J. DIFFERENTIAL GEOMETRY
12 (1977) 567-581

THE MORSE INDEX THEOREM IN THE CASE OF TWO VARIABLE END-POINTS

JOHN BOLTON

1. Introduction

Let W be a C^∞ complete positive-definite Riemannian manifold, and let P, Q be submanifolds of W. If $\gamma: [0, b] \to W$ is a geodesic of W intersecting P and Q orthogonally at $\gamma(0)$ and $\gamma(b)$ respectively, then γ may be thought of as a "stationary point" of the length function L acting on the space of paths from P to Q. If Ω_γ is the space of continuous piecewise-smooth vector fields along γ, which are orthogonal to γ and have initial vector tangential to P and final vector tangential to Q, then the Morse index form $I: \Omega_\gamma \times \Omega_\gamma \to \boldsymbol{R}$ is a symmetric bilinear map which is interpreted as the *hessian* of L. The index of I is the dimension of a maximal subspace of Ω_γ on which I is negative definite, so this is a measure of the number of *essentially different* directions in which γ can be deformed to obtain shorter paths from P to Q lying arbitrarily close to γ.

If Q is a point, the Morse index theorem says that the index of I is equal to the sum of the orders of the focal points of P along γ. (See e.g., [2, Chapter 11].)

In this paper we prove a Morse-type index theorem in the general case by defining the notion of a (P, Q)-focal point of signed order, and then obtaining an expression for the index of I as the sum of an initial term together with the signed orders of the (P, Q)-focal points. This is obtained in Theorem A in § 4.

Ambrose [1] and Morse [3] also have extensions to the general case. However the author feels that the present approach has advantages for two reasons. First, the initial term is easily computed because it depends only on the second fundamental forms S, T of P, Q respectively with respect to $\gamma'(0)$, $\gamma'(b)$ respectively. Secondly, the definition of (P, Q)-focal point is very natural and rather easier than, for instance, Ambrose's corresponding notion of a "conjugate point of P and Q".

The method of proof of Theorem A follows [1] and [2] in that an index function i is defind on $[0, b]$ and the discontinuities of i are analysed. Unlike [1] and [2] however the index function in our case is not necessarily nondecreasing. This makes it unlikely that the ad-hoc subdivisions of $[0, b]$ used in

Received October 27, 1975.

this paper can be avoided by using methods similar to those employed by Osborn in [4].

The simple nature of the initial term in Theorem A makes it interesting to obtain upperbounds on $c \in \boldsymbol{R}^+$ in order that there be no (P, Q)-focal points on $[0, c]$. This is the motivation behind Theorem B, which is stated and proved in § 5. This theorem is similar to some of the comparison theorems proved by Warner in [5], although the proof is rather different.

2. The index form

Notation will be as in § 1, with the additional assumptions that γ is parameterized by arc length and prolonged so that its domain of definition is $\boldsymbol{R}$. If B is a C^∞-manifold and if $b \in B$, then B_b will be the tangent space of B at b.

For each $t \in \boldsymbol{R}$, let $\gamma'(t)$ be the tangent vector of γ at $\gamma(t)$, and let

$$W_t = \{X \in W_{\gamma(t)} : \langle X, \gamma'(t)\rangle = 0\} .$$

For each $t \in \boldsymbol{R}$, $R(t)$ will be the Ricci transformation of W_t into itself given by

$$R(t)X = R(\gamma'(t), X)\gamma'(t) ,$$

where R is the curvature tensor of W.

Let V be the vector space of parallel vector fields along γ, which are orthogonal to γ. Then the evaluation map $V \to W_t$ which sends X to $X(t)$ is a linear isomorphism which will be used to identify W_t with V. For $t > 0$, Ω_γ^t will be the vector space of continuous piecewise-smooth maps $X: [0, t] \to V$ with $X(0) \in P_{\gamma(0)}$ and $X(t) \in Q_{\gamma(b)}$, and $\dot{X}$ will be the derivative of X. Then $I^t : \Omega_\gamma^t \times \Omega_\gamma^t \to \boldsymbol{R}$ will be given by

$$\begin{aligned} I^t(X, Y) = \int_0^t \langle RX - \ddot{X}, Y\rangle &+ \sum_i \langle \dot{X}(t_i^-) - \dot{X}(t_i^+), Y(t_i)\rangle \\ &+ \langle \dot{X}(t) - TX(t), Y(t)\rangle - \langle \dot{X}(0) - SX(0), Y(0)\rangle , \end{aligned}$$

where the sum is over the jumps t_i of $\dot{X}$ in $]0, t[$.

I^t is a symmetric bilinear map and is the Morse index form arising from the variational problem with end conditions S at 0, T at t as described below.

Suppose $\mathscr{Q}$ is a submanifold of W intersecting γ orthogonally at $\gamma(t_0)$, and suppose that the second fundamental form of $\mathscr{Q}$ with respect to $\gamma'(t_0)$ is equal to T (so, in particular $\mathscr{Q}_{\gamma(t_0)} = Q_{\gamma(b)}$). Consider a 1-parameter family of curves $\gamma_s(0 \le s \le \varepsilon)$ from P to $\mathscr{Q}$ converging to $\gamma = \gamma_0$ as $s \to 0$. Let X be the associated transverse vector field, i.e., $X(t)$ is tangential to the curve $s \mapsto \gamma_s(t)$ at $s = 0$. If $L(s)$ is the length of γ_s, then

$$\left.\frac{dL}{ds}\right|_{s=0} = 0 , \qquad \left.\frac{d^2L}{ds^2}\right|_{s=0} = I^{t_0}(X, X) .$$

It is in this manner that I has the interpretation of the hessian of L as mentioned in the introduction.

If $i(t)$, $a(t)$, $n(t)$ are the index, augmented index and nullity of I^t, it is well known [1, p. 65] that $a(t)$, $n(t)$ and $i(t)$ are finite, so that $a(t) = n(t) + i(t)$.

To prove Theorem A we study the way in which $i(t)$ changes as t goes from 0 to b. In § 3 it is shown that $a(t)$ is upper semi-continuous, and $i(t)$ is lower semi-continuous. Thus $a(t)$ and $i(t)$ are continuous (and hence locally constant) at all points where $n(t) = 0$. The jump discontinuities of $i(t)$ and $a(t)$ at a point with $n(t) \neq 0$ are evaluated in Propositions 1 and 2, and the proof of Theorem A is completed in § 4, where an expression for $i(0^+)$ is obtained.

