

## FOLIATIONS BY MANIFOLDS WITH BOUNDARY

J. L. NOAKES

### Introduction

We define a class of foliations admitting certain sorts of singularities. These foliations may be thought of as generalized Alexander decompositions and can also be used to construct examples of ordinary foliations. Our main result generalizes Bott's vanishing theorem [2] to our class of foliations. Another generalization of Bott's theorem to analytic foliations is given in [1] by Baum and Bott. However it appears from a remark in [3] that smooth foliations with singularities are less well understood.

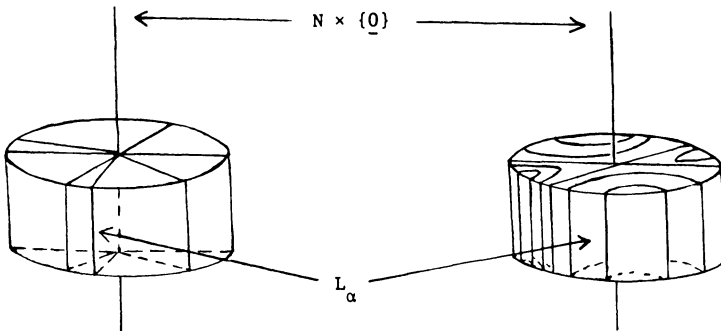
A (smooth)  $p$ -dimensional *foliation* of a smooth  $m$ -manifold  $M$  by *manifolds with boundary* is defined to be a set  $\{L_\alpha : \alpha \in A\}$  of smoothly immersed submanifolds of  $M$  such that

(1)  $S = \bigcup_A \partial L_\alpha$  is a closed  $(p - 1)$ -dimensional submanifold of  $M$ .

(2)  $\{L_\alpha - S : \alpha \in A\}$  is an ordinary smooth foliation of  $m - S$ .

We call  $L_\alpha$  a *leaf* and  $S$  the *singular manifold* of the foliation  $\{L_\alpha : \alpha \in A\}$ . The integer  $m - p$  is the *codimension* of the foliation.

**Example 1.** Let  $\{\lambda_\alpha : \alpha \in A\}$  be the set of orbits other than the orbit  $\{0\}$  of a nondegenerate linear vector field  $P$  on the open unit  $(m - p + 1)$ -ball  $e^{m-p+1}$ . Let  $N$  be a closed smooth  $(p - 1)$ -manifold. Then the path-components of the  $N \times (\lambda_\alpha \cup \{0\})$  give us a foliation of  $N \times e^{m-p+1}$  by manifolds with boundary. The singular manifold is  $N \times \{0\}$  and the foliation is said to be *trivial*.



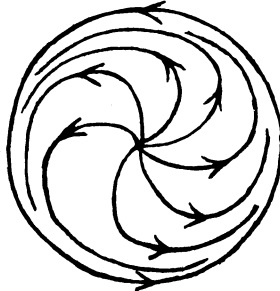
The illustration shows parts of two trivial foliations of  $S^1 \times e^2$  by 2-manifolds with boundary.

A foliation by manifolds with boundary is said to be *locally trivial* when the singular manifold  $S$  has an open neighborhood  $U$  such that  $\{L_\alpha \cap U : \alpha \in A\}$  corresponds under a diffeomorphism to a trivial foliation. In this case  $S$  must have a trivial normal bundle.

**Example 2.** Let  $M$  be compact. Then the orbits of a smooth vector field on  $M$  with nondegenerate zeros make up a locally trivial foliation of  $M$  by 1-manifolds with boundary.

**Example 3.** An Alexander decomposition [4, p. 379] gives rise to a locally trivial codimension-1 foliation of  $M$  by manifolds with boundary. Compact 3-manifolds admit Alexander decompositions, and we refer to [7] for more general results.

**Example 4.** Let  $D^{m-1}$  be the closed unit  $(m-1)$ -ball. Then for  $m > 3$  the construction of Reeb [4, p. 378] gives an ordinary smooth codimension-1 foliation of  $D^{m-1} \times S^1$  where  $\partial(D^{m-1} \times S^1) = S^{m-2} \times S^1$  is one of the leaves. Now a spiral vector field on  $D^2$



gives us a locally trivial codimension-1 foliation of  $S^{m-2} \times D^2$  by manifolds with boundary where  $\partial(S^{m-2} \times D^2) = S^{m-2} \times S^1$  is again one of the leaves. Identifying and smoothing the two foliations along  $S^{m-2} \times S^1$  we obtain a locally trivial codimension-1 foliation of  $S^m$  by manifolds with boundary. This foliation does not come from an Alexander decomposition because its leaves are not all diffeomorphic. Also when  $m$  is even  $S^m$  does not have a nonsingular  $(m-1)$ -plane field, and so  $S^m$  does not have an ordinary foliation of codimension 1.

