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THE FIRST PROPER SPACE OF Δ FOR p-FORMS IN COMPACT RIEMANNIAN MANIFOLDS OF POSITIVE CURVATURE OPERATOR

SHUN-ICHI TACHIBANA & SEIICHI YAMAGUCHI

Introduction

Let M^n be an n-dimensional Riemannian manifold, and denote the curvature tensor of M^n by $R_{kji}{}^h$. If there exists a positive constant k such that

$$(*) \qquad -R_{kjih}u^{kj}u^{ih} \geqslant 2ku_{ji}u^{ji}$$

holds for any 2-form u on an M^n everywhere, then the M^n is said to be of positive curvature operator. For a compact orientable M^n of positive curvature operator, M. Berger [1] and D. Meyer [9] have proved that its first $n-1$ Betti numbers $b_i(M^n)$, $i = 1, \cdots, n-1$, vanish. It has been also known that such a manifold is of constant curvature if its metric satisfies $\nabla_h R_{kji}{}^h = 0$, [10]. Let Δ denote the Laplacian operator. A nonzero p-form u satisfying $\Delta u = \lambda u$ with a constant λ is called a proper form of Δ corresponding to the proper value λ. S. Gallot and D. Meyer have discussed the proper value in compact M^n of positive curvature operator and obtained its lower bound as follows.

Theorem A, [6]. *In a compact Riemannian manifold M^n of positive curvature operator, the proper value λ of Δ for p-form u ($n \geqslant p \geqslant 1$) satisfies*

$$\lambda \geqslant p(n-p+1)k \qquad \text{if } du = 0,$$
$$\lambda \geqslant (p+1)(n-p)k \qquad \text{if } \delta u = 0.$$

Furthermore, Gallot [2], Gallot and Meyer [7] and the present authors [11], [14] discussed the case when λ actually takes the possible minimal values. In particular, the present authors showed that the Killing and the conformal Killing p-forms play essential roles in this field. On the other hand, one of the present authors has obtained

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Theorem B, [12]. *In a $2m$-dimensional compact conformally flat Riemannian manifold with positive constant scalar curvature $R = 2m(2m-1)k$, the proper value λ of Δ for m-forms satisfies*

$$\lambda \geqslant m(m+1)k,$$

and the following relations hold:

$$V^m_{m(m+1)k} = C^m = C^m(d) \oplus K^m, \quad (\textit{direct sum}).$$

Here and throughout this paper, V^p_λ, C^p etc. denote vector spaces with natural structure defined by

V^p_λ = *the proper space of p-forms corresponding to λ,*

C^p = *the space of all conformal Killing p-forms,*

$C^p(d)$ = *the space of all closed conformal Killing p-forms,*

K^p = *the space of all Killing p-forms,*

K^p_c = *the space of all special Killing p-forms with c.*

The purpose of this paper is to determine the first proper space of compact Riemannian manifold of positive curvature operator in terms of K^p, K^p_c and $C^p(d)$.

1. Preliminaries

Let M^n $(n > 1)$ be an n-dimensional Riemannian manifold. Throughout this paper, manifolds are assumed to be connected and of class C^∞. We denote respectively by g_{ji}, $R_{kji}{}^h$ and $R_{ji} = R_{hji}{}^h$ the metric, the curvature and the Ricci tensor of a Riemannian manifold. We shall represent tensors by their components with respect to the natural base, and shall use the summation convention. For a differential p-form

$$u = \frac{1}{p!} u_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

with skew symmetric coefficients $u_{i_1 \cdots i_p}$, the coefficients of its exterior differential du and the exterior codifferential δu are given by

$$(du)_{i_1 \cdots i_{p+1}} = \sum_{a=1}^{p+1} (-1)^{a+1} \nabla_{i_a} u_{i_1 \cdots \hat{i}_a \cdots i_{p+1}},$$