3. Jumps of i and a on $]0, b]$

We use $i(L)$, $a(L)$, $n(L)$ to denote the index, augmented index and nullity of a symmetric bilinear map L. If X is a scalar or vector valued function (resp. a vector field along γ), then $\dot{X}$ will be its derivative (resp. covariant derivative).

The following lemma is the tool used in the analysis of the discontinuities of $i(t)$ and $a(t)$.

Lemma 1. *Let U be a finite-dimensional vector space, and let $SB(U)$ be the vector space of real-valued symmetric bilinear maps on U. Let $K\colon]c, d[\to SB(U)$ be continuously differentiable at $t_0 \in]c, d[$, and let N be the null space of $K(t_0)$. Then $\exists \varepsilon > 0$ such that $\forall \mu \in]0, \varepsilon[$*

(i) $i(K(t_0 + \mu)) \geq i(K(t_0)) + i(\dot{K}(t_0) | N \times N)$,

(ii) $i(K(t_0 - \mu)) \geq i(K(t_0)) + i(-\dot{K}(t_0) | N \times N)$.

Proof. (i) Equip U with a scalar product, and if Z is a subspace of U let $(Z)_1$ be the unit sphere of Z. Let C be a subspace of U of dimension $i(K(t_0))$ on which $K(t_0)$ is negative definite, and let D be a subspace of N of dimension $i(\dot{K}(t_0) | N \times N)$ on which $\dot{K}(t_0)$ is negative definite. Since K is continuously differentiable at t_0, $\exists \varepsilon_1 > 0$ and an open neighborhood B of $(D)_1$ in $(D \oplus C)_1$ on which $K(t_0 + \mu)(X, X) < 0$ $\forall X \in B$, $\forall 0 < \mu < \varepsilon_1$. Now $(D \oplus C)_1 \backslash B$ is compact, so $\exists \varepsilon_2 > 0$ such that $K(t_0 + \mu)(Y, Y) < 0$ $\forall Y \in (D \oplus C)_1 \backslash B$, $\forall 0 < \mu < \varepsilon_2$. If $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, it is clear that $K(t_0 + \mu)(X, X) < 0$, $\forall X \in (D \oplus C)_1$, $\forall \mu \in]0, \varepsilon[$.

(ii) Apply (i) to L, where $L(t_0 + \mu) = K(t_0 - \mu)$ for all suitably small μ.

Let $t_0 \in]0, b]$. Following standard practice we construct a finite dimensional subspace B of $\Omega_\gamma^{t_0}$ such that $i(I^{t_0} | B \times B) = i(t_0)$ and $a(I^{t_0} | B \times B) = a(t_0)$.

If X is a smooth vector field along γ with $\ddot{X} = RX$ then X is called a *Jacobi field*. The set $\mathscr{J}$ of Jacobi fields which are everywhere orthogonal to γ is a vector space. If $X \in \mathscr{J}$ has $X(0) \in P_{\gamma(0)}$ and $\dot{X}(0) - SX(0) \perp P_{\gamma(0)}$, then X is called a *P-Jacobi field*. These arise as the transverse vector fields associated with variations of γ through geodesics intersecting P orthogonally (see [2, p. 222]).

A finite sequence $\{u_i\}$ with $0 < u_1 < \cdots < u_n < t_0$ is *strongly normal* in

$[0, t_0]$ under the following conditions:

(i) Each nontrivial P-Jacobi field has no zeros in $]0, u_1]$.

(ii) Each nontrivial Jacobi field X with $X(t_0) \in Q_{\gamma(b)}$ and $\dot{X}(t_0) - TX(t_0) \perp Q_{\gamma(b)}$ has no zeros in $[u_n, t_0[$.

(iii) For $i = 2, \cdots, n-1$, each nontrivial Jacobi field has at most one zero in $[u_{i-1}, u_{i+1}]$.

It follows from the Rauch comparison theorem and the extension due to Warner [5, Cor. 4.2] that strongly normal sequences exist, and moreover there are a finite sequence $\{u_i\}$ and $\varepsilon > 0$ such that $\{u_i\}$ is strongly normal in $[0, t]$ for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$. For all such t we set

$$B^t = \{X \in \Omega_\gamma^t : X \text{ is smooth with } \ddot{X} = RX \text{ except possibly at } u_1, \cdots, u_n ; SX(0) - \dot{X}(0) \perp P_{\gamma(0)}, TX(t) - \dot{X}(t) \perp Q_{\gamma(b)}\} .$$

Theorem (*For proof see* [1, p. 68]). *Assume $n \geq 1$. For each $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, $i(t) = i(I^t | B^t \times B^t)$ and $a(t) = a(I^t | B^t \times B^t)$.*

Let $H = W_{u_1} \oplus \cdots \oplus W_{u_n}$. The evaluation map $ev_t \colon B^t \to H$ given by

$$ev_t(X) = (X(u_1), \cdots, X(u_n))$$

is a linear isomorphism, so the map $J\colon]t_0 - \varepsilon, t_0 + \varepsilon[\to SB(H)$ given by

$$J(t)(\underset{\sim}{x}, \underset{\sim}{y}) = I^t(ev_t^{-1}(\underset{\sim}{x}), ev_t^{-1}(\underset{\sim}{y}))$$

is well-defined. Moreover, by the above theorem, $i(t) = i(J(t))$ and $n(t) = n(J(t))$. In the following lemma we do the computation necessary to apply Lemma 1.

Lemma. *J is smooth, and the derivative $\dot{J}(t_0)$ of J at t_0 is given by*

$$\dot{J}(t_0)(\underset{\sim}{x}, \underset{\sim}{y}) = \langle RX(t_0), Y(t_0)\rangle - \langle \dot{X}(t_0), \dot{Y}(t_0)\rangle ,$$

where $X = ev_{t_0}^{-1}(\underset{\sim}{x})$ and $Y = ev_{t_0}^{-1}(\underset{\sim}{y})$.

