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PROLONGATION OF CONNECTIONS TO BUNDLES OF INFINITELY NEAR POINTS

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Introduction

The purpose of this paper is to generalize the results of [5] to the bundles of infinitely near points of A-kinds in the sense of A. Weil [7], which generalizes the notions of p^r-jets in the sense of C. Ehresmann [1], [2]. Our results naturally generalizes the results of several authors, e.g., [4], [8], [10]. In fact, we have treated the same problem in the author's lecture notes (cf. [6, Part V]). However, in [6] we fully used the basis and structure constants of the local algebra A, and were obliged to consider (λ)-lifts of vector fields, 1-forms or tensor fields of type (p, q) with $p = 0$ or 1, where $\lambda = 0, 1, 2, \cdots, N$ and $N + 1 = \dim A$. Moreover, the geometric meaning of (λ)-lifts for $\lambda = 1, 2, \cdots, N$ are not so clear as that of (0)-lifts. In this paper, we shall essentially not use the basis and structure constants of the algebra A, and shall show that there exists essentially only one lift, which has a significant geometric meaning, and other (λ)-lifts can be derived naturally from that lift. Further, the proofs in [6] are much simplified, and some of results are somewhat sharpened (cf. [6, Theorem 6.6]).

In § 1, we explain the notion of local algebras and the infinitely near points of A-kind which will be simply called A-points. The covariant functor, which assigns to each manifold M its bundle M^A of infinitely near points, has many nice properties similar to the functor which assigns to M its tangent bundle $T(M)$. In particular, if G is a Lie group (acting on a manifold M), then G^A is also a Lie group (acting on M^A).

In § 2, by means of two different methods we define two A-module structures on the tangent space to M^A at each point of M^A, and we shall in fact show that these two A-module structures are essentially the same.

In § 3, we shall define the lift of vector fields and establish some relations between the lift of functions and the bracket of vector fields.

In § 4, § 5, we shall consider the lifting of covariant tensor fields and (1, 1)-tensor fields respectively. We shall prove that the lifting J^A of an almost complex structure J is integrable if and only if J is integrable.

In § 6, we shall first construct the prolongation of affine connections (Theorem 6.1), and next show that the prolonged affine connection ∇^A is locally

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affine symmetric if and only if ∇ is.

In § 7, we shall give a proof for the fact that if M is an affine symmetric space then M^A is also so. In such a manner, we obtain a method to construct a large number of affine symmetric spaces (resp. complex manifolds) from a given affine symmetric space (resp. complex manifold), (cf. [10, Introduction]).

In this paper, all manifolds and mappings (functions) are assumed to be differentiable of class C^∞, unless otherwise stated.

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1. Infinitely near points of A-kind

In this section we shall recall the notion of local algebras and infinitely near points of A-kind in the sense of A. Weil [7].

Definition 1.1. Let A be an associative algebra over the field $\boldsymbol{R}$ of real numbers with a unit (denoted by 1). We call A a *local algebra* if A is commutative and of finite dimension over $\boldsymbol{R}$, admitting a unique maximal ideal $\mathfrak{m}$ such that $A/\mathfrak{m}$ is of dimension 1 over $\boldsymbol{R}$ and that $\mathfrak{m}^{h+1} = (0)$ for a nonnegative integer h. The smallest h such that $\mathfrak{m}^{h+1} = (0)$ will be called the height of A. We shall identify the field $\boldsymbol{R}$ with the subspace of A consisting of all scalar multiples of the unit 1. Clearly A is the direct sum of $\boldsymbol{R}$ and $\mathfrak{m}$ as a vector space. If $a \in A$, the scalar $a_0 \in \boldsymbol{R}$, defined by $a \equiv a_0 \bmod \mathfrak{m}$, will be called the finite part of a. If $A/\mathfrak{m}$ is identified with $\boldsymbol{R}$, the map $a \to a_0$ is a homomorphism of A onto $\boldsymbol{R}$.

Let $\boldsymbol{R}[p] = \boldsymbol{R}[[X_1, \cdots, X_p]]$ be the algebra of all formal power series of p indeterminates $X_1, \cdots, X_p$, and let $\mathfrak{m}_p$ be the maximal ideal of $\boldsymbol{R}[p]$ consisting of all formal power series without constant terms. Let $\mathfrak{a}$ be an ideal of $\boldsymbol{R}[p]$ such that $\dim \boldsymbol{R}[p]/\mathfrak{a} < +\infty$. We see that $A = \boldsymbol{R}[p]/\mathfrak{a}$ is a local algebra with the maximal ideal $\mathfrak{m} = \mathfrak{m}_p/\mathfrak{a}$. Conversely, we know that every local algebra is isomorphic to such a local algebra (cf. [7, p. 112]).

Let M be a manifold of dimension n, and let $C^\infty(M)$ be the algebra of all differentiable functions on M. Take a point $x \in M$.

Definition 1.2. Let A be a local algebra with the maximal ideal $\mathfrak{m}$. An algebra homomorphism $x' \colon C^\infty(M) \to A$ will be called an *A-point* of M near to x (or *infinitely near point* to x on M of A-kind) if the finite part of $x'(f)$ is equal to $f(x)$, i.e.,

$$x'(f) \equiv f(x) \mod \mathfrak{m} \tag{1.1}$$

for every $f \in C^\infty(M)$. We denote by M_x^A the set of all A-points of M near to x and $M^A = \bigcup_{x \in M} M_x^A$, and define $\pi_A \colon M^A \to M$ by $\pi_A(M_x^A) = x$ for $x \in M$.

Remark 1.3. If $x' \in M_x^A$, and $f \in C^\infty(M)$ vanishes identically on a neighborhood of x, then we see that $x'(f) = 0$.

This remark shows that we can consider $x'(f)$ for any differentiable function f defined on a neighborhood of x if $x' \in M_x^A$.

Remark 1.4. If we take $\mathfrak{a} = (\mathfrak{m}_p)^{r-1}$, and $A = \boldsymbol{R}[p]/\mathfrak{a}$, then we see that the notion of A-points is nothing but the notion of p^r-jets (cf. [1], [2], [7]). In particular, if $\mathfrak{a} = (\mathfrak{m}_1)^2$, $D = \boldsymbol{R}[1]/\mathfrak{a}$, then the notion of D-points is nothing but the notion of tangent vectors on M. We denote by $\tau = \pi(X_1)$, where X_1 is the indeterminate in $\boldsymbol{R}[1]$ and $\pi: \boldsymbol{R}[1] \to D$ is the natural projection.

Let U be a coordinate neighborhood of x_0 in M with coordinate system $\{x_1, \cdots, x_n\}$. Take a basis $\{1 = B^0, B^1, \cdots, B^N\}$ of a local algebra A, where $B^1, \cdots, B^N$ span the maximal ideal $\mathfrak{m}$ of A. We define $x_{i,\lambda}: \pi_A^{-1}(U) \to R$ by

$$\sum_{\lambda=0}^{N} x_{i,\lambda}(x')B^\lambda = x'(x_i) , \tag{1.2}$$

for any $x' \in \pi_A^{-1}(U)$, where we have used Remark 1.3 for $f = x_i$ $(i = 1, \cdots, n)$. We see readily that the set M^A becomes a differentiable manifold of dimension $n(N+1)$ by the coordinate neighborhoods $\pi_A^{-1}(U)$ with coordinate system $\{x_{i,\lambda} | i = 1, \cdots, n; \lambda = 0, 1, \cdots, N\}$ induced by the coordinate system $\{x_1, \cdots, x_n\}$ on U. Clearly this differentiable structure on M^A does not depend on the choice of the basis $\{B^0, \cdots, B^N\}$ of A.

Definition 1.5. The differentiable manifold M^A defined above with the projection $\pi_A: M^A \to M$ will be called the *bundle of A-points* of M (or *bundle of infinitely near points* of M of A-kind).

Remark 1.6. The notion of bundle of D-kind is the same as that of tangent bundles. A tangent vector $X \in T_xM$ at x is identified with $x' \in M_x^D$ defined by $x'(f) = f(x) + (Xf)\cdot\tau$ for $f \in C^\infty(M)$.

