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GEOMETRY OF HOROSPHERES

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1. Introduction

Let M be a Hadamard manifold, i.e., a connected, simply connected, complete riemannian manifold of nonpositive curvature. To be more precise, assume that the sectional curvature K of M satisfies $-b^2 \leq K \leq -a^2$, where $0 \leq a < \infty$ and $0 \leq b \leq \infty$. If $p \in M$ and z is a point at infinity (cf. Eberlein-O'Neill [4], which we give as a general reference for Hadamard manifolds), there exists a *horosphere* through p with center z. This is defined as follows: Denote the geodesic ray from p to z by γ, and consider the geodesic spheres through p with center $\gamma(t)$, $t > 0$. As t goes to infinity, these spheres converge to the horosphere. More precisely, the horospheres are the level surfaces of the *Busemann function* $F = \lim F_t$, where F_t is defined by $F_t(p) = d(p, \gamma(t)) - t$. In the flat case ($a = b = 0$), horospheres are just affine hyperplanes, and in the case of constant negative curvature, using the Poincaré model we see that horospheres are euclidean spheres internally tangent to the boundary sphere, minus the point of tangency. The main purpose of this paper is to show that, to a certain extent, the geometry of horospheres in M may be compared with that in the spaces of constant curvature $-a^2$ and $-b^2$, respectively. We give two examples:

1. (Theorem 4.6). *If $\mathscr{H}$ is a horosphere and h denotes the distance in $\mathscr{H}$ with respect to the induced metric, then for all $p, q \in \mathscr{H}$*

$$\frac{2}{a} \sinh \frac{a}{2} d(p, q) \leq h(p, q) \leq \frac{2}{b} \sinh \frac{b}{2} d(p, q) ,$$

where d is the distance function of M.

2. (Theorem 4.9). *If γ is a geodesic tangent to a horosphere $\mathscr{H}$, and if p, q are the projections of $\gamma(\pm\infty)$ onto $\mathscr{H}$, then*

$$\frac{2}{b} \leq h(p, q) \leq \frac{2}{a} .$$

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We recall that we allow $a = 0$ and $b = \infty$, corresponding to the possibilities that there is only one or no curvature bound beside $K \leq 0$. In these cases all our inequalities have to be interpreted in the obvious way, and eventually become meaningless, e.g., $h(p, q) \leq 2/a$ for $a = 0$.

Here we give a brief account of the content of the paper. In § 2 we prove a comparison theorem for stable Jacobi fields, which is crucial for all the following estimates. Moreover, this theorem is of its own interest, especially because in contrast to former authors ([5, p. 117], [1, p. 132]) we get the optimal bounds. § 3 is concerned with the C^2-differentiability of Busemann functions. Combining this with the comparison theorem for stable Jacobi fields, we get estimates for the differential of the flow which moves M towards a given point at infinity. In § 4, the main part of the paper, we formulate and prove a series of geometric comparison theorems involving distances on horospheres and their relationship to geodesics, as indicated by the examples above.

Notation. All nontrivial geodesics are assumed to have unit speed. For $p, q \in M$ we denote by γ_{pq} the unique geodesic from p to q, and by $d(p, q)$ the distance between p and q. If m denotes a point different from p and q, then $\sphericalangle_m(p, q)$ denotes the angle in $[0, \pi]$ subtended by $\dot\gamma_{mp}(0)$ and $\dot\gamma_{mq}(0)$. Let $M(\infty)$ denote the points at infinity. If $p \in M$ and $z \in M(\infty)$, there exists a unique geodesic from p to z, denoted by γ_{pz}. Therefore the definition of angles $\sphericalangle_p(x, y)$ for $p \in M$ and $x, y \in M \cup M(\infty)$, $p \neq x, y$, makes sense.

2. Stable Jacobi fields

Definition 2.1. Let $\gamma\colon [0, \infty) \to M$ be a geodesic ray, and Y a Jacobi field along γ. Y is said to be *stable* if $\|Y(t)\|$ is bounded for $t \geq 0$.

Lemma 2.2 *Let $\gamma\colon [0, \infty) \to M$ be a geodesic ray, and let $v \in M_p$, $p = \gamma(0)$. Then there existis a unique stable Jacobi field Y along γ with $Y(0) = v$, and we shall denote this stable Jacobi field by Y_v.*

Proof. (i) Uniqueness follows immediately from the fact that in a Hadamard manifold the length of a Jacobi field is a convex function.

(ii) Denote by Y_n the unique Jacobi field along γ with $Y_n(0) = v$ and $Y_n(n) = 0$. Applying Rauch's comparison theorem to $Y_n - Y_m$ (comparison with the flat case), we get

$$\|Y_n'(0) - Y_m'(0)\| \leq \frac{1}{t}\|Y_n(t) - Y_m(t)\| .$$

Now by the convexity argument above, $\|Y_m(t)\|$ is monotone decreasing in the interval $[0, m]$, so that

$$\|Y_n'(0) - Y_m'(0)\| \leq \frac{1}{n}\|v\| , \qquad \text{for } n \leq m .$$

Thus $\{Y_n'(0)\}$ is a Cauchy sequence with limit w, say. If Y_v denotes the Jacobi field along γ with $Y_v(0) = v$ and $Y_v'(0) = w$, it follows immediately that Y_v, as the limit of the Y_n, is stable. This completes the proof of the lemma.