Proof. For $h \in]t_0 - \varepsilon, t_0 + \varepsilon[$ and for $\underset{\sim}{z} \in H$, let Z_h be the unique Jacobi field along γ such that Z_h and $ev_h^{-1}(\underset{\sim}{z})$ agree on $[u_n, h]$. Then the function $\underset{\sim}{Z}\colon]t_0 - \varepsilon, t_0 + \varepsilon[\times \boldsymbol{R} \to V$ given by $\underset{\sim}{Z}(h, t) = Z_h(t)$ is smooth. It follows that J is smooth and

$$\dot{J}(t_0)(\underset{\sim}{x}, \underset{\sim}{y}) = \left\langle -\frac{\partial}{\partial h}\frac{\partial \underset{\sim}{X}}{\partial t}, \underset{\sim}{Y}\right\rangle \bigg| (t_0, u_n) . \tag{1}$$

Now

$$\frac{\partial}{\partial t}\left[\left\langle \frac{\partial}{\partial h}\frac{\partial \underset{\sim}{X}}{\partial t}, \underset{\sim}{Y}\right\rangle - \left\langle \frac{\partial \underset{\sim}{X}}{\partial h}, \frac{\partial \underset{\sim}{Y}}{\partial t}\right\rangle\right]$$
$$= \left\langle \frac{\partial}{\partial t}\frac{\partial}{\partial h}\frac{\partial \underset{\sim}{X}}{\partial t}, \underset{\sim}{Y}\right\rangle + \left\langle \frac{\partial}{\partial h}\frac{\partial \underset{\sim}{X}}{\partial t}, \frac{\partial \underset{\sim}{Y}}{\partial t}\right\rangle$$

$$-\left\langle \frac{\partial}{\partial t}\frac{\partial \underset{\sim}{X}}{\partial h}, \frac{\partial \underset{\sim}{Y}}{\partial t}\right\rangle - \left\langle \frac{\partial \underset{\sim}{X}}{\partial h}, \frac{\partial}{\partial t}\frac{\partial \underset{\sim}{Y}}{\partial t}\right\rangle$$

$$= \left\langle \frac{\partial}{\partial h} R\underset{\sim}{X}, \underset{\sim}{Y}\right\rangle - \left\langle \frac{\partial \underset{\sim}{X}}{\partial h}, R\underset{\sim}{Y}\right\rangle = 0 \text{ by symmetry of } R\ .$$

Also, $(\partial \underset{\sim}{X}/\partial h)(t_0, u_n) = 0$, so from (1)

$$\dot{J}(t_0)(\underset{\sim}{x}, \underset{\sim}{y}) = \left[\left\langle -\frac{\partial}{\partial h}\frac{\partial \underset{\sim}{X}}{\partial t}, \underset{\sim}{Y}\right\rangle + \left\langle \frac{\partial \underset{\sim}{X}}{\partial h}, \frac{\partial \underset{\sim}{Y}}{\partial t}\right\rangle\right]\Bigg|(t_0, t_0)\ . \tag{2}$$

Writing $C(h) = (\partial \underset{\sim}{X}/\partial t)(h, h)$ and $D(h) = \underset{\sim}{X}(h, h)$, we see that if $U \in Q_{\gamma(b)}$ then

$$\langle C(h), U\rangle = \langle TD(h), U\rangle\ .$$

Differentiating this with respect to h, and then putting $\underset{\sim}{Y}(h, h) = U$ we get

$$\left\langle \frac{\partial^2 \underset{\sim}{X}}{\partial t^2} + \frac{\partial}{\partial h}\frac{\partial \underset{\sim}{X}}{\partial t}, \underset{\sim}{Y}\right\rangle\Bigg|(h, h) = \left\langle T\left[\frac{\partial \underset{\sim}{X}}{\partial h} + \frac{\partial \underset{\sim}{X}}{\partial t}\right], \underset{\sim}{Y}\right\rangle\Bigg|(h, h)\ .$$

This together with (2) gives

$$\begin{aligned}\dot{J}(t_0)(\underset{\sim}{x}, \underset{\sim}{y}) = [&\langle \partial \underset{\sim}{X}/\partial h, \partial \underset{\sim}{Y}/\partial t - T\underset{\sim}{Y}\rangle - \langle \partial \underset{\sim}{X}/\partial t, T\underset{\sim}{Y}\rangle \\ &+ \langle R\underset{\sim}{X}, \underset{\sim}{Y}\rangle]\,|\,(t_0, t_0)\ .\end{aligned} \tag{3}$$

Now if $N \perp Q_{\gamma(b)}$, then $\langle \underset{\sim}{X}, N\rangle\,|\,(h, h) = 0$ so that

$$\langle \partial \underset{\sim}{X}/\partial h + \partial \underset{\sim}{X}/\partial t, N\rangle\,|\,(h, h) = 0\ .$$

However, $(\partial \underset{\sim}{Y}/\partial t - T(\underset{\sim}{Y}))\,|\,(t_0, t_0)$ is orthogonal to $Q_{\gamma(b)}$, so from (3)

$$\dot{J}(t_0)(\underset{\sim}{x}, \underset{\sim}{y}) = [\langle R\underset{\sim}{X}, \underset{\sim}{Y}\rangle - \langle \partial \underset{\sim}{X}/\partial t, \partial \underset{\sim}{Y}/\partial t\rangle]\,|\,(t_0, t_0)$$

and this gives the answer needed to prove the lemma.

From the definition of I^t, it is clear that $X \in \Omega_\gamma^t$ is in the null space $\mathscr{J}^t$ of I^t if and only if each of the following two conditions holds.

(i) X is a P-Jacobi field.

(ii) $\dot{X}(t) - TX(t) \perp Q_{\gamma(b)}$.

If $\dim \mathscr{J}^t \neq 0$, then we call t a (P, Q)*-focal point* of order $n(t) = \dim \mathscr{J}^t$. Notice that if Q is a point then a (P, Q)-focal point is usually called a focal point of P along γ, while if both P and Q are points then a (P, Q)-focal point is just a conjugate point of $\gamma(0)$ along γ.

If $t \in\,]0, b]$ and $X, Y \in \mathscr{J}^t$, then

$$\dot{J}(t)(ev_t(X), ev_t(Y)) = \langle RX(t), Y(t)\rangle - \langle \dot{X}(t), \dot{Y}(t)\rangle\ ,$$

and this is independent of the choice of strongly normal sequence used to define H. Let $n_+(t)$ (resp. $n_-(t)$) be the dimension of a maximal subspace of $\mathscr{J}^t$ on which the symmetric bilinear map

$$(X, Y) \mapsto \langle RX(t), Y(t) \rangle - \langle \dot{X}(t), \dot{Y}(t) \rangle$$

is positive (resp. negative) definite. If t is a (P, Q)-focal point, we call $n_+(t)$ (resp. $n_-(t)$) the *positive* (resp. *negative*) *order* of t. Notice that if W has positive sectional curvatures at t, or if Q is a point, then $n_+(t) = 0$ and $n_-(t) = n(t)$.