Let $\Phi: M \to M'$ be a map of a manifold M into a manifold M'. Then the map $\Phi^A: M^A \to M'^A$ is defined by

$$(\Phi^A(x'))g = x'(g \circ \Phi) \tag{1.3}$$

for $x' \in M^A$ and $g \in C^\infty(M')$. Clearly Φ^A is differentiable.

Lemma 1.7. *Let $\pi_i: M_1 \times M_2 \to M_i$ $(i = 1, 2)$ be the projections. Then $(M_1 \times M_2)^A$ can be identified with $M_1^A \times M_2^A$ by the following identification*

$$x' = (\pi_1^A(x'), \pi_2^A(x')) \tag{1.4}$$

for $x' \in (M_1 \times M_2)^A$.

Proof. Straightforward verification.

Lemma 1.8. *Let $\Phi_1: M_1 \to M_1'$, $\Phi_2: M_2 \to M_2'$, $\Psi_1: M_1 \to M_1''$ and $\Phi_1': M_1' \to M$ be differentiable maps, $M_1, M_1', M_2, M_2', M_1'', M$ being manifolds. Then we have the following equalities:*

$$(\Phi_1' \circ \Phi_1)^A = \Phi_1'^A \circ \Phi_1^A , \qquad (\Phi_1 \times \Phi_2)^A = \Phi_1^A \times \Phi_2^A ,$$
$$(\Phi_1, \Psi_1)^A = (\Phi_1^A, \Psi_2^A) , \qquad (1_M)^A = 1_{M^A} ,$$

where 1_M *stands for the identity map of* M. *Further, if we denote by* π_i (resp. $\tilde{\pi}_i$) *the projection of* $M_1 \times M_2$ (*resp.* $M_1^A \times M_2^A$) *onto* M_i (*resp.* M_i^A) *for* $i = 1, 2$, *then we have* $\pi_i^A = \tilde{\pi}_i$ $(i = 1, 2)$.

Proof. Straightforward verification by using (1.3) and (1.4).

Lemma 1.9. $\boldsymbol{R}^A$ *can be identified with* A *by* $\boldsymbol{R}^A \ni x' \to x'(t) \in A$, *where* t *is the natural coordinates on* $\boldsymbol{R}$.

Proof. Straightforward verification (cf. [7]).

Lemma 1.10. *Let* A *and* B *be two local algebras. Then we can identify* A^B *with* $A \otimes B$ (cf. [7]).

Lemma 1.11. A, B *being as above, we can define canonically a diffeomorphism* $\psi: (M^A)^B \to M^{A\otimes B}$.

Proof. Take $x'' \in (M^A)^B$ and $f \in C^\infty(M)$. Since $f: M \to \boldsymbol{R}$ is a C^∞-map, we can consider the map $f^A: M^A \to \boldsymbol{R}^A = A$ (cf. Lemma 1.9). Hence using the map $(f^A)^B: (M^A)^B \to A^B = A \otimes B$ we can consider the map $x': C^\infty(M) \to A \otimes B$ defined by $x'(f) = (f^A)^B(x'') \in A \otimes B$, which is easily seen to be an $A \otimes B$-point on M. Thus we get a map $x'' \to x'$ from $(M^A)^B$ to $M^{A\otimes B}$, which can be verified to be a diffeomorphism (for detail, see [7]).

Corollary 1.12. A *and* B *being as above, we can identify* $x'' \in (M^A)^B$ *with* $x_1'' \in (M^B)^A$ *for elements* x'' *and* x_1'' *characterized by*

$$(f^A)^B(x'') = (f^B)^A(x_1'')$$

for every $f \in C^\infty(M)$, *where we have identified* $A \otimes B$ *with* $B \otimes A$.

Proof. Clear from the proof of Lemma 1.11.

Lemma 1.13. *Let* G *be a Lie group with group multiplication* μ. *Then* G^A *becomes a Lie group with group multiplication* $\mu^A: (G \times G)^A = G^A \times G^A \to G^A$.

Proof. Omitted (cf. [7]).

2. A-module structures on the tangent spaces of M^A

In this section, we define canonically an A-module structure on the tangent space of M^A at every point of M^A.

Let $\mu: R \times M^D \to M^D$ be the scalar multiplication of the tangent vectors of M, i.e., $\mu(t, X) = t \cdot X$ for $t \in \boldsymbol{R}$, $X \in M^D$. Since $\boldsymbol{R}^A = A$ and $(M^D)^A = (M^A)^D$ by our identification (cf. Corollary 1.12), the map $\mu^A: (\boldsymbol{R} \times M^D)^A = \boldsymbol{R}^A \times (M^D)^A \to (M^D)^A$ can be considered as the map $\mu^A: A \times (M^A)^D \to (M^A)^D$.

Definition 2.1. Put $\mu^A(a, x'') = a \cdot x''$ for $a \in A$, $x'' \in (M^A)^D$. We denote by π_D (resp. $\tilde{\pi}_D$) the projection $M^D \to M$ (resp. $(M^A)^D \to M^A$).

Lemma 2.2. *The notation being as above, we have*

(i) $\tilde{\pi}_D(a \cdot x'') = \tilde{\pi}_D(x'')$ *for every* $a \in A$ *and* $x'' \in (M^A)^D$,

(ii) *for any* $x' \in M^A$, *the tangent space* $(M^A)^D_{x'}$ *becomes an* A-*module by the multiplication* $(a, x'') \to a \cdot x''$ *for* $(a, x'') \in A \times (M^A)^D_{x'}$.

Remark 2.3. In fact, in the next section (cf. Corollary 3.10) we shall show that $(M^A)^D_{x'}$ is a free A-module for any $x' \in M^A$.

Proof of Lemma 2.2. (i) Consider the following diagram:

$$
\begin{array}{ccccccc}
\boldsymbol{R}^A \times (M^A)^D & \xrightarrow{1\times j} & \boldsymbol{R}^A \times (M^D)^A & \xrightarrow{\mu^A} & (M^D)^A & \xrightarrow{i} & (M^A)^D \\
\downarrow{\scriptstyle \tilde{\pi}_D\circ\tilde{\pi}_2} & & \downarrow{\scriptstyle (\pi_D\circ\pi_2)^A} & & \downarrow{\scriptstyle (\pi_D)^A} & & \downarrow{\scriptstyle \tilde{\pi}_D} \\
M^A & \longrightarrow & M^A & \longrightarrow & M^A & \longrightarrow & M^A
\end{array}
\tag{2.1}
$$

where $\tilde{\pi}_2: \boldsymbol{R}^A \times (M^A)^D \to (M^A)^D$ (resp. $\pi_2: \boldsymbol{R} \times M^D \to M^D$) is the projection and $j^{-1} = i: (M^D)^A \to (M^A)^D$ is the identification map (cf. Corollary 1.12). Since $\pi_D \circ \mu = \pi_D \circ \pi_2$, the middle rectangle of (2.1) is commutative. It is now sufficient to verify the commutativity of the right rectangle of (2.1), because $a\cdot x'' = (i \circ \mu^A \circ (1 \times j))(a, x'')$ for $(a, x'') \in A \times (M^A)^D$, and the commutativity of the left rectangle is implied by that of the right one.

Take $x'' \in (M^D)^A$ and put $x''_1 = i(x'') \in (M^A)^D$. Then for any $f \in C^\infty(M)$, we have

$$(f^D)^A(x'') = (f^A)^D(x''_1)\ , \tag{2.2}$$

(cf. Corollary 1.12), where we have identified: $A \otimes D = D \otimes A$. To show that $(\pi_D)^A(x'') = \tilde{\pi}_D(x''_1)$, it suffices to show

$$((\pi_D)^A(x''))(f) = (\tilde{\pi}_D(x''_1))(f) \tag{2.3}$$

for $f \in C^\infty(M^A)$. (2.3) is equivalent to

$$x''(f \circ \pi_D) = f^A(\tilde{\pi}_D(x''_1))\ . \tag{2.4}$$

Now, we know that $f^D \equiv f \circ \pi_D \pmod{\boldsymbol{R}\cdot\tau}$, where $D = \boldsymbol{R} \oplus \boldsymbol{R}\tau$. Therefore we have $(f^A)^D \equiv f^A \circ \tilde{\pi}_D \pmod{A \otimes \boldsymbol{R}\tau}$. Considering the A-components of (2.2) in $A \otimes D = A \oplus A \otimes \boldsymbol{R}\tau$, we obtain (2.4), since $(f^D)^A(x'') = x''(f^D) \equiv x''(f \circ \pi_D) \pmod{A \otimes \boldsymbol{R}\cdot\tau}$. Thus (i) is proved.

