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COMPLETENESS OF CURVATURE SURFACES OF AN ISOMETRIC IMMERSION

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Let M be a hypersurface in a euclidean space, let E_p be the null space of the second fundamental tensor of M at $p \in M$, denote by k the minimum value of the dimensions of the vector spaces E_p on M, and let G be the open subset of M on which this minimum occurs. Then it is well known from classical differential geometry that G is generated by k-dimensional totally geodesic submanifolds along which the normal space of M is constant. Moreover, if M is complete, then these generating submanifolds of G are also complete; this fact was proved first by S. S. Chern and R. K. Lashof [1] and later by many other authors.

In 1971 this theorem was generalized by D. Ferus [3] to submanifolds of higher codimension in arbitrary ambient spaces of constant curvature. The present paper is concerned with a further generalization. While in the above case the generating submanifolds of G may be interpreted as curvature surfaces corresponding to the principal curvature 0, now for an arbitrary principal curvature function λ of M the analogous problem will be considered. A first approach to this general situation was made by T. Otsuki [9] and the author [10]. But the proof of the completeness of the generating submanifolds was left until now. For solving this problem we shall modify the ideas of P. Dombrowski [2], who discovered a fundamental relation between Jacobi fields and so-called geodesic forms.

Applying our results to the case, where M is also a space of constant curvature which exceeds that of the ambient space, we can continue B. O'Neill's investigation [7] to obtain a result analogous to the one of B. O'Neill and E. Stiel [8] about spaces of the same constant curvature.

1. Statement of the principal results

Let $f: M \to N$ be an isometric immersion of a Riemannian manifold M into a Riemannian manifold N of constant curvature with $\dim N > \dim M$. Let $\nu(f)$ denote the normal bundle of f, $\nu^*(f)$ its dual, D the canonical covariant

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derivative of $\nu(f)$, and A the second fundamental form of f. In this article all manifolds, maps, vector bundles, ... are assumed to be of class C^∞, if no other assumptions are formulated explicitly.

Definition. A 1-form $\mu \in \nu_p^*(f)$ at $p \in M$ is called a *principal curvature* of f at p if the vector space

(1) $$E(\mu) := \{v \in T_pM \mid A_\eta v = \mu(\eta)\cdot v \text{ for all } \eta \in \nu_p(f)\}$$

is at least 1-dimensional.

Suppose now that there is given a continuous *principal curvature function* λ of f, i.e., a continuous section of the bundle $\nu^*(f)$ on M with $\dim E(\lambda_p) \geqslant 1$ for all $p \in M$ (see (1)), and let G be any open subset of M on which the function $p \mapsto \dim E(\lambda_p)$ is constant, say

(2) $$\dim E(\lambda_p) = k \text{ for all } p \in G.$$

For these data and hypotheses we obtain the following theorem, where the statements (i), (iv) and (v) are the main new results of this article.

Theorem 1. (i) *The principal curvature function λ is C^∞-differentiable on G.*

(ii) *The vector spaces $E(\lambda_p)$, $p \in G$, form a vector subbundle E of the tangent bundle $TM|G$ which is integrable.*

(iii) *If L denotes the foliation obtained by integrating E, and $g\colon L \hookrightarrow M$ its inclusion map, then all leaves of L are k-dimensional umbilical submanifolds of M, and $f \circ g\colon L \to N$ is an umbilical immersion into N.*

(iv) *If λ is covariant constant along E, i.e., if*

(3) $$(\nabla_X\lambda)(\eta) := X\cdot\lambda(\eta) - \lambda(D_X\eta) = 0$$

for all $X \in \Gamma(E)$ and $\eta \in \Gamma(\nu(f)|G)$ (see Remark (c)), if furthermore $c\colon J \to L$ is a geodesic of L with $\delta := \sup J < \infty$, and if $q := \lim_{t\to\delta} c(t)$ exists in M, then also $\dim E(\lambda_q) = k$; *see* (2).

(v) *If, in particular, G is the subset of M on which the function $p \mapsto$* $\dim E(\lambda_p)$ *is minimal (this subset is open, because $p \mapsto$* $\dim E(\lambda_p)$ *is upper-semicontinuous), λ is covariant constant along E (see Remark (c)), and M is complete, then all the leaves of L are also complete spaces.*

Remarks. (a) The leaves of L may be called the *curvature surfaces* of f in G corresponding to λ; for, at every point $p \in G$ the tangent space T_pL is contained in an eigenspace of each tensor A_η, $\eta \in \nu_p(f)$.

(b) If we denote by G_i the interior of the subset $\{p \in M \mid \dim E(\lambda_p) = i\}$ for every integer $i \leqslant \dim M$, then the subset $\bigcup G_i$ is dense in M, because the function $p \mapsto \dim E(\lambda_p)$ is upper-semicontinuous. Nearly the whole manifold M is therefore foliated by those curvature surfaces of f which correspond to λ.

(c) If $k \geqslant 2$, then λ is always covariant constant along E; see [10, Satz 2].

(d) If $k \geqslant 2$, G is the subset on which the function $p \mapsto \dim E(\lambda_p)$ is minimal, and M is complete, then *the leaves of* L *are* k*-dimensional space forms*, i.e., complete Riemannian manifolds of constant curvature. This follows from the fact that $f \circ g$ is umbilical; see (iii). If, moreover, N is a standard space, i.e., a euclidean space $\mathbf{R}^n$, or a euclidean sphere $S^n(r)$ of some radius r, or a hyperbolic space H^n_κ of some negative curvature κ, then every leaf of L is isometric to a standard space. For, $f \circ g$ maps every leaf K of L into a k-dimensional sphere S_K of N; see [10, Satz 2]. Since K is complete and S_K is simply connected, $f \circ g|K$ is in fact an isometry of K onto S_K.

(e) Every hypersurface of a surface of constant curvature with a global unit normal vector field ξ has global continuous principal curvature functions. For instance, define λ by means of the smallest eigenvalue of A_ξ at every $p \in M$.

