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REFLECTIVE SUBMANIFOLDS. IV. CLASSIFICATION OF REAL FORMS OF HERMITIAN SYMMETRIC SPACES

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1. Introduction

This note is a sequel to the author's previous papers [7], [8], [9] where we studied geodesic submanifolds of Riemannian symmetric spaces, which are fixed point sets of involutive isometries and are called *reflective submanifolds*. For a Hermitian symmetric space M, the fixed point sets of antiholomorphic involutive isometries (called complex *conjugations*) are called *real forms* of M. In the papers [3], [4], [5], the real forms of the bounded symmetric domains have been classified. Using the classification of reflective submanifolds ([8], [9]) we give here a classification of the real forms of compact and noncompact Hermitian symmetric spaces. The results will be useful in the study of the real points of compact Hermitian symmetric spaces and arithmetic quotients of bounded symmetric domains, when both of them are considered as algebraic varieties, [3]. Our present method is different from that of [3], [4], and [5], and is more elementary in the sense that we use only Lie theoretic machineries and avoid the use of Galois cohomology. Real forms belong to a special class of reflective submanifolds which are also self-complementary (see §2). So to classify the real forms in an irreducible Hermitian symmetric space M, we only need to look for them among the self-complementary reflective submanifolds of M.

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2. Reflective submanifolds

We recall here some facts from [7] and [8] and establish some notation. In this note Lie groups and their Lie algebras will be denoted respectively by capital Latin and the corresponding lower case German letters. For general terminology related to symmetric spaces we follow [6] closely.

Let $M = G/H$ be a simply connected Riemannian symmetric space with canonical involution σ and canonical decompostion $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ where $\mathfrak{m} =$

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$T_0(M)$ and $o \in M$ corresponds to the coset H. A connected submanifold $B = K/Q$ through o is said to be *reflective* if it is *a connected component of* the fixed point set of an involutive isometry ρ. A reflective submanifold is necessarily totally geodesic. If we let $\mathfrak{b} = T_0(B)$ and let $\mathfrak{b}^\perp$ be its orthogonal complement, then we have

$$\begin{aligned} &[[\mathfrak{b}, \mathfrak{b}], \mathfrak{b}] \subset \mathfrak{b}, \qquad [[\mathfrak{b}^\perp, \mathfrak{b}^\perp], \mathfrak{b}^\perp] \subset \mathfrak{b}^\perp, \\ &[[\mathfrak{b}, \mathfrak{b}^\perp], \mathfrak{b}] \subset \mathfrak{b}^\perp, \quad [[\mathfrak{b}, \mathfrak{b}^\perp], \mathfrak{b}^\perp] \subset \mathfrak{b}. \end{aligned} \tag{1}$$

A subspace $\mathfrak{b}$ of $\mathfrak{m}$ satisfying (1) is said to be *reflective*. There is a one-to-one correspondence between the reflective subspace $\mathfrak{b}$ of $\mathfrak{m}$ and the reflective submanifolds through o given by $T_0(B) = \mathfrak{b}$, [7, Theorem 3]. $\mathfrak{b}^\perp$ is also a reflective subspace, and corresponds to a reflective submanifold $B^\perp$. The pair $\{\mathfrak{b}, \mathfrak{b}^\perp\}$ (resp. $\{B, B^\perp\}$) is called a complementary pair of reflective subspaces (resp. submanifolds). If B is isometric to $B^\perp$, it is said to be *self-complementary*; in this case $\mathfrak{b}$ is also said to be self-complementary. Duality between compact and noncompact symmetric spaces preserves reflective subspaces, [8, Remarks 2–4].

3. Classification of real forms of irreducible Hermitian symmetric spaces

Let M be a Hermitian symmetric space. An involutive antiholomorphic isometry ρ of M is called a *complex conjugation* of M. In general the fixed point set of an involutive isometry of a Riemannian symmetric space is not connected. However, the fixed point set of a complex conjugation is connected, as we shall see later. For the moment, let us call any connected component of the fixed point set M^ρ of ρ, if not empty, a *real form* of M. Let B be a real form of M. Then B is a reflective submanifold. In fact, B is self-complementary since the complex structure J of M maps B isometrically onto its complementary partner. We will begin the classification with the case where M is irreducible, and will describe how to handle the general case later. For the remainder of this section, $M = G/H$ will denote an irreducible Hermitian symmetric space where G is the largest connected group of holomorphic isometries; the notation established in the previous section will also be preserved. We will also assume, unless the contrary will be stated, that all real forms go through the origin o (corresponding to the coset H) of M.

