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REFLECTIVE SUBMANIFOLDS. III. CONGRUENCY OF ISOMETRIC REFLECTIVE SUBMANIFOLDS AND CORRIGENDA TO THE CLASSIFICATION OF REFLECTIVE SUBMANIFOLDS

D. S. P. LEUNG

Introduction

This note is a sequel to the author's previous notes [3], [4] where we studied geodesic submanifolds of Riemannian symmetric spaces, which are the connected components of the fixed point sets of involutive isometries and are called *reflective submanifolds*. Here we shall prove that two isometric reflective submanifolds of a simply connected Riemannian symmetric space M are congruent under the full group of isometries of M. Since the second assertion in [3, Lemma 2.6] is incorrect, many of the reflective submanifolds of compact symmetric spaces listed in [4] should be replaced by their appropriate space forms. A list of the reflective submanifolds with the appropriate space form factors is given here. We also discuss in the note some facts which can be used to determine the connectivity of the fixed point sets of involutive isometries of a compact symmetric space. In our forthcoming papers, we plan to use the above results to study the various geometric significance of the reflective submanifolds. To begin with, we will classify all the real forms of Hermitian symmetric spaces. *For terminologies and notation related to reflective submanifolds and Lie groups, we follow* [4, §§1 *and* 2] *closely*.

1. Congruency of reflective submanifolds

Let $M = G/H$ be a simply connected irreducible compact symmetric space, $M^* = G^*/H$ its noncompact dual, and σ the canonical involution of M and M^*. For technical reasons, we will furthermore assume the Lie groups G and G^* to be simple and simply connected. If $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ is the canonical decomposition of M, then $\mathfrak{g}^* = i\mathfrak{m} + \mathfrak{h} \subset \mathfrak{g}^c$, $\mathfrak{g}^c$ being the complexification of $\mathfrak{g}$ or $\mathfrak{g}^*$. Suppose ρ is an involutive isometry of M. ρ induces through adjoint

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action an involutive automorphism of $(\mathfrak{g}, \mathfrak{h}, \sigma)$, denoted by ρ; then we have $\mathfrak{m} = \mathfrak{m}^+ + \mathfrak{m}^-$ and $\mathfrak{h} = \mathfrak{h}^+ + \mathfrak{h}^-$,, where the superscripts "+" and "−" refer to the $+1$ and -1 eigenspaces of ρ respectively. Extending by linearity over **C**, we have also an involutive automorphism of $(\mathfrak{g}^*, \mathfrak{h}, \sigma)$ and hence an involutive isometry of M^*. In fact all involutive isometries of M^* can be so induced. *When confusion is not likely, we will denote all the maps induced by the isometry ρ of M again by ρ.* Note that by restricting ρ to $\mathfrak{h}$, we have a symmetric Lie algebra $(\mathfrak{h}, \mathfrak{h}^+, \rho)$. Such symmetric Lie algebras play an important role in the classification of affine symmetric spaces in [1]. Similarly there is an involutive automorphism $\rho^\perp$ of $(\mathfrak{g}, \mathfrak{h}, \sigma)$ associated with $\mathfrak{m}^-$ such that its fixed point set on $\mathfrak{m}$ is $\mathfrak{m}^-$.

Now assume that M is of type I. It follows from [1, Lemmas 15.1 and 15.2] that if ρ_1 and ρ_2 are two involutive automorphisms of $(\mathfrak{g}^*, \mathfrak{h}, \sigma)$ *such that their associated symmetric Lie algebras $(\mathfrak{h}, \mathfrak{h}_1^+, \rho_1)$ and $(\mathfrak{h}, \mathfrak{h}_2^+, \rho_2)$ are isomorphic under an inner automorphism (resp. automorphism) α of $\mathfrak{h}$, then ρ_1 is conjugate to either ρ_2 or $\rho_2^\perp$ under an inner automorphism (resp. automorphism) which extends α.* It is easy to check that the above statement is also true if $\mathfrak{g}^*$ is replaced by $\mathfrak{g}$. Since G^* and G are both connected and simply connected by our assumption, every automorphism of $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) can be extended to an automorphism of G (resp. G^*).

