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CLOSED 2-FORMS AND AN EMBEDDING THEOREM FOR SYMPLECTIC MANIFOLDS

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The existence of universal connections was shown by Narasimhan and Ramanan [5], and Kostant [3] showed that any integral closed 2-form is the curvature form of a connection on some circle bundle. These results can be combined to show the existence of a universal closed 2-form with integral periods. In this paper we will use the symplectic structure of a complex projective space to give an elementary proof of this result; the precise statement is given in Theorem A. The result of Kostant is in fact a corollary of the existence of a universal closed 2-form, as is indicated below. Another immediate corollary of Theorem A is the result of Gromov [3] that closed symplectic manifolds can be symplectically immersed in CP^n, for large enough n; see Theorem B.

First we indicate why the proof which we are going to give here is a simple and natural generalization of an elementary fact about exact 2-forms. Consider the standard symplectic form $\Omega = \sum_{i=1}^n dx_i dy_i$ on R^{2n}. Any exact 2-form on a manifold M can be induced from Ω by a mapping to R^{2n} for some n, since any exact 2-form on M can be written in the form $\sum_{i=1}^k df_i \wedge dg_i$, where f_i, g_i are real valued functions on M. CP^n has a symplectic structure Ω_0 which is locally given by $\Omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. Furthermore, CP^n is the $2n$-skeleton of an Eilenberg-MacLane space of type $K(Z, 2)$. It is thus natural to expect that any closed 2-form with integral periods can be induced from Ω_0 by a map to CP^n, because there is some map to CP^n, for large n, which pulls back Ω_0 to within an exact 2-form of the given closed 2-form. The only complication that is met in CP^n to adjusting the map to account for the exact 2-form is that, unlike in R^{2n}, the symplectic charts on CP^n have finite radius, so the f_i, g_i's utilized would have to be bounded. The proof we give of Theorem A depends only on estimating the bounds on f_i, g_i as n becomes large.

A closed k-form on a manifold M will be said to be integral if its de Rham cohomology class is in the image of the canonical coefficient map $H^k(M; Z) \to H^k(M; R)$.

Complex projective space CP^n has a Kählerian structure, and we will denote its Kähler form by Ω_0^n. The 2-form Ω_0^n can be chosen to represent a generator in the image of $H^2(CP^n; Z) \to H^2(CP^n; R)$, and we can assume that $i^*(\Omega_0^{n+k}) = \Omega_0^n$ where i is the standard inclusion of CP^n in CP^{n+k}.

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Theorem A. *Let M be a closed manifold, and Ω an integral closed 2-form on M. Then there exists a map $f: M \to CP^n$, for n sufficiently large, such that $f^*(\Omega_0^n) = \Omega$.*

Since Ω_0^n is the curvature form of a connection on the canonical S^1 bundle over CP^n, a map to CP^n which induces a closed 2-form also induces an S^1 bundle. Hence we obtain

Theorem (*Kostant* [3]). *Every integral closed 2-form is the curvature form of a connection on an S^1 bundle.*

Definition. Let (M, Ω') and (N, Ω) denote two manifolds M, N with symplectic forms Ω', Ω respectively. A map $f: M \to N$ will be called a symplectic map from (M, Ω') to (N, Ω) if $f^*(\Omega) = \Omega'$.

Definition. Given a manifold M and a symplectic structure (N, Ω), a map $f: M \to N$ such that $f^*(\Omega)$ is a symplectic form on M will be said to be transverse to the symplectic form Ω.

Any submanifold M of CP^n such that the inclusion $i: M \to CP^n$ is transverse to Ω_0^n will support a symplectic structure, namely $i^*(\Omega_0^n)$, which is an integral closed 2-form. The converse is also true and resembles Kodaira's embedding theorem, but with Kählerian weakened to symplectic.

Suppose (M, Ω) is a symplectic structure. If Ω is an integral closed 2-form, then by Theorem A there is a map $f: M \to CP^n$ such that $f^*(\Omega_0^n) = \Omega$. Since Ω is a nondegenerate 2-forms f is automatically an immersion. This yields the result:

Theorem B (*Gromov* [2]). *If Ω is a symplectic structure on M, and Ω is an integral closed 2-form, then there exists a symplectic immersion of M into CP^n for sufficiently large n.*

Remark. This result can be improved to yield symplectic embeddings in the following way. Assume n is large enough so that the immersions can be approximated arbitrarily closely by embeddings. Choose an embedding $g: M \to CP^n$ so that $g^*(\Omega_0^n)$ is close to Ω. By Moser's theorem on the stability of symplectic forms [4], we conclude that there is a diffeomorphism F of M to itself such that $F^*(g^*(\Omega_0^n)) = \Omega$. Hence $g \circ F: M \to CP^n$ is the required symplectic embedding.