Proposition 2.3. *Let Y be a perpendicular Jacobi field along the geodesic $\gamma: \boldsymbol{R} \to M$ with $Y(0) = 0$. Assume as usual that the curvature K of M satisfies $-b^2 \leq K \leq -a^2$. Then for $0 < t \leq s$*

$$\frac{\sinh bt}{\sinh bs} \leq \frac{\|Y(t)\|}{\|Y(s)\|} \leq \frac{\sinh at}{\sinh as} .$$

A proof, using an idea of Bishop-Crittenden [2], may be found in Im Hof-Ruh [6]. The assumption of positively pinched curvature in [6] is not essential. The proof may be changed in an obvious way by replacing sin and cos by sinh and cosh, respectively. As a consequence we get

Theorem 2.4 (*Comparison theorem for stable Jacobi fields*). *Let $\gamma: [0, \infty) \to M$ be a geodesic ray, and Y_v the stable Jacobi field along γ with $Y_v(0) = v \in M_{\gamma(0)}$, $v \perp \dot{\gamma}(0)$. Then*

$$\|v\| e^{-bt} \leq \|Y_v(t)\| \leq \|v\| e^{-at} .$$

Proof. If Y_n denotes the Jacobi field along γ with $Y_n(0) = v$ and $Y_n(n) = 0$ as above, then $Y_n(t) \to Y_v(t)$ for fixed t. By the last proposition (applied to $Z_n(t) = Y_n(n - t)$), we get

$$\frac{\sinh b(n-t)}{\sinh bn} \leq \frac{\|Y_n(t)\|}{\|Y_n(0)\|} \leq \frac{\sinh a(n-t)}{\sinh an}$$

for $0 \leq t \leq n$. Thus

$$e^{-bt} = \lim_{n\to\infty} \frac{\sinh b(n-t)}{\sinh bn} \leq \frac{\|Y_v(t)\|}{\|v\|} \leq \lim_{n\to\infty} \frac{\sinh a(n-t)}{\sinh an} = e^{-at} .$$

This completes the proof.

3. Radial fields and radial flows

In this section we fix a point $z \in M(\infty)$ and consider the corresponding *radial field* Z defined by $Z(p) = \dot{\gamma}_{pz}(0)$. It is said to be radial in analogy to the radial field Z^q, which is given by $Z^q(p) = \dot{\gamma}_{pq}(0)$ for a fixed point $q \in M$ and $p \neq q$.

In the following we will strongly need that Z is continuously differentiable, a fact which has been proved by P.Eberlein in an unpublished paper [3]. For the convenience of the reader we give here a new proof, which is also considerably shorter. It is interesting to note that L. Green [5, p. 118] could show that actually Z is of class C^2, provided ∇R is bounded and the curvature is strictly $\frac{1}{4}$-pinched.

Proposition 3.1 (*Eberlein* [3]). *Let M be a Hadamard manifold, Z the radial field in direction of $z \in M(\infty)$, and F a Busemann function at z. Then $Z = -\operatorname{grad} F$, Z is C^1, and $\nabla_v Z = Y_v'(0)$ for all $v \in M_p$, where $p \in M$, and Y_v is the stable Jacobi field along γ_{pz} with $Y_v(0) = v$.*

The basic idea of the proof, going back to P. Eberlein, is to carry over statements for radial fields in the direction of finite points to the given field Z. If $q \in M$, and Z^q is the corresponding radial field, then we have $Z^q = -\operatorname{grad} F^q$, where F^q denotes the distance to q. Now if $v \in M_p$, an easy 2-parameter variation argument shows $\nabla_v Z^q = -\nabla_v \operatorname{grad} F^q = Y'(0)^\perp$, where Y is the Jacobi field along γ_{pq} with $Y(0) = v$, $Y(d(p,q)) = 0$, and $\perp$ denotes the component orthogonal to $\dot\gamma_{pq}(0)$. If q is replaced by a point at infinity, F^q has to be replaced by the Busemann function F, and Y by the stable Jacobi field "vanishing at z".