Theorem 1.1. *Let $M = G/H$ be a simply connected irreducible symmetric space of type* I *or* III, *and assume that G is connected and simply connected. Then any two isometric reflective submanifolds B_1 and B_2 defined by involutive isometries ρ_1 and ρ_2 respectively are congruent by an element of the full group of isometries of M. Furthermore, if the associated symmetric Lie algebras $(\mathfrak{h}, \mathfrak{h}_1^+, \rho_1)$ and $(\mathfrak{h}, \mathfrak{h}_2^+, \rho_2)$ are related by an inner automorphism of $\mathfrak{h}$, B_1 and B_2 are congruent by an element of G.*

Proof. We can assume that B_1 and B_2 both go through the origin of M. Since B_1 and B_2 are isometric, the associated symmetric Lie algebras $(\mathfrak{h}, \mathfrak{h}_1^+, \rho_1)$ and $(\mathfrak{h}, \mathfrak{h}_2^+, \rho_2)$ are isomorphic and hence related by automorphism α_* of $\mathfrak{h}$. Using results of the previous paragraph we therefore conclude that there is an automorphism α_* of $\mathfrak{g}$, which leaves $\mathfrak{h}$ invariant, such that $\rho_1 = \bar{\alpha}_*\rho_2\bar{\alpha}_*^{-1}$. The assumption that G is simply connected implies that $\bar{\alpha}_*$ is induced by an automorphism $\bar{\alpha}$ of G which leaves H invariant. Hence $\bar{\alpha}$ is an element of the full group of isometry of M, [6, §8.8].

Therefore we have $\rho_1 = \bar{\alpha}\rho_2\bar{\alpha}^{-1}$ as required. The last statement of the theorem is obvious.

Theorem 1.2. *Let M be a simply connected irreducible symmetric space of type* II *or* IV. *Then any two isometric reflective submanifolds of M are congruent by an element of the full group of isometries of M.*

Proof. We can write $M = G/H$, where either G is the product of two simple compact Lie groups with H as its diagonal or G is a simple complex Lie group with H as a maximal compact subgroup of G. Furthermore, we will assume that G is connected and simply connected. It follows that H is connected and simply connected (cf. [2, Chapter VI]). With this preparation, it is clear that we only need to prove the theorem at the infinitesimal level. Let $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ be the canonical decomposition, and σ the canonical involution as usual.

In the compact case, we have $\mathfrak{g} = \mathfrak{i} + \mathfrak{i}$ where $\mathfrak{i}$ is a simple Lie algebra. From the proof of [4, Theorem 3.3], we know that every complementary pair of reflective subspaces of $\mathfrak{m}$ is determined by an involutive automorphism of $\mathfrak{m}$ which is in turn determined by the isomorphic type of its fixed point set. Therefore the involutive automorphisms of $\mathfrak{i}$ associated with two isometric reflective submanifolds must be related by an automorphism of $\mathfrak{i}$. An argument similar to the last part of the proof of Theorem 1.1 finishes the proof in this case.

When M is noncompact, we have $\mathfrak{g} = i\mathfrak{h} + \mathfrak{h}$. Let ρ be an involutive automorphism of $\mathfrak{h}$. Extension by linearity over $\mathbf{C}$ gives also an involutive automorphism of $(\mathfrak{g}, \mathfrak{h}, \sigma)$. Let $\mathfrak{h} = \mathfrak{h}^+ + \mathfrak{h}^-$ be the decomposition of $\mathfrak{h}$ into eigenspaces with respect to ρ as usual. It is easy to check that the reflective subalgebra pair [4, p. 329] associated with ρ in this case is $\{(i\mathfrak{h}^+ + \mathfrak{h}^+, \mathfrak{h}^+, \sigma), (i\mathfrak{h}^- + \mathfrak{h}^+, \mathfrak{h}^+, \sigma)\}$. By duality, it follows from [4, Theorem 3.3] that every complementary pair of reflective subspaces (or algebras) can be obtained in this way. Hence every complementary pair of reflective subspaces of $\mathfrak{m}$ is determined by an involutive automorphisms $\mathfrak{u}$ of $\mathfrak{h}$ and $\mathfrak{u}$ is determined by the isomorphic type of fixed point set of $\mathfrak{u}$. One can now finish the proof as in the compact case.