(f) If in the assertions (iv) and (v) we had omitted the assumption that λ is covariant constant, these assertions would not be true. (Examples: the compact surfaces of revolution in $\mathbf{R}^3$ which are not spheres.)

The statements (ii) and (iii) of Theorem 1 are immediate consequences of the statement (i) and of [10, §1, Bemerkung (v) and Satz 2]. The proof of (i), (iv) and (v) will be given in §§3 and 4 of this article. The assertion (v) will be deduced from (iv), and the proof of (iv) is essentially based on the investigations of §2, where we collect those arguments which do not depend on the special situation of the problem and may also be applied to other "spherical foliations".

There are isometric immersions without any principal curvature; for instance the immersion of the Veronese surface into S^4. The following theorem, however, describes an important class of isometric immersions with just one distinguished principal curvature function.

Theorem 2. *Let* M (*resp.* N) *be an* m-dim. (*resp.* n-dim.) *Riemannian manifold of constant curvature* κ_M (*resp.* κ_N), *and* f: $M \to N$ *an isometric immersion. If* $\kappa_M - \kappa_N > 0$ *and* $m < n \leqslant 2m - 2$, *then the following two statements are true*:

(i) *At every* $p \in M$ *there exists exactly one principal curvature* λ_p *of* f *with* $\dim E(\lambda_p) \geqslant 2$; *see* (1); *moreover, one has* $\dim E(\lambda_p) \geqslant 2m - n + 1 \geqslant 3$. (B. O'Neill)

(ii) *The function* $p \mapsto \lambda_p$ *is a continuous principal curvature function*; *and if* E *is defined as in Theorem* 1(ii), *then* λ *is covariant constant along* E, *and therefore the whole Theorem* 1 *applies to the map* f.

The assertion (i) of Theorem 2 is due to B. O'Neill, who studied the second fundamental form of such isometric immersions in detail (see [7, Theorem 1]) and also showed that M is foliated by curvature surfaces in the neighborhood of "regular points" (see [7, Theorem 2]). But his results about the question whether a point is regular or not do not allow us to deduce a global statement about the curvature surfaces. In [4] W. Henke treated the case $n = m + 2$ and obtained the completeness result by a method adapted just to this case.

The proof of Theorem 2(ii) is given in the last section of this article.

2. The tangent bundle of a spherical foliation

In this section we shall deal with a little more general situation. The essential assertion is Proposition 3(vi) which together with Propositions 4 and 5(iv) will give the main step of the proof of Theorem 1(iv). It should be mentioned that the system (9) of differential equations is a modification of the Jacobi equation adapted to our situation.

A foliation L of a Riemannian manifold G will be called a *spherical foliation* of G, iff every leaf of L is an extrinsic sphere of G, i.e., an umbilical submanifold of G with a mean curvature vector field which is parallel with respect to the normal connection of L in G; see [6]. For instance, every totally geodesic foliation is a special spherical foliation.

Proposition 1. *Let G denote a Riemannian manifold, ∇ its Levi-Civita covariant derivative, E a subbundle of the tangent bundle TG of G with* $\dim E_p \geqslant 1$, *and P: $TG \to E$ and Q: $TG \to E^{\perp}$ the orthogonal projections. Then E is the tangent bundle of a spherical foliation L of G, if and only if there exists a section $H \in \Gamma(E^{\perp})$ with*

$$(4) \qquad Q\nabla_X Y = \langle X, Y\rangle H \textit{ and } Q\nabla_X H = 0 \textit{ for all } X, Y \in \Gamma(E).$$

($E^{\perp}$ denotes the orthogonal complement of E in TG. If E is the tangent bundle of a foliation L, then $E_p^{\perp}$ is the normal space of L at p for all $p \in G$.)

For the proof, which will not be given in detail, one has to remark that (4) implies $Q[X, Y] = Q\nabla_X Y - Q\nabla_Y X = 0$ for all $X, Y \in \Gamma(E)$, i.e., the integrability of E, that every $X \in \Gamma(E)$ may be considered as a vector field of L, and that $H \circ g$ ($g =$ the inclusion map $L \hookrightarrow G$) is the mean curvature vector field of L. The concept of spherical foliations is useful for us because of the following two propositions, the first of which is obtained as an immediate consequence of [10, Satz 2 and its proof] and of Proposition 1. (In [10] the "normal field" H was denoted by $\bar{Z}$.)

Proposition 2. *If f: $G \to N$ is an isometric immersion of a Riemannian manifold G into a space N of constant curvature with* $\dim N > \dim G$, *λ is a differentiable principal curvature function of f, for which the vector spaces $E(\lambda_p)$ (see* (1)*) form a vector subbundle E of TG, and furthermore λ is covariant constant along E, then E is the tangent bundle of a spherical foliation of G.*

Proposition 3. *Let E be the tangent bundle of a spherical foliation L of a Riemannian manifold G, let ∇, P, Q and H be as in Proposition* 1, *and let R be* *the Riemannian curvature tensor of G. Let $\hat{\nabla}$ denote the covariant derivative of G defined by*

$$(5) \qquad \hat{\nabla}_X Y := \nabla_X Y - \langle X, Y \rangle H + \langle Y, H \rangle X,$$

and B the tensor field of type (2,1) *on G defined by*

$$(6) \qquad B(X, Y) := Q\hat{\nabla}_Y PX = Q\nabla_Y PX - \langle PX, PY \rangle H$$

for all vector fields X, Y on G. Furthermore, let J be an open interval of **R** *with* $0 \in J$, $c: J \to L$ *a path in L with* $\langle \dot{c}, \dot{c} \rangle = 1$, *and* $w \in E^{\perp}_{c(0)}$. *Then the following hold*:

(i) $\hat{\nabla}$ *is metric with respect to the Riemannian metric* $\langle\ ,\ \rangle$ *of G, i.e.,*

$$X \cdot \langle Y, Z \rangle = \langle \hat{\nabla}_X Y, Z \rangle + \langle Y, \hat{\nabla}_X Z \rangle \text{ for all } X, Y, Z \in \Gamma(TG).$$

(ii) *E is auto-$\hat{\nabla}$-parallel, i.e.,* $\hat{\nabla}_X Y \in \Gamma(E)$ *for all* $X, Y \in \Gamma(E)$, *and therefore* $P\hat{\nabla}_X Z = \hat{\nabla}_X PZ$ *and* $Q\hat{\nabla}_X Z = \hat{\nabla}_X QZ$ *for all* $X \in \Gamma(E)$ *and* $Z \in \Gamma(TG)$.