Proof. Let γ be a geodesic with $\gamma(\infty) = z$, and $p_n = \gamma(n)$, $n \in N$. If F_n is defined by $F_n(p) = d(p_n, p) - n$, then $F = \lim F_n$ is a Busemann function with respect to z, and $Z_n = -\operatorname{grad} F_n$ is the radial field in the direction of p_n. Z_n is defined and C^∞ on $M - \{p_n\}$. We will show (i) the fields Z_n converge uniformly on compact sets to Z, and (ii) for any vector field V on M the covariant derivatives $\nabla_V Z_n$ converge uniformly on compact sets to Y_V, where $Y_V(p) = Y_v'(0)$ and Y_v is the stable Jacobi field along γ_{pz} with $Y_v(0) = v = V(p)$. This proves the uniform convergence of the first and second derivatives of the functions F_n on compact sets. Thus F is C^2, $\operatorname{grad} F = \lim \operatorname{grad} F_n = -Z$, Z is C^1, and $\nabla_v Z = -\nabla_v \operatorname{grad} F = -\lim \nabla_v \operatorname{grad} F_n = Y_v'(0)$.

Let $K \subset M$ be compact and $n_0 \in N$, such that $p_n \notin K$ for all $n \geq n_0$.

(i) Let $p \in K$ and $n \geq n_0$. Then $\|Z_n - Z\|(p) = \|\dot\gamma_{pp_n}(0) - \dot\gamma_{pz}(0)\|$ goes to zero uniformly on K, if the angles $\sphericalangle_p(p_n, z)$ do so. But this is an easy consequence of the uniform boundedness of the distances $d(p_n, \gamma_{pz})$ by the constant $\max\{d(\gamma(0), p) \mid p \in K\}$.

(ii) Next, consider for $p \in K$ and $n > n_0$

$$\|\nabla_V Z_n - Y_V\|(p) = \|\nabla_v \operatorname{grad} F_n - Y_v'(0)\| = \|Y_{pp_n}'^{\perp}(0) - Y_v'(0)\| ,$$

where $v = V(p)$, Y_{pp_n} is the Jacobi field along γ_{pp_n} with $Y_{pp_n}(0) = v$ and $Y_{pp_n}(d(p, p_n)) = 0$, and $\perp$ denotes the component orthogonal to $\dot\gamma_{pp_n}(0)$, which also depends on n. But an easy computation shows

$$\|Y_{pp_n}'(0) - Y_{pp_n}'^{\perp}(0)\| = \frac{\|v - v^\perp\|}{d(p, p_n)} \leq \frac{\|v\|}{d(p, p_n)} ,$$

which goes to zero uniformly on K as n tends to infinity. Thus it is enough to show $\|Y_{pp_n}'(0) - Y_v'(0)\| \to 0$ uniformly on K. For $T > 0$ let $X_{pp_n}^T$ be the Jacobi field along γ_{pp_n} with $T_{pp_n}^T(0) = V(p)$, $X_{pp_n}^T(T) = 0$, and define X_{pz}^T analogously. Then

$$\|Y'_{pp_n}(0) - Y'_v(0)\| \leq \|Y'_{pp_n}(0) - X^{T'}_{pp_n}(0)\| + \|X^{T'}_{pp_n}(0) - X^{T'}_{pz}(0)\|$$
$$+ \|X^{T'}_{pz}(0) - Y'_v(0)\| .$$

By Rauch's comparison theorem

$$\|Y'_{pp_n}(0) - X^{T'}_{pp_n}(0)\| \leq \frac{1}{T}\|Y_{pp_n}(T)\| \leq \frac{1}{T}\|V(p)\| ,$$

if n is sufficiently large so that $d(p, p_n) > T$. The same argument yields $\|X^{T'}_{pz}(0) - Y'_v(0)\| \leq \|V(p)\|/T$. Thus the problem is reduced to show that, for fixed T, the difference $\|X^{T'}_{pp_n}(0) - X^{T'}_{pz}(0)\|$ goes to zero uniformly on K, as n tends to infinity. Using a lower curvature bound on $K_T = \{p \in M \mid d(p, K) \leq T\}$ it is clear that $d(q_n(p), q(p)) \to 0$ uniformly on K, where $q_n(p) = \gamma_{pp_n}(T)$, $q(p) = \gamma_{pz}(T)$. By the differentiable dependence of Jacobi fields and their derivatives on the boundary values, the result now follows.

The radial flow. Now we want to study the flow generated by the vector field Z, which we call the *radial flow* (with respect to a fixed $z \in M(\infty)$) and denote by ψ or $\{\psi_t\}$. Since the geodesics going to z are the integral curves of Z, this vector field is obviously complete, and ψ is given by $\psi = \pi \circ \varphi \circ (1_{\boldsymbol{R}} \times Z) : \boldsymbol{R} \times M \to M$, where φ denotes the geodesic flow, and π the canonical projection. Proposition 3.1 implies immediately that ψ is C^1.

The following properties of ψ_{t*} are infinite versions of the lemma of Gauss and the comparison theorem of Rauch.