We now apply Lemma 1 to the above. Statements (i) and (ii) of the following proposition are immediate, while (iii) and (iv) use the fact that $a(t) + i(-J(t)) = \dim H$.

Proposition 1. *Let* $t \in]0, b]$. *Then* $\exists \varepsilon > 0$ *such that* $\forall \mu \in]0, \varepsilon[$

(i) $i(t + \mu) \geq i(t) + n_-(t)$,

(ii) $i(t - \mu) \geq i(t) + n_+(t)$,

(iii) $a(t + \mu) \leq a(t) - n_+(t)$,

(iv) $a(t - \mu) \leq a(t) - n_-(t)$.

It follows that i and a are locally constant at any t with $n(t) = 0$. We call t a *nondegenerate* (P, Q)-focal point if $n_-(t) + n_+(t) = n(t) > 0$. Clearly, if W has positive sectional curvatures at the (P, Q)-focal point t, then t is nondegenerate, while if Q is a point then all (P, Q)-focal points are nondegenerate.

Proposition 2. *If t is a nondegenerate (P, Q)-focal point, then the inequalities of Proposition* 1 *are equalities, and t is an isolated (P, Q)-focal point.*

Proof. Let ε be as in Proposition 1 and let $\mu \in]0, \varepsilon[$. From Proposition 1 we have

$$a(t) - n_+(t) \geq a(t + \mu) \geq i(t + \mu) \geq i(t) + n_-(t) \ ,$$
$$a(t) - n_-(t) \geq a(t - \mu) \geq i(t - \mu) \geq i(t) + n_+(t) \ .$$

Since $a(t) = i(t) + n(t)$, the hypothesis of Proposition 2 implies that all the above inequalities are equalities. The result now follows.

4. Calculation of $i(0^+)$

If t is sufficiently small and positive, then $i(t)$ (resp. $a(t)$) is equal to $i(I^t | \mathscr{J}_t \times \mathscr{J}_t)$ (resp. $a(I^t | \mathscr{J}_t \times \mathscr{J}_t)$) where

$$\mathscr{J}_t = \{X \in \mathscr{J} : X(0) \in P_{\gamma(0)}, X(t) \in Q_{\gamma(b)}\} \ .$$

For a proof of this see [1, p. 64].

If $X, Y \in \mathscr{J}_t$, then

$$I^t(X, Y) = \langle \dot{X}(t) - TX(t), Y(t) \rangle - \langle \dot{X}(0) - SX(0), Y(0) \rangle \ . \tag{4}$$

If P and Q were both hypersurfaces, each $\mathscr{J}_t = \mathscr{J}$ and the right hand side of (4) would be defined on $\mathscr{J}$ for all $t \in \boldsymbol{R}$. This would make it possible to compute $i(0^+)$ by using Lemma 1, so we begin this section by considering this case.

Let $\tilde{P}$, $\tilde{Q}$ be hypersurfaces of W, which intersect γ orthogonally at $\gamma(0)$, $\gamma(b)$ respectively. Let $\tilde{S}$, $\tilde{T}$ be the second fundamental forms of $\tilde{P}$, $\tilde{Q}$ with respect to $\gamma'(0)$, $\gamma'(b)$, respectively, and let $\tilde{i}(t)$ (resp. $\tilde{a}(t)$) be the index (resp. augmented index) of the corresponding index form $\tilde{I}^t$. We compute $\tilde{i}(0^+)$ in terms of $\tilde{S}$, $\tilde{T}$ and $R(0)$, and later use this to compute $i(0^+)$ in the general case.

Lemma 3. *Let $\tilde{N}$ be the null space of $\tilde{S} - \tilde{T}$. Then $\exists \varepsilon > 0$ such that $\forall \mu \in]0, \varepsilon[$*

(i) $\tilde{i}(\mu) \geq i(\tilde{S} - \tilde{T}) + i(L \mid \tilde{N} \times \tilde{N})$,

(ii) $\tilde{a}(\mu) \leq i(\tilde{S} - \tilde{T}) + a(L \mid \tilde{N} \times \tilde{N})$,

where $L \in SB(V)$ is given by

$$L(X, Y) = \langle RX - \tilde{T}\tilde{T}X, Y \rangle .$$

Proof. Let $J : \boldsymbol{R} \to SB(\mathscr{J})$ be given by

$$J(t)(X, Y) = \langle \dot{X}(t) - \tilde{T}X(t), Y(t) \rangle - \langle \dot{X}(0) - \tilde{S}X(0), Y(0) \rangle .$$

As already remarked, $\exists \varepsilon > 0$ such that for $0 < t < \varepsilon$, $i(J(t)) = \tilde{i}(t)$ and $a(J(t)) = \tilde{a}(t)$.

Clearly

$$\dot{J}(0)(X, Y) = \langle RX(0) - \tilde{T}\dot{X}(0), Y(0) \rangle + \langle \dot{X}(0) - \tilde{T}X(0), \dot{Y}(0) \rangle ,$$

and the null space N of $J(0)$ is given by

$$N = \{X \in \mathscr{J} : \tilde{S}(X) = \tilde{T}(X)\} .$$

We now show that

(a) $i(\dot{J}(0) \mid N \times N) = i(L \mid \tilde{N} \times \tilde{N})$,

(b) $a(\dot{J}(0) \mid N \times N) = a(L \mid \tilde{N} \times \tilde{N})$.

Let

$$U = \{X \in \mathscr{J} : \dot{X}(0) = \tilde{S}X(0)\} ,$$

and

$$N_1 = \{X \in \mathscr{J} : X(0) = 0\} .$$

Then $\mathscr{J} = U \oplus N_1$, and $l : U \to V$ given by $l(X) = X(0)$ is a linear isomorphism. Let $N_2 = l^{-1}(\tilde{N})$. Then $N = N_1 \oplus N_2$. Also $\dot{J}(0)$ is positive definite on N_1, and

$$\dot{J}(0)(X, Y) = L(l(X), l(Y)) \qquad \text{for } X, Y \in N_2 .$$

Further, if $X \in N_1$, $Y \in N_2$, then

$$\dot{J}(0)(X, Y) = -\langle \tilde{T}\dot{X}(0), Y(0)\rangle + \langle \dot{X}(0), \tilde{T}Y(0)\rangle = 0 .$$

The proof of (a) and (b) is now clear, and the lemma follows from Lemma 1.

We now return to the general case in which the end manifolds P, Q are not necessarily hypersurfaces.