(iii) *H is $\hat{\nabla}$-parallel along E, i.e.,*

$$\hat{\nabla}_X H = \nabla_X H + \langle H, H \rangle X = 0 \text{ for all } X \in \Gamma(E),$$

and therefore $\langle H \circ c, H \circ c \rangle$ *is a constant function and*

$$(7) \qquad \nabla_\partial H \circ c = -\langle H \circ c, H \circ c \rangle \cdot \dot{c}$$

where, as always in this article, ∂ *denotes the canonical unit vector field of* **R**.

(iv) *The following three statements are pairwise equivalent*:

$$(8) \qquad \begin{array}{ll} \text{(a)} & c \text{ is a geodesic of } L, \\ \text{(b)} & c \text{ is a } \hat{\nabla}\text{-geodesic of } G, \\ \text{(c)} & \nabla_\partial \dot{c} = H \circ c. \end{array}$$

(v) $QR(X, Y)Z = R(X, QY)Z$ *for all* $X, Z \in \Gamma(E)$ *and* $Y \in \Gamma(TG)$.

(vi) *If c is a geodesic of L, and* (U_1, U_2, U_3) *with* $U_i \in \Gamma(c^*TG)$ *is the solution of the homogeneous system of linear differential equations*

$$(9) \qquad \begin{aligned} \nabla_\partial U_1 &= U_2, \\ \nabla_\partial U_2 &= R(\dot{c}, U_1)\dot{c} + U_3, \\ \nabla_\partial U_3 &= R(\dot{c}, U_1)\nabla_\partial \dot{c} - \langle \nabla_\partial \dot{c}, \nabla_\partial \dot{c} \rangle U_2 - 2\langle U_3, \nabla_\partial \dot{c} \rangle \dot{c} \end{aligned}$$

with $U_1(0) = w$, $U_2(0) = B(\dot{c}(0), w)$ *and* $U_3(0) = \nabla_w H$, *then* $Y := QU_1$ *is a solution of the linear differential equation*

$$\hat{\nabla}_\partial Y = B(\dot{c}, Y).$$

Remark. The suitableness of the covariant derivative $\hat{\nabla}$ and the tensor field B is shown by the assertions (ii) and (iii) of this property and above all by Proposition 5(iii).

PROOF OF PROPOSITION 3. Using (4), (5), $P + Q = \mathrm{id}_{TG}$ and the integrability of E we can obtain the statements (i), $\cdots$, (v) by a simple computation.

For (v) we first prove $R(X, Y)Z \in \Gamma(E)$ for all $X, Y, Z \in \Gamma(E)$ and then use curvature identities. To verify (vi) let (U_1, U_2, U_3) be the solution of (9) with the prescribed initial values. Then the proof will be finished by the uniqueness theorem for ordinary differential equations, and the following statements (10), (11), (12).

(10) $U := U_1$ is the solution of the differential equation

$$\nabla_\partial \nabla_\partial U = R(\dot{c}, U)\dot{c} + \nabla_U H$$

with $U(0) = w$ and $(\nabla_\partial U)_0 = B(\dot{c}(0), w)$, and the function $\langle \dot{c}, \nabla_\partial U \rangle$ vanishes identically on J.

(11) $Y := QU$ is the solution of the differential equation

$$\nabla_\partial \nabla_\partial Y = R(\dot{c}, Y)\dot{c} + Q\nabla_Y H - 2(\langle \nabla_\partial Y, \nabla_\partial \dot{c}\rangle \dot{c} + \langle Y, \nabla_\partial \dot{c}\rangle \nabla_\partial \dot{c})$$

with $Y(0) = w$ and $(\nabla_\partial Y)_0 = B(\dot{c}(0), w) - \langle w, (\nabla_\partial \dot{c})_0\rangle \dot{c}(0)$.

(12) If $\tilde{Y} \in \Gamma(c^*TG)$ is the solution of the differential equation $\hat{\nabla}_\partial \tilde{Y} = B(\dot{c}, \tilde{Y})$ with $\tilde{Y}(0) = w$, then $\tilde{Y}$ also satisfies the differential equation and the initial conditions of (11).

Now we proceed to prove the above statements (10), (11), (12).