Proposition 3.2. (i) *If a vector $u \in M_p$ is parallel to $Z(p)$, then $\psi_{t*}(u)$ is parallel to $Z(\psi_t(p))$, and $\|\psi_{t*}(u)\| = \|u\|$.*

(ii) *If a vector $v \in M_p$ is orthogonal to $Z(p)$, then $\psi_{t*}(v)$ is orthogonal to $Z(\psi_t(p))$, and the following inequalities hold*

$$\|v\| e^{-bt} \leq \|\psi_{t*}(v)\| \leq \|v\| e^{-at} \qquad \textit{for } t \geq 0 .$$

Proof. (i) It is enough to show $\psi_{t*}(Z(p)) = Z(\psi_t(p))$, but this is true, since the geodesics going to z are the integral curves of Z.

(ii) We recall that $Z = -\operatorname{grad} F$, where F is a Busemann function at z, and that the horospheres centered at z are the level surfaces of F. Therefore the complements $M_p^{\perp} = \{v \in M_p \mid v \perp Z(p)\}$ are the tangent spaces of the horospheres, and ψ_t maps horospheres onto parallel horospheres. This implies the first part of (ii). In order to prove the inequalities, we now compute $\psi_{t*}(v)$ for $v \in M_p$ explicitely. By definition $\psi_{t*}(v) = \pi_* \circ \varphi_{t*} \circ Z_*(v)$. We use the identification $TSM \cong SM \oplus TM \oplus TM$ given by $\pi_S \times \pi_* \times K$, where SM denotes the unit tangent bundle, $\pi_S : TSM \to SM$ is the canonical projection, $\pi_* : TSM \to TM$ is the differential of $\pi : SM \to M$, and $K : TSM \to TM$ is the connection map. Then $Z_*(v) = (Z(p), v, \nabla_v Z) = (Z(p), Y_v(0), Y'_v(0))$, where Y_v is the stable Jacobi field along γ_{pz} with initial value $Y_v(0) = v$, (compare Proposition 3.1). Therefore we get $\varphi_{t*} \circ Z_*(v) = (Z(\psi_t(p)), Y_v(t), Y'_v(t))$ and

$\psi_{t*}(v) = Y_v(t)$. By the comparison theorem for stable Jacobi fields we conclude $\|v\| e^{-bt} \leq \|\psi_{t*}(v)\| \leq \|v\| e^{-at}$.

4. Distances on horospheres

Generalities. Since Busemann functions, and therefore horospheres, are at least C^2, the notions of distance and geodesic curves are defined with respect to the induced metric. As level surfaces of a Busemann function, horospheres are closed and therefore complete; in particular, we always have minimal geodesics joining two points. In the case of constant negative curvature horospheres are flat, but in the other symmetric spaces of rank 1 and negative curvature this is no longer true. In these spaces horospheres may be represented as nonabelian nilpotent Lie groups with a left invariant metric, and therefore have curvatures of both signs (J. Wolf [9]) and even conjugate points (J. O'Sullivan [7]).

In the following we will estimate some distances on horospheres arising in special geometric situations. Still assuming $-b^2 \leq K \leq -a^2$, we will use as comparison manifolds the spaces H_a and H_b of constant curvature $-a^2$ and $-b^2$, respectively.

Two asymptotic geodesics. Let γ_0 be a geodesic, and denote by $\mathscr{H}_t$ the horosphere trhough $\gamma_0(t)$ with center $\gamma_0(\infty)$. Obviously we have $\mathscr{H}_t = \psi_t(\mathscr{H}_0)$, where ψ_t is the radial flow in the direction of $\gamma_0(\infty)$. Now consider an asymptotic geodesic γ_1, and choose the origin $\gamma_1(0)$ on $\mathscr{H}_0$. Then $\gamma_1(t) \in \mathscr{H}_t$, and we can define $h(t)$ to be the $\mathscr{H}_t$-distance of $\gamma_0(t)$ and $\gamma_1(t)$. As a first application of Proposition 3.2 we give an estimate for $h(t)$.

Propositson 4.1. *For $t \geq 0$ we have*

$$h(0)e^{-bt} \leq h(t) \leq h(0)e^{-at} .$$

Proof. Let $\mu_0 : [0, 1] \to \mathscr{H}_0$ be a minimal $\mathscr{H}_0$-geodesic joining $\gamma_0(0)$ and $\gamma_1(0)$. Then $\mu_t = \psi_t \circ \mu_0$ is a curve on $\mathscr{H}_t$ from $\gamma_0(t)$ to $\gamma_1(t)$, and we have

$$h(t) \leq l(\mu_t) = \int_0^1 \|\dot{\mu}_t\| = \int_0^1 \|\psi_{t*}\dot{\mu}_0\| \leq e^{-at} \int_0^1 \|\dot{\mu}_0\| = e^{-at}h(0) .$$

The proof of the inequality on the left hand side is similar.

