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# A SINGULAR MAP OF A CUBE ONTO A SQUARE

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An example is given of a transformation $F$ of class $C^1$ on a cube in $R^3$, *of rank at most* 1 *everywhere*, onto a square. With merely verbal changes, the example operates from $R^{n+1}$ to $R^n$ for $n = 3, 4, 5 \dots$. The construction begins with a Cantor set $C(\beta)$ in $R^3$; $C(\beta)$ can be found by the standard method but the one outlined in the first paragraph leads quickly to a system of boundaries $B(n_1, \dots, n_k)$, the main geometrical curiosity of this example.

We learned of this kind of problem from Kevin Grasse and Felix Albrecht; it was stated by M. Hirsch (*Differential topology*, Graduate Texts in Math. Vol. 33, Springer, Berlin, 1976, p. 74).

**A system of cubes.** For each number $\beta$ in (0, 1/2) we define a method of constructing 8 subcubes in each cube in $R^3$. Let the larger cube be defined by the inequalities $|x_i - c_i| \leqslant L/2$, $1 \leqslant i \leqslant 3$. Then the subcubes are defined by $|x_i - c_i \pm L/4| \leqslant \beta L/2$, so there are 8 in all; any two have a distance $\geqslant L/2 - \beta L$, and all have a distance $\geqslant L/4 - \beta L/2$ from the boundary of the large cube.

Beginning with the cube $I_0$: $|x_i| \leqslant 1$, we define cubes $I(n_1, \dots, n_k)$, wherein $I(n_1)$ are the 8 cubes obtained from $I_0$, etc., and each $n_k = 1, 2, \dots, 8$. Distinct cubes $I(n_1, \dots, n_k)$ and $I(n'_1, \dots, n'_k)$ have a distance at least $\beta^{k-1} - 2\beta^k$. In the case of cubes $I(n_1, \dots, n_k)$ and $I(n'_1, \dots, n'_j)$ with $k \leqslant j$, the situation is more complicated. When the cubes are disjoint we have the lower bound $\beta^{k-1}(1 - 2\beta)$ found above; when the larger contains the smaller, the distance between their boundaries exceeds $\beta^k(1/2 - \beta)$. We denote the boundary of $I_0$ by $B_0$, and the boundary of $I(n_1, \dots, n_k)$ by $B(n_1, \dots, n_k)$. The Cantor set defined by the cubes is denoted $C(\beta)$ and we require $\beta^2 > 1/8$, for reasons to appear presently.

**A mapping of** $C(\beta)$. Let $R_0$ be any closed cube in $R^2$, and the rectangles $R(n_1, \dots, n_k)$ be defined by this variant of the process used above. When $k$ is even (or $R = R_0$) we divide $R(n_1, \dots, n_k)$ by 7 vertical lines into 8 congruent rectangles; when $k$ is odd we divide by horizontal lines. Thus $R(n_1, \dots, n_k)$ has diameter $\leqslant C2^{-3k/2}$.

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We specify that $C(\beta) \cap I(n_1, \ldots, n_k)$ be mapped into $R(n_1, \ldots, n_k)$ by our transformation $\Phi$ of $C(\beta)$ onto $R_0$, and we shall now prove

$$\|\Phi(x) - \Phi(y)\| \leqslant C'\|x - y\|^{\lambda}, \lambda = -3ln2/2ln\beta > 1.$$

Indeed, if $k$ is the largest integer such that $x$ and $y$ belong to the same cube $I(n_1, \ldots, n_k)$, then $\|x - y\| \geqslant \beta^k(1 - 2\beta)$ and

$$\|\Phi(x) - \Phi(y)\| \leqslant C(2^{-3/2})^k = C'\left[\beta^k(1 - 2\beta)\right]^{\lambda} \leqslant C\|x - y\|^{\lambda}.$$

**Extension of the mapping.** This is accomplished in two stages; in the first (easy) one we define $F$ on each $B(n_1, \ldots, n_k)$; on each boundary $F$ is constant and its value is in $R(n_1, \ldots, n_k)$. In the second stage we define $F$ in the sets $I(n_1, \ldots, n_k) - \bigcup I(n_1, \ldots, n_k, n_{k+1})$ and $I_0 - \bigcup I(n_1)$. To avoid excessive notation we write $J_0$ for a large cube, $J_m$ $(1 \leqslant m \leqslant 8)$ for its progeny, and $a_m = F(J_m)$, $0 \leqslant m \leqslant 8$.

Let $f$ be a mapping of the interval $T = [0, 1]$ into $R_0$, with $f(m/8) = a_m$. Moreover $f$ is of class $C^1(T)$ and $\|f'\| \leqslant 9 \max\|a_m - a_{m+1}\|$, $0 \leqslant m < 7$. (We can confine $f$ to the convex hull of $a_0, \ldots, a_8$, and construct $f$ by an explicit formula to obtain the estimate for $f'$.) Let $g$ be a mapping of class $C^1(R^3)$ so that $g = m/8$ on a neighborhood of $\partial J_m$, $0 \leqslant m < 8$, $g(R^3) \subseteq T$, and $\|Dg\| \leqslant C''\beta^{-k}$. This can be accomplished first for $I_0 = J_0$, and for smaller cubes by the similarity of $I_0$ and $J_0$ in the ratio $\beta^{-k}$. Finally we set $F = f \circ g$ in the set $J_0 - J_1 - \cdots - J_8$.

Clearly the mapping $F$ is of class $C^1$ on $I_0 - C(\beta)$, since $DF = 0$ on all the boundaries. Moreover, $\|DF\| \leqslant \|Dg\| \cdot \|f'\| = 0(\beta^{-k}\beta^{\lambda k})$ on $I(n_1, \ldots, n_k) - C(\beta)$, so $\|DF\|$ tends to 0 on approach to $C(\beta)$, while $f$ is a continuous extension of $\Phi$ to $I_0$. To conclude that $DF = 0$ at $C(\beta)$ we use the inequality on $\|\Phi(x) - \Phi(y)\|$ found before for $x, y$ in $C(\beta)$.

Now plainly $DF$ has rank 0 on $C(\beta)$ and the boundaries $B(n_1, \ldots, n_k)$, and in the intermediate regions $DF = Df \circ Dg$ has rank at most 1 by construction.

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