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SECTIONAL CURVATURES AND QUASI-SYMMETRIC DOMAINS

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Introduction

This paper continues our study [1], [2] of the curvature properties of the class of homogeneous Kähler metrics arising from admissible forms on normal j-algebras. As is well-known, this class includes the Bergman metrics on homogeneous bounded domains. In §1 we derive new necessary conditions for nonpositive sectional curvature. Then in §2 we use these and the results of J. Dorfmeister [3] to prove our main result: If a quasi-symmetric domain (in the sense of Satake [6]) has nonpositive sectional curvature in the Bergman metric, then it is symmetric. This should be contrasted with the result of Zelow (Lundquist) [7], [8] that quasi-symmetric domains have holomorphic sectional curvature bounded above by a negative constant. §3 gives an improvement of a result in [2] and a correction.

1. Fix a normal j-algebra $(\mathfrak{s}, j)$ with admissible form ω. This means that $\mathfrak{s}$ is a finite dimensional real split solvable Lie algebra with almost complex structure j such that $[X, Y] + j[jX, Y] + j[X, jY] = [jX, jY]$ and ω is a linear form on $\mathfrak{s}$ such that the bilinear form $\langle X, Y\rangle = \omega[jX, Y]$ is symmetric, positive-definite, and j-invariant. Let $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$, and let $\mathfrak{a}$ be the orthogonal complement of $\mathfrak{n}$ in $\mathfrak{s}$. By the basic structure theorem of Pyatetskii-Shapiro [5], $\mathfrak{a}$ is a commutative subalgebra, and $\mathfrak{n}$ can be represented as the orthogonal (cf. [1]) direct sum of the root spaces $\mathfrak{n}_\alpha = \{X \in \mathfrak{n}: [H, X] = \alpha(H)X, H \in \mathfrak{a}\}$ with $[\mathfrak{n}_\alpha, \mathfrak{n}_\beta] \subset \mathfrak{n}_{\alpha+\beta}$. If $\varepsilon_1, \cdots, \varepsilon_R$ are the roots whose root spaces are mapped into $\mathfrak{a}$ by j, then $R = \dim \mathfrak{a}$, the roots $\varepsilon_1, \cdots, \varepsilon_R$ are linearly independent, and, with proper labelling, all roots are of the form $\frac{1}{2}\varepsilon_k, \varepsilon_k, 1 \leqslant k \leqslant R$; $\frac{1}{2}(\varepsilon_m \pm \varepsilon_n)$, $1 \leqslant m < n \leqslant R$. Further, $j\mathfrak{n}_{\frac{1}{2}\varepsilon_k} = \mathfrak{n}_{\frac{1}{2}\varepsilon_k}$ and $j\mathfrak{n}_{\frac{1}{2}(\varepsilon_m+\varepsilon_n)} = \mathfrak{n}_{\frac{1}{2}(\varepsilon_m-\varepsilon_n)}$, $m < n$. Since each root space $\mathfrak{n}_{\varepsilon_k}$ is one-dimensional, we will once and for all fix $X_k \in \mathfrak{n}_{\varepsilon_k}$ so that $\varepsilon_k(jX_l) = \delta_{kl}$. Also we set $E = \sum X_k$. Note that $\mathfrak{a}$ does not depend on the choice of admissible form ω [2]; hence the same is true of the root space decomposition and the X_k.

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If S is a connected, simply-connected, Lie group with Lie algebra $\mathfrak{s}$, then $\langle\ ,\ \rangle$ induces a left invariant Riemannian metric (also denoted $\langle\ ,\ \rangle$) on S which is Kähler with respect to the left invariant complex structure induced by j. The associated Levi-Civita connection ∇ is computed by

$$2\langle \nabla_X Y, Z\rangle = \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle + \langle [Z, Y], X\rangle, \quad X, Y, Z \in \mathfrak{s}. \tag{1}$$

Since the metric is left-invariant Kähler, one has

$$\nabla_X(jY) = j(\nabla_X Y), \quad \langle \nabla_X Y, Z\rangle = -\langle Y, \nabla_X Z\rangle, \quad X, Y, Z \in \mathfrak{s}. \tag{2}$$