(ii) Let $\mu_0: \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ and $\mu_A: A \times A \to A$ be the multiplication in $\boldsymbol{R}$ and A respectively. We see easily that $(\mu_0)^A = \mu_A$. The equality $(t\cdot s)\cdot X = t\cdot(s\cdot X)$ for $t, s \in \boldsymbol{R}$ and $X \in M^D$ can be written as $\mu \circ (\mu_0 \circ \pi_{12}, \pi_3) = \mu \circ (\pi_1, \mu \circ \pi_{23})$, where $\pi_{12}: \boldsymbol{R} \times \boldsymbol{R} \times M \to \boldsymbol{R} \times \boldsymbol{R}$, $\pi_3: \boldsymbol{R} \times \boldsymbol{R} \times M \to M$ etc. denote the natural projection. Then by the functoriality of $\mu \to \mu^A$, etc. (cf. Lemma 1.8) it follows that $\mu^A \circ (\mu_A \circ \tilde{\pi}_{12}, \tilde{\pi}_3) = \mu^A \circ (\tilde{\pi}_1, \mu^A \circ \tilde{\pi}_{23})$, where $\tilde{\pi}_{12}: A \times A \times M^A \to A \times A$ etc. denote the natural projection similar to π_{12} etc. Thus we get the associativity: $(a\cdot b)\cdot x'' = a\cdot(b\cdot x'')$ for $a, b \in A$, $x'' \in M^A$.

The distributivity $(a + b)\cdot x'' = a\cdot x'' + b\cdot x''$, $a\cdot(x' + x'') = a\cdot x' + a\cdot x''$ are similarly proved by using the addition $\alpha: M^D \oplus M^D \to M^D$ of tangent vectors, where $M^D \oplus M^D$ denotes the Whitney sum of the tangent bundles M^D with itself.

Remark 2.4. We can prove (ii) of Lemma 2.1 more quickly by using the local coordinate system around $x'' \in (M^D)^A$ and the local expression of μ^A by coordinates. In fact, taking a coordinate system $\{x_1, \cdots, x_n\}$ around x, we see that

$$x_{i,\varepsilon,\lambda}(a \cdot x'') = \begin{cases} \sum a_\kappa x_{i,\varepsilon,\nu}(x'')C_\lambda^{\kappa,\nu} & (\varepsilon = 1)\ , \\ x_{i,\varepsilon,\lambda}(x'') & (\varepsilon = 0) \end{cases}$$

for $a = \sum a_\kappa B^\kappa$, $B^\kappa \cdot B^\nu = \sum C_\lambda^{\kappa,\nu} B^\lambda$ (cf. (1.2)).

We want to give another interpretation of the A-module structure on the tangent space $T_{x'}(M^A)$ with $x' \in M^A$. Let $L \in T_{x'}(M^A)$ be a tangent vector at $x' \in M^A$. Then there exists a curve $t \to x'_t$ on M^A such that $x'_0 = x'$ and that

$$L\tilde{f} = \left.\frac{d\tilde{f}(x'_t)}{dt}\right|_{t=0} \tag{2.5}$$

for $\tilde{f} \in C^\infty(M^A)$. We define $L' \colon C^\infty(M) \to A$ by

$$L'f = \left.\frac{d(x'_t(f))}{dt}\right|_{t=0} . \tag{2.6}$$

Lemma 2.5. *The map $L' \colon C^\infty(M) \to A$ is well-defined and linear, and has the property*

$$L'(f \cdot g) = L'f \cdot x'(g) + x'(f) \cdot L'g \tag{2.7}$$

for $f, g \in C^\infty(M)$.

Proof. Since $x'_t(f) = f^A(x'_t)$, $x'_t(f)$ is differentiable with respect to t and $\left.\frac{dx'_t(f)}{dt}\right|_{t=0} = L(f^A)$, (cf. (2.5)). If another curve x''_t on M^A satisfies $x''_0 = x'$ and $L\tilde{f} = \left.\frac{d\tilde{f}(x''_t)}{dt}\right|_{t=0}$, then we have $\left.\frac{dx'_t(f)}{dt}\right|_{t=0} = \left.\frac{dx''_t(f)}{dt}\right|_{t=0}$. Thus L' is well-defined.

(2.7) can be verified directly.

Definition 2.6. We denote by $T'_{x'}(M^A)$ the set of all linear map $L' \colon C^\infty(M) \to A$ such that (2.7) holds for every $f, g \in C^\infty(M)$.

Remark 2.7. For $L' \in T'_{x'}M^A$, we can define $L'h$ for any C^∞-function h around x.

Thus we have obtained a map $j \colon T_{x'}M^A \to T'_{x'}M^A$ by $j(L) = L'$, (cf. (2.6)).

Lemma 2.8. *The map j is a bijective linear map.*

Proof. Let $L_1, L_2 \in T_{x'}M^A$. For $f \in C^\infty(M)$ we have $(L_1 + L_2)'f = (L_1 + L_2)f^A = L_1 f^A + L_2 f^A = L'_1 f + L'_2 f = (L'_1 + L'_2)f$. Similarly $(\alpha L_1)'f = (\alpha L_1)f^A = \alpha(L_1 f^A) = \alpha(L'_1 f) = (\alpha L'_1)f$ for $\alpha \in \boldsymbol{R}$. Thus j is linear.

To prove the bijectivity of j, we first prove

(2.7) $$\dim T'_{x'}M^A \leq \dim M^A .$$

In fact, take a coordinate system $\{x_1, \cdots, x_n\}$ around x, and consider the linear map $g\colon T'_{x'}M^A \to A^n$ by $g(L') = (L'x_1, \cdots, L'x_n)$. We show first that g is injective. Take L'_1 and $L'_2 \in T'_{x'}M^A$ and assume $L'_1 x_i = L'_2 x_i$ for every $i = 1, \cdots, n$. For any $f \in C^\infty(M)$ we can find a polynomials P, Q of $x_1, \cdots, x_n$ and $g \in C^\infty(M)$ such that

$$f = P + g \cdot Q$$

holds on some neighborhood of x, where Q is homogeneous and of degree $\geq$ height of A. Then we have

$$\begin{aligned} L'_1 f &= L'_1(P) + L'_1(g)x'(Q) + x'(g)L'_1(Q) \\ &= L'_1(P) + x'(g)L'_1(Q) = L'_2(P) + x'(g)L'_2(Q) = L'_2 f , \end{aligned}$$

where we have used the fact that $x'(f_1 \cdots f_h) = 0$ for $f_i \in C^\infty(M)$ with $f_i(x) = 0$. Thus $L'_1 = L'_2$, which proves the injectivity of g. Therefore we get (2.7).

To prove the injectivity of j, it suffices to show that $L'f = 0$ for every $f \in C^\infty(M)$ implies $L = 0$. Now $\sum L(x_{i,\lambda}) \cdot B^\lambda = L(x_i{}^A) = L'x_i = 0$, which implies $L(x_{i,\lambda}) = 0$ for any $i = 1, \cdots, n$; $\lambda = 0, \cdots, N$. Thus $L = 0$. The injectivity and the inequality (2.7) imply the bijectivity of j.

Remark 2.9. $T'_{x'}M^A$ becomes canonically an A-module, i.e., for $a \in A$ and $L' \in T'_{x'}(M^A)$ we define $a \cdot L' \in T'_{x'}(M^A)$ by

$$(a \cdot L')f = a \cdot (L'f)$$

for $f \in C^\infty(M)$.

Lemma 2.10. *For any $a \in A$ and $L \in T_{x'}(M^A)$, we have*

(2.8) $$(a \cdot L)' = a \cdot L' .$$

(Cf. Definition 2.1 *for* $a \cdot L$).