$$(\delta u)_{i_2 \cdots i_p} = -\nabla^h u_{h i_2 \cdots i_p},$$

where $\nabla^h = g^{hj}\nabla_j$, ∇_j denotes the operator of covariant differentiation, and $\hat{i}_a$ means i_a to be deleted. For p-forms u and v the inner product $\langle u, v\rangle$, the lengths $|u|$ and $|\nabla u|$ are given by

$$\langle u, v\rangle = \frac{1}{p!} u_{i_1 \cdots i_p} v^{i_1 \cdots i_p}, \quad |u|^2 = \langle u, u\rangle,$$

$$|\nabla u|^2 = \frac{1}{p!} \nabla_h u_{i_1 \cdots i_p} \nabla^h u^{i_1 \cdots i_p}.$$

Denoting by $\Delta = d\delta + \delta d$ the Laplacian operator, we have $\nabla f = -\nabla^r \nabla_r f$ for function f and

$$(1.1) \qquad (\Delta u)_{i_1 \cdots i_p} = -\nabla^r \nabla_r u_{i_1 \cdots i_p} + H(u)_{i_1 \cdots i_p}$$

as the coefficients of Δu, where $H(u)_{i_1 \cdots i_p}$ are the coefficients of $H(u)$ given by

$$H(u)_i = R_{ir} u^r,$$

$$H(u)_{i_1 \cdots i_p} = \sum_{a=1}^{p} R_{i_a}{}^r u_{i_1 \cdots r \cdots i_p} + \sum_{a<b} R_{i_a i_b}{}^{rs} u_{i_1 \cdots r \cdots s \cdots i_p}, \quad n \geqslant p \geqslant 2.$$

In the second term on the right-hand side of the last above equation the subscripts r and s are in the positions of i_a and i_b respectively, and we shall use similar arrangements of indices without special notice. (1.1) may be written as

$$(1.2) \qquad \Delta u = -\nabla^r \nabla_r u + H(u).$$

The quadratic form $F_p(u)$ of u is denfined by

$$F_p(u) = \langle H(u), u\rangle$$

$$= \frac{1}{(p-1)!}\left(R_{rs} u^{ri_2 \cdots i_p} u^s{}_{i_2 \cdots i_p} + \frac{p-1}{2} R_{rsjh} u^{rsi_3 \cdots i_p} u^{jh}{}_{i_3 \cdots i_p}\right),$$

and it appears in the following well-known formula which is valid for any p-form u:

$$(1.3) \qquad \frac{1}{2}\Delta(|u|^2) = \langle \Delta u, u\rangle - |\nabla u|^2 - F_p(u).$$

2. The Killing and the conformal Killing p-forms

A p-form v ($p \geqslant 1$) is said to be Killing if it satisfies

$$(2.1) \qquad \nabla_h v_{ji_2 \cdots i_p} + \nabla_j v_{hi_2 \cdots i_p} = 0.$$

Any Killing p-form is coclosed, and it is easy to see that (2.1) is equivalent to the following equation:

$$(2.2) \qquad (dv)_{hi_1 \cdots i_p} = (p+1)\nabla_h v_{i_1 \cdots i_p}.$$

It is known [13, (1.4)] that a Killing p-form v satisfies

$$p\nabla^r\nabla_r v + H(v) = 0. \tag{2.3}$$

Hence, if we take account of (1.2), it follows that

$$p\Delta v = (p+1)H(v). \tag{2.4}$$

A Killing p-form v is said to be special with c, if it satisfies

$$\nabla_h\nabla_j v_{i_1\cdots i_p} + c\left(g_{hj}v_{i_1\cdots i_p} + \sum_{a=1}^{p}(-1)^a g_{hi_a}v_{ji_1\cdots\hat{i}_a\cdots i_p}\right) = 0 \tag{2.5}$$

with a constant c.

For example, any Killing p-form in the sphere of positive constant sectional curvature k is special with $c = k$.