Corollary. *Given a symplectic structure (M, Ω), there is, for large enough n, an embedding $f; M \to CP^n$ transverse to Ω_0^n, such that $f^*(\Omega_0^n)$ can be made arbitrarily close to Ω in the following sense: given a norm $\| \quad \|$ on closed 2-forms and an $\varepsilon > 0$, there are a real number k and an embedding f such that $\|k \cdot f^*(\Omega_0^n) - \Omega\| < \varepsilon$.*

Proof. Choose a collection of integral closed 2-forms α_i, $1 \leq i \leq d$, which define a basis for $H^2(M; R)$. Any symplectic form Ω can be written as $\Omega = \sum_{i=1}^d r_i \alpha_i + d\omega$ for some 1-form ω and real numbers r_i. Choose rational numbers q_i such that $\Omega' = \sum_{i=1}^d q_i \alpha_i + d\omega$ satisfies $\|\Omega - \Omega'\| < \varepsilon$. There is an integer D such that $D\Omega'$ is an integral 2-form. By Theorem B, $D\Omega' = f^*(\Omega_0^n)$ for some embedding $f: M \to CP^n$. The corollary follows by setting $k = 1/D$.

Before beginning the proof of Theorem A, we need to establish several notations. C^n will denote n-dimensional complex space, $\langle\ ,\ \rangle$ the usual Hermitian inner product on C^n, and $|\ \ |$ the corresponding norm.

We will consider CP^n as the complex lines in C^{n+1} passing through the origin, and also as the quotient space of the unit sphere S^{2n+1} in C^{n+1} by the action of the complex numbers of norm equal to 1.

Given two points p_1, p_2 in CP^n we denote by $\alpha(p_1, p_2)$ the angle between them viewed as real two-dimensional planes in C^{n+1}, $(\cos\alpha = |\langle p_1, p_2\rangle|/(|p_1|\cdot|p_2|)$ where we are now considering p_1, p_2 as points in C^{n+1}).

For each p in CP^n, we make a choice of x in S^{2n+1} which represents p. Where it creates no confusion we will speak of x in CP^n, and where necessary we will denote the class of x in CP^n by $[x]$.

For each p in CP^n the above choice of x allows us to choose a complex hyperplane T_x in C^{n+1} which passes through x and is orthogonal to x with respect to the Hermitian metric. T_x can be identified with the tangent space to CP^n at $[x]$. Let D_x be the subset of CP^n consisting of those complex lines in C^{n+1} which intersect T_x. The mapping from D_x to T_x given by sending a point in D_x to its point of intersection with T_x will be denoted by $S(x)$. For $\varepsilon > 0$, $T_x(\varepsilon)$ will denote all points y in T_x such that $|y - x| < \varepsilon$, and $S^{-1}(x)(T_x(\varepsilon))$ will be denoted by $V(x, \varepsilon)$.

Let $z = (z_0, \cdots, z_n)$ be complex coordinates on C^{n+1}. We can think of C^n as all points z in C^{n+1} with $z_0 = 1$. Let $B^n(r)$ denote all points $(z_1, \cdots, z_n)$ in C^n such that $\sum_{i=1}^n z_i\bar{z}_i < r^2$.

One can identify T_x with C^n by choosing some unitary transformation of C^{n+1} which sends x to $(1, 0, \cdots, 0)$ in C^{n+1}. Composing this map with the mapping $(z_1, \cdots, z_n) \to (1 + \sum_{i=1}^n z_i\bar{z}_i)^{-1/2}\cdot(z_1, \cdots, z_n)$ yields a diffeomorphism $H\colon T_x \to B^n(1)$. Consider the closed 2-form $\sum_{i=1}^n dx_i \wedge dy_i$ on $B^n(1)$ where $z_i = x_i + \sqrt{-1}y_i$. One can show that the Kähler form Ω_0^n on D_x satisfies $\Omega_0^n = S^*(x)\circ H^*(x)(\pi^{-1}\sum_{i=1}^n dx_i \wedge dy_i)$, by using the fact that $\Omega_0^n = (i/2\pi)\partial\bar{\partial}\log(1 + \sum_{i=1}^n z_i\bar{z}_i)$ on the hyperplane $z_0 = 1$ viewed as a holomorphic cross-section of the canonical line bundle over CP^n; see Chern [1] for details of the Kähler structures of CP^n. One can think of $H(x)\circ S(x)\colon D_x \to B^n(1)$ as a symplectic chart for CP^n.

