$$

from which for any $\lambda \geqslant 0$ we obtain the inequality

$$\chi + \frac{\lambda^2}{\lambda^2 - 4}p \geqslant \frac{1}{8\pi^2}\int_M \frac{4 + \lambda^2}{4 - \lambda^2}L^2\Omega \geqslant 0. \tag{24}$$

Therefore inequalities (3) follow from (17), (18), (23) and (24).

Remarks. 1. A result similar to Theorem 2 holds also in the case of nonpositive holomorphic curvature, but the pinching condition $-A \leqslant H(X) \leqslant \lambda A$ with $\lambda \leqslant 0$ must be substituted for $\lambda A \leqslant H(X) \leqslant A$ with $\lambda \geqslant 0$. It is easy to see that the proof is similar.

2. From (4′), (19) and (20) it follows

$$(25) \quad \chi = \frac{1}{8\pi^2}\int_M \Big[L^2 + \tfrac{1}{2}\big(H(e_1) + H(e_2) - \tfrac{1}{2}S\big)^2 + 2H(e_1)H(e_2) \Big]\Omega,$$

which is nonnegative when holomorphic curvature has the same sign everywhere. This is the theorem of R. Bishop and S. I. Goldberg, which is a special case of Theorem 2 in this paper. It is easy to see that $\chi = 0$ forces M to be flat.

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