Transvecting (2.5) with g^{hj}, we have $\nabla^r\nabla_r v + (n-p)cv = 0$, from which it follows that $H(v) = p(n-p)cv$ by virtue of (2.3). Substituting the last equation into (2.4) we obtain

$$\Delta v = (p+1)(n-p)cv,$$

which shows that v is proper corresponding to $(p+1)(n-p)c$. Hence

Lemma 2.1. *In any n-dimensional Riemannian manifold, we have*

$$K_c^{\,p} \subset V^p_{(p+1)(n-p)c} \quad (n \geqslant p \geqslant 1),$$

where c is any constant.

Next, let w be a closed p-form ($p > 1$) such that δw is special Killing with c, i.e., $w \in d^{-1}(0) \cap \delta^{-1}(K_c^{\,p-1})$. Since $\delta w \in K_c^{\,p-1}$, we have $\Delta\delta w = p(n-p+1)c\delta w$ by Lemma 2.1. Applying d to both sides of the last equation we obtain $\Delta\Delta w = p(n-p+1)c\Delta w$ because of $d\Delta = \Delta d$. Hence

Lemma 2.2. *In any n-dimensional Riemannian manifold, we have*

$$\Delta\left(d^{-1}(0) \cap \delta^{-1}\left(K_c^{\,p-1}\right)\right) \subset V^p_{p(n-p+1)c} \quad (p > 1),$$

where c is any constant.

A p-form w ($p \geqslant 1$) is said to be conformal Killing [7, (1.1)], if there exists a $(p-1)$-form θ called the associated form such that

$$\begin{aligned}&\nabla_h w_{ji_2\cdots i_p} + \nabla_j w_{hi_2\cdots i_p}\\ &\quad = 2\theta_{i_2\cdots i_p}g_{hj} - \sum_{a=2}^{p}(-1)^a\left(\theta_{hi_2\cdots\hat{i}_a\cdots i_p}g_{ji_a} + \theta_{ji_2\cdots\hat{i}_a\cdots i_p}g_{hi_a}\right).\end{aligned} \tag{2.6}$$

For a conformal Killing p-form w, the following equations hold [8, (1.2), (2.4)]

$$\delta w = -(n-p+1)\theta, \tag{2.7}$$

$$(dw)_{hi_1\cdots i_p} = (p+1)\left(\nabla_h w_{i_1\cdots i_p} + \sum_{a=1}^{p}(-1)^a\theta_{i_1\cdots\hat{i}_a\cdots i_p}g_{hi_a}\right), \tag{2.8}$$

$$p\nabla^r\nabla_r w + H(w) + \frac{2p-n}{n-p+1}d\delta w = 0. \tag{2.9}$$

It should be noticed that (2.8) is equivalent to (2.6).

From (2.6) and (2.7) we have

$$K^p = C^p \cap \delta^{-1}(0).$$

On the other hand, a simple calculation shows

$$K_c^p \subset d^{-1}(C^{p+1}(d)) \tag{2.10}$$

to be valid for any constant c.

Now we can prove

Lemma 2.3. *In any n-dimensional Riemannian manifold, we have*

$$K^p \cap V^p_{(p+1)(n-p)c} \cap d^{-1}(C^{p+1}(d)) = K_c^p \quad (n > p)$$

for any constant c.

Proof. The left-hand side includes the right-hand side, because of Lemma 2.1 and (2.10). Conversely, let v be a Killing p-form such that $w = dv$ is conformal Killing and $\Delta v = (p+1)(n-p)cv$. Then we have

$$w_{i_1\cdots i_{p+1}} = (dv)_{i_1\cdots i_{p+1}} = (p+1)\nabla_{i_1}v_{i_2\cdots i_{p+1}}, \tag{2.11}$$

$$\nabla_h w_{i_1\cdots i_{p+1}} + \sum_{a=1}^{p}(-1)^a\theta_{i_1\cdots \hat{i}_a\cdots i_{p+1}}g_{hi_a} = 0, \tag{2.12}$$

$$\Delta v = \delta dv = \delta w = -(n-p)\theta = (p+1)(n-p)cv \tag{2.13}$$

by virtue of (2.7) and (2.8). If we write out (2.12) in terms of v making use of (2.11) and (2.13), it is seen that (2.5) holds. q.e.d.