Let  $V, W$  be connected closed smooth manifolds of dimensions  $r+1, s+1$  respectively. We remove small open discs  $e^{r+1}, e^{s+1}$  from  $V, W$  to obtain manifolds  $V_1, W_1$  with boundaries  $S^r, S^s$  respectively. We then identify and smooth  $V_1 \times S^s, S^r \times W_1$  along their common boundary  $S^r \times S^s$  to obtain a smooth  $(r+s+1)$ -manifold  $V \cdot W$ . For instance  $S^{r+1} \cdot S^{s+1}$  is  $S^{r+s+1}$ .

**Proposition 1.** *Let  $s$  be odd. Then  $V \cdot W$  has a locally trivial codimension- $s$  foliation by manifolds with boundary.*

The construction is given in §1.

**Example 5.** A sphere has locally trivial foliations of all odd codimensions.

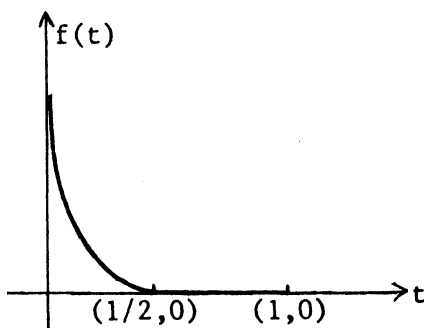
**Proposition 2.** *Suppose that  $M$  has a locally trivial foliation of codimension  $q$ , and let  $N$  be a smooth  $n$ -manifold admitting a never-zero vector field. Then  $M \times N$  has an ordinary smooth foliation of codimension  $n + q$ .*

The construction is given in §2.

**Example 6.** Let  $N$  be as in Proposition 2. There is an  $(m + 1)$ -frame field on  $S^m \times N$ , and it follows from theorems of Thurston [5], [6] that  $S^m \times N$  has ordinary smooth foliations of all codimensions  $\leq m + 1$ . We add to this picture in the following way. It follows from Proposition 2 and Example 5 that  $S^m \times N$  has ordinary smooth foliations of all codimensions  $n + q$  where  $0 \leq q \leq m$  is odd. Examining our constructions, we find that  $(V \cdot W) \times N$  has an ordinary smooth foliation of codimension  $s + n$ . Consequently  $S^m \times N$  actually has ordinary smooth foliations of all codimensions  $> n - 1$ . This is done in §3.

We next define the notion of tangency for locally trivial foliations  $\{L_\alpha : \alpha \in A\}$  by manifolds with boundary. Let  $\epsilon_{\mathbf{R}}$  be the trivial real line bundle and let  $f : (0, 1] \rightarrow \mathbf{R}^1$  be a smooth map satisfying

- (1)  $f(t) = 0$  for  $1/2 \leq t \leq 1$ ,
- (2)  $\lim_{t \rightarrow 0^+} |f^{(r)}(t)| = \infty$  for  $r \geq 0$ .



Let  $E_1 \subset T(M - S)$  be the tangent bundle of the ordinary foliation  $\{L_\alpha - S : \alpha \in A\}$  of  $M - S$  and, identifying  $U$  with  $S \times e^{m-p+1}$ , let  $E_2$  be the bundle  $TS \times \epsilon_{\mathbf{R}} \rightarrow S \times e^{m-p+1}$  over  $U$ . Again let  $P$  denote a linear vector field on  $e^{m-p+1}$  giving rise to the trivial foliation on  $S \times e^{m-p+1}$ .