2. Corrigenda to [4]

Some of the reflective submanifolds listed in [4] for symmetric spaces of type I should be replaced by their appropriate nontrivial space forms. We will next describe how these space form factors can be computed.

Using the methods described in [4, §4] one can obtain the infnitesimal classification of reflective submanifolds of every noncompact irreducible symmetric space and hence by duality also the infinitesimal classification in the compact case. To compute the required space form factors, it is more convenient to write a given compact simply connected irreducible symmetric space as $M = G/G^\sigma$, where σ is the canonical involution, and G is simply connected and simple. Let B be a reflective submanifold of M through its origin defined by an involutive automorphism ρ of G which commutes with σ. Let $M^* = G^*/H$ be the dual of M, and use here the notation in §1. Using [1, Table II] and the infinitesimal structure of B, we can determine $\mathfrak{g}^{*\rho}$. In fact, using the notation in [4, §§2 and 4], we have $\mathfrak{g}^{*\rho} = \mathfrak{l}^* + \mathfrak{a}$ where $\mathfrak{l}^*$ is the Lie algebra of the largest subgroup K^* of G^* which acts effectively on B^*, the reflective submanifold of M^* determined by ρ. $\mathfrak{l}^*$ can be determined by the method described in [4, §4]. By duality we can then obtain $\mathfrak{g}^\rho = \mathfrak{k} + \mathfrak{a}$. Since G is simply connected by our assumption, this determines G^ρ which is the fixed point set of ρ and is connected [2, Theorem 7.2]. Again using the infinitesimal structure of B, we can determine the structure of the largest connected subgroup K of G^ρ which acts effectively on B. Then $B = K/K^\sigma$. When K is the product of simply connected compact Lie groups, and possibly also the circle group, using the infinitesimal structure of B we can simply determine K^σ. When K is the almost product of simply connected Lie groups and possibly also the circle group, using [4, Lemma 2.7] we can still determine the structure of $K/(K^\sigma)_0$, $(K^\sigma)_0$ being the identity component of K^σ. However, in general we have $K^\sigma = \Gamma(K^\sigma)_0$ where Γ is a discrete finite subgroup of K^σ. In this case $B = [K/(K^\sigma)_0]/\Gamma$. To compute Γ, the following observation is useful.

Lemma 2.1. *Let $\bar{K}$ be a compact Lie group, N a finite subgroup of the centralizers of $\bar{K}$, and σ an involutive automorphism of $\bar{K}$ such that its fixed point set $\bar{K}^\sigma$ is connected and $\sigma(N) = N$. Then σ induces an automorphism of $K = \bar{K}/N$. The number of components of K^σ is equal to the number of cosets of $N\bar{K}^\sigma$ in $\bar{K}$ of the form $gN\bar{K}^\sigma$, where $\sigma(g) = ng$ with $n \in N$.*

When exceptional Lie groups, which are not easily representable in terms of matrices, enter into the description of the group K, the following lemma will be useful in the computation of the group Γ. In the following lemma we use the terminology in [5] and recall it briefly here. Let G be a compact simple Lie group with involution σ. Then $M = G/G^\sigma$ is global symmetric space of the compact type. Let $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ be its canonical decomposition, and $\mathfrak{t}$ a maximal abelian subalgebra of $\mathfrak{g}$ such that $\mathfrak{u} = \mathfrak{t} \cap \mathfrak{m}$ is a maximal abelian subspace of $\mathfrak{m}$. Let $\Lambda_0(G)$, $\Lambda(G)$ and $\Lambda_1(G)$ be respectively the root lattice, unit lattice and central lattice of G in $\mathfrak{t}$, and put

$$\Lambda_0(M) = \Lambda_0(G) \cap \mathfrak{u}, \qquad \Lambda(M) = \Lambda(G) \cap \mathfrak{u} \quad \text{and} \quad \Lambda_1(M) = \Lambda_1(G) \cap \mathfrak{u}.$$

It is proved in [5] that the center $Z(M)$ of M is isomorphic to $\Lambda_1(M)/\Lambda(M)$, and the fundamental group of M is isomorphic to $\Lambda(M)/\Lambda_0(M)$.