There is a natural inclusion $\bar{i}\colon CP^n \to CP^{n+1}$ given by the inclusion $i\colon C^{n+1} \to C^{n+2}$ defined by identifying C^{n+1} as the first $n + 1$ coordinates of C^{n+2}. The choices made above can be made compatible with the inclusion of CP^n in CP^{n+1} in the following sense. For a point $[x]$ in CP^n we can choose $T_x, D_x, S(x), H(x)$ as above. We can also let $i(x) \in C^{n+2}$ represent $\bar{i}[x]$, and we have $T_x = T_{i(x)} \cap C^{n+1}$, and $S(i(x))\circ\bar{i} = i\circ S(x)\colon D_x \to T_{i(x)}$. One can also choose $H(i(x))$ so that $H(i(x))\circ i = i\circ H(x)\colon T_x \to B^{n+1}(1)$. With these choices,

$$\frac{1}{\pi}\sum_{i=1}^{n+1} dx_i \wedge dy_i = ((H(i(x))\circ S(i(x)))^{-1})^*(\Omega_0^{n+1})$$

on B^{n+1}, and also

$$\frac{1}{\pi}\sum_{i=1}^{n} dx_i \wedge dy_i = \pi_1^*((H(x)\circ S(x))^{-1})^*(\Omega_0^n) ,$$

where π_1 is the projection of $B^{n+1}(1)$ onto $B^n(1)$ defined by the projection of C^{n+1} onto the first n coordinates.

Proof of Theorem A. The function f will be constructed in stages; the jth stage will be denoted f_j, where $0 \le j \le p$ for some p to be chosen later. Choose $f_0 \colon M \to CP^n$ for n sufficiently large, so that $f_0^*(\Omega_0^n)$ and Ω are cohomologous. This can be done since CP^n can be taken to be the $2n$-skeleton of an Eilenberg-MacLane space of type $K(Z, 2)$. Hence $\Omega = f_0^*(\Omega_0^n) + d\omega$ for some 1-form ω on M.

We need a couple of lemmas before we can construct the f_j's.

Lemma 1. *Given $R > \varepsilon > 0$, there exists a $\delta > 0$ such that*

$$V(x, \varepsilon, \delta) = \{y \in CP^n \mid \alpha(y, x') < \delta \text{ for some } x' \in V(x, \varepsilon)\} \subset S^{-1}(x)(T_x(R)) .$$

Furthermore, δ can be chosen independently of n.

Proof of Lemma 1. The lemma follows easily from the facts that $T_x(\varepsilon) \subset T_x(R)$ and that, for $0 \le \theta \le \frac{1}{2}\pi$,

$$\{y \in D_x \mid \alpha(x, y) < \theta\} = S^{-1}\{z \in T_x \mid \cos\theta < |z|^{-1}\} .$$

From now on we fix a choice of ε, R, δ satisfying Lemma 1. We also choose a $\rho > 0$ such that $1 - \rho > \cos^2\delta$.

Lemma 2. *Given a 1-form ω on a closed manifold M, a finite open cover $\{W_i\}$ of M, an $R > 0$, and a ρ such that $1 > \rho > 0$, there exist real valued functions h_k, t_k, $1 \le k \le p$ such that*

(1) $\sum_{k=1}^{p} dh_k \wedge dt_k = d\omega$,

(2) *each pair (h_k, t_k) has support contained in some element of the cover $\{W_i\}$,*

(3) $\prod_{k=1}^{p} (1 + K^2(h_k^2 + t_k^2)) < 1/(1 - \rho)$, *where $K^2 = 1 + R^2$,*

(4) $h_k^2 + t_k^2 + R^2/(1 + R^2) < 1$.

Proof of Lemma 2. There exists some choice of functions $\bar{h}_k, \bar{t}_k$, $1 \le k \le \bar{p}$, such that $\sum_{k=1}^{\bar{p}} d\bar{h}_k \wedge d\bar{t}_k = d\omega$. This can be seen by choosing a partition of unity $\{\varphi_k\}$ subordinate to some finite coordinate cover $\{U_i\}$ of M. Then $d\omega = d(\sum \varphi_k\omega)$, and $d(\varphi_k\omega) = \sum_{i=1}^{m} d\bar{h}_k^i \wedge d\bar{t}_k^i$ for each k and some choice of $\bar{h}_k^i$, $\bar{t}_k^i$ with support in U_i, where $m =$ dimension of M. Hence (1) can be satisfied. Now choose a partition of unity $\{\Psi_i\}$, $0 \le i \le c$, subordinate to $\{W_i\}$. Then