Lemma. *Fix $m < n$ and $U_n \in \mathfrak{n}_{\frac{1}{2}\varepsilon_n}$. Let*

$$\pi: \mathfrak{n}_{\frac{1}{2}\varepsilon_m} \to j[U_n, \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}] = [jU_n, \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}]$$

be orthogonal projection. Then for any $U_m, V_m \in \mathfrak{n}_{\frac{1}{2}\varepsilon_m}$, one has

$$\langle [U_m, U_n], [V_m, U_n]\rangle = \frac{1}{2\omega(X_n)}\langle U_n, U_n\rangle\langle \pi U_m, \pi V_m\rangle.$$

In particular, $[U_m, U_n] = 0$ if and only if $\langle U_m, j[U_n, \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}]\rangle = 0$. (We allow the possibility that some of these root spaces are trivial.)

Proof. Take any $Z \in \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}$. Note that $j[U_n, Z] = [jU_n, Z]$ (cf. [1, Formula (9)]). Further [5, p. 63, Formula (46)],

$$\langle [U_n, Z], [U_n, Z]\rangle = \frac{1}{2\omega(X_n)}\langle U_n, U_n\rangle\langle Z, Z\rangle,$$

which by polarizing in Z implies

$$\langle [U_n, Z], [U_n, Z']\rangle = \frac{1}{2\omega(X_n)}\langle U_n, U_n\rangle\langle Z, Z'\rangle, \quad Z, Z' \in \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}. \tag{3}$$

Now for any $Z, Z' \in \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}$, we have $j[U_n, [jU_n, Z]] \in \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}$ and

$$\begin{aligned}\langle j[U_n, [jU_n, Z]], Z'\rangle &= -\omega[[U_n, [jU_n, Z]], Z'] \\ &= \omega[[[jU_n, Z], Z'], U_n] \\ &\quad + \omega[[Z', U_n], [jU_n, Z]] \\ &= \langle j[U_n, Z'], [jU_n, Z]\rangle \\ &= \langle [jU_n, Z'], [jU_n, Z]\rangle \\ &= \frac{1}{2\omega(X_n)}\langle U_n, U_n\rangle\langle Z, Z'\rangle.\end{aligned}$$

Thus we have

$$j[U_n, [jU_n, Z]] = \frac{1}{2\omega(X_n)}\langle U_n, U_n\rangle Z, \tag{4}$$

$$U_n \in \mathfrak{n}_{\frac{1}{2}\varepsilon_n},\ Z \in \mathfrak{n}_{\frac{1}{2}(\varepsilon_m-\varepsilon_n)}.$$

Take any $U_m \in \mathfrak{n}_{\frac{1}{2}\varepsilon_m}$. Then $[U_m, U_n] \in \mathfrak{n}_{\frac{1}{2}(\varepsilon_m+\varepsilon_n)}$, and for any $Z \in \mathfrak{n}_{\frac{1}{2}(\varepsilon_m-\varepsilon_n)}$,

$$\begin{aligned}\langle [U_m, U_n], jZ\rangle &= -\omega[Z, [U_m, U_n]]\\ &= \omega[U_m, [U_n, Z]] + \omega[U_n, [Z, U_m]]\\ &= \langle j[U_n, Z], U_m\rangle.\end{aligned}$$

This proves that $[U_m, U_n] = 0$ if and only if $\langle U_m, j[U_n, \mathfrak{n}_{\frac{1}{2}(\varepsilon_m-\varepsilon_n)}]\rangle = 0$. Thus if we write $\pi U_m = j[U_n, Z]$ for some $Z \in \mathfrak{n}_{\frac{1}{2}(\varepsilon_m-\varepsilon_n)}$, then

$$[U_m, U_n] = [\pi U_m, U_n] = -[U_n, [jU_n, Z]].$$

Using (4) and then (3), we have

$$\begin{aligned}\langle [U_m, U_n], [U_m, U_n]\rangle &= \frac{1}{4\omega(X_n)^2}\langle U_n, U_n\rangle^2\langle Z, Z\rangle\\ &= \frac{1}{2\omega(X_n)}\langle U_n, U_n\rangle\langle [jU_n, Z], [jU_n, Z]\rangle\\ &= \frac{1}{2\omega(X_n)}\langle U_n, U_n\rangle\langle \pi U_m, \pi U_m\rangle.\end{aligned}$$

Polarizing in U_m gives the result.