Remark. Combining the above result with Theorem 4.6 below we immediately get, for $d(t) = d(\gamma_0(t), \gamma_1(t))$ and $t \geq 0$,

$$\left(\frac{2}{b} \operatorname{arcsinh} \frac{b}{2}h(0)\right)e^{-bt} \leq d(t) \leq h(0)e^{-at} .$$

As H. Karcher remarked, this can be improved by a different method to

$$d(0)e^{-bt} \leq d(t) \leq \left(\frac{2}{a} \sinh \frac{a}{2}d(0)\right)e^{-at} .$$

Two estimates for the Busemann function with geometric applications. We consider a Busemann function F at an infinite point z. To compare F with Busemann functions in spaces of constant curvature, we study the restriction $f = F \circ \gamma$ for a given geodesic γ. While f measures the deviation of γ from a fixed horosphere with center z, the derivative $f' = \langle \dot{\gamma}, \operatorname{grad} F \rangle$ measures the angle between γ and the horospheres centered at z.

In the following, f_a and f_b denote functions defined analogously in the spaces H_a and H_b, respectively.

Lemma 4.2. *Given that f, f_a, f_b are as described above. Assume $f(0) = f_a(0) = f_b(0)$ and $f'(0) = f'_a(0) = f'_b(0)$. Then $f'_a(s) \leq f'(s) \leq f'_b(s)$ for $s \geq 0$ and $f_a(s) \leq f(s) \leq f_b(s)$ for $s \in \boldsymbol{R}$.*

Proof. For $s > 0$ consider the triangle Δ determined by $p = \gamma(0)$, $q = \gamma(s)$ and z. The angles $\alpha = \sphericalangle_p(q, z)$ and $\beta = \sphericalangle_q(p, z)$ satisfy $\cos \alpha = -f'(0)$ and $\cos \beta = f'(s)$. Let σ be the geodesic ray from p to z, and denote by $\Delta(t)$ the triangle determined by p, q and $\sigma(t)$. The angle $\beta(t) = \sphericalangle_q(p, \sigma(t))$ converges to β as soon as t tends to ∞.

Now consider the analogous data in H_a and H_b. For sufficiently large t, Toponogov's comparison theorem implies $\beta_a(t) \geq \beta(t) \geq \beta_b(t)$, so that $\beta_a \geq \beta \geq \beta_b$ and therefore $f'_a(s) \leq f'(s) \leq f'_b(s)$. The second statement of the proposition follows by integration. (If $s \leq 0$, consider the inverse geodesic $\gamma_-(s) = \gamma(-s)$ and observe $f'_a(s) \geq f'(s) \geq f'_b(s)$).

Lemma 4.3. *Given that f, f_a, f_b are as before. Assume $f(0) = f_a(0) = f_b(0)$ and $f(l) = f_a(l) = f_b(l)$. Then*

$$f_b(s) \leq f(s) \leq f_a(s) \qquad \textit{for } s \in [0, l] \ .$$

Proof. Fix $s \in [0, l]$ and look at the triangles $\Delta_1 = (\gamma(0), \gamma(s), z)$ and $\Delta_2 = (\gamma(1), \gamma(s), z)$. In one of thet riangles, say in Δ_1, the angle β at $\gamma(s)$ is not smaller than the corresponding angle β_a in H_a. Suppose for the moment that β equals β_a. Then Lemma 4.2, applied to Δ_1, implies $f(s) \leq f_a(s)$. This is a fortiori if $\beta > \beta_a$.

The proof of the inequality on the left hand side is similar.

Remark. Since in the flat case f_0 is linear, the above lemma gives another proof of the convexity of F.

For the geometric applications consider triangles Δ with two vertices $p, q \in M$ and one vertex z at infinity. Such a triangle gives rise to the following data: $l = d(p, q)$, $\alpha = \sphericalangle_p(q, z)$, and $\beta = \sphericalangle_q(p, z)$. The lengths of the infinite sides are not defined. However, we can measure their difference. We define $d = F(q) - F(p)$, where F is a Busemann function at z. This (oriented) difference is independent of the choice of F. Now we reformulate Lemmas 4.2 and 4.3 as comparison theorems for triangles with one vertex at infinity.

Proposition 4.4. *Given that Δ is as described above. In the spaces H_a and H_b there exist unique triangles Δ_a and Δ_b (up to isometries) with $l = l_a = l_b$ and $\alpha = \alpha_a = \alpha_b$. For these triangles we have*

$$d_a \leq d \leq d_b \quad \textit{and} \quad \beta_a \leq \beta \leq \beta_b \ .$$

Proof. Existence and uniqueness of Δ_a and Δ_b are obvious.

Let γ be the geodesic ray with $\gamma(0) = p$ and $\gamma(l) = q$, and assume that F is normalized such that $f(0) = 0$, where again $f = F \circ \gamma$. Then $f(l) = d$, $f'(l) = \cos \beta$, and Lemma 4.2 applies.