Next we shall prove

Lemma 2.4. *In any n-dimensional Riemannian manifold, we have*

$$C^p(d) \cap V^p_{p(n-p+1)c} \cap \delta^{-1}(K^{p-1}) \subset \delta^{-1}(K_c^{p-1}) \; (p > 1),$$

for any constant c.

Proof. Let w be a p-form in the left-hand side set, then w is closed conformal Killing such that $\Delta w = p(n-p+1)cw$ and $v = \delta w$ is Killing. Since $\Delta w = d\delta w = dv$, we have

$$p\nabla_{i_1}v_{i_2\cdots i_p} = p(n-p+1)cw_{i_1\cdots i_p},$$

and, in the consequence of (2.7) and (2.8),

$$\begin{aligned}\nabla_h \nabla_{i_1} v_{i_2 \cdots i_p} &= (n-p+1)c\nabla_h w_{i_1 \cdots i_p} \\ &= -(n-p+1)c \sum_{a=1}^{p} (-1)^a \theta_{i_1 \cdots \hat{i}_a \cdots i_p} g_{hi_a} \\ &= c \sum_{a=1}^{p} (-1)^a v_{i_1 \cdots \hat{i}_a \cdots i_p} g_{hi_a},\end{aligned}$$

which shows that $v = \delta w$ is special with c. q.e.d.

For a closed conformal Killing p-form w, (2.9) becomes

$$p\nabla^r \nabla_r w + H(w) + \frac{2p-n}{n-p+1}\Delta w = 0.$$

If we take account of (1.2), we have

$$(n-p)\Delta w = (n-p+1)H(w) \quad \text{for } w \in C^p(d), \tag{2.14}$$

which is useful in the next section.

When $n = 2p$, (2.14) reduces to (2.4) without the assumption "closed".

S. Gallot and D. Meyer [6] have proved that the inequality

$$|\nabla u|^2 \geqslant \frac{1}{p+1}|du|^2 + \frac{1}{n-p+1}|\delta u|^2 \quad (n \geqslant p \geqslant 1) \tag{2.15}$$

holds for any p-form u. As they did not discuss when the equality holds in (2.15), we shall formulate the inequality as follows, containing the case of equality and with a new proof.

Lemma 2.5. *For any p-form u in a Riemannian manifold, the inequality* (2.15) *holds, where the equality sign holds if and only if the p-form u is conformal Killing.*

Proof. Let us define a tensor field t by

$$\begin{aligned}t_{hi_1 \cdots i_p} &= \nabla_h u_{i_1 \cdots i_p} - \frac{1}{p+1}(du)_{hi_1 \cdots i_p} \\ &\quad - \frac{1}{n-p+1}\sum_{a=1}^{p} (-1)^a (\delta u)_{i_1 \cdots \hat{i}_a \cdots i_p} g_{hi_a}.\end{aligned}$$

Then we know by virtue of (2.7) and (2.8) that t vanishes identically if and only if u is conformal Killing. If we put $|t|^2 = (1/p!)t_{hi_1 \cdots i_p} t^{hi_1 \cdots i_p}$, it is easy to see

$$|t|^2 = |\nabla u|^2 - \frac{1}{p+1}|du|^2 - \frac{1}{n-p+1}|\delta u|^2,$$

which proves our assertion.

3. Theorems

In the remainder of this paper, we assume that M^n is compact and of positive curvature operator. Taking its orientable double covering, if necessary, we may consider M^n as orientable without loss of generality. As M^n is of positive curvature operator, the inequality (*) in Introduction is valid for a positive constant k which will be fixed throughout the paper.

We shall give some theorems for M^n as applications of the results in §1 and §2. Those theorems would be meaningful if we take account of Gallot's works [3], [4].

It is known [9] that any p-form u satisfies

$$F_p(u) \geqslant p(n-p)k|u|^2 \tag{3.1}$$

by virtue of (*).