Theorem 1. *Suppose $m < n$. If the sectional curvatures of $\langle\ ,\ \rangle$ are nonpositive, then for any $U_n \in \mathfrak{n}_{\frac{1}{2}\varepsilon_n}$, the space $[U_n, \mathfrak{n}_{\frac{1}{2}(\varepsilon_m-\varepsilon_n)}]$ is j-invariant.*

Proof. Suppose there is a (necessarily nonzero) $U_n \in \mathfrak{n}_{\frac{1}{2}\varepsilon_n}$ such that $[U_n, \mathfrak{n}_{\frac{1}{2}(\varepsilon_m-\varepsilon_n)}]$ is not j-invariant. Then there is a nonzero $U_m \in \mathfrak{n}_{\frac{1}{2}\varepsilon_m}$ such that $\langle U_m, j[U_n, \mathfrak{n}_{\frac{1}{2}(\varepsilon_m-\varepsilon_n)}]\rangle = 0$, but $\langle U_m, [U_n, \mathfrak{n}_{\frac{1}{2}(\varepsilon_m-\varepsilon_n)}]\rangle \neq 0$. We compute the sectional curvature of $U_m \wedge U_n$ by

$$\langle R(U_m, U_n)U_n, U_m\rangle = -\langle \nabla_{U_n}U_n, \nabla_{U_m}U_m\rangle + \langle \nabla_{U_m}U_n, \nabla_{U_m}U_n\rangle$$

since $[U_m, U_n] = 0$ and the torsion vanishes. From [1, Formula (15)], it follows that $\nabla_{U_m}U_m \in j\mathfrak{n}_{\varepsilon_m}$, $\nabla_{U_n}U_n \in j\mathfrak{n}_{\varepsilon_n}$ so that $\langle \nabla_{U_n}U_n, \nabla_{U_m}U_m\rangle = 0$. There is a $Z \in \mathfrak{n}_{\frac{1}{2}(\varepsilon_m-\varepsilon_n)}$ with $\langle U_m, [U_n, Z]\rangle \neq 0$, so $\langle \nabla_{U_m}U_n, Z\rangle = \frac{1}{2}\langle [Z, U_n], U_m\rangle \neq 0$. Thus the sectional curvature of $U_m \wedge U_n$ would be positive.

The following extends Theorem 3 of [1].

Corollary. *Suppose $m < n$, and $\frac{1}{2}\varepsilon_n$ is a root (i.e., $\mathfrak{n}_{\frac{1}{2}\varepsilon_n} \neq 0$). If the sectional curvatures of $\langle\ ,\ \rangle$ are nonpositive, then $\mathfrak{n}_{\frac{1}{2}(\varepsilon_m-\varepsilon_n)}$ has even dimension (possibly zero).*

Proof. For nonzero $U_n \in \mathfrak{n}_{\frac{1}{2}\varepsilon_n}$, the map ad U_n: $\mathfrak{n}_{\frac{1}{2}(\varepsilon_m-\varepsilon_n)} \to \mathfrak{n}_{\frac{1}{2}\varepsilon_m}$ is injective [5, p. 61]. If $\mathfrak{n}_{\frac{1}{2}(\varepsilon_m-\varepsilon_n)}$ is odd dimensional, so is $[U_n, \mathfrak{n}_{\frac{1}{2}(\varepsilon_m-\varepsilon_n)}]$ which obviously cannot then be j-invariant.

We now give some material which will help relate our computations in normal j-algebras to the work of Dorfmeister [3]. As we shall see in the next section, the following definitions really occur already in [3]. Theorem 2 is implicit in [3].

Definition. Let $\mathfrak{L} = \Sigma\, \mathfrak{n}_{\varepsilon_k} + \Sigma_{k<l}\, \mathfrak{n}_{\frac{1}{2}(\varepsilon_k+\varepsilon_l)}$. For $Y, Z \in \mathfrak{L}$, we define a product by $YZ = -j\nabla_Y Z$, and denote the restriction of $\langle\ ,\ \rangle$ to $\mathfrak{L}$ by σ. Next, let $\mathfrak{U} = \Sigma\, \mathfrak{n}_{\frac{1}{2}\varepsilon_k}$. We define a form ρ on $\mathfrak{U}$ by $4\rho(U, V) = \langle U, V\rangle + i\langle U, jV\rangle$ for $U, V \in \mathfrak{U}$. Lastly, for $Y \in \mathfrak{L}$, we define an endomorphism $\phi(Y)$ of $\mathfrak{U}$ by $\phi(Y)U = -2j\nabla_Y U$.