Now let $\tilde{L}$ be a simple and simply connected compact Lie group, N be a subgroup of the center of $\tilde{L}$ and $L = \tilde{L}/N$. Let σ be an involution of L leaving N invariant, and denote the involution induced on L also by σ. Then

$Z(\tilde{L}/\tilde{L}^\sigma)$ is a subgroup of the center $Z(\tilde{L})$ of $\tilde{L}$ (cf. [5]).

Lemma 2.2. *The fundamental group $\pi_1(L/L^\sigma)$ of L/L^σ is isomorphic to $Z(\tilde{L}/\tilde{L}^\sigma) \cap N$.*

Proof. Put $\tilde{M} = \tilde{L}/\tilde{L}^\sigma$ and $M = L/L^\sigma$. Then we have $\pi_1(M) = \Lambda(M)/\Lambda_0(M)$. Since $\tilde{M}$ and $\tilde{G}$ are simply connected, $\Lambda_0(\tilde{M}) = \Lambda(\tilde{M})$ and $\Lambda_0(\tilde{G}) = \Lambda(G)$. We also have $\exp \Lambda_1(\tilde{M}) = Z(\tilde{M})$, $\exp \Lambda(G) = N$, and $\exp \Lambda(M)$ is isomorphic to $\pi_1(M)$. Now

$$\Lambda(G) \cap \Lambda_1(\tilde{M}) = \Lambda(G) \cap \Lambda_1(\tilde{G}) \cap \mathfrak{u} = \Lambda(G) \cap \mathfrak{u} = \Lambda(M).$$

Therefore we have

$$\exp \Lambda(M) = \exp \Lambda(G) \cap \exp \Lambda_1(\tilde{M}) = N \cap Z(\tilde{M}).$$

Remark 2.3. Let $L^\sigma = (L^\sigma)_0 \cdot \Gamma$. We can compute the finite group Γ as follows. Using [4, Lemma 2.7] we can compute the discrete group $\tilde{\Gamma}$ such that $L^\sigma/(L^\sigma)_0 = (\tilde{L}/\tilde{L}^\sigma)/\tilde{\Gamma}$ where $\tilde{\Gamma} = \pi_1(L^\sigma/(L^\sigma)_0)$. Then $\Gamma = \pi_1(M)/\tilde{\Gamma}$.

We now list below all the isometric types of reflective submanifolds of dimension greater than zero in a simply connected compact irreducible symmetric space of Type I. Note that because of the results of §1 and duality, the list in fact gives a classification of reflective submanifolds of irreducible simply connected symmetric spaces of Type I and III up to isometries. For notation on compact Lie groups we follow [6]. We denote by $\mathbf{Z}_p$ the multiplicative group of the pth roots of unity; for every pair of positive integers p and q, we put $m(p, q) = pq/(p, q)^2$, where (p, q) is the greatest common divisor of p and q. We also use the following notation for the Grassman manifolds:

$$G^0_{r,r+s}(\mathbf{R}) = \frac{SO(r+s)}{SO(r) \times SO(s)}, \qquad G^u_{r,r+s}(\mathbf{R}) = \frac{O(r+s)}{O(r) \times O(s)},$$

$$G_{r,r+s}(\mathbf{C}) = \frac{SU(r+s)}{S(U(r) \times U(s))}, \qquad G_{r,r+s}(Q) = \frac{Sp(r+s)}{Sp(r) + Sp(s)}.$$

In our notation $G_{r,r}(F) = G_{0,r}(F)$, where $F = \mathbf{R}$, $\mathbf{C}$ or Q, is simply a single point. Let M_1 and M_2 be two Riemannian manifolds, Γ be a finite discrete group, and $i_1: \Gamma \to I(M_1)$, $i_2: \Gamma \to I(M_2)$ be two monomorphism of Γ into the group of isometries of M_1 and M_2 respectively. The quotient space $(M_1 \times M_2)/\Delta(i_1(\Gamma) \times i_2(\Gamma))$, $\Delta(i_1(\Gamma) \times i_2(\Gamma))$ being the diagonal of the product $i_1(\Gamma) \times i_2(\Gamma)$, is called the *almost product* of M_1 and M_2. For simplicity the above almost product is usually denoted by $(M_1 \times M_2)/\Gamma$. M_1 and M_2 are both naturally embedded in $(M_1 \times M_2)/\Gamma$ with their images intersecting at a finite number of points.