We define a smooth vector bundle  $E$  over  $M$  to be the bundle  $E_1 \cup E_2 / \sim$  where  $\sim$  identifies  $(0, (Pv) \cos \theta) \in T(S \times e^{m-p+1})_{(x,v)}$  with the positive

vector of length  $\|v\|$  in  $(\{0\} \times \varepsilon_{\mathbf{R}})_{(x,v)}$  for  $(x, v) \in S \times (e^{m-p+1} - \{0\})$ . Here  $\tan \theta = df_{\|v\|}(1)$  with  $\pi/2 < \theta \leq \pi$ . We say that  $E$  is *tangent* to  $\{L_\alpha : \alpha \in A\}$ . To justify this terminology we define a map  $\rho : E \rightarrow TM$  of smooth vector bundles over  $M$  by the following two conditions:

- (1) Over  $m - S$   $\rho$  is the inclusion of  $E_1$  in  $TM|_{M - S}$ .
- (2) Over  $S$ ,  $\rho$  is projection to the first factor  $E_2|_S = (TS) \oplus \varepsilon_{\mathbf{R}} \rightarrow TS$  followed by inclusion in  $TM|_S$ .

Then  $\rho(E_x)$  is  $(TL_\alpha)_x$  or  $(T\partial L_\alpha)_x$  according as  $x \in \text{Int } L_\alpha$  or  $x \in \partial L_\alpha$ .

Now we change our point of view. Let  $E$  be any smooth  $p$ -plane bundle over  $M$ .

**Theorem.** *Suppose that  $E$  is tangent to a locally trivial foliation of  $M$  by manifolds with boundary. Then the real Pontrjagin ring of the stable difference  $TM/E$  is trivial in dimensions greater than  $2(m - p)$ .*

Taking  $S$  to be empty we recover Bott's result [2] for ordinary smooth foliations. Our proof is along the same lines as Bott's, but care is needed near the singular manifold  $S$ . A weaker result can be obtained by applying Bott's result together with Proposition 2.

**Example 7.** Let  $M$  be  $(\mathbf{C}P^n) \times (S^1 \times S^1)$ , and  $H$  the Hopf complex line bundle over  $\mathbf{C}P^n$ . Then  $(n + 1)H \simeq (TCP^n) \oplus \varepsilon_{\mathbf{C}}$  where  $\varepsilon_{\mathbf{C}}$  is the trivial complex line bundle. For  $1 \leq r \leq n$  let  $E$  be the bundle over  $M$  induced from  $rH$  by projection onto  $\mathbf{C}P^n$ . Then  $E$  is a smooth sub-bundle of  $TM$  and Bott's theorem asserts that for  $\frac{1}{2}(n + 3) \leq r \leq n$ ,  $E$  is not tangent to an ordinary smooth foliation—nor is any bundle stably equivalent to  $E$ . Our theorem asserts for the same values of  $r$  that  $E$  is not tangent to a locally trivial foliation by manifolds with boundary.

If  $E$  is not a sub-bundle of  $TM$ , then there is often no map  $\rho : E \rightarrow TM$  of smooth vector bundles and no closed  $(p - 1)$ -dimensional submanifold  $S$  of  $M$  satisfying

- (1)  $\dim \rho(E_x) = p$  or  $p - 1$  according as  $x \in M - S$  or  $x \in S$ ,
- (2)  $E$  is tangent to a trivial foliation in some neighborhood of  $S$ .

The obstructions could be the subject of a future note, but these vanish in Example 7.

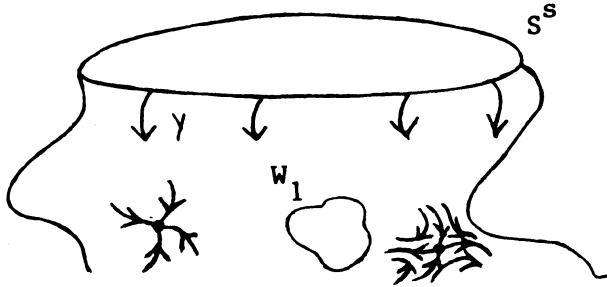
The author wishes to thank Professor J. R. Vanstone and the referee for helpful and encouraging comments, and also Professor S. Halperin for improving Proposition 2.