Lemma 3.1. *Let* $\mathfrak{b}$ *be a self-complementary reflective subspace of* $\mathfrak{m}$. *The reflective submanifold* B *corresponding to* $\mathfrak{b}$ *is a real form if and only if the complex structure* J *maps* $\mathfrak{b}$ *and* $\mathfrak{b}^\perp$ *isometrically onto each other.*

Proof. Suppose B is a real form. Let ρ denote the involutive antiholomorphic isometry which leaves B point-wise fixed. Then we have

$$\rho_*|\mathfrak{b} = \mathrm{id}, \ \rho_*|b^\perp = -\mathrm{id}. \tag{4}$$

Since ρ is antiholomorphic, we have

(5) $$\rho_* J = -J\rho_*.$$

It follows that

(6) $$J\mathfrak{b} \subset \mathfrak{b}^\perp, J\mathfrak{b}^\perp \subset b.$$

Being an isometry, J must map $\mathfrak{b}$ and $\mathfrak{b}^\perp$ isometrically onto each other.

On the other hand, suppose J maps $\mathfrak{b}$ isometrically onto $\mathfrak{b}^\perp$. Then we have $\mathfrak{m} = \mathfrak{b} + J\mathfrak{b}$. Define $\rho_*: \mathfrak{m} \to \mathfrak{m}$ by

(7) $$\rho_*(x + Jy) = x - Jy, \quad x, y \in \mathfrak{b}.$$

One can easily check that (5) holds again in this case and that ρ_* preserves the curvature tensor of M restricted to $\mathfrak{m}$. By [10, 8.1.1] and the fact that M is globally symmetric, ρ_* can be extended to a global isometry ρ. Since J is invariant under G, (5) implies that ρ is antiholomorphic. Obviously ρ is involutive. If B is the reflective submanifold corresponding to $\mathfrak{b}$, then B is a real form.

Definition. A subspace $\mathfrak{b}$ of $\mathfrak{m}$ which satisfies the condition of Lemma 3.1 is called a *real form* of $\mathfrak{m}$.

Next we will prove a number of lemmas which will be used to decide which of the self-complementary reflective submanifolds of M are indeed real forms.

For an irreducible Hermitian symmetric space $M = G/H$, it is well known that $H = L \cdot T^1$, where $T^1 = \{\exp 2\pi i t v | t \in R\}$ for a suitably chosen element v in $\mathfrak{h}$, is a circle group and H is the centralizer of T^1 in G. Furthermore, there exists an element $j \in T^1$ such that $J = \mathrm{Ad} j$ (cf. [10, §8.7]). Let $\mathfrak{b}$ be a self-complementary reflective subspace of $\mathfrak{m}$. The irreducibility of M implies that

$$\mathfrak{h} = [\mathfrak{b}, \mathfrak{b}] + [\mathfrak{b}, \mathfrak{b}^\perp] + [\mathfrak{b}^\perp, \mathfrak{b}^\perp].$$

In fact $\mathfrak{e} = [\mathfrak{b}, \mathfrak{b}] + [\mathfrak{b}^\perp, \mathfrak{b}^\perp]$ is the Lie algebra of the largest subgroup of G which leaves B invariant. Let us put $\mathfrak{f} = [\mathfrak{b}, \mathfrak{b}^\perp]$. Then we have

$$\mathfrak{h} = \mathfrak{e} + \mathfrak{f} \text{ (a direct sum of vector spaces only)},$$

$$[\mathfrak{e}, \mathfrak{e}] \subset \mathfrak{e}, \quad [\mathfrak{f}, \mathfrak{f}] \subset \mathfrak{e},$$

$$[\mathfrak{f}, \mathfrak{e}] \subset \mathfrak{f}, \quad [\mathfrak{f}, \mathfrak{b}] \subset \mathfrak{b}^\perp, \quad [\mathfrak{f}, \mathfrak{b}^\perp] \subset \mathfrak{b}.$$

Lemma 3.2. *Either $v \in \mathfrak{e}$ or $v \in \mathfrak{f}$, $\mathfrak{b}$ is a real form if and only if $v \in \mathfrak{f}$.*

Proof. Suppose $v = v_1 + v_2$, $v_1 \in \mathfrak{e}$ and $v_2 \cap \mathfrak{f}$. Since v lies in the centralizer of $\mathfrak{h}$, we have

$$0 = [v, \mathfrak{e}] = [v_1, \mathfrak{e}] + [v_2, \mathfrak{e}].$$

Since $[v_1, \mathfrak{e}] \subset \mathfrak{e}$ and $[v_2, \mathfrak{e}] \subset \mathfrak{f}$, we must have

$$0 = [v_1, \mathfrak{e}] = [v_2, \mathfrak{e}].$$

Similarly one can show that

$$0 = [v_1, \mathfrak{f}] = [v_2, \mathfrak{f}].$$

Since the dimension of the centralizer of $\mathfrak{h}$ is one, we must have $v_1 = 0$ or $v_2 = 0$. The last part of the lemma follows now easily from Lemma 3.1. q.e.d.