A I $SU(n)/SO(n)$

(1_r) $\{[(SU(r)/SO(r)) \times (SU(n-r)/SO(n-r))]/\mathbf{Z}_2 \times S^1\}/\mathbf{Z}_{m(r,n-r)}$, $0 < r < n$;

(2_r) $G^u_{r,n}$, $0 < r < n$;

(3) $Sp(n/2)/U(n/2)$;

(4) $SU(n/2)$, ((3) & (4): n even).

Complementary pairs: $\{(1_r), (2_r)\}$ and $\{(3), (4)\}$.

A II $SU(2n)/Sp(n)$, $n > 1$

(1_r) $\{[(SU(2r)/Sp(r)) \times (SU(2n-2r)/Sp(n-r))] \times S^1\}/\mathbf{Z}_{m(r,n-r)}$, $0 < r < n$;

(2_r) $G_{r,n}(Q)$, $0 < r < n$;

(3) $SO(2n)/U(n)$.

(4) $SU(n)$.

Complementary pairs: $\{(1_r), (2_r)\}$ and $\{(3), (4)\}$.

A III $SU(p+q)/S(U(p) \times U(q))$

$(1_{r,s})$ $G_{r,r+s}(\mathbf{C}) \times G_{p-r,p+q-r-s}(\mathbf{C})$, $rs \neq 0$, $0 \leqslant r \leqslant [p/2]$, $0 \leqslant s \leqslant q$;

(2) $G^u_{p,p+q}(\mathbf{R})$;

(3) $(p, q$ even$)\, G_{p/2,(p+q)/2}(\mathbf{Q})$;

(4) $SO(2p)/U(p)$;

(5) $Sp(p)/U(p)$;

(6) $[SU(p) \times S^1]/\mathbf{Z}_p$, ((4), (5) and (6): $p = q$).

Complementary pairs: $\{(1_{r,s}), (1_{r,q-s})\}$, $q \neq 2s$, and $\{(4), (5)\}$.

Self-complementary spaces: $(1_{r,s})$, $p = 2r$ or $q = 2s$, (2), (3) and (6).

B.D.I. $SO(p+q)/SO(p) \times SO(q)$, $p + q > 4$

$(1_{r,s})$ $[G^0_{r,s}(\mathbf{R}) \times G^0_{p-r,p+q-r-s}(\mathbf{R})]/\mathbf{Z}_2$, $0 < r \leqslant [p/2]$, $0 < s \leqslant q$;

$(1_{0,s})$ $G^0_{p,p+q-s}(\mathbf{R})$, $0 < s \leqslant q$;

$(1_{r,0})$ $G^0_{p-r,p+q-r}(\mathbf{R})$, $0 < r \leqslant [p/2]$;

(2) $(p, q$ even$)$: $G_{p/2,(p+q)/2}(\mathbf{C})$;

(3) $(p = q)$: $\{[SU(p)/SO(p)] \times S^1\}/\mathbf{Z}_p$;

(4) $(p = q)$: $SO(p)$.

Complementary pairs: $\{(1_{r,s}), (1_{r,q-s})\}$, $q \neq 2s$, and $\{(3), (4)\}$.

Self-complementary spaces: $(1_{r,s})$, $p = 2r$ or $q = 2s$, and (2).

D III $SO(2n)/U(n)$, $n > 2$

(1_r) $[SO(2r)/U(r)] \times [SO(2n-2r)/U(n-r)]$, $0 < r < n$;

(2_r) $G_{r,n}(\mathbf{C})$, $0 < r < n$;

(3) $SO(n)$;

(4) $\{[SU(n)/Sp(n/2)] \times S^1\}/\mathbf{Z}_n$ (n even).