For any subspace $B \subset V$, $p_B: V \to B$ will be the orthogonal projection onto B, and $B^\perp$ will be $\ker p_B$. Let $U = P_{\gamma(0)} \cap Q_{\gamma(b)}$, and write $S - T$ as an abbreviation for the map

$$p_U \circ (S \,|\, U - T \,|\, U): U \to U .$$

We will construct symmetric linear maps $\tilde{S}$, $\tilde{T}$ on V such that

(i) $p_{P_{\gamma(0)}} \circ \tilde{S} \,|\, P_{\gamma(0)} = S$ and $p_{Q_{\gamma(b)}} \circ \tilde{T} \,|\, Q_{\gamma(b)} = T$,

(ii) index (resp. null space) of $\tilde{S} - \tilde{T}$ = index (resp. null space) of $S - T$,

(iii) $\tilde{T} \,|\, U$ depends only on S and T.

$\tilde{P}$, $\tilde{Q}$ will then be chosen to have $\tilde{S}$, $\tilde{T}$ as second fundamental forms with respect to $\gamma'(0)$, $\gamma'(b)$ respectively.

It is clear that (i) implies that $\tilde{i}(t) \geq i(t)$ and $\tilde{a}(t) \geq a(t)$ for all $t > 0$. We later show that for sufficiently small positive t, $\tilde{i}(t) \leq i(t)$. This will yield the desired expression for $i(0^+)$ which, by Lemma 3, depends only on S, T and $R(0)$.

Construction of $\tilde{S}$ and $\tilde{T}$. Let C (resp. D) be the orthogonal complementary subspace of $P_{\gamma(0)}$ (resp. $Q_{\gamma(b)}$) in $P_{\gamma(0)} + Q_{\gamma(b)}$. Then

$$P_{\gamma(0)} + Q_{\gamma(b)} = U \oplus C \oplus D ,$$

and $C \oplus D$ is orthogonal to U.

Define $S_1, T_1: V \to V$ by the requirements that

(a) $\operatorname{Im} T_1 \subset D$ and $\operatorname{Im} S_1 \subset C$,

(b) $p_{C\oplus D} \circ (S \circ p_{P_{\gamma(0)}} - T \circ p_{Q_{\gamma(b)}}) = T_1 - S_1$.

Let S_1^*, T_1^* be the adjoints of S_1, T_1, and let $\lambda \in \boldsymbol{R}$. Put

$$\tilde{S} = S_1 + S_1^* + \lambda p_{P^\perp_{\gamma(0)}} + S \circ p_{P_{\gamma(0)}} ,$$
$$\tilde{T} = T_1 + T_1^* - \lambda p_{Q^\perp_{\gamma(b)}} + T \circ p_{Q_{\gamma(b)}} .$$

It is clear that $\tilde{S}$ and $\tilde{T}$ are symmetric and that (i) and (iii) are satisfied. Also, if N is the null space of $S - T$, then $\forall X \in N$

$$\begin{aligned}(\tilde{S} - \tilde{T})X &= (S_1 - T_1)X + (S_1^* - T_1^*)X + SX - TX \\ &= -p_{C\oplus D}(SX - TX) + SX - TX + (S_1^* - T_1^*)X \\ &= p_U SX - p_U TX + (S_1^* - T_1^*)X \\ &= 0 , \qquad \text{since } X \in N \subset U .\end{aligned}$$

It is a consequence of (i) that $i(\tilde{S} - \tilde{T}) \geq i(S - T)$, so the following lemma, together with a countup of dimensions, shows that (ii) is also true.

Lemma . *Let $S - T$ be positive definite on $G \subset U$. Then, for suitably large λ, $\tilde{S} - \tilde{T}$ is positive definite on $G \oplus U^{\perp}$. (We henceforth assume that $\tilde{S}/\tilde{T}$ are defined using such a λ.)*

Proof. Recall that $(B)_1$ denotes the unit sphere of a normed vector space B. If $Z \in (G)_1$, then by (i)

$$\langle(\tilde{S} - \tilde{T})X, X\rangle = \langle(S - T)X, X\rangle > 0 \ .$$

Thus there is an open neighborhood D of $(G)_1$ in $(G \oplus U^{\perp})_1$ such that $(\tilde{S} - \tilde{T})X, X > 0$, $\forall X \in D$. If $Y \in (G \oplus U^{\perp})_1$, then there are $Y_1 \in G$, $Y_2 \in C$, $Y_3 \in D$, $Y_4 \in (P_{\gamma(0)} + Q_{\gamma(b)})^{\perp}$ such that $Y = Y_1 + Y_2 + Y_3 + Y_4$. Thus

$$\langle(\tilde{S} - \tilde{T})Y, Y\rangle = \lambda(\langle Y_2, Y_2\rangle + \langle Y_3, Y_3\rangle + 2\langle Y_4, Y_4\rangle) + K(Y) \ ,$$

where K is a continuous function on $(G \oplus U^{\perp})_1$. If $H = (G \oplus U^{\perp})_1 \backslash D$, then H is compact so we may choose $\varepsilon, \mu \in \boldsymbol{R}^+$ such that

$$\varepsilon = \inf_{Y \in H} \{\langle Y_2, Y_2\rangle + \langle Y_3, Y_3\rangle + 2\langle Y_4, Y_4\rangle\} > 0 \ ,$$
$$\mu = \sup_{Y \in H} \{|K(Y)|\} \ .$$

Choose $\lambda > \mu/\varepsilon$. Then

$$\langle(\tilde{S} - \tilde{T})X, X\rangle > 0 \ , \qquad \forall X \in (C \oplus U^{\perp})_1 \ ,$$

and the lemma is established.

So far, i, n, n_+, n_- have been integer-valued functions defined on the positive real numbers. We now extend their domains of definition to the nonnegative reals as follows.

Let $i(0) = i(S - T)$ and $n(0) = n(S - T)$. Let $n_+(0)$ (resp. $n_-(0)$) be the dimension of a maximal subspace of N on which $p_N \circ (R - \tilde{T}\tilde{T} | N)$ is positive (resp. negative) definite, where N is the null space of $S - T$. If $n(0) \neq 0$, then we say that 0 is a (P, Q)-focal point of order $n(0)$, while 0 is a *nondegenerate* (P, Q)-focal point if $n_-(0) + n_+(0) = n(0) > 0$. Notice that these definitions are independent of the choice of λ used in the definition of $\tilde{S}$ and $\tilde{T}$. Also, if W has positive sectional curvatures at $\gamma(0)$, then $n_+(0) = 0$ and $n_-(0) = n(0)$, while if $P_{\gamma(0)} \cap Q_{\gamma(b)} = \{0\}$, then $i(0) = n(0) = 0$.