Theorem 2. *$\mathfrak{L}$ with the above product is a commutative (nonassociative) algebra with identity $E = \Sigma X_k$, and $\{X_1, \cdots, X_R\}$ is a complete orthogonal system of idempotents. Further, each X_k is primitive (cf. [3, p. 94]) in the sense that $\mathbf{R}X_k = \{Y \in \mathfrak{L}: YX_k = Y\}$.*

Proof. Note that $\mathfrak{L}$ is an abelian ideal in the Lie algebra $\mathfrak{s}$. From (1) and (2), it is easy to see that $\nabla_Y Z \in j\mathfrak{L}$ for $Y, Z \in \mathfrak{L}$, and the definition gives a commutative product on $\mathfrak{L}$. From (1) and (2) one also computes the following covariant derivatives (cf. [2, Formulas (3)–(5), (8)–(11), (13)–(19)]):

$$\nabla_{X_k} X_l = \delta_{kl}(jX_k), \tag{5}$$

$$\nabla_{X_k} Y = \frac{1}{2}(\delta_{kl} + \delta_{km})jY \quad \text{for } Y \in \mathfrak{n}_{\frac{1}{2}(\varepsilon_l+\varepsilon_m)}, \tag{6}$$

$$\nabla_Y Y = \frac{1}{2}\langle Y, Y\rangle\left(\frac{jX_k}{\omega(X_k)} + \frac{jX_l}{\omega(X_l)}\right) \quad \text{for } Y \in \mathfrak{n}_{\frac{1}{2}(\varepsilon_k+\varepsilon_l)},\quad k<l, \tag{7}$$

$$\langle \nabla_Y Z, \mathfrak{a}\rangle = 0 \quad \text{for } Y \in \mathfrak{n}_{\frac{1}{2}(\varepsilon_k+\varepsilon_l)},\quad Z \in \mathfrak{n}_{\frac{1}{2}(\varepsilon_m+\varepsilon_n)}, \quad (k, l) \neq (m, n). \tag{8}$$

Then the theorem follows easily.

2. We continue the notation of §1. Given the normal j-algebra $(\mathfrak{s}, j)$ there is associated canonically a homogeneous Siegel domain [5, pp. 66–73] whose construction we recall briefly. Note that $j\mathfrak{L} = \mathfrak{a} + \Sigma_{k<l}\, \mathfrak{n}_{\frac{1}{2}(\varepsilon_k-\varepsilon_l)}$ is a subalgebra of $\mathfrak{s}$ with $[j\mathfrak{L}, \mathfrak{L}] \subset \mathfrak{L}$. Then $jY \to (\operatorname{ad} jY)|\mathfrak{L}$ is a faithful representation of $j\mathfrak{L}$ on the vector space $\mathfrak{L}$. Let K be the orbit of E under the action of the group generated by $\exp\{(\operatorname{ad} jY)|\mathfrak{L}: Y \in \mathfrak{L}\}$. Then K is a regular cone in $\mathfrak{L}$. We consider $\mathfrak{L}^{\mathbf{C}} \oplus \mathfrak{U}$ as a complex vector space where $iU = jU$ for $U \in \mathfrak{U}$. As a manifold, $\mathfrak{L}^{\mathbf{C}} \oplus \mathfrak{U}$ has a complex structure, and we denote the complex structure operator on each real tangent space by J. Any element $V \in \mathfrak{L}^{\mathbf{C}} \oplus \mathfrak{U}$

determines a real vector field V^0 on $\mathfrak{L}^{\mathbf{C}} \oplus \mathfrak{U}$ by translation, and we have $(iV)^0 = J(V^0)$. We have the K-Hermitian form $F\colon \mathfrak{U} \times \mathfrak{U} \to \mathfrak{L}^{\mathbf{C}}$ defined by $F(U, V) = \frac{1}{4}\{[jU, V] + i[U, V]\}$ which determines the Siegel domain $D = \{(Z, U) \in \mathfrak{L}^{\mathbf{C}} \oplus \mathfrak{U}\colon \operatorname{Im} Z - F(U, U) \in K\}$. The Lie group S acts simply transitively on D by affine transformations. For our purposes, we only need to explicitly know the action of $\exp Y$ for $Y \in \mathfrak{L}$; this is given by translations, $(\exp Y) \cdot (Z, U) = (Z + Y, U)$.