Complementary pair: $\{(1_r), (2_r)\}$.

Self-complementary spaces: (3) and (4).

C I $Sp(n)/U(n), n > 2$.
(1_r) $[Sp(r)/U(r)] \times [Sp(n-r)/U(r)], 0 < r < n$;
(2_r) $G_{r,n}(\mathbf{C}), 0 < r < n$;
(3) $\{[SU(n)/SO(n)] \times S^1\}/\mathbf{Z}_n$.
(4) (n even) $Sp(n/2)$.
Complementary pair: $\{(1_r), (2_r)\}$.
Self-complementary spaces: (3) and (4).

C II $Sp(p+q)/Sp(p) \times Sp(q), p+q > 2$.
$(1_{r,s})$ $G_{r,r+s}(Q) \times G_{p-r,p+q-r-s}(Q), rs \neq 0, 0 \leqslant s \leqslant [p/q], 0 \leqslant r \leqslant q$;
(2) $G_{p,p+q}(\mathbf{C})$;
(3) $\{[SU(2p)/Sp(p)] \times S^1\}/\mathbf{Z}_{2p}$;
(4) $Sp(p)$, ((3) and (4): $p = q$).
Complementary pairs: $\{(1_{r,s}), (1_{r,q-s})\}, q \neq 2s$ and $\{(3), (4)\}$.
Self-complementary spaces: $(1_{r,s}), p = 2r$ or $q = 2s$, and (2).

E I $E_6/\{Sp(4)/\mathbf{Z}_2\}$
(1) $F_4/\{[Sp(1) \times Sp(3)]/\mathbf{Z}_2\}$;
(2) $[SU(6)/Sp(3)]/\mathbf{Z}_3$;
(3) $[G^0_{5,10} \times S^1]/\mathbf{Z}_4$;
(4) $G_{2,4}(Q)/\mathbf{Z}_2$;
(5) $[Sp(4)/U(4)]/\mathbf{Z}_2$;
(6) $\{[(SU(6)/SO(6))/\mathbf{Z}_3] \times [SU(2)/SO(2)]\}/\mathbf{Z}_2$.
Complementary pairs: $\{(1), (2)\}$, $\{(3), (4)\}$ and $\{(5), (6)\}$.

E II $Ad(E_6)/\{([SU(6)/\mathbf{Z}_3] \times SU(2))/\mathbf{Z}_2\}$
(1) $SO(10)/U(5)$;
(2) $G^0_{4,10}(\mathbf{R})$;
(3) $G_{2,6}(\mathbf{C})$;
(4) $\{G_{3,6}(\mathbf{C}) \times [SU(2)/SO(2)]\}/\mathbf{Z}_2$;
(5) $G_{1,4}(Q)$;
(6) $F_4/\{[Sp(1) \times Sp(3)]/\mathbf{Z}_2\}$;
(7) $[Sp(4)/U(4)]/\mathbf{Z}_2$.
Complementary pairs: $\{(2), (3)\}$ and $\{(5), (6)\}$.
Self-complementary spaces: (1), (4) and (7).

E III $E_6/\{[\mathrm{Spin}(10) \times T^1]/\mathbf{Z}_4\}$
(1) $F_4/\mathrm{Spin}(9)$;
(2) $G^0_{2,10}(\mathbf{R})$;
(3) $G_{2,6}(\mathbf{C})$;
(4) $G_{2,4}(Q)/\mathbf{Z}_2$;
(5) $G_{1,6}(\mathbf{C}) \times [SU(2)/SO(2)]$;

(6) $SO(10)/U(5)$.
Complementary pair: $\{(5), (6)\}$.
Self-complementary spaces: (1), (2), (3), and (4).

E IV E_6/F_4
(1) $[SU(6)/Sp(3)]/\mathbf{Z}_3$;
(2) $G_{1,4}(Q)$;
(3) $\{[SO(10)/SO(9)] \times S^1\}/\mathbf{Z}_4$.
(4) $F_4/\text{Spin}(9)$.
Complementary pairs: $\{(1), (2)\}$ and $\{(3), (4)\}$.