$$\sum_{k=1}^{\bar{p}} d\bar{h}_k \wedge d\bar{t}_k = \sum_{k=1}^{\bar{p}}\sum_{j=1}^{c}\sum_{i=1}^{c} d(\Psi_i\bar{h}_k) \wedge d(\Psi_j\bar{t}_k) ,$$

and (2) can also be satisfied by taking the $\Psi_i\bar{h}_k$ as the h_k's and the $\Psi_j\bar{t}_k$ as the

t_k's. By replacing h_k and t_k by N copies of h_k/N and t_k/N respectively, and using the fact that $\lim_{n\to\infty}(1+n^{-2})^n = 1$, we see that we can choose the h_k's and t_k's to satisfy condition (3). By a similar argument, the h_k's and t_k's can be chosen small enough so that condition (4) is satisfied as well, and the proof of the lemma is complete.

M has an open cover given by $\{f_0^{-1}(V(x,\varepsilon))\}$, $[x]\in CP^n$. Fix a finite subcover $\{W_i\}$ of this cover. Fix a choice of $\{h_k, t_k\}$, $1\le k\le p$, satisfying Lemma 2 applied to our fixed choices of $\varepsilon, R, \delta, \rho, \{W_i\}$, and such that

$$\frac{1}{\pi}\sum_{i=1}^{p}(dh_k\wedge dt_k) = d\omega \qquad \text{where } d\omega = \Omega - f_0^*(\Omega_0^n)\ .$$

For each k, $1\le k\le p$, we choose a W_k in the cover $\{W_i\}$, such that the support of h_k and t_k are contained in W_k. Recall that $W_k = f_0^{-1}(V(x_k,\varepsilon))$ for some $x_k\in C^{n+1}$.

For each j, $1\le j\le p$, let us assume the two induction hypotheses:

(i) There is a map $f_{j-1}\colon M\to CP^{n+j-1}$ such that

$$f_{j-1}^*(\Omega_0^{n+j-1}) = f_0^*(\Omega_0^n) + \frac{1}{\pi}\sum_{k=1}^{j-1}(dh_k)\wedge(dt_k)\ .$$

(ii) $f_i(W_j)\subset V(x_j,R)$, for all $i\le j-1$.

If we show that (i) is true for f_p, we will be done since

$$f_p^*(\Omega_0^{n+p}) = f_0^*(\Omega_0^n) + \frac{1}{\pi}\sum_{i=1}^{p}(dh_k)\wedge(dt_k) = f_0^*(\Omega_0^n) + d\omega = \Omega\ .$$

We already have (i) and (ii) satisfied for $j=1$; (i) is true vacuously and (ii) follows from the fact that $V(x_j,\varepsilon)\subset V(x_j,R)$. Hence it suffices to show that given f_{j-1} satisfying (i) and (ii) there is an f_j satisfying (i) and (ii). Define f_j as follows:

(a) On $M-W_j$, set $f_j=\bar{i}\circ f_{j-1}$ where $\bar{i}\colon CP^{n+j-1}\to CP^{n+j}$ is the inclusion.

(b) On W_j, we define first a map $g_j\colon W_j\to B^{n+j}(1)$ given by $\pi_1 g_j = H(x_j)\circ S(x_j)\circ f_{j-1}$ with values in $B^{n+j-1}(1)$, and by $\pi_2 g_j = h_j + \sqrt{-1}\,t_j$ with values in $B^1(1)$, where π_1,π_2 are the projections of $B^{n+j}(1)$ onto $B^{n+j-1}(1)$ and $B^1(1)$ respectively, induced by the projections of C^{n+j} onto its first $n+j-1$ coordinates and last coordinate respectively.

We can now define $f_j = S^{-1}(i(x_j))\circ H^{-1}(i(x_j))\circ g_j$, (we are taking the choices of $H(x)$, $H(i(x))$, to be compatible in the sense described just before the beginning of the proof of Theorem A).