Proposition 4.5. *Given that Δ is as above. In H_a and H_b there exist unique triangles Δ_a and Δ_b (up to isometries) with $l = l_a = l_b$ and $d = d_a = d_b$, and for these triangles we have*

$$\alpha_b \leq \alpha \leq \alpha_a \quad \textit{and} \quad \beta_b \leq \beta \leq \beta_a \ .$$

Proof. With the same notation as above, Lemma 4.3 implies

$$f_b'(0) \leq f'(0) \leq f_a'(0) \quad \text{and} \quad f_b'(l) \geq f'(l) \geq f_a'(l) \ .$$

These give the estimates for α and β, since $f'(0) = -\cos \alpha$ and $f'(l) = \cos \beta$.

Distances on horospheres. Our next aim is to compare the $\mathscr{H}$-distance $h(p, q)$ of two points p, q on a given horosphere $\mathscr{H}$ with their usual distance $d(p, q)$. If moreover p and q lie on a different horosphere $\mathscr{H}'$, then their $\mathscr{H}'$-distance $h'(p, q)$ may be different from $h(p, q)$. However, the following theorem gives estimates independent of the chosen horosphere.

Theorem 4.6. *Assume $p, q \in \mathscr{H}$ and denote their $\mathscr{H}$-distance by $h(p, q)$. Then*

$$\frac{2}{a} \sinh \frac{a}{2} d(p, q) \leq h(p, q) \leq \frac{2}{b} \sinh \frac{b}{2} d(p, q) \ .$$

Proof. First we prove the inequality on the left hand side. We choose in H_a two points p_a and q_a lying on a horosphere $\mathscr{H}_a$, such that their $\mathscr{H}_a$-distance $h_a(p_a, q_a)$ equals $h(p, q)$. Let $\gamma_a \colon [0, 1] \to H_a$ be the geodesic from p_a to q_a, and $\mu_a \colon [0, 1] \to \mathscr{H}_a$ the projection of γ_a onto $\mathscr{H}_a$ along the geodesics orthogonal to $\mathscr{H}_a$. Then $l(\mu_a) = h_a(p_a, q_a)$ and $\gamma_a(s) = \psi^a(-f_a(s), \mu_a(s))$, where ψ^a denotes the radial flow associated with $\mathscr{H}_a$, F_a is the Busemann function vanishing on $\mathscr{H}_a$, and $f_a = F_a \circ \gamma_a$.

Now let $\mu \colon [0, 1] \to \mathscr{H}$ be a minimal $\mathscr{H}$-geodesic from p to q satisfying $\|\dot\mu\| = \|\dot\mu_a\|$, and define the curve $\gamma \colon [0, 1] \to M$ from p to q by $\gamma(s) = \psi(-f_a(s), \mu(s))$, where ψ denotes the radial flow associated with $\mathscr{H}$. Since

$$\dot\gamma_a(s) = -f_a'(s) \operatorname{grad} F_a + \psi^a_{t*}\dot\mu_a(s) \ , \qquad \dot\gamma(s) = -f_a'(s) \operatorname{grad} F + \psi_{t*}\dot\mu(s) \ ,$$

where $t = -f_a(s) \geq 0$ and F denotes the Busemann function vanishing on $\mathscr{H}$, Proposition 3.2 implies $\|\dot\gamma(s)\| \leq \|\dot\gamma_a(s)\|$, and hence $l(\gamma) \leq l(\gamma_a)$. Now we have $d(p, q) \leq l(\gamma) \leq l(\gamma_a) = d(p_a, q_a)$. A computation in hyperbolic geometry shows (cf. [8, p. 106]) : $h_a(p_a, q_a) = (2/a) \sinh (a/2) d(p_a, q_a)$. Therefore we get

$$\frac{2}{a}\sinh\frac{a}{2}d(p,q) \leq \frac{2}{a}\sinh\frac{a}{2}d(p_a,q_a) = h_a(p_a,q_a) = h(p,q)\ .$$

In order to prove the other inequality we start with the geodesic $\gamma\colon [0,1] \to M$ from p to q and its projection $\mu\colon [0,1] \to \mathscr{H}$. Then $h(p,q) \leq l(\mu)$, and (with the same notation as above) $\gamma(s) = \psi(-f(s), \mu(s))$. Now we choose two points p_b and q_b in H_b lying on a horosphere $\mathscr{H}_b$ such that $h_b(p_b,q_b) = l(\mu)$. Let $\mu_b\colon [0,1] \to \mathscr{H}_b$ be the $\mathscr{H}_b$-geodesic from p_b to q_b, and consider the curve $\gamma_b\colon [0,1] \to H_b$ from p_b to q_b defined by $\gamma_b(s) = \psi^b(-f(s), \mu_b(s))$. As before, Proposition 3.2 implies $d(p_b,q_b) \leq l(\gamma_b) \leq l(\gamma) = d(p,q)$, and therefore

$$h(p,q) \leq l(\mu) = h_b(p_b,q_b) = \frac{2}{b}\sinh\frac{b}{2}d(p_b,q_b) \leq \frac{2}{b}\sinh\frac{b}{2}d(p,q)\ .$$

Projection of a geodesic onto a horosphere. Let $\mathscr{H}$ be a horosphere with center z, and F the Busemann function vanishing on $\mathscr{H}$. Then the projection $\eta\colon M \to \mathscr{H}$ along the geodesics going to z is defined by $\eta = \psi\circ(F \times 1_M)$, where ψ is the radial flow in the direction of z. Now given a geodesic γ, we estimate the length of its projection curve $\mu = \eta\circ\gamma$ and the $\mathscr{H}$-distance between its endpoints.