Proof. To make the several identifications more clear, we introduce the following notation. For $L \in T_{x'}M^A$ the identified element in $(M^A)^D_{x'}$ will be denoted by L^*, and conversely for $K \in (M^A)^D_{x'}$ the corresponding element in $T_{x'}M^A$ will be denoted by $^*K$. Similarly for $S \in T'_{x'}M^A$, we denote $'S = j^{-1}(S)$. Further for $L^* \in (M^A)^D$ the corresponding element in $(M^D)^A$ will be denoted by L^*_1. Then (2.8) means more precisely

(2.8)′ $$({}^*(a \cdot L^*))' = a \cdot L' .$$

Now (2.8)′ is equivalent to

(2.8)″ $$\mu^A(a, L^*_1) = ({}'(aL'))^*_1 ,$$

which is equivalent to

$$(2.8)''' \qquad (f^D)^A(\mu^A(a, L_1^*)) = (f^A)^D(('(aL'))^*)$$

for $f \in C^\infty(M)$.

The left hand side of (2.8)''' is equal to

$$(f^D \circ \mu)^A(a, L_1^*) = (a, L_1^*)(f^D \circ \mu) ,$$

while the right hand side of (2.8)''' is equal to

$$\begin{aligned}('(aL'))^*(f^A) &= f^A(x') + '(aL')f^A \cdot \tau \\ &= f^A(x') + (a \cdot L')f \cdot \tau = f^A(x') + a \cdot L'f \cdot \tau .\end{aligned}$$

Therefore it remains to verify

$$(2.9) \qquad (a, L_1^*)(f^D \circ \mu) = f^A(x') + a \cdot L'f \cdot \tau$$

for $f \in C^\infty(M)$.

Now, since L^* and L_1^* are corresponding elements in $(M^A)^D$ and $(M^D)^A$, we have

$$(2.10) \qquad (f^D)^A(L_1^*) = (f^A)^D(L^*)$$

for $f \in C^\infty(M)$.

Put $K = (a, L_1^*)$. Then we have $K(g \circ \pi_1) = a(g)$, $K(g' \circ \pi_2) = L_1^*(g')$ for $g \in C^\infty(\boldsymbol{R})$, $g' \in C^\infty(M^D)$. Next, we have, for $(t, X) \in \boldsymbol{R} \times M^D$,

$$\begin{aligned}(f^D \circ \mu)(t, X) &= f^D(tX) = f(\pi X) + (tX)f \cdot \tau \\ &= (f \circ \pi) \circ \pi_2(t, X) + (1 \circ \pi_1(t, X)) \cdot \pi_2(t, X)f \cdot \tau \\ &= (f \circ \pi) \circ \pi_2(t, X) + (1 \circ \pi_1) \cdot (f' \circ \pi_2)(t, X) \cdot \tau ,\end{aligned}$$

where $f' \in C^\infty(M^D)$ is defined by $f'(X) = Xf$ for $X \in M^D$. Hence we have

$$(2.11) \qquad \begin{aligned}K(f^D \circ \mu) &= K((f \circ \pi) \circ \pi_2) + K(1 \circ \pi_1) \cdot K(f' \circ \pi_2) \cdot \tau \\ &= L_1^*(f \circ \pi) + a \cdot L_1^* f' \cdot \tau .\end{aligned}$$

On the other hand, from (2.10) we get

$$\begin{aligned}(f^D)^A(L_1^*) &= L_1^*(f^D) = L_1^*(f \circ \pi + f' \cdot \tau) = L_1^*(f \circ \pi) + L_1^* f' \cdot \tau , \\ (f^A)^D(L^*) &= L^*(f^A) = f^A(x') + Lf^A \cdot \tau = f^A(x') + L'f \cdot \tau ,\end{aligned}$$

which imply

$$(2.12) \qquad L_1^*(f \circ \pi) = f^A(x') , \qquad L_1^* f' = L'f .$$

Combining (2.10), (2.11) and (2.12), we get (2.9).

3. Lifting of vector fields

We denote by $\mathscr{T}_0^1(M)$ the set of all vector fields on M. Take $X \in \mathscr{T}_0^1(M)$. The corresponding $X'\colon M \to M^D$ is defined by

$$X'(x)f = f(x) + (X(x)f)\cdot\tau \in D \tag{3.1}$$

for $f \in C^\infty(M)$ and $x \in M$. The map X' induces a map $X'^A\colon M^A \to (M^D)^A$. Consider the map $\tilde{X} = i \circ X'^A\colon M^A \to (M^A)^D$, where $i\colon (M^D)^A \to (M^A)^D$ is the identification map. The commutativity of the right triangle of the diagram (2.1) implies that $\tilde{X}(x') \in (M^A)^D_{x'}$ for every $x' \in M^A$. Hence by Remark 1.6 we obtain a tangent vector in $T_{x'}(M^A)$ corresponding to $\tilde{X}(x')$, which we denote by $X^A(x')$. Thus we obtain a vector field $X^A \in \mathscr{T}_0^1(M^A)$.

Definition 3.1. The vector field X^A is called the *lift* of X to M^A.

Remark 3.2. Any $X \in \mathscr{T}_0^1(M^A)$ can be extended to a derivation of $C^\infty(M^A, A)$ by

$$X\tilde{f} = X(\textstyle\sum f_i B^i) = \sum (Xf_i)\cdot B^i ,$$

where $\{1, B^1, \cdots, B^N\}$ is a basis of A, and $\tilde{f} = \sum f_i B^i$ with $f_i \in C^\infty(M^A)$.

Lemma 3.3. *For any* $X \in \mathscr{T}_0^1(M)$ *and* $f \in C^\infty(M)$, *we have*

$$(Xf)^A = X^A f^A . \tag{3.2}$$

Proof. We have to show

$$(Xf)^A(x') = (X^A f^A)(x') \tag{3.3}$$

for $x' \in M^A$. Put $x'' = X'^A(x') \in (M^D)^A$. Then $x_1'' = (X^A(x'))'$ is the element corresponding to x'' in $(M^A)^D$ (for the notation $(\quad)'$ see Remark 1.6). Using Corollary 1.12 we have

$$\begin{aligned}(f^D)^A(x'') &= (f^A)^D(x_1'') = x_1''(f^A) = (X^A(x'))'f^A \\ &= f^A(x') + X^A(x')f^A\cdot\tau .\end{aligned} \tag{3.4}$$

The left hand side of (3.3) is equal to

$$((f^D)^A \circ X'^A)(x') = (f^D \circ X')^A(x') = x'(f^D \circ X') . \tag{3.5}$$

Since

$$\begin{aligned}(f^D \circ X')(x) &= f^D(X'(x)) = X'(x)(f) = f(x) + (Xf)(x)\cdot\tau \\ &= (f + Xf\cdot\tau)(x) ,\end{aligned}$$

(3.5) is equal to

$$x'(f) + x'(Xf)\cdot\tau = f^A(x') + (Xf)^A(x')\cdot\tau . \tag{3.6}$$

Comparing (3.4) and (3.6), we obtain (3.3).

Lemma 3.4. *The map $X \to X^A$ is linear.*

Proof. Take $f \in C^\infty(M)$ and $X, Y \in \mathcal{T}^1_0(M)$. Then by Lemma 3.3 we have $(X + Y)^A f^A = ((X + Y)f)^A = (Xf + Yf)^A = (Xf)^A + (Yf)^A = X^A f^A + Y^A f^A = (X^A + Y^A)f^A$, and therefore $((X + Y)^A)'f = (X + Y)^A f^A = (X^A + Y^A)f^A = (X^A + Y^A)'f$, which implies $((X + Y)^A)' = (X^A + Y^A)'$ (for the notation $(\ )'$ see Lemme 2.8). Hence we get $(X + Y)^A = X^A + Y^A$. Similarly, $(\alpha \cdot X)^A = \alpha \cdot X^A$ for $\alpha \in R$.

Lemma 3.5. *For any $f \in C^\infty(M)$ and $X \in \mathcal{T}^1_0(M)$, we have*

$$(f \cdot X)^A = f^A \cdot X^A \ , \tag{3.7}$$

equivalently,

$$(fX)^A(x') = f^A(x')X^A(x')$$

for every $x' \in M^A$ (cf. Definition 2.1).