Choosing $(iE, 0) \in D$ as a base point, we identify S with D by $g \to g \cdot (iE, 0)$. The complex structure J on $T_{(iE,0)}D$ pulls back to j, and the Bergman metric on D pulls back to a left invariant metric on S which, by results of Koszul [4], comes from an admissible form on $\mathfrak{s}$. From now on we assume ω is that form, and $\langle\ ,\ \rangle$ will denote both the Bergman metric on D and the induced left invariant metric on S (as well as the corresponding bilinear form on $\mathfrak{s}$).

An element $Y \in \mathfrak{s}$ is, as usual, both a tangent vector to S at the identity element e and a left invariant vector field on S. However, Y also determines a right invariant vector field on S as well as the vector field on D given by the action of S, namely, $(Z, U) \to d/dt(\exp tY) \cdot (Z, U)|_{t=0}$. These two vector fields agree under the diffeomorphism $S \simeq D$ and will both be denoted Y^*. One has

$$(9) \qquad \begin{aligned} &\langle Y^*, Z^*\rangle_g = \langle Y^*, Z^*\rangle_{g\cdot(iE,0)},\\ &(jY)^*_e \quad = j(Y^*_e) \text{ which agrees with } J(Y^*{}_{(iE,0)}). \end{aligned}$$

Also for $Y, Z \in \mathfrak{s}$, one has $[Y^*, Z]_e = 0$, so

$$(10) \qquad (\nabla_Y Z)_e = (\nabla_{Y^*} Z)_e = (\nabla_Z Y^*)_e + [Y^*, Z]_e = (\nabla_{Z^*} Y^*)_e.$$

Since for $Y \in \mathfrak{L}$, $\exp Y$ acts on D by translation, one has

$$(11) \qquad Y^* = Y^0 \quad \text{for } Y \in \mathfrak{L}.$$

Now Dorfmeister [3, pp. 12–13] defines a bilinear form σ on $\mathfrak{L}$ so that

$$(12) \qquad \sigma(Y, Z) = \langle Y^0, Z^0\rangle_{(iE,0)}.$$

Then (9), (11), (12) show that σ is the restriction of $\langle\ ,\ \rangle$ from $\mathfrak{s}$ to $\mathfrak{L}$. For $Y_1 \in \mathfrak{L}$, let Y_1^0 also denote the translation invariant vector field on the vector space $\mathfrak{L}$, and for $Y_2, Y_3 \in \mathfrak{L}$, consider the function on K defined by $X \to \langle Y_2^0, Y_3^0\rangle_{(iX,0)}$. Then [3, p. 14] defines a commutative product on $\mathfrak{L}$ by requiring

$$(13) \qquad \sigma(Y_1 Y_2, Y_3) = -\frac{1}{2} Y_1^0 \cdot \{X \to \langle Y_2^0, Y_3^0\rangle_{(iX,0)}\}|_E.$$

Using (11), (9), (10), (2), the vanishing of the torsion and commutativity of the Lie algebra $\mathfrak{L}$, one has

$$\begin{aligned}\sigma(Y_1Y_2, Y_3) &= -\frac{1}{2}(iY_1)^0 \cdot \langle Y_2^0, Y_3^0\rangle|_{(iE,0)} \\ &= -\frac{1}{2}(jY_1)^*\langle Y_2^*, Y_3^*\rangle|_e \\ &= -\frac{1}{2}\{\langle \nabla_{(jY_1)^*}Y_2^*, Y_3^*\rangle + \langle Y_2^*, \nabla_{(jY_1)^*}Y_3^*\rangle\}|_e \\ &= -\frac{1}{2}\{\langle \nabla_{Y_2}(jY_1), Y_3\rangle + \langle Y_2, \nabla_{Y_3}(jY_1)\rangle\}|_e \\ &= -\frac{1}{2}\{\langle \nabla_{Y_2}(jY_1), Y_3\rangle + \langle \nabla_{Y_1}(jY_2), Y_3\rangle\}|_e \\ &= -\langle j\nabla_{Y_1}Y_2, Y_3\rangle.\end{aligned}$$

Since $j\nabla_{Y_1}Y_2$ is in $\mathfrak{L}$, this shows $Y_1Y_2 = -j\nabla_{Y_1}Y_2$ ([7] has a similar formula for quasi-symmetric domains but the definition of the product is formulated differently).