For (10), for abbreviation and in accordance with (8) set

$$h := H \circ c = \nabla_\partial \dot{c}. \tag{13}$$

Now choose any path $\gamma\colon]-1, 1[\to G$ with $\gamma(0) = c(0)$ and $\dot{\gamma}(0) = w$, and denote by Z the section in E along γ with $Z_0 = \dot{c}(0)$, which is parallel with respect to the metric covariant derivative of E, induced by ∇, i.e., Z satisfies the equation $P\nabla_\partial Z = 0$. Thus $\|Z_s\| = 1$ for all $s \in]-1, 1[$. If $V_s\colon J_s \to L$ denotes the maximal geodesic in L with $V_s(0) = \gamma(s)$ and $\dot{V}_s(0) = Z_s$, then we get $\langle \dot{V}_s, \dot{V}_s\rangle = 1$ for all $s \in]-1, 1[$. Since, according to (iv), the V_s's are $\hat{\nabla}$-geodesics in G at the same time, we obtain a C^∞-map $V\colon (t, s) \mapsto V_s(t)$ defined on an open subset $\subset \mathbf{R}^2$, which contains $J \times \{0\}$. To study the infinitesimal variation along c induced by V, denote the two canonical vector fields of $\mathbf{R}^2$ by ∂_1 and ∂_2, and the maps $t \mapsto (t, s)$, $s \mapsto (0, s)$ of $\mathbf{R}$ into $\mathbf{R}^2$ by τ_s, σ_0 respectively, and set $X_1 := V_*\partial_1$, $X_2 := V_*\partial_2$, $\tilde{U}_1 := X_2 \circ \tau_0$ (this is the infinitesimal variation), $\tilde{U}_2 := (\nabla_{\partial_2} X_1) \circ \tau_0$, and $\tilde{U}_3 := (\nabla_{\partial_2}\nabla_{\partial_1} X_1) \circ \tau_0$. Of course, one has $V \circ \tau_s = V_s$, $X_1 \circ \tau_s = \dot{V}_s$ (especially $V \circ \tau_0 = c$, $X_1 \circ \tau_0 = \dot{c}$), $V \circ \sigma_0 = \gamma$, and $X_1 \circ \sigma_0 = Z$. Since

$$(\nabla_{\partial_1} W) \circ \tau_s = \nabla_\partial (W \circ \tau_s) \text{ for all } W \in \Gamma(V^*TG), \tag{$*$}$$

and $[\partial_1, \partial_2] = 0$, we obtain the differential equations

$$\nabla_\partial \tilde{U}_1 = \tilde{U}_2,\ \nabla_\partial \tilde{U}_2 = R(\dot{c}, \tilde{U}_1)\dot{c} + \tilde{U}_3,$$
$$\nabla_\partial \tilde{U}_3 = R(\dot{c}, \tilde{U}_1)h + (\nabla_{\partial_2}\nabla_{\partial_1}\nabla_{\partial_1}X_1) \circ \tau_0.$$

To calculate the last term, by means of (*) we derive $\nabla_{\partial_1}X_1 = H \circ V$ from (8) and $\nabla_{\partial_1}H \circ V = -\langle H \circ V, H \circ V\rangle X_1$ from (7). Hence the definition of $\tilde{U}_3$ yields

$$\tilde{U}_3 = (\nabla_{\partial_2}H \circ V) \circ \tau_0 = (\nabla_{X_2}H) \circ \tau_0 = \nabla_{\tilde{U}_1}H,$$
$$(\nabla_{\partial_2}\nabla_{\partial_1}\nabla_{\partial_1}X_1) \circ \tau_0 = -(\nabla_{\partial_2}\langle H \circ V, H \circ V\rangle X_1) \circ \tau_0$$
$$= -2\langle \tilde{U}_3, h\rangle\dot{c} - \langle h, h\rangle \tilde{U}_2.$$

Thus $(\tilde{U}_1, \tilde{U}_2, \tilde{U}_3)$ is a solution of (9), $\tilde{U}_3 = \nabla_{\tilde{U}_1}H$, and $\nabla_\partial\nabla_\partial\tilde{U}_1 = R(\dot{c}, \tilde{U}_1)\dot{c} + \nabla_{\tilde{U}_1}H$. Moreover, since $\langle X_1, X_1\rangle_{(t,s)} = \langle \dot{V}_s(t), \dot{V}_s(t)\rangle = 1$ for all t and s, we have $\langle \dot{c}, \nabla_\partial \tilde{U}_1\rangle = \langle \dot{c}, \tilde{U}_2\rangle = \langle X_1, \nabla_{\partial_2}X_1\rangle \circ \tau_0 = 0$. To complete the proof of (10) it therefore suffices to show $\tilde{U}_i(0) = U_i(0)$. For instance,

$$\tilde{U}_2(0) = \nabla_{\partial_2 \circ \sigma_0(0)}X_1 = \nabla_{\dot{\sigma}_0(0)}X_1 = (\nabla_\partial X_1 \circ \sigma_0)_0 = (\nabla_\partial Z)_0$$
$$= (Q\nabla_\partial Z)_0 = (Q\hat{\nabla}_\partial Z)_0 = B(Z_0, \dot{\gamma}(0)) = U_2(0).$$

The verificaton of $\tilde{U}_1(0) = U_1(0)$ is similar; and this implies $\tilde{U}_3(0) = U_3(0)$ because of $\tilde{U}_3 = \nabla_{\tilde{U}_1}H$.

For (11), the assertion (ii) "$Q\hat{\nabla}_X Z = \hat{\nabla}_X QZ$ for all $X \in \Gamma(E)$ and $Z \in \Gamma(TG)$" is, by (5), equivalent to

(14) $$\nabla_X QZ = Q\nabla_X Z - \langle Z, H\rangle X - \langle X, Z\rangle H, \quad \text{for all } X \in \Gamma(E) \text{ and } Z \in \Gamma(TG).$$

Applying this to the vector fields U and $\nabla_\partial U \in \Gamma(c^*TG)$, and bearing (13) and $Y = QU$ in mind we get

(15) $$\nabla_\partial Y = Q\nabla_\partial U - \langle Y, h\rangle\dot{c} - \langle \dot{c}, U\rangle h,$$

and by means of (10) and (15),

$$\nabla_\partial Q\nabla_\partial U = Q\nabla_\partial\nabla_\partial U - (\langle \dot{c}, U\rangle\langle h, h\rangle + \langle \nabla_\partial Y, h\rangle)\dot{c}.$$

Since, according to (13) and (7), we have $\nabla_\partial h = -\langle h, h\rangle\dot{c}$, the last two equations yield

(16) $$\nabla_\partial\nabla_\partial Y = Q\nabla_\partial\nabla_\partial U - 2(\langle \nabla_\partial Y, h\rangle\dot{c} + \langle Y, h\rangle h).$$

Furthermore, from (10), assertion (v) and (4) it follows that

$$Q\nabla_\partial\nabla_\partial U = R(\dot{c}, QU)\dot{c} + Q\nabla_{PU}H + Q\nabla_{QU}H = R(\dot{c}, Y)\dot{c} + Q\nabla_Y H,$$

so that (16) becomes the required differential equation for Y. To compute the initial value $(\nabla_\partial Y)_0$ use (15) and (10) again.