Proof. Let $\mu: \boldsymbol{R} \times M^D \to M^D$ be the scalar multiplication of tangent vectors. Identifying X with its corresponding $X': M \to M^D$ (cf. Definition 3.1), we have

$$(f \cdot X)^A = (\mu \circ (f, X))^A = \mu^A \circ (f, X)^A = \mu^A \circ (f^A, X^A) = f^A \cdot X^A \ .$$

Definition 3.6. For $\tilde{X} \in \mathcal{T}^1_0(M^A)$, we define a map $\tilde{X}': C^\infty(M) \to C^\infty(M^A, A)$ by

$$(\tilde{X}'f)(x') = (\tilde{X}(x'))'f = (j(\tilde{X}(x')))f$$

for $f \in C^\infty(M)$ and $x' \in M^A$ (cf. Remark 2.7).

Remark 3.7. By Lemma 2.8, we have

$$(g \cdot \tilde{X})' = g \cdot \tilde{X}'$$

for $g \in C^\infty(M^A, A)$, $\tilde{X} \in \mathcal{T}^1_0(M^A)$.

Lemma 3.8. *For $a \in A$, $X \in \mathcal{T}^1_0(M)$ and $f \in C^\infty(M)$, we have*

$$(a \cdot X^A)f^A = a \cdot (Xf)^A \ . \tag{3.8}$$

Proof. We have

$$(a \cdot X^A)f^A = (a \cdot X^A)'f = a \cdot ((X^A)'f) = a \cdot (X^A f^A) = a \cdot (Xf)^A \ ,$$

where we have used Lemma 2.5, Lemma 3.3 and Remark 3.7.

Lemma 3.9. *Let $\{x_1, \cdots, x_n\}$ be a coordinate system on some neighborhood of M. Then we have*

$$B^\lambda \cdot (\partial/\partial x_i)^A = \partial/\partial x_{i,\lambda} \tag{3.9}$$

for $i = 1, \cdots, n$; $\lambda = 0, \cdots, N$.

Proof. We have $(B^\lambda(\partial/\partial x_i)^A)(x_j)^A = B^\lambda(\partial x_j/\partial x_i)^A = B^\lambda(\delta_{ij})^A = B^\lambda \cdot \delta_{ij}$. On the other hand we have

$$(\partial/\partial x_{i,\lambda})(x_j)^A = (\partial/\partial x_{i,\lambda})\Big(\sum_\nu x_{j,\nu}B^\nu\Big) = \delta_{ij}B^\lambda .$$

Hence we get (3.9).

Corollary 3.10. *For any* $x' \in M^A$, *the* A*-module* $T_{x'}M^A$, *is a free* A*-module.*

Proof. Take $X_i = (\partial/\partial x_i)^A_{x'}$ $(i = 1, \cdots, n)$. Then $\{X_1, \cdots, X_n\}$ is a free A-basis of $T_{x'}M^A$.

Lemma 3.11. *For any* $X, Y \in \mathscr{T}^1_0(M)$ *we have*

$$[X^A, Y^A] = [X, Y]^A .$$

Proof. For any $f \in C^\infty(M)$, we have

$$\begin{aligned}[X^A, Y^A]f^A &= X^A Y^A f^A - Y^A X^A f^A = (XYf - YXf)^A \\ &= ([X, Y]f)^A = [X, Y]^A f^A .\end{aligned}$$

Hence we have

$$[X^A, Y^A]'f = [X^A, Y^A]f^A = [X, Y]^A f^A = ([X, Y]^A)'f ,$$

which implies $[X^A, Y^A]' = ([X, Y]^A)'$ and hence we get $[X^A, Y^A] = [X, Y]^A$.

Lemma 3.12. *For any* $a, b \in A$ *and* $X, Y \in \mathscr{T}^1_0(M)$ *we have*

$$[aX^A, bY^A] = (a\cdot b)\cdot[X, Y]^A . \tag{3.10}$$

Proof. We calculate as follows: for any $f \in C^\infty(M)$

$$\begin{aligned}[aX^A, bY^A]f^A &= (aX^A)(bY^A)f^A - (bY^A)(aX^A)f^A \\ &= (a\cdot X^A)(b\cdot(Yf)^A) - bY^A(a\cdot(Xf)^A) \\ &= b\cdot(aX^A(Yf)^A) - a\cdot(bY^A\cdot(Xf)^A) \\ &= b\cdot a\cdot(XYf)^A - a\cdot b(YXf)^A \\ &= (a\cdot b)\cdot([X, Y]f)^A = (a\cdot b)([X, Y]^A f^A) \\ &= ((ab)[X, Y]^A)f^A .\end{aligned}$$

By the same argument as in Lemma 3.10 we get (3.10).

Remark 3.13. We can verify that if $\{\Phi^t\}$ is a one-parameter group of diffeomorphisms on M generated by a vector field X, then the one-parameter group $\{(\Phi^t)^A\}$ induces the vector field X^A.

4. Lifting of covariant tensor fields

Take $f \in C^\infty(M)$. Since $f^A: M^A \to A$ is an A-valued function, we can consider $df^A: T(M^A) \to A$. On the other hand, since $df: M^D \to \boldsymbol{R}$ is a function, we can consider $(df)^A: (M^D)^A \to \boldsymbol{R}^A = A$.

Lemma 4.1. *Identifying $T(M^A) = (M^A)^D$ with $(M^D)^A$, we have*

$$(df)^A = df^A . \tag{4.1}$$

Proof. Let π (resp. $\tilde{\pi}$) be the projection $\pi: \boldsymbol{R}^D = \boldsymbol{R} \oplus \boldsymbol{R}\cdot\tau \to \boldsymbol{R}\cdot\tau = \boldsymbol{R}$ (resp. $\tilde{\pi}: A^D = A \oplus A\cdot\tau \to A\cdot\tau = A$). Then we have, by definition, $df = \pi \circ f^D$, $df^A = \tilde{\pi} \circ (f^A)^D$. Hence $(df)^A = \pi^A \circ (f^D)^A$. Then the commutative diagram

$$\begin{array}{ccccc} (M^A)^D & \xrightarrow{(f^A)^D} & A^D & \xrightarrow{\tilde{\pi}} & A \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle 1} \\ (M^D)^A & \xrightarrow[(f^D)^A]{} & D^A & \xrightarrow[\pi^A]{} & A \end{array}$$

proves (4.1). q.e.d.

Take a 1-form $\theta \in \mathscr{T}^0_1(M)$. Then θ can be considered as a function $\theta: M^D \to \boldsymbol{R}$. Hence $\theta^A: (M^D)^A \to A$ is an A-valued function on $(M^D)^A = (M^A)^D$. To prove that θ^A is in fact a 1-form on M^A, we shall first prove

Lemma 4.2. *Take $\theta_1, \theta_2 \in \mathscr{T}^0_1(M)$. Then we have*

$$(\theta_1 + \theta_2)^A = \theta_1{}^A + \theta_2{}^A . \tag{4.2}$$

Proof. Let $\alpha: \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ (resp. $\alpha_A: A \times A \to A$) be the addition in $\boldsymbol{R}$ (resp. A). Then we know that $\alpha_A = \alpha^A$, and therefore that

$$\begin{aligned}(\theta_1 + \theta_2)^A &= (\alpha \circ (\theta_1, \theta_2))^A = \alpha^A \circ (\theta_1{}^A, \theta_2{}^A) \\ &= \alpha_A(\theta_1{}^A, \theta_2{}^A) = \theta_1{}^A + \theta_2{}^A .\end{aligned}$$

Lemma 4.3. *For $f \in C^\infty(M)$ and $\theta \in \mathscr{T}^0_1(M)$, we have*

$$(f\cdot\theta)^A = f^A\cdot\theta^A . \tag{4.3}$$

Proof. Let $\mu_0: \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ (resp. $\mu_A: A \times A \to A$) be the multiplication in $\boldsymbol{R}$ (resp. in A). Then we know $(\mu_0)^A = \mu_A$, and therefore $(f\cdot\theta)^A = (\mu_0 \circ (f, \theta))^A = (\mu_0)^A \circ (f^A, \theta^A) = \mu_A \circ (f^A, \theta^A) = f^A\cdot\theta^A$.