E V $E_7/\{SU(8)/\mathbf{Z}_2\}$
(1) $SO(12)/U(6)$;
(2) $E_6/\{[(SU(6)/\mathbf{Z}_3) \times SU(2)]/\mathbf{Z}_2\}$;
(3) $\{G^0_{6,12} \times [SU(2)/SO(2)]\}/\mathbf{Z}_2$;
(4) $G_{4,8}(\mathbf{C})/\mathbf{Z}_2$;
(5) $[SU(8)/Sp(4)]/\mathbf{Z}_2$;
(6) $\{[E_6/(Sp(4)/\mathbf{Z}_2)] \times S^1\}/\mathbf{Z}_3$;
(7) $[SU(8)/SO(8)]/\mathbf{Z}_2$.
Complementary pairs: $\{(1),(2)\}$, $\{(3), (4)\}$ and $\{(5), (6)\}$.
Self-complementary space: (7).

E VI $E_7/\{[\text{Spin}(12) \times SU(2)]/\mathbf{Z}_2\}$
(1) $E_6/\{[\text{Spin}(10) \times SO(2)]/\mathbf{Z}_2\}$;
(2) $G^0_{4,12}(\mathbf{R})$;
(3) $G_{4,8}(\mathbf{C})/\mathbf{Z}_2$;
(4) $G_{2,8}(\mathbf{C})$;
(5) $E_6/\{[(SU(6)/\mathbf{Z}_3) \times SU(2)]/\mathbf{Z}_2\}$;
(6) $[SO(12)/U(6)] \times [SU(2)/SO(2)]$.
Complementary pairs: $\{(4), (5)\}$.
Self-complementary spaces: (1), (2), (3) and (6).

E VII $Ad(E_7)/\{[E_6 \times SO(2)]/\mathbf{Z}_3\}$
(1) $[SU(8)/Sp(4)]/\mathbf{Z}_2$;
(2) $\{G^0_{2,12}(\mathbf{R}) \times [SU(2)/SO(2)]\}/\mathbf{Z}_2$;
(3) $E_6/\{[\text{Spin}(10) \times T^1]/\mathbf{Z}_2\}$;
(4) $G_{2,8}(\mathbf{C})$;
(5) $SO(12)/U(6)$;
(6) $[(E_6/F_4) \times S^1]/\mathbf{Z}_3$.
Complementary pairs: $\{(2), (3)\}$ and $\{(4), (5)\}$.
Self-complementary spaces: (1) and (6).

E VIII $E_8/SO(16)$
(1) $E_7/\{[\mathrm{Spin}(12) \times SU(2)]/\mathbf{Z}_2\}$;
(2) $G^u_{8,16}$;
(3) $SO(16)/U(8)$
(4) $\{(E_7/[(SU(8)/\mathbf{Z}_2)]) \times [SU(2)/SO(2)]\}/\mathbf{Z}_2$.
Complementary pair: $\{(3), (4)\}$.
Self-complementary spaces: (1) and (2).

E IX $E_8/\{[E_7 \times SU(2)]/\mathbf{Z}_2\}$.
(1) $SO(16)/U(8)$;
(2) $G^u_{4,16}$;
(3) $E_7/\{[\mathrm{Spin}(12) \times SU(2)]/\mathbf{Z}_2\}$.
(4) $\{(E_7/[(E_6 \times SO(2))/\mathbf{Z}_3])\} \times \{[SU(2)/SO(2)]\}/\mathbf{Z}_2$.
Complementary pairs: $\{(2), (3)\}$.
Self-complementary spaces: (1) and (4).

F I $F_4/\{[Sp(1) \times Sp(3)]/\mathbf{Z}_2\}$
(1) $G^0_{4,9}(\mathbf{R})$;
(2) $G_{1,3}(\mathbf{Q})$;
(3) $\{[Sp(3)/U(3)] \times [SU(2)/SO(2)]\}/\mathbf{Z}_2$.
Complementary pair: $\{(1), (2)\}$.
Self-complementary space: (3).

F II $F_4/\mathrm{Spin}(9)$
(1) $SO(9)/SO(8)$;
(2) $G_{1,3}(Q)$.
Self-complementary spaces: (1) and (2).