### 1. Proof of Proposition 1

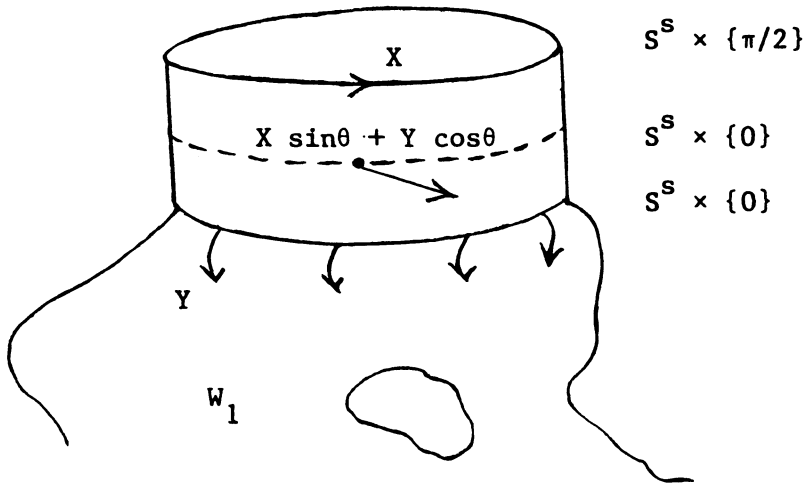
Here we are guided by Example 4. When  $s = 1$  we apply the Reeb construction [4, p. 378] to obtain an ordinary smooth codimension- $s$  foliation

of  $V_1 \times S^s$ . This is possible because the Hopf bundle is orientable. Note that on  $\partial(V_1 \times S^s) = S^r \times S^s$  a leaf of this foliation is either the whole of  $S^r \times S^s$  when  $s = 1$  or the product of  $S^r$  with a fibre of the Hopf bundle when  $s \geq 3$ .

Let  $Y$  be the gradient vector field of a Morse function  $f: W \rightarrow \mathbf{R}^1$ . We obtain  $W_1$  by removing a small open disc containing a relative minimum of  $f$  and no other critical points.



If  $s = 1$ , let  $X$  be a never-zero vector field on  $S^s$ . If  $s = 2n + 1 \geq 3$ , an orientation of the Hopf bundle  $S^s \rightarrow CP^n$  defines a never-zero vector field  $X$  on  $S^s$  whose orbits are the fibres. Changing  $Y$  to  $X$  through a collar on  $S^s$



and smoothing, we obtain a smooth vector field  $Z$  on  $W_1$  which is  $X$  on  $\partial W_1 = S^s$  and whose zeroes are the same as those of  $Y$ . As in Example 4, where  $W = S^2$ ,  $Z$  gives us a locally trivial codimension- $s$  foliation of  $S^r \times W_1$ . This foliation agrees on  $\partial(S^r \times W_1) = S^r \times S^s$  with the foliation defined on  $V_1 \times S^s$ . Identifying and smoothing these foliations along  $S^r \times S^s$  we obtain a locally trivial codimension- $s$  foliation of  $V \cdot W$ .

## 2. Proof of Proposition 2

Let  $Y$  be the negative of the identity map on  $e^{m-p+1}$ , regarded as a vector field, and let  $X$  be a smooth never-zero vector field on  $N$ . Let  $e'$  be the open ball of radius  $\frac{1}{2}$ , and let  $Z$  be a never-zero vector field on  $e^{m-p+1} \times N$  which agrees with  $X$  near  $\{0\} \times N$ , and with  $Y$  on  $(e^{m-p+1} - e') \times N$ . Then the orbits of  $Z$  give an ordinary smooth 1-dimensional foliation of  $e^{m-p+1} \times N$ , and from this we obtain an ordinary smooth  $p$ -dimensional foliation of  $S \times e^{m-p+1} \times N = U \times N$ . Let  $U' = S \times e'$ . Then on  $(U - U') \times N$  the leaves of this foliation are those of the product with  $N$  of the restriction of the singular foliation to  $M - U'$ . Identifying these foliations over  $(U - U') \times N$  we obtain an ordinary smooth  $p$ -dimensional foliation of  $M \times N$ .

## 3. Example 6

We put ourselves in the situation of §1, except that  $s$  need no longer be odd, let  $X$  be a never-zero vector field on  $N$  and note that the Reeb construction applies to  $V_1 \times N$  (all that is really needed in §1 is the existence of a never-zero vector field on  $S^s$ ). So we obtain an ordinary smooth codimension- $n$  foliation of  $V_1 \times N$  whose leaves on  $\partial(V_1 \times N) = S^r \times N$  are the products of  $S^r$  with the orbits of  $X$ . We multiply by  $S^s$  to obtain a codimension- $(n + s)$  foliation of  $V_1 \times S^s \times N$ .