Remark. In [3], the product is normalized by choosing the base point so that the Bergman kernel function has the value 1 there. However, our definition of the Bergman kernel on D is ambiguous up to a constant multiple in that we have not specified a volume form on the vector space $\mathfrak{L}^{\mathbf{C}} \oplus \mathfrak{U}$. We can always make this choice to achieve the desired normalization.

Finally, [3, pp. 15–16] defines a Hermitian form ρ on $\mathfrak{U}$ and a map ϕ: $\mathfrak{L} \to \mathrm{Sym}(\mathfrak{U}, \rho)$ by

$$\rho(U, V) = \sigma(F(U, V), E), \tag{14}$$

$$\sigma(F(U, V), Y) = \rho(\phi(Y)U, V), \quad Y \in \mathfrak{L}, \tag{15}$$

where σ is extended $\mathbf{C}$ linearly to $\mathfrak{L}^{\mathbf{C}}$ (note that σ on $\mathfrak{L}^{\mathbf{C}}$ is not then identifiable with $\langle\ ,\ \rangle$ on $\mathfrak{L} \oplus j\mathfrak{L}$). For $U, V \in \mathfrak{U}$, one has

$$\begin{aligned}\langle [U, V], E\rangle &= \omega[jE, [U, V]] = -\omega[U, [V, jE]] - \omega[V, [jE, U]] \\ &= \frac{1}{2}\omega[U, V] - \frac{1}{2}\omega[V, U] = -\langle jU, V\rangle,\end{aligned}$$

which shows that

$$\rho(U, V) = \tfrac{1}{4}\{\langle U, V\rangle + i\langle U, jV\rangle\}. \tag{16}$$

Then (15), (16), (1) and $[\mathfrak{L}, \mathfrak{U}] = 0$ imply that

$$\langle \phi(Y)U, V\rangle = \langle [jU, V], Y\rangle = -2\langle \nabla_Y(jU), V\rangle \quad \text{for } Y \in \mathfrak{L},$$

which means that (14) and (15) agree with the definitions in §1 since $\nabla_Y(jU) \in \mathfrak{U}$. We will extend ϕ linearly to $\mathfrak{L}^{\mathbf{C}}$ so that (15) still holds. Then for $Y \in \mathfrak{L}$, $U \in \mathfrak{U}$, one has $\phi(iY)U = j\phi(Y)U = 2\nabla_Y U$.

Now we are ready for our main result. Recall that the Siegel domain D is quasi-symmetric if either of the following equivalent conditions hold: (1) The cone K is self dual with respect to the inner product σ or (2) $\mathfrak{L}$ is a Jordan algebra with respect to the product defined by (13). (This is the formulation in [3, p. 77]. The notion was introduced by Satake [6] with a different but equivalent definition.)

Theorem 3. *Suppose D is a quasi-symmetric Siegel domain whose sectional curvatures (in the Bergman metric) are nonpositive. Then D is symmetric.*

Proof. A quasi-symmetric Siegel domain is homogeneous and so is equivalent to the canonical domain associated to some normal j-algebra $(\mathfrak{s}, j)$. Using the previous notation, $\{X_1, \cdots, X_R\}$ is a complete orthogonal set of primitive idempotents for the Jordan algebra $\mathfrak{L}$.