Proposition 4.7. *Let $\mathscr{H}$ be a horosphere and γ a geodesic starting on $\mathscr{H}$. Assume $\beta \leq \pi/2$, where β denotes the angle between $\dot\gamma(0)$ and $\operatorname{grad}_{\gamma(0)} F$. Denote by $l(s)$ the length of $\mu|[0,s]$, and by $h(s)$ the $\mathscr{H}$-distance between $\mu(0) = \gamma(0)$ and $\mu(s)$. Then for $s \geq 0$*

$$\frac{1}{b}\left(\frac{\sin\beta}{\coth bs + \cos\beta}\right) \leq h(s) \leq l(s) \leq \frac{1}{a}\left(\frac{\sin\beta}{\coth as + \cos\beta}\right)\ .$$

Proof. First we prove the inequality on the right hand side. Consider the same data as above in H_a and fix $s > 0$. Lemma 4.2 implies $f(s) \geq f_a(s)$ and $\beta(s) \leq \beta_a(s)$, where f denotes the restriction $F\circ\gamma$, and $\beta(s)$ the angle between $\dot\gamma(s)$ and $\operatorname{grad}_{\gamma(s)} F$. Now we compute $\|\dot\mu(s)\|$. We decompose $\dot\gamma(s)$ into $\cos\beta(s) \cdot\operatorname{grad}_{\gamma(s)} F$ and an orthogonal part $\dot\gamma^\perp(s)$ of length $\sin\beta(s)$. Then $\dot\mu(s) = \eta_*(\dot\gamma(s)) = \eta_*(\dot\gamma^\perp(s)) = \psi_{t*}(\dot\gamma^\perp(s))$ for $t = f(s)$. Therefore Proposition 3.2 implies $\|\dot\mu(s)\| \leq \sin\beta(s)\cdot e^{-af(s)}$. Similarly we get $\|\dot\mu_a(s)\| = \sin\beta_a(s)\cdot e^{-af_a(s)}$, which together with $f(s) \geq f_a(s)$ and $\beta(s) \leq \beta_a(s)$ yields $\|\dot\mu(s)\| \leq \|\dot\mu_a(s)\|$ and, by integration, $h(s) \leq h_a(s)$, where $h_a(s)$ is the corresponding function for H_a, as usual. Now an easy computation in H_a gives $h_a(s) = a^{-1}\sin\beta\,(\coth as + \cos\beta)^{-1}$.

Next we prove the inequality on the left hand side. Consider the same data as above in H_b, and assume $h(s) < h_b(s)$ for a certain $s > 0$. By Lemma 4.2 we have $f(s) \leq f_b(s)$. In H_b there is a unique point q_b with $F_b(q_b) = f_b(s)$ and $\eta_b(q_b) = \mu_b(h(s))$. Denote by γ_b' the geodesic segment from $\mu_b(0)$ to q_b, and by s_b' its length. The assumption $h(s) < h_b(s)$ implies $s_b' < s$. Now consider the curve γ' in M lying over an $\mathscr{H}$-geodesic from $\mu(0)$ to $\mu(s)$ with $F\circ\gamma' =$

$F_b \circ \gamma'_b$, and denote its length by s'. Proposition 3.2 implies (as in the proof of Theorem 4.6) $s' \leq s'_b$. By its construction the curve γ' joins $\mu(0)$ to a point q with the properties $F(q) = f_b(s)$ and $\eta(q) = \mu(s)$. Since $f(s) \leq f_b(s)$, the convexity of the distance function $d(\mu(0), \cdot)$ implies $s \leq d(\mu(0), q) \leq s'$, which contradicts $s' \leq s'_b < s$. Hence $h(s) \geq h_b(s)$.

From now on we assume $a > 0$, i.e., the curvature of M is bounded away from zero. In this case the point $\mu(\infty)$ is defined to be the intersection with $\mathscr{H}$ of the unique geodesic from $\gamma(\infty)$ to z. Denote by l the length of μ, and by h the $\mathscr{H}$-distance between $\mu(0)$ and $\mu(\infty)$.

Corollary 4.8. *Assume $\beta \leq \pi/2$ as before. Then*

$$\frac{1}{b}\,\frac{\sin\beta}{1+\cos\beta} \leq h \leq l \leq \frac{1}{a}\,\frac{\sin\beta}{1+\cos\beta}\,.$$

Finally we look at the special case, where γ is tangent to $\mathscr{H}$. Then the entire geodesic γ lies outside of $\mathscr{H}$, and we may apply the above results to the whole projection $\mu = \eta \circ \gamma$.