Let  $w^{(1)}, \dots, w^{(k)}$  be the critical points of  $f$  in  $W_1$ . Then changing  $Y$  to  $X$  within small neighborhoods of the  $\{w^{(j)}\} \times N$ , as well as through a collar on  $S^s \times N$ , we obtain a never-zero vector field  $Z$  on  $W_1 \times N$  whose orbits on  $\partial(W_1 \times N) = S^s \times N$  are the orbits of  $X$ . Multiplying by  $S^s$  we obtain an ordinary smooth foliation of  $S^r \times W_1 \times N$  which agrees on  $\partial(S^r \times W_1 \times N) = S^r \times S^s \times N$  with the foliation defined above on  $V_1 \times S^s \times N$ . Identifying and smoothing these foliations along  $S^r \times S^s \times N$  we obtain an ordinary smooth codimension- $(s + n)$  foliation of  $(V \cdot W) \times N$ .

Of course it is trivial that  $(V \cdot W) \times N$  admits ordinary smooth foliations of codimensions  $n, n - 1$ .

## 4. The main result

To prove our theorem we first define an inclusion  $\iota : E \rightarrow (TM) \oplus \varepsilon_{\mathbb{R}}$  of smooth vector bundles over  $M$  by the following three conditions:

(1) Over  $M - U$ ,  $\iota$  is the inclusion of  $E_1|_{M - U}$  in  $TM|_{M - U}$  followed by the inclusion in  $(TM) \oplus \varepsilon_{\mathbb{R}}|_{M - U}$ .

(2) Over  $U - S$ ,  $\iota$  is given by  $\iota(w, (Pv) \cos \theta) = (w, (Pv) \cos \theta, \|v\| \sin \theta)$  for  $(w, (Pv) \cos \theta) \in (E_1)_{(x,v)}$ . Again  $\tan \theta = df_{\|v\|}(1)$  with  $\pi/2 < \theta \leq \pi$ .

(3) Over  $S$ ,  $\iota$  is the inclusion  $E_2|S = (TS) \oplus \epsilon_{\mathbf{R}} \rightarrow (TM) \oplus \epsilon_{\mathbf{R}}|S$ . Then the bundle map  $\rho$  of the Introduction is  $\iota$  followed by projection onto the first factor.

We choose a Riemannian metric on  $TS$  and take the usual Riemannian metric on  $Te^{m-p+1}$ ,  $\epsilon_{\mathbf{R}}$ . Since we are identifying  $U$  with  $S \times e^{m-p+1}$  this gives us a Riemannian metric on  $TU$  which, by contracting  $U$  if necessary, we extend to the whole of  $TM$ . Adding in the Riemannian metric on  $\epsilon_{\mathbf{R}}$  we obtain one on  $(TM) \oplus \epsilon_{\mathbf{R}}$ .

Let  $F$  be the orthogonal complement of  $\iota(E)$  in  $(TM) \oplus \epsilon_{\mathbf{R}}$ . From the construction of the metric and from the definition of  $\iota$  we see that  $F|U$  is  $S \times F_2 \rightarrow S \times e^{m-p+1}$  where  $F_2$  is an  $(m-p+1)$ -plane bundle over  $e^{m-p+1}$ . Let  $e' \subset e^{m-p+1}$  be the closed ball of radius  $1/2$ , and let  $U' \subset U$  by  $S \times e'$ . Let  $\tilde{F}_2$  be the normal bundle of the ordinary 1-dimensional foliation of  $e^{m-p+1} - e'$  by the orbits of the linear vector field  $P$ . Then we also see that  $F_2|e^{m-p+1} - e' = F_2 \oplus \epsilon_{\mathbf{R}}$ .

We choose a basic connection [2]  $\tilde{\nabla}_2$  and take the direct sum with the trivial flat connection  $\nabla_e$  on  $\epsilon_{\mathbf{R}}$  to obtain a connection on  $F_2|e^{m-p+1} - e'$ . We extend this to a connection  $\nabla_2$  on the whole of  $F_2$ . Now we define a connection  $\nabla$  on  $F|U$  (which is  $S \times F_2 \rightarrow S \times e^{m-p+1}$ ) in the following way.

Let  $Z$  be a cross-section of  $F|U$ . Then if  $X$  is a cross-section of  $\{0\} \times Te^{m-p+1}$  we define  $\nabla_X Z| \{x\} \times e^{m-p+1} = (\nabla_2)_X(Z| \{x\} \times e^{m-p+1})$ . If  $X$  is a cross-section of  $TS \times \{0\}$ , we define  $\bar{X}, \bar{Z}$  to be the vector fields on  $U \times \mathbf{R}^1$  given by

$$\bar{X}(x, v, t) = (X(x, v), 0) \in T(S \times e^{m-p+1} \times \mathbf{R}^1),$$

$$\bar{Z}(x, v, t) = Z(x, v) \in (F|U) \times \mathbf{R}^1 \subset ((TU) \oplus \epsilon_{\mathbf{R}}) \times \mathbf{R}^1 = T(U \times \mathbf{R}^1).$$

Now we define  $\nabla_X Z = \pi_F[\bar{X}, \bar{Z}]$  where  $\pi_F$  is the orthogonal projection from  $T(U \times \mathbf{R}^1)|U \times \{0\} = (TU) \oplus \epsilon_{\mathbf{R}}$  onto  $F$ , and the Lie bracket is taken on  $U \times \mathbf{R}^1$ .