Now let us integrate (1.3) for a p-form u over M^n. Then it follows from (2.15) and (3.1) that

$$\begin{aligned}(\Delta u, u) &\geqslant \|\nabla u\|^2 + p(n-p)k\|u\|^2 \\ &\geqslant \frac{1}{p+1}\|du\|^2 + \frac{1}{n-p+1}\|\delta u\|^2 + p(n-p)k\|u\|^2,\end{aligned} \tag{3.2}$$

where (,) and $\|\ \|$ denote the global inner product and the global length respectively. For a proper form u corresponding to a proper value λ, (3.2) becomes

$$\lambda\|u\|^2 \geqslant \frac{1}{p+1}\|du\|^2 + \frac{1}{n-p+1}\|\delta u\|^2 + p(n-p)k\|u\|^2,$$

and making use of $(\Delta u, u) = \|du\|^2 + \|\delta u\|^2$ we have

$$p\{\lambda - (p+1)(n-p)k\}\|u\|^2 \geqslant \frac{2p-n}{n-p+1}\|\delta u\|^2, \tag{3.3}$$

$$(n-p)\{\lambda - p(n-p+1)k\}\|u\|^2 \geqslant \frac{n-2p}{p+1}\|du\|^2. \tag{3.4}$$

If the equality is valid in (3.3) or (3.4), u is conformal Killing by virtue of Lemma 2.5.

Remark 1. Theorem A in Introduction follows from these inequalities.

Remark 2. $(p+1)(n-p)k \geqslant p(n-p+1)k$ if and only if $n \geqslant 2p$.

Remark 3. When $n = 2p$, (3.3) and (3.4) both reduce to the following single inequality

$$\lambda \geqslant p(p+1)k. \tag{3.5}$$

First we shall prove

Theorem 3.1. *In a compact Riemannian manifold M^n of positive curvature* *operator, we have*

$$V^p_{(p+1)(n-p)k} \cap \delta^{-1}(0) = K_k{}^p \quad (n > p \geqslant 1).$$

Proof. Let $v \in K_k{}^p$ for $n \geqslant p \geqslant 1$. Then $\delta v = 0$, and by Lemma 2.1 we have $v \in V^p_{(p+1)(n-p)k}$ which proves $V^p_{(p+1)(n-p)k} \cap \delta^{-1}(0) \supset K_k{}^p$.

Conversely, let v be a p-form satisfying

$$\Delta v = (p + 1)(n - p)kv, \tag{3.6}$$

and assume that v is coclosed. (It should be noticed that because of (3.3), v is necessarily coclosed if $2p > n$.) Then the equality holds in (3.3) and hence by Lemma 2.5, v is Killing. If we operate d to both sides of (3.6) and put $w = dv$, then it follows that $\Delta w = (p + 1)(n - p)kw$. Next applying (3.4) to the closed $(p + 1)$-form w, we find that w is closed conformal Killing. Therefore $v \in K^p \cap V^p_{(p+1)(n-p)k} \cap d^{-1}(C^{p+1}(d))$, and hence we have $v \in K_k{}^p$ by Lemma 2.3. q.e.d.

As a corollary we have

Theorem 3.2. *In a compact Riemannian manifold M^n of positive curvature operator,*

$$V^p_{(p+1)(n-p)k} = K_k{}^p$$

holds for $2p > n > p \geqslant 1$.

Next we shall prove

Theorem 3.3. *In a compact Riemannian manifold M^n of positive curvature operator, we have*

$$V^p_{p(n-p+1)k} \cap d^{-1}(0) = C^p(d) \cap \delta^{-1}(K_k{}^{p-1}) \qquad (n > p > 1).$$