Proposition 3. *If $n(0) = 0$ or if 0 is a nondegenerate (P, Q)-focal point, then $\exists \varepsilon > 0$ such that there are no (P, Q)-focal points on $]0, \varepsilon[$ and $\forall \mu \in]0, \varepsilon[$, $i(\mu) = i(0) + n_-(0)$.*

Proof. Let $\varepsilon > 0$ be as in Lemma 3. Then that lemma, together with property (ii) of $\tilde{S}$ and $\tilde{T}$, shows that $\forall \mu \in]0, \varepsilon[$

$$i(0) + n_-(0) \geq \tilde{a}(\mu) \geq \tilde{i}(\mu) \geq i(0) + n_-(0) \ .$$

Thus the above inequalities are equalities, so there are no $(\tilde{P}, \tilde{Q})$-focal points on $]0, \varepsilon[$. We already know that $a(\mu) \leq \tilde{a}(\mu)$, so it remains to show that for sufficiently small positive μ, $i(\mu) \geq i(0) + n_-(0)$.

Let V_1, V_2 be subspaces of $P_{\gamma(0)}$, $Q_{\gamma(b)}$ respectively, such that $V_1 \cap V_2 = \{0\}$ and $V_1 \oplus V_2 = P_{\gamma(0)} + Q_{\gamma(b)}$. Define $L: U \to V$ as follows. For each $X \in U$, there are unique elements $v_1 \in V_1$, $v_2 \in V_2$ such that

$$S_1X + T_1X = v_2 - v_1 .$$

Set

$$LX = v_1 + S_1X = v_2 - T_1X .$$

It follows from the Rauch comparison theorem for submanifolds [5, p, 351] that $\exists \varepsilon_1 > 0$ such that for all $h \in]-\varepsilon_1, \varepsilon_1[\setminus\{0\}$ and all $X \in U$ there are unique Jacobi fields $\underset{\sim}{X}_h$, $\mathscr{X}_h$ along γ such that

(a) $\underset{\sim}{X}_h$ is a P-Jacobi field with $\underset{\sim}{X}_h(\frac{1}{2}h) = X + \frac{1}{2}hLX$,

(b) $\mathscr{X}_h(h) \in Q_{\gamma(b)}$, $\dot{\mathscr{X}}_h(h) - T\mathscr{X}_h(h) \perp Q_{\gamma(b)}$, $\mathscr{X}_h(\frac{1}{2}h) = X + \frac{1}{2}hLX$.

Now define $J:]-\varepsilon_1, \varepsilon_1[\to SB(U)$ by

$$\begin{aligned} J(h)(X, Y) &= \langle \dot{\underset{\sim}{X}}_h(\tfrac{1}{2}h) - \dot{\mathscr{X}}_h(\tfrac{1}{2}h), \underset{\sim}{Y}_h(\tfrac{1}{2}h)\rangle \qquad \text{for } h \neq 0 , \\ J(0)(X, Y) &= \langle (S - T)X, Y\rangle . \end{aligned}$$

Notice that for $h \in]0, \varepsilon_1[$, $J(h)(X, Y) = I^h(\mathfrak{X}_h, \mathfrak{Y}_h)$ where $\mathfrak{X}_h | [0, \frac{1}{2}h] = \underset{\sim}{X}_h | [0, \frac{1}{2}h]$, $\mathfrak{X}_h | [\frac{1}{2}h, h] = \mathscr{X}_h | [\frac{1}{2}h, h]$, and similarly for $\mathfrak{Y}_h$. In the following lemma we do the calculation necessary for applying Lemma 1.

Lemma. *J is smooth, and*

$$\dot{J}(0)(X, Y) = \langle (R - \tilde{T}\tilde{T})X, Y\rangle ,$$

for X, Y in the null space N of $S - T$.

Proof of Lemma. Let m be the dimension of P, and d the dimension of V. The following ranges of indices will be used:

$$1 \leq A, B, \cdots, \leq d , \quad 1 \leq i, j, \cdots, \leq m , \quad m + 1 \leq \alpha, \beta, \cdots, \leq d .$$

We shall also employ the summation convention whereby repeated indices are summed over their respective ranges.

Let $X, Y \in U$, and pick an orthonormal basis $e_1, \cdots, e_d$ for V such that $X = xe_1$ for some $x \in \boldsymbol{R}$, and $\{e_1, \cdots, e_m\}$ spans $P_{\gamma(0)}$. Let u_i and v_α be the Jacobi fields with $u_i(0) = e_i$, $\dot{u}_i(0) = \tilde{S}e_i$, and $v_\alpha(0) = 0$, $\dot{v}_\alpha(0) = e_\alpha$. Since $v_\alpha(0) = 0$, the vector fields w_α given by

$$w_\alpha(t) = v_\alpha(t)/t \qquad \text{for } t \neq 0,\ w_\alpha(0) = \dot{v}_\alpha(0) = e_\alpha$$

are smooth, and $\{u_1(t), \cdots, u_m(t), w_{m+1}(t), \cdots, w_d(t)\}$ are linearly independent for $t \in]-\varepsilon_1, \varepsilon_1[$. Thus we can uniquely define smooth functions $a_A :]-\varepsilon_1, \varepsilon_1[\to \boldsymbol{R}$ by the requirement that

$$a_i(h)u_i(\tfrac{1}{2}h) + a_\alpha(h)w_\alpha(\tfrac{1}{2}h) = X + \tfrac{1}{2}hLX \ .$$

Clearly $a_\alpha(0) = 0$, so the functions $b_\alpha :]-\varepsilon_1, \varepsilon_1[\to \boldsymbol{R}$ given by

$$b_\alpha(h) = a_\alpha(h)/h \qquad \text{for } h \neq 0,\ b_\alpha(0) = \dot{a}_\alpha(0)$$

are smooth. Also

$$(5) \quad a_i(h)u_i(\tfrac{1}{2}h) + 2b_\alpha(h)v_\alpha(\tfrac{1}{2}h) = X + \tfrac{1}{2}hLX \ , \qquad \text{for } h \in]-\varepsilon_1, \varepsilon_1[\ .$$

However, by definition of ε_1 we have that $\forall h \in]-\varepsilon_1, \varepsilon_1[\setminus\{0\}$, $\forall t \in \boldsymbol{R}$,

$$(6) \qquad \underset{\sim}{X}_h(t) = a_i(h)u_i(t) + 2b_\alpha(h)v_\alpha(t) \ .$$

So, if we define $\underset{\sim}{X}_0(t) = a_i(0)u_i(t) + 2b_\alpha(0)v_\alpha(t)$, then $\underset{\sim}{X}(h, t) = \underset{\sim}{X}_h(t)$ is a smooth vector-valued function of two variables.