Lemma 4.4. *For any $\theta \in \mathscr{T}^0_1(M)$, we have $\theta^A \in \mathscr{T}^0_1(M^A)$.*

Proof. Since the problem is local, we can assume that $\theta = \sum g_i\, df_i$ with $g_i, f_i \in C^\infty(M)$. By (4.1), (4.2), (4.3) we have $\theta^A = \sum g_i{}^A df_i{}^A$, which is a 1-form on M^A.

Lemma 4.5. *For $\theta \in \mathscr{T}^0_1(M)$ and $X \in \mathscr{T}^1_0(M)$, we have*

$$(4.4) \qquad (\theta(X))^A = \theta^A(X^A) .$$

Proof. The function $\theta(X)\colon M \to \boldsymbol{R}$ can be written as $\theta(X) = \theta \circ X$, where $X\colon M \to M^D$ and $\theta\colon M^D \to R$. Hence we have $(\theta(X))^A = (\theta \circ X)^A = \theta^A \circ X^A = \theta^A(X^A)$.

Lemma 4.6. *For $\theta \in \mathscr{T}^0_1(M)$, $a \in A$ and $\tilde{X} \in T_{x'}(M^A)$, we have*

$$(4.5) \qquad \theta^A(a \cdot \tilde{X}) = a \cdot \theta^A(\tilde{X}) .$$

Proof. Since $\theta(t \cdot X) = t\theta(X)$ for $t \in \boldsymbol{R}$ and $X \in T(M)$ we have $(\theta \circ \mu)(t, X) = \theta(tX) = \mu_0(t, \theta(X)) = \mu_0(1 \times \theta)(t, X)$. Hence $(\theta \circ \mu)^A = (\mu_0)^A \circ (1 \times \theta^A) = \mu_A \circ (1 \times \theta^A)$, which implies (4.5).

Since θ^A is an A-valued 1-form on M^A, we can consider it as $\theta^A \in \mathscr{T}^0_1(M^A) \otimes A$. We can easily verify

Lemma 4.7. *$\mathscr{T}^0_*(M^A) \otimes A$ becomes an associative graded algebra over A with the multiplication*:

$$(K_1 \otimes a_1) \otimes (K_2 \otimes a_2) = K_1 \otimes K_2 \otimes (a_1 a_2)$$

for $K_1, K_2 \in \mathscr{T}^0_*(M^A) = \sum_q \mathscr{T}^0_q(M^A)$ and $a_1, a_2 \in A$.

Lemma 4.8. *The map $L\colon \mathscr{T}^0_*(M) \to \mathscr{T}^0_*(M^A) \otimes A$ defined by $L(\theta_1 \otimes \cdots \otimes \theta_q) = \theta_1{}^A \otimes \cdots \otimes \theta_q{}^A$ for $\theta_i \in \mathscr{T}^0_1(M)$ is an algebra homomorphism.*

Proof. Let $L\colon (\mathscr{T}^0_1(M))^q \to \mathscr{T}^0_q(M^A) \otimes A$ be defined by $L(\theta_1, \cdots, \theta_q) = \theta_1{}^A \otimes \cdots \otimes \theta_q{}^A$. It is easily checked that $L(f_1\theta_1, \cdots, f_q\theta_q) = (f_1 \cdots f_q)^A L(\theta_1, \cdots, \theta_q)$ for $f_i \in \mathscr{T}^0_0(M)$, $\theta_i \in \mathscr{T}^0_1(M)$. Hence there exists a map $L\colon \mathscr{T}^0_q(M) \to \mathscr{T}^0_q(M^A) \otimes A$ such that $L(\theta_1 \otimes \cdots \otimes \theta_q) = L(\theta_1, \cdots, \theta_q)$. Now it is easy to see that L is an algebra homomorphism.

5. Lifting of (1, 1)-tensor fields

Let $K \in \mathscr{T}^1_1(M)$ be a (1, 1)-tensor field on M. Then K can be considered as a map $K\colon M^D \to M^D$ such that $\pi \circ K = \pi$. Then $K^A\colon (M^D)^A \to (M^D)^A$ can be considered as $K^A\colon (M^A)^D \to (M^A)^D$.

Lemma 5.1. *K^A is a (1, 1)-tensor field on M^A.*

Proof. Since the problem is local, we assume $K = \sum \theta_i \otimes Y^i$ with $\theta_i \in \mathscr{T}^0_1(M)$ and $Y^i \in \mathscr{T}^1_0(M)$. Then

$$\begin{aligned} K(X) = \sum \theta_i(X) Y^i &= \sum \mu(\theta_i(X), (Y^i \circ \pi)(X)) \\ &= (\alpha_r \circ (\mu \circ (\theta_1, Y^1 \circ \pi), \cdots, \mu \circ (\theta_r, Y^r \circ \pi)))(X) , \end{aligned}$$

where $\alpha_r\colon \boldsymbol{R}^r \to \boldsymbol{R}$ is the addition $\alpha_r(a_1, \cdots, a_r) = a_1 + \cdots + a_r$ for $a_i \in \boldsymbol{R}$. Hence we have

$$K^A = (\alpha_r)^A \circ (\mu^A \circ (\theta_1{}^A, (Y^1)^A \circ \pi^A), \cdots, \mu^A \circ (\theta_r{}^A, (Y^r)^A \circ \pi^A)) ,$$

which implies

$$K^A(\tilde{X}) = \sum \theta_i{}^A(\tilde{X})\cdot(Y^i)^A$$

for $\tilde{X} \in (M^A)^D$. Thus $K^A \in \mathcal{T}^1_1(M^A)$.

Lemma 5.2. *For $K \in \mathcal{T}^1_1(M)$, $X \in \mathcal{T}^1_0(M)$ and $a \in A$, we have*

$$K^A(a\cdot X^A) = a\cdot(K(X))^A \ .$$

Proof. As before, we can assume $K = \sum \theta_i \otimes Y_i$. Then

$$\begin{aligned} K^A(a\cdot X^A) &= \sum \theta_i{}^A(aX^A)\cdot(Y^i)^A = \sum a\cdot\theta_i{}^A(X^A)(Y^i)^A \\ &= a\cdot\sum(\theta_i(X))^A(Y^i)^A = a\cdot(\sum \theta_i(X)Y^i)^A = a\cdot(K(X))^A \ . \end{aligned}$$

Theorem 5.3. *Let $J \in \mathcal{T}^1_1(M)$ be an almost complex structure on M. Then J^A is an almost complex structure on M^A. Moreover, J^A is integrable if and only if J is.*

Proof. Let I be the (1, 1)-tensor field of identity maps of T_xM for $x \in M$. Since $I = \sum dx_i \otimes \partial/\partial x_i$ locally, we get, for $\tilde{X} \in (M^A)^D$,

$$\begin{aligned} I^A(\tilde{X}) &= \sum (dx_i)^A(\tilde{X}) \otimes \left(\frac{\partial}{\partial x_i}\right)^A = \sum dx_i{}^A(\tilde{X})\cdot\frac{\partial}{\partial x_{i,0}} \\ &= \sum dx_{i,\lambda}(\tilde{X})B^\lambda\frac{\partial}{\partial x_{i,0}} = \sum dx_{i,\lambda(\tilde{X})}\cdot\frac{\partial}{\partial x_{i,\lambda}} = \tilde{X} \ , \end{aligned}$$

where we have used (3.9) and (4.5). Thus we have $J^A \circ J^A = (J \circ J)^A = (-I)^A = -I^A = -\tilde{I}$, where $\tilde{I}$ is the (1, 1)-tensor field of identity maps of M^A. Hence J^A is an almost complex structure on M^A.