For (12), at first the definitions (5), (6) and the assertion (ii) imply

$$
\begin{aligned}
&B(V, W) = 0, B(X, QW) = B(X, W),\\
(17)\qquad &Q(\hat{\nabla}_X W - B(X, W)) = Q[X, W],\\
&\text{for all } X \in \Gamma(E),\ V \in \Gamma(E^{\perp}),\ W \in \Gamma(TG).
\end{aligned}
$$

Using (4), (5), (6), (14), (17), and assertion (v), by a straightforward computation we get, for all $X, Z \in \Gamma(E)$ and $V \in \Gamma(E^{\perp})$,

$$
\begin{aligned}
&\nabla_X(B(Z, V) - \langle V, H\rangle Z) - B(\nabla_X Z, V)\\
&\quad - B(Z, \hat{\nabla}_X V - B(X, V)) - \langle \hat{\nabla}_X V - B(X, V), H\rangle Z\\
(18)\qquad &\quad = R(X, V)Z + \langle X, Z\rangle Q\nabla_V H + (\langle Z, \nabla_V H\rangle X - \langle X, \nabla_V H\rangle Z)\\
&\qquad -2\langle \nabla_X V, H\rangle Z - (\langle V, \nabla_X Z\rangle H + \langle V, H\rangle \nabla_X Z).
\end{aligned}
$$

Now let $\tilde{Y} \in \Gamma(c^* TG)$ be the solution of $\hat{\nabla}_\partial \tilde{Y} = B(\dot{c}, \tilde{Y})$ with $\tilde{Y}(0) = w$. Then the assertion (ii) and (17) show $\hat{\nabla}_\partial Q\tilde{Y} = B(\dot{c}, Q\tilde{Y})$ and $Q\tilde{Y}(0) = w$. The uniqueness theorem for ordinary differential equations therefore implies $Q\tilde{Y} = \tilde{Y}$, that means $\tilde{Y} \in \Gamma(c^* E^{\perp})$. Thus by the usual technic (for instance using local frame fields for the vector bundles E and $E^{\perp}$), from (18) we may deduce

$$
\begin{aligned}
&\nabla_\partial(B(\dot{c}, \tilde{Y}) - \langle \tilde{Y}, h\rangle \dot{c}) - B(\nabla_\partial \dot{c}, \tilde{Y})\\
&\quad = R(\dot{c}, \tilde{Y})\dot{c} + Q\nabla_{\tilde{Y}} H - 2\langle \nabla_\partial \tilde{Y}, h\rangle \dot{c} - (\langle \tilde{Y}, \nabla_\partial \dot{c}\rangle h + \langle \tilde{Y}, h\rangle \nabla_\partial \dot{c}).
\end{aligned}
$$

Because of (13) and (17) one has $B(\nabla_\partial \dot{c}, \tilde{Y}) = 0$; and by (5) and the assumption about $\tilde{Y}$ we get $\nabla_\partial \tilde{Y} = B(\dot{c}, \tilde{Y}) - \langle \tilde{Y}, h\rangle \dot{c}$. Using (13) again we see that $\tilde{Y}$ satisfies the differential equation and the initial conditions of (11). Hence the proof of Proposition 3(vi) is complete.

Remark. As a corollary of (7) and (8) we obtain: If the hypotheses of Proposition 3 are fulfilled and c is a geodesic of L, then c satisfies the following differential equation of third order:

$$
(19)\qquad \nabla_\partial \nabla_\partial \dot{c} + \langle \nabla_\partial \dot{c}, \nabla_\partial \dot{c}\rangle \dot{c} = 0.
$$

Thus c is either a geodesic or a circle of G; see [6].

For the proof of Theorem 1(iv) it will be important to notice that in a Riemannian manifold geodesics and circles either have infinite length or run to the "boundary" of the manifold; this is the content of

Proposition 4. *If M is a Riemannian manifold, $p \in M$, and v, w are vectors of T_pM with $\|v\| = 1$ and $\langle v, w\rangle = 0$, then there exists a unit speed path c: $I \to M$, defined on an open interval $I \subset \mathbf{R}$, which satisfies the differential equation* (19) *and the initial conditions $0 \in I$, $c(0) = p$, $\dot{c}(0) = v$, and $(\nabla_\partial \dot{c})_0 = w$, and is maximal in the following sense: If $t^+ := \sup I < \infty$ (resp. $t^- := \inf I > -\infty$), then for every compact subset K of M there exists an*

$\varepsilon \in \mathbf{R}_+$, *such that* $\varepsilon < t^+$ *and* $c(]t^+ - \varepsilon, t^+[) \cap K = \varnothing$ (*resp.* $\varepsilon < -t^-$ *and* $c(]t^-, t^- + \varepsilon[) \cap K = \varnothing$). *Moreover, if* $\tilde{c}: J \to M$ *is any other unit speed path, which satisfies the differential equation* (19) *and the preceding initial conditions, then* $J \subset I$ *and* $\tilde{c} = c|J$.