Theorem 4.9. *Let γ be a geodesic tangent to a horosphere $\mathscr{H}$. Denote by $\bar{l}$ the length of the projection curve μ, and by $\bar{h}$ the $\mathscr{H}$-distance between $\mu(-\infty)$ and $\mu(+\infty)$. Then*

$$\frac{2}{b} \leq \bar{h} \leq \bar{l} \leq \frac{2}{a}\,.$$

Proof. The inequalities $2/b \leq \bar{l} \leq 2/a$ follow from Corollary 4.8, by observing $\beta = 0$. Here we prove $\bar{h} \geq 2/b$.

For $s > 0$ there are points $p = \gamma(-s_-)$ and $q = \gamma(s_+)$, such that $s_- + s_+ = 2s$ and $f(-s_-) = f(s_+)$. Consider the same situation in H_b, and look at the points $p_b = \gamma_b(-s)$ and $q_b = \gamma_b(s)$. Then $f_b(-s) = f_b(s)$, and Lemma 4.2 implies $f(s_+) \leq f_b(s)$ (provided $s_+ \leq s_-$; otherwise, we have $f(-s_-) \leq f_b(-s)$). Denote by $\bar{h}(s)$ the $\mathscr{H}$-distance between $\eta(p)$ and $\eta(q)$, and assume $\bar{h}(s) < 2h_b(s)$.

Now choose points p'_b and q'_b on the horosphere $\mathscr{H}'_b = F_b^{-1}(f_b(s))$, such that the $\mathscr{H}_b$-distance between $\eta_b(p'_b)$ and $\eta_b(q'_b)$ is $\bar{h}(s)$, and join p'_b and q'_b by the geodesic segment γ'_b. The assumption $\bar{h}(s) < 2h_b(s)$ implies that the $\mathscr{H}'_b$-distance between p'_b and q'_b is strictly smaller than the $\mathscr{H}'_b$-distance between p_b and q_b. Therefore $l(\gamma'_b) = d(p'_b, q'_b) < d(p_b, q_b) = 2s$. Now define a curve γ' over an $\mathscr{H}$-geodesic from $\eta(p)$ to $\eta(q)$ by $F \circ \gamma' = F_b \circ \gamma'_b$. As in the proof of the previous proposition we have $l(\gamma') \leq l(\gamma'_b)$. Denote the endpoints of γ' by p' and q'. Since $\eta(p') = \eta(p)$, $\eta(q') = \eta(q)$, and $F(p') = F(q') = f_b(s) \geq f(s_+) = F(p) = F(q)$, we get $2s = d(p, q) \leq d(p', q') \leq l(\gamma')$. This contradicts $l(\gamma') \leq l(\gamma'_b) < 2s$; hence $\bar{h}(s) \geq 2h_b(s) = 2\,(b \coth bs)^{-1}$, according to Proposition 4.7. Passing to the limit we get $\bar{h} \geq 2/b$.

As an application of Theorem 4.9 we give the following example. Let T:

$M \to M$ be a parabolic isometry (cf. [4]) with fixed point z, and $\mathscr{H}$ a horosphere with center z. Denote by h_T the $\mathscr{H}$-displacement function of T restricted to $\mathscr{H}$.

Proposition 4.10. *Assume* $a > 0$. *If* $\inf h_T \geq 2/a$ $(\sup h_T < 2/b)$, *then for all* $x \in M(\infty)$, $x \neq z$, *the geodesic* γ *from* x *to* $T(x)$ *intersects* $\mathscr{H}$ (*does not intersect* $\mathscr{H}$).

Proof. Denote by σ the geodesic from x to z, and assume that $\sigma(0)$ lies on the horosphere $\mathscr{H}_0$, which has center z and is tangent to γ. Then $T \circ \sigma$ joins $T(x)$ to $T(z) = z$, and $T \circ \sigma(0)$ lies on $\mathscr{H}_0$ (cf. [4, p. 83]). Let $\mathscr{H}_t$ be the horosphere parallel to $\mathscr{H}_0$ and containing the points $\sigma(t)$ and $T \circ \sigma(t)$, and denote by $h(t)$ the $\mathscr{H}_t$-distance between these two points. According to Theorem 4.9 we have $2/b \leq h(0) \leq 2/a$, and Proposition 4.1 implies $h(t) \leq h(0)e^{-at}$ for $t \geq 0$ and $h(t) \geq h(0)e^{-at}$ for $t \leq 0$. Now for $t > 0$ $\mathscr{H}_t$ does not meet γ and $h(t) \leq h(0)e^{-at} < h(0) \leq 2/a$, whereas for $t \leq 0$, $\mathscr{H}_t$ intersects γ and $h(t) \geq h(0)e^{-at} \geq h(0) \geq 2/b$. This completes the proof.

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