By property (4) of Lemma 2 we have that $|(\pi_2 g_j)|^2 < (1-R^2/(1+R^2))$ in $B^1(1)$. By induction hypothesis (ii) applied to f_{j-1} and by the fact that $H(x_j)(T_{x_j}(R))\subset B^{n+j-1}R(1+R^2)^{-1/2}$ we have that $|\pi_1(g_j)|^2 < R^2/(1+R^2)$ in $B^{n+j-1}(1)$. Hence we can conclude that $g_j\colon W_j\to B^{n+j}(1)$ is well defined, and consequently that f_j is well defined on W_j. By Lemma 2, part (2), we can conclude that f_j is well defined on all of M. On W_j

$$\begin{aligned} f_j^*(\Omega_0^{n+j}) &= g_j^*((H(i(x)\circ S(I(x)))^{-1})^*(\Omega_0^{n+1}) = g_j^*\Big(\frac{1}{\pi}\sum_{i=1}^{n+j} dx_i\wedge dy_i\Big) \\ &= (\pi_1 g_j)^*\Big(\frac{1}{\pi}\sum_{i=1}^{n+j} dx_i\wedge dy_i\Big) + (\pi_2 g_j)^*\Big(\frac{1}{\pi}\sum_{i=1}^{n+j} dx_i\wedge dy_i\Big) \\ &= (H(x_j)\circ S(x_j)\circ f_{j-1})^*\Big(\frac{1}{\pi}\sum_{i=1}^{n+j-1} dx_i\wedge dy_i\Big) + \frac{1}{\pi}(dh_j\wedge dt_j) \\ &= f_{j-1}^*\Big(S^*(x_j)\circ H^*(x_j)\Big(\frac{1}{\pi}\sum_{i=1}^{n+j-1} dx_i\wedge dy_i\Big)\Big) + \frac{1}{\pi}(dh_j\wedge dt_j) \\ &= f_{j-1}^*(\Omega_0^{n+j-1}) + \frac{1}{\pi}(dh_j\wedge dt_j)\ . \end{aligned}$$

This equality follows from the compatibility conditions on $H(x_j)$ and $H(i(x_j))$ discussed just before the beginning of the proof of Theorem A. Hence we have shown that induction hypothesis (i) is satisfied for f_j. Therefore we will be done if we can show that $f_j(W_k) \subset V(x_k, R)$ for all $k > j$. For any $x \in W_k$ and $0 \leq i \leq j$, set $A_i = S(x_{i+1})(f_i(x))$ and $B_i = S(x_{i+1})(f_{i+1}(x))$. We consider the A_i, B_i as all contained in C^{n+j}, (note that A_i is a scalar multiple of B_{i-1}). We now add another induction hypothesis for each j, $1 \leq j \leq p$,

(iii) $\langle B_i - A_i, A_{i'}\rangle = 0$ for all $i' \leq i \leq j-1$.

If hypothesis (iii) is true for $j-1$, it is seen to hold for j, since $B_j - A_j$ is perpendicular to C^{n+j} in C^{n+j+1}, using the construction of f_j as above, and by the compatibility conditions given before the proof of Theorem A. (Hypothesis (iii) is vacuously satisfied for f_0.)

Given A_i, B_i as above and our fixed ρ, we will show that $\cos^2\alpha_{j-1} > 1-\rho$, where $\alpha_i = \alpha([A_0], [B_i])$. We have

$$\cos^2\alpha_i = \left(\frac{|\langle A_0, B_i\rangle|}{|A_0|\cdot|B_i|}\right)^2 = \left(\frac{|\langle A_0, A_i\rangle|}{|A_0|\cdot|B_i|}\right)^2$$

by induction hypothesis (iii), and this expression is equal to $(\cos^2\alpha_{i-1})|A_i|^2/|B_i|^2$. Since $|B_i|^2 = |A_i|^2 + |B_i - A_i|^2$ and $|A_i| \geq 1$, we have that $|A_i|^2/|B_i|^2 \geq 1/(1+|B_i - A_i|^2)$. However $|B_i - A_i|^2 \leq K^2(h_k{}^2 + t_k{}^2)$ with $K^2 = 1 + R^2$, by the construction of f_{i+1}, the definition of the map $H(x_{i+1})$, and the fact that B_i and A_i are in $T_{x_{i+1}}(R)$. Hence we have $\cos^2\alpha_i \geq \cos^2\alpha_{i-1}\cdot(1+K^2(h_k{}^2+t_k{}^2))^{-1}$, and so

$$\cos^2\alpha_{j-1} \geq \prod_{k=1}^{j-1}(1 + K^2(h_k{}^2 + t_k{}^2)^{-1})\ ,$$

which is greater than $1-\rho$ by part (3) of Lemma 2. Since we chose ρ such that $1 - \rho > \cos^2 \delta$, we have $\alpha_i < \delta$. Since A_0 is contained in $V(x_k, \varepsilon)$, we get that B_{j-1} is contained in $V(x_k, \varepsilon, \delta)$ which is contained in $V(x_k, R)$ by Lemma 1. Hence $f_j(x)$ is contained in $V(x_k, R)$ for all x in W_k. This shows that f_j satisfies induction hypothesis (ii), and the proof of Theorem A is complete.

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