Proof. If c is a solution of (19) and $\langle \dot{c}, \dot{c} \rangle = 1$, then $\langle \nabla_\partial \dot{c}, \nabla_\partial \dot{c} \rangle$ is a constant function. If $w = 0$, then the path c is a geodesic. Thus from now on we assume $\omega := \|w\| > 0$. In this case the path c has to be a circle. By [6] we know that a unit speed path $c: I \to M$ with $0 \in I$ is a circle, if and only if its development in the tangent euclidean space $T_{c(0)}M$ is a circle in the ordinary euclidean sense. Thus we shall get the desired solution c by "enveloping" the curve

$$C: \mathbf{R} \to T_pM, \; t \mapsto \omega^{-1}\sin(\omega t)v + \omega^{-2}(1 - \cos(\omega t))w$$

into M. For this let u_0 denote an orthonormal base of T_pM, and denote the induced linear isomorphism $\mathbf{R}^m \to T_pM$ ($m := \dim M$), as usual, also by u_0. Then it is easy to see that a path $c: I \to M$ with $0 \in I$ and $c(0) = p$ has the development $C|I$ if and only if the horizontal lift u of c in the bundle of linear frames with $u(0) = u_0$ is a solution of the ordinary differential equation

$$\dot{u}(t) = B\big(u_0^{-1}C'(t)\big)_{u(t)}, \tag{20}$$

where C' denotes the ordinary derivative of C in the vector space T_pM, and $B(\xi)$ the standard horizontal vector field corresponding to $\xi \in \mathbf{R}^m$; see [5]. (The author was pointed to the differential equation (20) by B. Wettstein, Zürich, who learned it by corresponding with K. Nomizu about the "envelopment" of curves.) Since every $u(t)$, $t \in I$, is an orthonormal frame and the orthogonal group $O(m)$ is compact, Proposition 4 follows from the theory of ordinary differential equations.

3. The proof of Theorem 1(i)

Let $f: M \to N$ be an isometric immersion of a Riemannian manifold M into a space N of constant curvature with $r := \dim N - \dim M > 0$.

1st step. Let p_0 be a point of M and U a neighborhood of p_0, on which there exist such sections $\eta_1, \cdots, \eta_r$ of $\nu(f)$ that $\eta_1(p), \cdots, \eta_r(p)$ is a basis of $\nu_p(f)$ for each $p \in U$. Abbreviate A_{η_j} to A_j. Furthermore, denote the spectrum of an operator T by $\sigma(T)$ and set $\sigma_j := \sigma(A_{j|p_0})$.

Now let $\varepsilon > 0$ be a constant with the following property:

(21) For all $j \in \{1, \cdots, r\}$ and all distinct eigenvalues $z, w \in \sigma_j$, the open discs $B_\varepsilon(z)$ and $B_\varepsilon(w)$ in $\mathbf{C}$ of radius ε and centers z, w respectively have no common points.

Since the A_j are continuous sections of the bundle $\mathrm{Hom}(TM, TM)|U$, their eigenvalues depend also continuously on the points of U. We may therefore assume U to be so small that we have

$$\sigma(A_{j|p}) \subset \bigcup_{z \in \sigma_j} B_\varepsilon(z) \tag{22}$$

for all $j \in \{1, \cdots, r\}$ and $p \in U$. Now define

$$P_{(j,z)|p} := -\frac{1}{2\pi i} \oint_{|\zeta - z| = \varepsilon} \left(A_{j|p} - \zeta \cdot \mathrm{id}_{T_pM}\right)^{-1} d\zeta$$

for all $J \in \{1, \cdots, r\}$, $z \in \sigma_j$, and $p \in U$. By this construction we get orthogonal C^∞-projections $P_{(j,z)}$ of $TM|U$ onto subbundles which we denote by $F_{(j,z)}$ (see e.g. [10, p. 172], where this functional analytical idea was also used). Then it is known that, because of (22),

(23) the fibre $F_{(j,z)|p}$ is the sum of the eigenspaces of $A_{j|p}$ corresponding to the eigenvalues $w \in \sigma(A_{j|p}) \cap B_\varepsilon(z)$.

To relate these vector bundles with the principal curvatures of f (see (1)) define

$$F_\varphi := \bigcap_{j=1}^{r} F_{(j,\varphi(j))} \text{ for all } \varphi \in \Pi := \prod_{j=1}^{r} \sigma_j. \tag{24}$$

As F_φ is the kernel of the vector bundle homomorphism

$$TM|U \to \bigoplus_{j=1}^{r} TM|U,\ v \mapsto (v - P_{(j,\varphi(j))}v)_{j=1,\cdots,r},$$

the function $p \mapsto \dim F_{\varphi|p}$ is upper-semicontinuous. Thus we may further assume U to be so small that

$$\dim F_{\varphi|p} \leqslant \dim F_{\varphi|p_0} \tag{25}$$

for all $\varphi \in \Pi$ and $p \in U$. Moreover,

(26) if the function $p \mapsto \dim F_{\varphi|p}$ is constant on U, then F_φ is a vector subbundle of $TM|U$.

2nd step. Now let λ be a continuous principal curvature function of f, and G an open subset of M on which the function $p \mapsto \dim E(\lambda_p)$ is constant, and let p_0 be a point of G. Using the notation of the 1st step we define $\varphi_0 \in \Pi$ (see (24)) by $\varphi_0(j) := \lambda_{p_0}(\eta_j(p_0))$.

Since the functions $\lambda(\eta_j)$ are continuous on U, we may assume

$$U \subset G \text{ and } \lambda_p(\eta_j(p)) \in B_\varepsilon(\varphi_0(j))$$

for all $p \in U$ and $j \in \{1, \cdots, r\}$. By (23) we see that the eigenspace of $A_{j|p}$

corresponding to $\lambda_p(\eta_j(p))$ is contained in $F_{(j,\varphi_0(j))|p}$. According to (1) we therefore get

$$E(\lambda_p) \subset F_{\varphi_0|p} \text{ for all } p \in U.$$

On the other hand (23), (21) and (1) show $F_{\varphi_0|p_0} = E(\lambda_{p_0})$. Using the assumption $U \subset G$ and (25) we obtain

$$\dim E(\lambda_{p_0}) = \dim E(\lambda_p) \leqslant \dim F_{\varphi_0|p} \leqslant \dim F_{\varphi_0|p_0} = \dim E(\lambda_{p_0}),$$

i.e., $E(\lambda_p) = F_{\varphi_0|p}$ for all $p \in U$. Thus, because of (26), the vector spaces $E(\lambda_p)$ form a C^∞-bundle on U, namely, F_{φ_0}. Therefore for all $X \in \Gamma(F_{\varphi_0})$ and $\eta \in \Gamma(\nu(f)|U)$ we have $\langle A_\eta X, X\rangle = \lambda(\eta) \cdot \langle X, X\rangle$ which proves the differentiability of λ at p_0.