Let  $F_1$  be the normal bundle of the ordinary foliation  $\{L_\alpha \cap (M - S) : \alpha \in A\}$  of  $M - S$ . Then we see that over  $U - U'$ ,  $\nabla$  is the direct sum  $\tilde{\nabla}_1 \oplus \nabla_e$  where  $\tilde{\nabla}_1$  is a basic connection on  $F_1|U - U'$ . We extend  $\tilde{\nabla}_1$  to a basic connection  $\nabla_1$  on  $F_1|M - U'$ . Now  $F|M - U' = (F_1|M - U') \oplus \epsilon_{\mathbf{R}}$ , and we extend  $\nabla$  to a connection on the whole of  $F$  by defining  $\nabla|M - U'$  to be  $\nabla_1 \oplus \nabla_e$ .

Let  $Z$  be a cross-section of  $F$ . Then if  $X, Y$  are cross-sections of  $T(M - U')$  tangent to the  $L_\alpha$ , we argue as in [2] that the curvature  $K(X, Y)Z$  with respect to  $\nabla$  is zero. Similarly, if  $X, Y$  are cross-sections of  $TU$ , which are zero in the  $e^{m-p+1}$  directions, then  $K(X, Y)Z = 0$ . We also wish to prove that  $K(X, Y)Z = 0$  when  $X$  is a cross-section of  $TU$ , which is zero in the

$e^{m-p+1}$  directions, and  $Y$  is a cross-section of  $TU$ , which is zero in the  $S$  directions. For this we go into more detail.

Since  $K$  is a tensor, we may compute  $K(X, Y)Z$  at any point under the assumption that  $X(x, v)$  does not depend on  $v$  and that  $Y(x, v)$ ,  $Z(x, v)$  do not depend on  $x$ . It follows that the Lie bracket  $[X, Y]$  on  $U$  is zero so that  $K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_X \nabla_Y Z$  since the Lie bracket  $[\bar{X}, \bar{Z}]$  on  $U \times \mathbf{R}^1$  is also zero. But  $\nabla_Y Z|_{\{x\}} \times e^{m-p+1}$  is  $(\nabla_Y)_Y(Z|_{\{x\}} \times e^{m-p+1})$  and so  $\nabla_Y Z$  does not depend on  $x$ , since neither  $Y$  nor  $Z$  do. Therefore  $\nabla_X \nabla_Y Z = \pi_F[\bar{X}, \overline{\nabla_Y Z}] = 0$ .

This shows that for each  $y \in M$  there is a subspace  $H_y$  of  $TM_y$  of dimension  $\geq p$  such that  $K(X, Y)Z = 0$  for  $X, Y \in H_y$  and  $z \in F_y$ . Our theorem now follows from the Chern-Weil construction of the Pontrjagin ring.

### References

- [1] P. Baum & R. Bott, *Singularities of holomorphic foliations*, J. Differential Geometry **7** (1972) 279–342.
- [2] R. Bott, *On a topological obstruction to integrability*, Proc. Sympos. Pure Math., Vol. 16, Amer. Math. Soc., 1970, 127–131.
- [3] R. P. Buemi, *An obstruction to certain non-integrable 2-plane fields*, Topology **16** (1977) 173–176.
- [4] H. B. Lawson, Jr., *Foliations*, Bull. Amer. Math. Soc. **80** (1974) 369–418.
- [5] W. P. Thurston, *The theory of foliations of codimensions greater than one*, Comment. Math. Helv. **49** (1974) 214–231.
- [6] ———, *Existence of codimension-one foliations*, Ann. of Math. (2) **104** (1976) 249–268.
- [7] H. E. Winkelkemper, *Manifolds as open books*, Bull. Amer. Math. Soc. **79** (1973) 45–51.

UNIVERSITY OF WESTERN AUSTRALIA