Proof. Let $w \in C^p(d) \cap \delta^{-1}(K_k{}^{p-1})$ for $n > p > 1$. As $w \in d^{-1}(0) \cap \delta^{-1}(K_k{}^{p-1})$, by Lemma 2.2 we have

$$\Delta\Delta w = p(n - p + 1)k\Delta w. \tag{3.8}$$

On the other hand, as $w \in C^p(d)$ we have (2.14), i.e.,

$$(n - p)\Delta w = (n - p + 1)H(w). \tag{3.9}$$

Now let us put $\alpha = \Delta w - p(n - p + 1)kw$. Then it follows from (3.8), (3.9) and (3.1) that

$$\begin{aligned}\|\alpha\|^2 &= (\Delta w, \Delta w) - 2p(n - p + 1)k(\Delta w, w) + p^2(n - p + 1)^2k^2\|w\|^2 \\ &= p(n - p + 1)^2k\left(-\frac{1}{n - p}F_p(w) + pk\|w\|^2\right) \leqslant 0.\end{aligned}$$

Thus we obtain $\alpha = 0$ which shows $w \in V^p_{p(n-p+1)k}$.

Conversely, let us consider $w \in V^p_{p(n-p+1)k}$, and assume that w is closed. (If $n > 2p$, because of (3.4) w is necessarily closed.) Then w is conformal Killing by virtue of (3.4). Applying δ to both sides of $\Delta w = p(n-p+1)kw$ and putting $v = \delta w$ we have $\Delta v = p(n-p+1)kv$. Thus v is a coclosed $(p-1)$-form and satisfies (3.3) with the equality. Therefore v is Killing and hence we have $w \in C^p(d) \cap V^p_{p(n-p+1)k} \cap \delta^{-1}(K^{p-1})$, which together with Lemma 2.4 gives $w \in \delta^{-1}(K_k{}^{p-1})$. q.e.d.

As a corollary we have

Theorem 3.4. *In a compact Riemannian manifold M^n of positive curvature operator.*

$$V^p_{p(n-p+1)k} = C^p(d) \cap \delta^{-1}(K_k{}^{p-1}),$$

holds for $n > 2p > 2$.

Remark 4. If we take account of (2.10), the assertion of Theorem 3.2 can be written as

$$V^p_{(p+1)(n-p)k} = K_k{}^p \cap d^{-1}(C^{p+1}(d)).$$

Remark 5. Gallot [2] has determined $V^p_{p(n-p+1)k} \cap d^{-1}(0)$ in a way different from ours.

The case of $n = 2p$ is special; for the case we can have

Theorem 3.5. *In a compact Riemannian manifold M^{2m} $(m > 1)$ of positive curvature operator, the following direct sum holds:*

$$V^m_{m(m+1)k} = K_k{}^m \oplus \left(C^m(d) \cap \delta^{-1}(K_k{}^{m-1})\right).$$

Proof. Let $u \in V^m_{m(m+1)k}$. u is written uniquely as $u = v + w$, where $\delta v = 0$ and $dw = 0$, because the mth Betti number vanishes. Then we have

$$\Delta u = \Delta v + \Delta w = m(m+1)ku = m(m+1)k(v+w).$$

As $\delta\Delta v = \Delta\delta v = 0$, $d\Delta w = \Delta\, dw = 0$ and the decomposition of Δu is unique, we obtain

$$v \in V^m_{m(m+1)k} \cap \delta^{-1}(0), \quad w \in V^m_{m(m+1)k} \cap d^{-1}(0).$$

Consequently, using Theorems 3.1 and 3.3. and taking account of (3.5) we see that $v \in K_k{}^m$, $w \in C^m(d) \cap \delta^{-1}(K_k{}^{m-1})$. The converse is evident.

Remark 6. In [14] there have been given the converse parts of Theorems 3.1 and 3.3 with further results.

Remark 7. Yanamoto [15] has conjectured that in any Riemannian manifold the dual $*u$ of a conformal Killing p-form u would be conformal Killing, and has proved the conjecture for $n = 3$. In a compact Riemannian manifold of positive curvature operator we know that $u \in K_k{}^{n-p}$ implies $*u \in C^p(d) \cap \delta^{-1}(K_k{}^{p-1})$ and vice versa.

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OCHANOMIZU UNIVERSITY
SCIENCE UNIVERSITY OF TOKYO