Differentiating (5) with respect to h at $h = 0$ we have

$$(7) \quad \dot{a}_i(0)u_i(0) + \tfrac{1}{2}a_i(0)\dot{u}_i(0) + 2\dot{b}_\alpha(0)v_\alpha(0) + b_\alpha(0)\dot{v}_\alpha(0) = \tfrac{1}{2}LX \ .$$

Putting $h = 0$ in (5) we get that

$$(8) \qquad a_1(0) = x,\ a_i(0) = 0 \qquad \text{for } i = 2, \cdots, m \ ,$$

so from (7)

$$(9) \qquad \dot{a}_i(0)e_i + \tfrac{1}{2}\tilde{S}X + b_\alpha(0)e_\alpha = \tfrac{1}{2}LX \ .$$

Thus, if $K(t) = \langle(\partial\underset{\sim}{X}/\partial t)(t, \tfrac{1}{2}t), \underset{\sim}{Y}(t, \tfrac{1}{2}t)\rangle$, then K is smooth, and from (6)

$$\begin{aligned}\dot{K}(0) = &\langle \dot{a}_i(0)\dot{u}_i(0) + \tfrac{1}{2}a_i(0)\ddot{u}_i(0) + b_\alpha(0)\ddot{v}_\alpha(0) + 2\dot{b}_\alpha(0)\dot{v}_\alpha(0), Y\rangle \\ &+ \langle a_i(0)\dot{u}_i(0) + 2b_\alpha(0)\dot{v}_\alpha(0), \tfrac{1}{2}LY\rangle \ ,\end{aligned}$$

which becomes, in consequence of (8) and (9),

$$(10) \qquad \begin{aligned}\dot{K}(0) = &\tfrac{1}{2}(\langle LX - \tilde{S}X, \tilde{S}Y\rangle + \langle RX, Y\rangle + \langle \tilde{S}X, LY\rangle) \\ &+ \langle b_\alpha(0)e_\alpha, LY - \tilde{S}Y\rangle \ .\end{aligned}$$

However, $b_\alpha(0)e_\alpha$ is orthogonal to $P_{\gamma(0)}$, so

$$\langle b_\alpha(0)e_\alpha, LY\rangle = \langle b_\alpha(0)e_\alpha, S_1Y\rangle = \langle b_\alpha(0)e_\alpha, \tilde{S}Y\rangle \ .$$

Thus from (10)

$$(11) \qquad \dot{K}(0) = \tfrac{1}{2}(\langle LX - \tilde{S}X, \tilde{S}Y\rangle + \langle RX, Y\rangle + \langle \tilde{S}X, LY\rangle) \ .$$

Using similar techniques it can be shown that if

$$H(h) = \langle \dot{\mathscr{X}}_h(\tfrac{1}{2}h), \mathscr{Y}_h(\tfrac{1}{2}h)\rangle \qquad \text{for } h \neq 0 ,$$
$$H(0) = \langle \tilde{T}X, Y\rangle ,$$

then H is smooth and

$$\dot{H}(0) = \tfrac{1}{2}(\langle LX + \tilde{T}X, \tilde{T}Y\rangle - \langle RX, Y\rangle + \langle \tilde{T}X, LY\rangle) . \tag{12}$$

Since $J(h)(X, Y) = K(h) - H(h)$ we see that J is smooth and

$$\dot{J}(0)(X, Y) = \langle RX, Y\rangle + \tfrac{1}{2}(\langle \tilde{S}X - \tilde{T}X, LY\rangle + \langle LX - \tilde{S}X, \tilde{S}Y\rangle - \langle LX + \tilde{T}X, \tilde{T}Y\rangle) .$$

Thus, if $X, Y \in N$ then

$$\dot{J}(0)(X, Y) = \langle RX, Y\rangle - \langle \tilde{T}X, \tilde{T}Y\rangle ,$$

as was required to prove the lemma.

Returning to the proof of Proposition 3, we see that the above lemma, together with Lemma 1, shows that $\exists \varepsilon_2 > 0$ such that $\forall \mu \in]0, \varepsilon_2[$

$$i(J(\mu)) \geq i(J(0)) + n_-(0) .$$

However, as already remarked, $J(\mu)(X, Y) = I^\mu(\mathfrak{X}_\mu, \mathfrak{Y}_\mu)$, so that $i(J(\mu)) \leq i(\mu)$. Thus

$$i(0) + n_-(0) \leq i(J(\mu)) \leq i(\mu) \leq a(\mu) \leq \tilde{a}(\mu) \leq i(0) + n_-(0) ,$$

and the proof of the proposition is complete.

Propositions 1, 2 and 3 are combined to give the main result of the paper:

Theorem A. *Let P, Q be submanifolds of W, and let γ be a geodesic of W intersecting P and Q orthogonally at $\gamma(0)$ and $\gamma(b)$ respectively. If P, Q have only nondegenerate (P, Q)-focal points on $[0, b]$, then these (P, Q)-focal points are finite in number. Further, the index $i(b)$ and the augmented index $a(b)$ of the index form of this configuration are given by*

$$i(b) = i(S - T) + \sum_{0\leq t<b} n_-(t) - \sum_{0<t\leq b} n_+(t) ,$$
$$a(b) = i(S - T) + \sum_{0\leq t\leq b} n_-(t) + \sum_{0<t<b} n_+(t) ,$$

where S, T are the second fundamental forms of P, Q with respect to $\gamma'(0)$, $\gamma'(b)$, respectively.

5. A comparison theorem

In view of Theorem A it is desirable to obtain an estimate of the distance from P to the first (P, Q)-focal point. If $S - T$ is positive definite (e.g., if

$P_{\gamma(0)} \cap Q_{\gamma(b)} = \{0\}$), then methods similar to those employed by Warner [5, Proof of Theorem 3.2] may be used to yield such information. However, these methods depend on I^t being positive definite for small t, and as we have seen, this is not the case for general S, T.

In this section we illustrate a method of finding such estimates using the idea of *translates of* S as employed by Ambrose in [1]. The principal drawback in our use of this construction is that we must assume that P is a hypersurface of W.