Next, J is integrable if and only if

$$J[X, Y] = [JX, Y] + [X, JY] + J[JX, JY] \tag{5.1}$$

for every $X, Y \in \mathcal{T}^1_0(M)$. Using Lemmas 3.12 and 5.2 we have

$$\begin{aligned} J^A[ax^A, bY^A] &= J^A(ab[X, Y]^A) = (ab)(J[X, Y])^A \\ &= (ab)\{[J^AX^A, Y^A] + [X^A, J^AY^A] + J^A[J^AX^A, J^AY^A]\} \\ &= [J^A(aX^A), bY^A] + [aX^A, J^A(bY^A)] + J^A[J^A(aX^A), J^A(bY^A)] \end{aligned}$$

for $a, b \in A$. Since $T_{x'}M^A$ is a free A-module (Corollary 3.10), we conclude that J^A is integrable. Conversely, if J^A is integrable, we get

$$(J[X, Y])^A = ([JX, Y] + [X, JY] + J[JX, JY])^A$$

for $X, Y \in \mathcal{T}^1_0(M)$, which implies (5.1), and hence J is integrable.

6. Prolongations of affine connections

Let ∇ be the covariant differentiation defined by an affine connection of M. In the sequel, for the sake of convenience of notation, we shall denote by $\nabla(X, K)$ the covariant differentiation of a tensor field K on M with respect to $X \in \mathscr{T}_0^1(M)$, i.e.,

$$\nabla(X, K) = \nabla_X K .$$

Theorem 6.1. *There exists one and only one affine connection on M^A whose covariant differentiation $\tilde{\nabla}$ satisfies the following condition*

$$\tilde{\nabla}_{aX^A} bY^A = (ab)(\nabla_X Y)^A \tag{6.1}$$

for every $X, Y \in \mathscr{T}_0^1(M)$ and $a, b \in A$.

Proof. Take a coordinate neighborhood U with coordinate system $\{x_1, \cdots, x_n\}$ and let $\Gamma_{ij}{}^k$ be the connection components of ∇ with respect to $\{x_1, \cdots, x_n\}$, i.e.,

$$\nabla\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \sum \Gamma_i{}^k{}_j \frac{\partial}{\partial x_k} \tag{6.2}$$

for $i, j = 1, \cdots, n$. Let $\Gamma'_{ij}{}^k$ be the connection components of ∇ with respect to another coordinate system $\{y_1, \cdots, y_n\}$ on U. Then we have the following equalities:

$$\Gamma'_i{}^k{}_j = \sum \frac{\partial x_b}{\partial y_i} \frac{\partial x_c}{\partial y_j} \frac{\partial y_k}{\partial x_a} \Gamma_b{}^a{}_c + \sum \frac{\partial^2 x_a}{\partial y_i \partial y_j} \frac{\partial y_k}{\partial x_a} \tag{6.3}$$

for $i, j = 1, 2, \cdots, n$ (cf. for instance [3, p. 27]). Let $\{x_{i,\lambda} \mid i = 1, \cdots, n; \lambda = 0, 1, \cdots, N\}$ (resp. $\{y_{i,\lambda}\}$) be the induced coordinate system on $\pi_A{}^{-1}(U)$. Define $\tilde{\Gamma}_{(i,\lambda)}{}^{(k,\nu)}{}_{(j,\mu)}$ by

$$\sum \tilde{\Gamma}_{(i,\lambda)}{}^{(k,\nu)}{}_{(j,\mu)} B^\nu = B^\lambda B^\mu (\Gamma_{ij}{}^k)^A \tag{6.4}$$

for $i, j, k = 1, \cdots, n$; $\lambda, \mu, \nu = 0, 1, \cdots, N$, where $\{B^0 = 1, B^1, \cdots, B^N\}$ is a basis of A as in § 1.

We shall now prove that there exists a connection $\tilde{\nabla}$ whose connection components with respect to $\{x_{i,\lambda}\}$ are given by (6.4). To prove this we have to prove the following equalities (6.5) similar to (6.3):

$$\begin{aligned} \tilde{\Gamma}'_{(i,\nu)}{}^{(k,\lambda)}{}_{(j,\mu)} &= \sum \frac{\partial x_{b,\beta}}{\partial y_{i,\nu}} \frac{\partial x_{c,\gamma}}{\partial y_{j,\mu}} \frac{\partial y_{k,\lambda}}{\partial x_{a,\alpha}} \tilde{\Gamma}_{(b,\beta)}{}^{(a,\alpha)}{}_{(c,\gamma)} \\ &\quad + \sum \frac{\partial^2 x_{a,\alpha}}{\partial y_{i,\nu} \partial y_{j,\mu}} \frac{\partial y_{k,\lambda}}{\partial x_{a,\alpha}} \end{aligned} \tag{6.5}$$

for $i, j, k = 1, \cdots, n$; $\lambda, \mu, \nu = 0, 1, \cdots, N$, where $\tilde{\Gamma}'_{(i,\nu)}{}^{(k,\lambda)}{}_{(j,\mu)}$ denote the connection components of $\tilde{\nabla}$ with respect to the coordinate system $\{y_{i,\lambda}\}$. Denoting the right hand side of (6.5) by $\tilde{\Gamma}^*_{(i,\nu)}{}^{(k,\lambda)}{}_{(j,\mu)}$ and using Lemmas 3.8 and 3.9, we calculate as follows:

$$\begin{aligned}
&\sum \tilde{\Gamma}^*_{(i,\nu)}{}^{(k,\lambda)}{}_{(j,\mu)} B^\lambda \\
&= \frac{\partial x_{b,\beta}}{\partial y_{i,\nu}} \frac{\partial x_{c,\gamma}}{\partial y_{j,\mu}} \left(\frac{\partial y_k}{\partial x_a}\right)^A B^\alpha \tilde{\Gamma}_{(b,\beta)}{}^{(a,\alpha)}{}_{(c,\gamma)} + \frac{\partial^2 x_{a,\alpha}}{\partial y_{i,\nu} \partial y_{j,\mu}} \left(\frac{\partial y_k}{\partial x_a}\right)^A B^\alpha \\
&= \frac{\partial x_{b,\beta}}{\partial y_{i,\nu}} \frac{\partial x_{c,\gamma}}{\partial y_{j,\mu}} \left(\frac{\partial y_k}{\partial x_a}\right)^A B^\beta B^\gamma (\Gamma_b{}^a{}_c)^A + \frac{\partial}{\partial y_{i,\nu}} \left(\frac{\partial x_{a,\alpha}}{\partial y_{j,\mu}} B^\alpha\right) \left(\frac{\partial y_k}{\partial x_a}\right)^A \\
&= \left(\frac{\partial x_b}{\partial y_i}\right)^A B^\nu \left(\frac{\partial x_c}{\partial y_j}\right)^A B^\mu \left(\frac{\partial y_k}{\partial x_a}\right)^A (\Gamma_b{}^a{}_c)^A + \frac{\partial}{\partial y_{i,\nu}} \left(\frac{\partial x_a}{\partial y_j}\right)^A B^\mu \left(\frac{\partial y_k}{\partial x_a}\right)^A \\
&= \left(\frac{\partial x_b}{\partial y_i}\right)^A B^\nu \left(\frac{\partial x_c}{\partial y_j}\right)^A B^\mu \left(\frac{\partial y_k}{\partial x_a}\right)^A (\Gamma_b{}^a{}_c)^A + \left(\frac{\partial^2 x_a}{\partial y_i \partial y_j}\right)^A B^\nu B^\mu \left(\frac{\partial y_k}{\partial x_a}\right)^A \\
&= B^\nu B^\mu (\Gamma'_i{}^k{}_j)^A = \tilde{\Gamma}'_{(i,\nu)}{}^{(k,\lambda)}{}_{(j,\mu)} B^\lambda ,
\end{aligned}$$

which implies (6.5).

Thus we have proved the existence of $\tilde{\nabla}$ whose connection components with respect to $\{x_{i,\lambda}\}$ are given by (6.4).