As a first step, we will next give a classification of the real forms up to isometric types. According to [8, Remark 2.4] duality between compact and noncompact spaces preserves reflective submanifolds. Since in the Hermitian case, duality preserves also complex structure and therefore real forms. We will only classify the real forms of irreducible compact Hermitian symmetric spaces, and can then obtain the classification in the noncompact case by duality.

Theorem 3.4. *Up to isometric types, the real forms of compact irreducible Hermitian symmetric spaces are given as follows.* (*We put here* $G^0_{r,r+s}(\mathbf{R}) = SO(r+s)/SO(r) \times SO(s)$, $G^u_{r,r+s}(\mathbf{R}) = O(r+s)/O(r) \times O(s)$ *and* $G_{r,r+s}(Q) = Sp(r+s)/Sp(r) \times Sp(s)$.)

M	Real forms
$SU(p+q)/S(U(p) \times U(q))$	
$p = q = 1$ or $p < q$	
p, q not both even	$G^u_{p,p+q}(\mathbf{R})$,
p, q both even	$G^u_{p,p+q}(\mathbf{R})$; $G_{p/2,(p+q)/2}(Q)$.
$p = q \geqslant 2$	
p odd	$G^u_{p,2p}(\mathbf{R})$; $\{SU(p) \times S^1\}/\mathbf{Z}_p$,
p even	$G^u_{p,2p}(\mathbf{R})$; $\{SU(p) \times S^1\}/\mathbf{Z}_p$; $G_{p/2,p}(Q)$.
$SO(p+2)/SO(p) \times SO(2)$	
$p \neq 2$	$[G^0_{1,k+1}(\mathbf{R}) \times G^0_{1,p-k+1}(\mathbf{R})]/\mathbf{Z}_2$, $0 < k \leqslant [p/2]$; $G^0_{1,p+1}$.
$SO(2n)/U(n)$, $n \geqslant 3$	
n odd	$SO(n)$,
n even	$SO(n)$; $\{[SU(n)/Sp(n/2)] \times S^1\}/\mathbf{Z}_n$.
$Sp(n)/U(n)$	
n odd	$\{[SU(n)/SO(n)] \times S^1\}/\mathbf{Z}_n$,
n even	$\{[SU(n)/SO(n)] \times S^1\}/\mathbf{Z}_n$; $Sp(n/2)$.
$E_6/\{[\mathrm{Spin}(10) \times T^1]/\mathbf{Z}_4\}$	$F_4/\mathrm{Spin}(9)$; $G_{2,4}(Q)/Z_2$,
$\mathrm{Ad}(E_7)/\{E_6 \times T^1/\mathbf{Z}_3\}$	$[SU(8)/Sp(4)]/\mathbf{Z}_2$; $\{[E_6/F_4] \times S^1\}/\mathbf{Z}_3$.

Proof. We can obtain the table by computations using the results on self-complementary spaces in [9, §2], the structure of the largest group of isometries leaving a reflective submanifold invariant (cf. [9, §2]), and Lemmas 3.1 and 3.2. Indeed, let $M = G/H$ be the ambient space, and $B = K/Q$ be a self-complementary reflective submanifold of M determined by an involutive automorphism ρ of G. From the structure of the largest subgroup of G which leaves B invariant, we can determine the structure of G^ρ. In fact, $G^\rho = K \times A$ or $K \cdot A$, where A is the subgroup of G^ρ which leaves B pointwise fixed. From this we can conclude that $H^\rho = Q \times A$ or $Q \cdot A$. This determines the involutive automorphisms $\rho|H$ up to automorphism of H. In particular, since $\rho(T^1) = T^1$, the involutive automorphism $\rho|T^1$ is uniquely determined. In fact, if we identify T^1 with the set of complex number of modulus one, ρ is either the identity mapping or complex conjugation. From Lemma 3.2 and the definition of an almost complex structure, it follows easily that B is a real form if and only if $\rho|T^1$ the complex conjugation. Using this it is straightforward to decide which of the self-complementary reflective submanifolds of M are real forms. q.e.d.

Next we will show that Theorem 4 in fact gives a classification of real forms of Hermitian symmetric spaces up to holomorphic equivalence. We begin with the following lemma.