Let t_0 be the first focal point of P along γ, and let $t \in [0, t_0[$. For each $X \in V$, there is a unique P-Jacobi field $\mathscr{X}$ such that $\mathscr{X}(t) = X$. Let $S_t(X) = \dot{\mathscr{X}}(t)$. This defines a map $S_t \colon V \to V$ which may be shown to be an element of the space $SL(V)$ of symmetric linear maps from V to V (See [1, p. 54]). Notice that if P is not a hypersurface, then the above breaks down at $t = 0$.

Lemma 4. *The map* $\underset{\sim}{S} \colon [0, t_0[\to SL(V)$ *given by* $\underset{\sim}{S}(t) = S_t$ *is smooth and satisfies the Riccati equation*

$$\dot{\underset{\sim}{S}}(t) = R(t) - \underset{\sim}{S}^2(t) .$$

Proof. The smoothness of $\underset{\sim}{S}$ follows from the theory of solutions of ordinary differential equations. Let $\mathscr{X}$ be a P-Jacobi field. Then $(\underset{\sim}{S}\mathscr{X})(t) = \dot{\mathscr{X}}(t)$, so by differentiating we get

$$(\dot{\underset{\sim}{S}}\mathscr{X})(t) + (\underset{\sim}{S}\dot{\mathscr{X}})(t) = \ddot{\mathscr{X}}(t) ,$$

which gives

$$(\dot{\underset{\sim}{S}}\mathscr{X})(t) = (R\mathscr{X})(t) - (\underset{\sim}{S}\underset{\sim}{S}\mathscr{X})(t) .$$

Hence the lemma is proved.

If $L \in SL(V)$, let $L^{\#} \in SL(Q_{\gamma(b)})$ be given by $L^{\#}(X) = p_{Q_{\gamma(b)}} LX$.

Theorem B. *Assume that P is a hypersurface of W and that*

(i) *each eigenvalue of S has modulus $\leq \Lambda$,*

(ii) *each eigenvalue of $S^{\#} - T$ has modulus $\geq \Omega > 0$,*

(iii) *for each positive t, each eigenvalue of $R(t)$ has modulus $\leq \Theta$.*

Then the first (P, Q)-focal point occurs at or after t_1, where t_1 is the smallest positive solution of the equation

$$\begin{aligned} \cot \Theta^{1/2} t &= \Omega^{-1}\Theta^{-1/2}(\Theta + \Lambda^2 + \Lambda\Omega) && \text{if } \Theta > 0 , \\ t &= \Omega\Lambda^{-1}(\Lambda + \Omega)^{-1} && \text{if } \Theta = 0 . \end{aligned}$$

Proof. For $\theta > 0$, let τ_θ be the smallest positive solution of

$$\cot \theta^{1/2} t = \Lambda \theta^{-1/2} ,$$

and let $\tau_0 = \Lambda^{-1}$. It follows from the Rauch comparison theorem for submanifolds [5, Corollary 4.2(a)] that the first focal point of P occurs at or after τ_Θ and this occurs after t_1. Thus $\underset{\sim}{S}$ is defined on $[0, t_1]$. Let

$$D = \{(\theta, t) \in \boldsymbol{R}(\geq 0) \times \boldsymbol{R}(\geq 0) : t < \tau_\theta\} ,$$

and define $g : D \to \boldsymbol{R}$ by

$$g(\theta, t) = \theta^{1/2} \cot (\theta^{1/2} t + K) , \qquad \text{where } K = \cot^{-1} (-\Lambda \theta^{-1/2}) .$$

Then g is continuous and negative on D, and $\partial g / \partial t = -\theta - g^2$.

Lemma. *If X, Y are unit vectors in V and if $t < \tau_\Theta$, then*

$$(d/dt)\langle S_t X, Y\rangle \leq |(\partial g / \partial t)(\theta, t)| .$$

Proof of Lemma. Let $\|\ \|$ be the norm on V associated with $\langle\ ,\ \rangle$, and for $L \in SL(V)$ let

$$\|L\| = \sup \{\|LZ\| : Z \in V \text{ and } \|Z\| = 1\} .$$

Then

$$\left|\frac{d}{dt}\langle S_t X, Y\rangle\right| = |\langle R(t)X - S_t S_t X, Y\rangle| \leq \Theta + \|S_t\|^2 .$$

To establish the lemma it is enough to show that $\|S_t\| \leq |g(\theta, t)|$ for $\theta > \Theta$, $t < \tau_\theta$. Since $\|S_0\| \leq |g(\theta, 0)|$ it suffices to show that if $\theta > \Theta$, $t_2 \in [0, \tau_\theta[$, $Z \in V$ are such that $\|Z\| = 1$, $0 < \|S_{t_2}\| = |g(\theta, t_2)|$, then $(d/dt)\,\|S_t Z\| < |\partial g / \partial t|$ at (θ, t_2). However, this is clear because in this case

$$\begin{aligned} \frac{d}{dt}\|S_t Z\| &= \langle R(t)Z - S_t S_t Z, S_t Z\rangle\, \|S_t Z\|^{-1} && \text{at } t = t_2 \\ &\leq \Theta + \|S_t\|^2 < |\partial g / \partial t| && \text{at } (\theta, t_2) . \end{aligned}$$

Returning to the proof of Theorem B, we note that if $t_2 \in [0, t_1[$ is a (P, Q)-focal point, then $\exists X \in Q_{\gamma(b)}$ such that $\|X\| = 1$ and $S^{\#}_{t_2} X = TX$. However, from the lemma it is clear that if $Y \in Q_{\gamma(b)}$, then

$$\langle S_0 X - S_{t_2} X, Y\rangle \leq |g(\Theta, 0) - g(\Theta, t_2)| < |g(\Theta, 0) - g(\Theta, t_1)| = \Omega .$$

Since $S_0 = S$, we now have a contradiction of hypothesis (ii). This completes the proof of Theorem B.

Similar theorems may be proved using the above methods.

References

[1] W. Ambrose, *The index theorem in Riemannian geometry*, Ann. of Math. **73** (1961) 49–86.
[2] R. L. Bishop & R. J. Crittenden, *Geometry of manifolds*, Academic Press, New York, 1964.
[3] M. Morse, *The calculus of variations in the large*, Amer. Math. Soc. Colloq. Publ. Vol. 18, 1934.
[4] H. Osborn, *The Morse index theorem*, Proc. Amer. Math. Soc. **18** (1967) 759–762.
[5] F. W. Warner, *Extension of the Rauch comparison theorem to submanifolds*, Trans. Amer. Math. Soc. **122** (1966) 341–356.

UNIVERSITY OF DURHAM, ENGLAND