4. The proof of Theorem 1(iv) and (v)

Let f: $M \to N$, λ and G be as described at the beginning of §1. In particular, suppose

(27) $$\dim E(\lambda_p) = k \text{ for all } p \in G.$$

By Theorem 1(i) and (ii), λ is differentiable on G and the $E(\lambda_p)$ form an integrable subbundle E of $TM|G = TG$. Let us suppose λ to be *covariant constant along* E (see (3)). Then, according to Proposition 2, E is the tangent bundle of a spherical foliation L of G, and Proposition 3 is applicable. Henceforth let P, Q, H, $\hat{\nabla}$ and B be as described in Propositions 1 and 3.

Furthermore, denote by $\hat{\eta}$ the continuous section of $\nu(f)$ characterized by

(28) $$\lambda(\eta) = \langle \eta, \hat{\eta}\rangle \text{ for all } \eta \in \Gamma(\nu(f)),$$

and let $\hat{\alpha}$ be the continuous bilinear vector bundle map $TM \times_M TM \to \nu(f)$ defined by

(29) $$\hat{\alpha}(X, Y) = \alpha(X, Y) - \langle X, Y\rangle\hat{\eta} \text{ for all } X, Y \in \Gamma(TM).$$

Since λ is differentiable on G, $\hat{\eta}$ and $\hat{\alpha}$ are also so. First we show how to control the variation of $E(\lambda_p)$ along a path in L by means of $\hat{\alpha}$.

Proposition 5. *If D denotes the covariant derivative of $\nu(f)$, then the following statements are true*:

(i) $E(\lambda_p) = \{v \in T_pM \,|\, \hat{\alpha}(v, w) = 0$ *for all* $w \in T_pM\}$ *for all* $p \in M$.

(ii) $D_X\hat{\eta} = 0$ *for all* $X \in \Gamma(E)$.

(iii) $D_X\hat{\alpha}(Z, Y) = \hat{\alpha}(\hat{\nabla}_X Z, Y) + \hat{\alpha}(Z, \hat{\nabla}_X Y - B(X, Y))$ *for all* $X \in \Gamma(E)$ *and* $Y, Z \in \Gamma(TM|G)$.

(iv) *If* c: $J \to L$ *is a path,* $Z \in \Gamma(c^*TM)$ *a* $\hat{\nabla}$*-parallel vector field, and* $Y \in \Gamma(c^*TM)$ *is a solution of the differential equation* $\hat{\nabla}_\partial Y = B(\dot{c}, Y)$, *then the section* $\hat{\alpha}(Z, Y) \in \Gamma(c^*\nu(f))$ *is* D*-parallel.*

Proof. First, as a consequence of the symmetry of α and the well known relation between α and A, we get

$$\hat{\alpha}(v, w) = \hat{\alpha}(w, v), \tag{30}$$

$$\langle A_\eta v - \lambda_p(\eta)v, w\rangle = \langle \hat{\alpha}(v, w), \eta\rangle \tag{31}$$

for all $p \in M$, v, $w \in T_pM$ and $\eta \in \nu_p(f)$. Thus (i) is immediately obtained by (1) and (31); and (ii) is exactly the covariant constancy of λ along E expressed by $\hat{\eta}$ (use (28) and (3)).

For (iii), the statement (i) and formula (30) shows

$$\hat{\alpha}(Z, QY) = \hat{\alpha}(QZ, Y) = \hat{\alpha}(Z, Y) \text{ for all } Y, Z \in \Gamma(TM|G). \tag{32}$$

Therefore by Proposition 3(ii) it suffices to prove (iii) for $Z \in \Gamma(E^\perp)$. In this case, using (30), (32), (17), (5), (ii), (i) and the Codazzi equation we get, for all $X \in \Gamma(E)$ and $Y \in \Gamma(TM|G)$,

$$\begin{aligned}
D_X\hat{\alpha}(Z, Y) &- (\hat{\alpha}(\hat{\nabla}_X Z, Y) + \hat{\alpha}(Z, \hat{\nabla}_X Y - B(X, Y))) \\
&= D_X(\alpha(Y, Z) - \langle Y, Z\rangle\hat{\eta}) - \hat{\alpha}(Y, \nabla_X Z) - \hat{\alpha}([X, Y], Z) \\
&= (D_X\alpha(Y, Z) - \alpha([X, Y], Z) - \alpha(Y, \nabla_X Z)) \\
&\qquad - (X\langle Y, Z\rangle - \langle [X, Y], Z\rangle - \langle Y, \nabla_X Z\rangle)\hat{\eta} \\
&= (D_Y\alpha(X, Z) - \alpha(X, \nabla_Y Z)) - \langle \nabla_Y X, Z\rangle\hat{\eta} = \langle X, Z\rangle D_Y\hat{\eta} = 0.
\end{aligned}$$

For (iv), by applying (iii) to sections $Y, Z \in \Gamma(c^*TM)$, one obtains

$$D_\partial\hat{\alpha}(Z, Y) = \hat{\alpha}(\hat{\nabla}_\partial Z, Y) + \hat{\alpha}(Z, \hat{\nabla}_\partial Y - B(\dot{c}, Y)).$$

Proof of Theorem 1(iv). Let $c: J \to L$ be a geodesic of L with $\delta := \sup J < \infty$ for which $q := \lim_{t\to\delta} c(t)$ exists in M. We may assume J to be an open interval, $0 \in J$ and $\langle \dot{c}, \dot{c}\rangle = 1$. Furthermore since, by the Remark before Proposition 4, c is a solution of the differential equation (19), Proposition 4 shows that we may continue c to a differentiable path $I \to M$, where I is an open interval containing J and δ. This continuation will also be denoted by c.