Lemma 3.5. *Let $M = G/H$ be a Hermitian symmetric space, and B_1 and B_2 be two real forms left fixed by the involutive isometries ρ_1 and ρ_2 respectively. If B_1 and B_2 are isometric, then they are mapped onto each other by some elements of G (the identity component of the group of isometries of M).*

Proof. The lemma is a corollary of [9, Theorem 1.1]. (We use here the notation in [9].) In fact, using the computations of [1, Chapter III], it is easy to check that for all the ρ which define real forms in Hermitian symmetric spaces, $\rho|\mathfrak{h}$ is determined up to inner automorphisms of $\mathfrak{h}$ by the isomorphic type of the fixed point set $\mathfrak{h}^\rho$.

Theorem 3.5. *The table of Theorem 3.4 in fact gives a classification of the real forms of irreducible compact Hermitian symmetric spaces up to holomorphic transformations of M, and also gives a classification in the noncompact case by duality.*

Proof. By Lemma 3.5 and the remark before Theorem 3.4, we can conclude that Theorem 3.4 in fact gives a classification of the real forms of M (compact or noncompact) up to isometries of M. Now G is a subgroup of the group of holomorphic transformations of M, and no two real forms or a fixed M in the table of Theorem 3.4 are diffeomorphic to each other. Therefore Theorem 3.4 actually gives a classification of real forms up to holomorphic transformations. q.e.d.

Next we will give a proof of the fact that the fixed point set of the conjugation of an irreducible Hermitian symmetric space is connected. We begin with the following lemma.

Lemma 3.7. *Let $M = G/H$ be a Hermitian symmetric space with G acting effectively on M as holomorphic isometries. If $k \in G$ leaves a real form B of M pointwise fixed, then k must be the identity element of G.*

Proof. We can assume o (the coset H) $\in B$. Then k_* leaves $T_0(B)$ pointwise fixed, and $k_*J = Jk_*$. Therefore k_* also leaves $JT_0(B)$ pointwise fixed. Since $T_0(M) = T_0(B) + JT_0(B)$, k_* leaves $T_0(M)$ pointwise fixed. Hence g must be the identity transformation.

Theorem 3.8. *Let M be an irreducible Hermitian symmetric space. Then the fixed point set M^ρ of a complex conjugation ρ is connected.*

Proof. When M is noncompact, the theorem follows from [9, Lemma 3.1]. Now assume that $M = G/H$ is compact with G simply connected. For a given reflective submanifold B of M, it is easy to compute the largest subgroup of G which leaves B invariant (cf. [9, §2]). Using the list of reflective submanifolds in [9], the list of real forms and [9, Lemma 3.2], we can conclude that for a complex conjugation ρ, if M^ρ is not connected, then any two components of M^ρ must be isometric. Let $B = K/Q$ and B' are two isometric components of M^ρ, by Lemma 3.5, there is a $g \in G$ such that $gB = B'$. Using [9, Lemma 3.1] and the notation there, we can conclude that $k = g^{-1}\rho(g)$ leaves B pointwise fixed. Lemma 3.7 then implies that k is the identity element of K and hence that g is in the fixed point set G^ρ of ρ. Again using Lemma 3.7 we can conclude that $G^\rho = K$, since G^ρ is connected [2, Theorem 7.2]. Therefore $g \in K$, and M^ρ is connected. q.e.d.

Theorem 3.8 was also known to H. A. Jaffee.

4. Classification of real forms of Hermitian symmetric spaces

Let M be a Hermitian symmetric space, ρ a complex conjugation of M, and M^ρ the fixed point set of ρ. If M is noncompact, then it follows from [6, Theorem 9.2] that M^ρ is nonempty. However, when M is compact, M^ρ could be empty. In fact, for a given irreducible compact Hermitian symmetric space, one can classify all its fixed point free complex conjugations (cf. [10, §9.58]). Now let

$$M = M_0 \times M_1 \times \cdots \times M_r$$

where M_0 is C^m, and M_l, $l = 1, \cdots, r$, are irreducible Hermitian symmetric spaces. If M^ρ is not empty, then and we can show without difficulty that it is holomorphically and isometrically equivalent to

$$M_0^\rho \times M_1^{\rho_1} \times \cdots M_r^{\rho_r}$$

where ρ_i, $i = 0, 1, \cdots, r$, is a complex conjugation of M_i such that its fixed point set $M_i^{\rho_i}$ is not empty. The fixed point set of complex conjugation of C^m is isometric to $\mathbf{R}^m$ and is connected. It follows from Theorem 3.8 that this is also true for any irreducible Hermitian symmetric space. Therefore we can conclude that M^ρ, if not empty, is also connected. Using the classification of real forms of irreducible Hermitian symmetric spaces, we can easily obtain a classification of real forms of all Hermitian symmetric spaces.

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