G $G_2/SO(4)$
(1) $[S^3 \times S^3]/\mathbf{Z}_2$.
Self-complementary spaces: (1).

Remarks. (1) The assertion of [4, Lemma 2.6] is not true in general. Neither is the assertion about the connectivity of Q_1 and Q_2 [4, p. 335] true in general.

(2) The compact Lie group $Sp(n)$ has been left out in [4, Theorem 3.3] (see [4, erratum]).

(3) In the table for [4, Theorem 3.3], (3) and (4) of E_6 should be (3): $\{[\mathrm{Spin}(10) \times SO(2)]/\mathbf{Z}_4\}$ and (4) $E_6/\{[\mathrm{Spin}(10) \times SO(2)]/\mathbf{Z}_4\}$ respectively; (1) and (2) of E_7 should be (1): $\{\mathrm{Spin}(12) \times SU(2)\}/\mathbf{Z}_2$ and (2): $E_7/\{[\mathrm{Spin}(12) \times SU(2)]/\mathbf{Z}_2\}$ respectively.

(4) In [4, Lemma 3.2] as well as in its proof, "$SU(n) \times T^1$" should be "$[SU(n) \times T^1]/\mathbf{Z}_n$".

3. Connectivity of fixed point sets of involutive isometries

Now we make a few observations which will be useful when needed in determining the structures of the fixed point set of involutive isometries of Riemannian symmetric spaces.

Lemma 3.1. *Let M be a noncompact Riemannian symmetric space. Then the fixed point set M^ρ of an involutive isometry ρ of M must be connected.*

Proof. Suppose M^ρ is not connected. Let B_1 and B_2 be two disjoint component of M^ρ. Let α: $[0, b] \to M$, $\alpha(0) \in B_1$ and $\alpha(b) \in B_2$, be a unit speed geodesic which realizes the distance between B_1 and B_2. Since M is simply connected and of nonpositive sectional curvature, there is only one unit speed geodesic joining $\alpha(0)$ and $\alpha(b)$ such that $\rho(\alpha(0)) = \alpha(0)$ and $\rho(\alpha(b)) = \alpha(b)$. Therefore $\alpha([0, b])$ is left pointwise fixed by ρ, contradicting our assumption on B_1 and B_2.

Lemma 3.2. *Let $M = G/H$ be a compact irreducible simply connected Riemannian symmetric space. For $i = 1, 2$ let L_i be the largest connected subgroup of G which leaves the reflective submanifold B_i of M invariant. If B_1 and B_2 are left fixed by the same involutive isometry, then L_1 and L_2 must be isomorphic.*

Proof. Suppose B_1 and B_2 are left fixed by the involutive isometry ρ (also considered as involutive and automorphism of G). It follows from the proof of Theorem 2.1 [4 that the Lie algebras of L_1 and L_2 are both isomorphic to $\mathfrak{g}^\rho$. Therefore L_1 and L_2 are both isomorphic to G^ρ.

Lemma 3.3. *Let $M = G/H$ as in Lemma 3.2 with G simply connected. Let B be a reflective submanifold of M through the origin defined by an involutive isometry ρ (also an involution of G). For $g \in G$, gB belongs to the fixed point set of ρ if and only if $g^{-1}\rho(g)$ lies in the subgroup of G which leaves B pointwise fixed.*

Proof. Let $B = K/Q$, where K is the largest connected subgroup of G which leaves B invariant and acts effectively on it. K is left fixed by ρ. Now suppose $g \in G$ is such that gB is also left fixed by the isometry ρ. Then

$$\rho(gkH) = gkH$$

for all $k \in K$. Therefore we have

$$g^{-1}\rho(g) \in \bigcap_{k \in K} k^{-1}Hk.$$

The subgroup $\bigcap_{k\in K} k^{-1}Hk$ of course leaves B pointwise fixed. Conversely, suppose $g = g^{-1}\rho(g)$ leaves B pointwise fixed. Then

$$\rho(gkH) = gqkH = gkH$$

for all $k \in K$. Therefore gB is left fixed by ρ.

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