To prove $\dim E(\lambda_q) = k$ (see (27)) we first remember that the function $p \mapsto \dim E(\lambda_p)$ is upper-semicontinuous. Therefore one has $\dim E(\lambda_q) \geqslant k$, and thus it is sufficient to construct a linear isomorphism

$$\phi: T_qM \to T_{c(0)}M \text{ with } \phi(E(\lambda_q)) \subset E(\lambda_{c(0)}).$$

To this end we define $\phi(v) := Z(0)$ for $v \in T_qM$ using the solution $Z \in \Gamma(c^*TM)$ of the linear differential equation

$$\nabla_\partial Z - \langle \dot{c}, Z\rangle\nabla_\partial\dot{c} + \langle Z, \nabla_\partial\dot{c}\rangle\dot{c} = 0 \text{ with } Z(\delta) = v. \tag{33}$$

(It is to be emphasized that the solution Z exists on the entire interval I.) In this way we get a linear map ϕ which is injective, since every solution of (33) vanishing at any point does vanish identically. Hence ϕ is an isomorphism.

Now suppose $v \in E(\lambda_q)$ and let Z be the vector field along c satisfying (33). To prove $\phi(v) \in E(\lambda_{c(0)})$ we use Proposition 5(i), according to which we have only to check $\hat{\alpha}(Z(0), w) = 0$ for each $w \in E(\lambda_{c(0)})^{\perp}$. For this purpose we denote by (U_1, U_2, U_3), $U_i \in \Gamma(c^*TM)$, the solution of the system (9) of differential equations (which is also defined on the entire interval I) with the initial values $U_1(0) = w$, $U_2(0) = B(\dot{c}(0), w)$ and $U_3(0) = \nabla_w H$. Since $c|J$ is a unit speed geodesic of L, Proposition 3(vi) yields that $Y := Q(U_1|J)$ is a solution of the differential equation $\hat{\nabla}_\partial Y = B(\dot{c}, Y)$. On the other hand, Proposition 3(iv) and the formulas (5) and (33) show that $Z|J$ is a $\hat{\nabla}$-parallel vector field. By Proposition 5(iv) we therefore see that $\hat{\alpha}(Z|J, Y) = \hat{\alpha}(Z, U_1)|J$ is D = parallel (see (32)). Furthermore $\hat{\alpha}(Z, U_1)$ is continuous and $\hat{\alpha}(Z(\delta), U_1(\delta)) = 0$, since $v = Z(\delta) \in E(\lambda_q)$. Thus the parallel section $\hat{\alpha}(Z, U_1)|J$ must vanish identically, and especially $\hat{\alpha}(Z(0), w) = 0$.

Proof of Theorem 1(v). Now suppose M to be complete, let k be the minimal value of the function $p \mapsto \dim E(\lambda_p)$, see (27), and denote the open subset $\{p \in M | \dim E(\lambda_p) = k\}$ by G. Then we have to prove the completeness of each leaf K of L. By using the theorem of Hopf and Rinow, it suffices to show that if $c: J \to K$ is a maximal unit speed geodesic of K, then $\delta := \sup J = \infty$.

Assume the existence of such a geodesic c with $\delta < \infty$. If we denote by d the distance function on M induced by the Riemannian metric of M, then we have $d(c(t_1), c(t_2)) \leqslant |t_1 - t_2|$ for all $t_1, t_2 \in J$, i.e, c is a uniformly continuous map into the complete metric space (M, d). Hence there exists $q := \lim_{t \to \delta} c(t)$ on M, so that the hypotheses of Theorem 1(iv) are fulfilled. Thus $\dim E(\lambda_q) = k$, i.e., $q \in G$. But then q has to be a point of K, and c would be a unit speed geodesic of K ending at an interior point of K. This is a contradiction to the maximality of c.

5. The proof of Theorem 2(ii)

We have only to prove that the section $\lambda: p \mapsto \lambda_p$, defined in Theorem 2(i), is continuous. The other assertions of Theorem 2(ii) are obvious because of Theorem 1 and the Remark (c) of §1.

To prove the continuity of λ, let p_0 be any point of M, and use the notation and the construction described in the 1st step of the proof of Theorem 1(i). Obviously it suffices to prove the continuity of the functions $\lambda(\eta_1), \cdots, \lambda(\eta_r)$ at p_0. For this let $\varepsilon > 0$ be given. We may assume (21) to be satisfied. Then define $\varphi_0 \in \Pi$ (see (24)) by $\varphi_0(j) := \lambda_{p_0}(\eta_j(p_0))$ and show

$$E(\lambda_p) \subset F_{\varphi_0|p} \text{ for all } p \in U. \tag{34}$$

For (34), if $p \in U$ is fixed, then there exists exactly one $\varphi \in \Pi$ with

$$\lambda_p(\eta_j(p)) \in B_\varepsilon(\varphi(j));$$

see (21), (22), and (24). By means of (1) and (23) we get $E(\lambda_p) \subset F_{\varphi|p}$, and thus, in consequence of (25),

$$2 \leqslant \dim E(\lambda_p) \leqslant \dim F_{\varphi|p} \leqslant \dim F_{\varphi|p_0}. \tag{35}$$

On the other hand, if μ denotes the principal curvature of f at p_0 characterized by $\mu(\eta_j(p_0)) = \varphi(j)$ for $j = 1, \ldots, r$, then (21) and (23) yield $F_{\varphi|p_0} = E(\mu)$. Hence (35) shows $\mu = \lambda_{p_0}$, i.e., $\varphi = \varphi_0$. Thus (34) is verified.

The continuity of $\lambda(\eta_j)$ at p_0 follows now immediately; for by (34), (24) and (23) we obtain

$$\lambda_p(\eta_j(p)) \in B_\varepsilon(\varphi_0(j)) = B_\varepsilon(\lambda_{p_0}(\eta_j(p_0)))$$

for all $j = 1, \ldots, r$ and $p \in U$.

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