Let $\mathfrak{U}_k = \phi(X_k)\mathfrak{U}$. For $U \in \mathfrak{n}_{\frac{1}{2}\varepsilon_l}$, we have (cf. [2, Formula (10) with $H = jX_k$])

$$\nabla_{X_k}(jU) = j\nabla_U X_k = \nabla_U(jX_k) = -\frac{1}{2}\varepsilon_l(jX_k)U = -\frac{1}{2}\delta_{kl}U.$$

Thus $\mathfrak{U}_k$ is just $\mathfrak{n}_{\frac{1}{2}\varepsilon_k}$. Fix $m < n \leqslant R$ and take $U_m \in \mathfrak{n}_{\frac{1}{2}\varepsilon_m}$, $U_n \in \mathfrak{n}_{\frac{1}{2}\varepsilon_n}$. Since the sectional curvatures are nonpositive, Theorem 1 says $[U_n, \mathfrak{n}_{\frac{1}{2}(\varepsilon_m - \varepsilon_n)}]$ is j-invariant. Then the projection π of the lemma commutes with j. Thus $\phi(F(U_n, U_m))U_n = -\frac{1}{2}\{\nabla_{[jU_n, U_m]}(jU_n) - \nabla_{[U_n, U_m]}U_n\}$ is in $\mathfrak{n}_{\frac{1}{2}\varepsilon_m}$, and for any $V_m \in \mathfrak{n}_{\frac{1}{2}\varepsilon_m}$ the lemma gives

$$\begin{aligned}
4\langle\phi(F(U_n, U_m))U_n, V_m\rangle &= -\langle[jU_n, U_m], [V_m, jU_n]\rangle \\
&\quad + \langle[U_n, U_m], [V_m, U_n]\rangle \\
&= \langle[jU_m, U_n], [jV_m, U_n]\rangle \\
&\quad -\langle[U_m, U_n], [V_m, U_n]\rangle \\
&= \frac{1}{2\omega(X_n)}\langle U_n, U_n\rangle\{\langle\pi jU_m, \pi jV_m\rangle \\
&\qquad -\langle\pi U_m, \pi V_m\rangle\} \\
&= 0.
\end{aligned}$$

Thus $\phi(F(U_n, U_m))U_n = 0$. By [3, Satz 3.4, part (3), p. 95] (note that the ordering of the idempotents is not relevant in that theorem), the domain is symmetric.

3. We take this opportunity to improve Theorem 5 of [2]. Again we retain the previous notation where $(\mathfrak{s}, j)$ is a normal j-algebra, D is the associated Siegel domain, and ω gives the Bergman metric on D.

Theorem 4. *Suppose $\frac{1}{2}(\varepsilon_k + \varepsilon_l)$, $k < l$, is a root and D is quasi-symmetric. Then $\omega(X_k) = \omega(X_l)$.*

Proof. $\mathfrak{L}$ with the product defined previously is a Jordan algebra. For $Y \in \mathfrak{n}_{\frac{1}{2}(\varepsilon_k+\varepsilon_l)}$ and X_m, using (5), (6), (7) and an easy computation we can show

$$(Y^2X_m)Y = \frac{1}{4}\langle Y, Y\rangle(\delta_{km} + \delta_{lm})\left(\frac{\delta_{km}}{\omega(X_k)} + \frac{\delta_{lm}}{\omega(X_l)}\right)Y,$$

$$Y^2(X_mY) = \frac{1}{4}\langle Y, Y\rangle(\delta_{km} + \delta_{lm})\left(\frac{1}{2\omega(X_k)} + \frac{1}{2\omega(X_l)}\right)Y.$$

Then the Jordan condition, $(Y^2X_m)Y = Y^2(X_mY)$, shows $\omega(X_k) = \omega(X_l)$.

Remark. Theorem 4 shows that an example, given in [2], of a homogeneous Siegel domain with negative holomorphic sectional curvature is not quasi-symmetric. Thus the sign of the holomorphic sectional curvature does not characterize the quasi-symmetric domains (compare [7], [8], [9]).

We would also like to add a correction to [2]. The condition that the eigenvalues of the adjoint representation of $\mathfrak{s}$ are real should be included in the defintion of normal j-algebras. The remark made after that definition that this condition could be omitted was based on the author's misreading of the cited notes of Rossi. Then to justify the comments made in the succeeding paragraph, reference should also be made to I. I. Pyatitskii-Shapiro, Izv. Akad. Nauk. SSSR Ser. Math. **26** (1962) 107–124, or H. Shima, J. Math. Soc. Japan **25** (1973) 422–445.

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