Next, we shall prove (6.1) for $X = \partial/\partial x_i$, $Y = \partial/\partial x_j$, and $a = B^\lambda$, $b = B^\mu$. We calculate as follows:

$$\begin{aligned}
\tilde{\nabla}\left(B^\lambda \left(\frac{\partial}{\partial x_i}\right)^A, B^\mu \left(\frac{\partial}{\partial x_j}\right)^A\right) &= \tilde{\nabla}\left(\frac{\partial}{\partial x_{i,\lambda}}, \frac{\partial}{\partial x_{j,\mu}}\right) = \tilde{\Gamma}_{(i,\lambda)}{}^{(k,\nu)}{}_{(j,\mu)} \frac{\partial}{\partial x_{k,\nu}} \\
&= \tilde{\Gamma}_{(i,\lambda)}{}^{(k,\nu)}{}_{(j,\mu)} B^\nu \left(\frac{\partial}{\partial x_k}\right)^A = B^\lambda B^\mu (\Gamma_i{}^k{}_j)^A \left(\frac{\partial}{\partial x_k}\right)^A \\
&= B^\lambda B^\mu \left(\Gamma_i{}^k{}_j \frac{\partial}{\partial x_k}\right)^A = B^\lambda B^\mu \left(\nabla\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\right)^A ,
\end{aligned}$$

which proves (6.1) for $X = \partial/\partial x_i$, $Y = \partial/\partial x_j$ and $a = B^\lambda$, $b = B^\mu$ and hence for arbitrary $a, b \in A$.

Now put $X_i = \partial/\partial x_i$ for $i = 1, \cdots, n$. We shall prove (6.1) for $X = fX_i$, $Y = X_j$ with $f \in \mathscr{T}^0_0(U)$. We calculate as follows:

$$\begin{aligned}
\tilde{\nabla}(aX^A, bY^A) &= \tilde{\nabla}(a(f \cdot X_i)^A, b \cdot X_j{}^A) \tilde{\nabla}(af^A \cdot X_i{}^A, b \cdot X_j{}^A) \\
&= af^A \tilde{\nabla}(X_i{}^A, bX_j{}^A) = abf^A(\nabla(X_i, X_j))^A \\
&= ab(f\nabla(X_i, X_j))^A = ab(\nabla(fX_i, X_j))^A = ab(\nabla(X, Y))^A ,
\end{aligned}$$

which proves our assertion. Therefore we see that (6.1) holds for $X \in \mathscr{T}^1_0(M)$ and $Y = X_j$ with $j = 1, \cdots, n$. Next we prove that (6.1) holds for $X \in \mathscr{T}^1_0(M)$ and $Y = f \cdot X_j$ with $f \in \mathscr{T}^0_0(U)$ and $j = 1, \cdots, n$. We calculate as follows:

$$\begin{aligned}\tilde{\nabla}(a\cdot X^A, b\cdot Y^A) &= \tilde{\nabla}(a\cdot X^A, b(f\cdot X_j)^A) = \tilde{\nabla}(aX^A, bf^A\cdot X_j{}^A) \\ &= f^A\nabla_{aX^A}\, bX_j{}^A + aX^A(bf^A)X_j{}^A = f^Aab(\nabla_X Y_j)^A + ab(Xf)^A X_j{}^A \\ &= ab(f\nabla_X Y_j + Xf\cdot X_j)^A = ab(\nabla_X f Y_j)^A = ab(\nabla(X, Y))^A ,\end{aligned}$$

where we have used Lemmas 3.5 and 3.8.

Thus we have proved (6.1) for any $X, Y \in \mathscr{T}^1_0(M)$ and $a, b \in A$. The uniqueness of such $\tilde{\nabla}$ follows from Lemma 3.9.

Definition 6.2. The unique affine connection $\tilde{\nabla}$ in Theorem 6.1 will be called the *prolongation* of ∇ to M^A and will be denoted by $\tilde{\nabla} = \nabla^A$.

Theorem 6.3. *Let T and R (resp. $\tilde{T}$ and $\tilde{R}$) be the torsion and curvature tensor fields of ∇ (resp. $\tilde{\nabla} = \nabla^A$). Then according as $T = 0$, $\nabla T = 0$, $R = 0$ or $\nabla R = 0$, we have $\tilde{T} = 0$, $\tilde{\nabla}\tilde{T} = 0$, $\tilde{R} = 0$ or $\tilde{\nabla}\tilde{R} = 0$ and vice versa. In particular, if M is locally affine symmetric with respect to ∇, so is M^A with respect to $\tilde{\nabla} = \nabla^A$.*

Proof. First we prove

$$\tilde{T}(aX^A, bY^A) = ab(T(X, Y))^A \tag{6.6}$$

for $X, Y \in \mathscr{T}^1_0(M)$ and $a, b \in A$.

In fact, by the definition of $\tilde{T}$, Lemma 3.12 and (6.1) we get

$$\begin{aligned}\tilde{T}(aX^A, bY^A) &= \tilde{\nabla}_{aX^A}\, bY^A - \tilde{\nabla}_{bY^A}\, aX^A - [aX^A, bY^A] \\ &= (ab)(\nabla_X Y - \nabla_Y X - [X, Y])^A = ab(T(X, Y))^A .\end{aligned}$$

Thus we see that $T = 0$ if and only if $\tilde{T} = 0$ (cf. Corollary 3.10).

Similarly we know that $\tilde{R}(aX^A, bY^A, cZ^A) = (abc)(R(X, Y, Z))^A$ for $X, Y, Z \in \mathscr{T}^1_0(M)$ and $a, b, c \in A$, from which we see that $R = 0$ if and only if $\tilde{R} = 0$. The proof for the case ∇T and ∇R is similar.

7. Affine symmetric spaces

Lemma 7.1. *Let Φ be a diffeomorphism of M onto M', and let $X \in \mathscr{T}^1_0(M)$ and $a \in A$. Then we have*

$$(T\Phi^A)(aX^A) = a((T\Phi)X)^A . \tag{7.1}$$

Proof. Take $f \in C^\infty(M)$. We have

$$\begin{aligned}(\Phi^A)^D(aX^A)f^A &= (aX^A)(f^A\circ\Phi^A) = (aX^A)(f\circ\Phi)^A = a\cdot X^A(f\circ\Phi)^A \\ &= a\cdot(X(f\circ\Phi))^A = a\cdot((\Phi^D X)f)^A = a\cdot((\Phi^D X)^A f^A) \\ &= (a(\Phi^D X)^A)f^A ,\end{aligned}$$

from which follows (7.1).

Lemma 7.2. *Let ∇ (resp. ∇') be an affine connection on M (resp. on M')*

and let Φ be a diffeomorphism of M onto M' transforming ∇ onto ∇'. Then Φ^A transforms ∇^A onto ∇'^A.

Proof. Take $X, Y \in \mathcal{T}_0^1(M)$. Then we have, for $a, b \in A$,

$$T\Phi^A(\nabla^A_{aX^A} bY^A) = T\Phi^A(ab\nabla_X Y)^A = ab(T\Phi(\nabla_X Y))^A = ab(\nabla'_{T\Phi X} T\Phi Y)^A$$
$$= \nabla'^A_{a(T\Phi X)^A} b(T\Phi Y)^A = \nabla'_{T\Phi^A(aX^A)} T\Phi^A(bY^A) ,$$

where we have used Lemma 7.1. Since X, Y, a, b are arbitrary, Lemma 7.2 follows.

Lemma 7.3. *Let $X \in \mathcal{T}_0^1(M)$, and $x_0 \in M$. Assume $X_{x_0} = 0$. Then $(X^A)_{\tilde{x}_0} = 0$, where $\tilde{x}_0 \in M^A$ is defined by $\tilde{x}_0(f) = f(x_0)$ for $f \in C^\infty(M)$.*

Proof. Let Φ^t be a local one-parameter group of local diffeomorphisms around x_0 generated by X. Then X^A generates the local group $(\Phi^t)^A$ around $\tilde{x}_0$ (cf. Remark 3.13). Since $\Phi^t(x_0) = x_0$, we get $(\Phi^t)^A(\tilde{x}_0) = \tilde{x}_0$ and therefore $(X^A)_{\tilde{x}_0} = 0$.

Lemma 7.4. *Let $\Phi: M \to M$ be a diffeomorphism such that there exist $x_0 \in M$ and $\alpha \in \boldsymbol{R}$ with $\Phi(x_0) = x_0$ and $T_{x_0}\Phi = \alpha \cdot 1_{T_{x_0}M}$. Then $T_{\tilde{x}_0}\Phi^A = \alpha \cdot 1_{T_{\tilde{x}_0}M^A}$.*

Proof. Let $\{x_1, \cdots, x_n\}$ be a local coordinate system around x_0. By Lemma 7.1 we have $T\Phi^A(\partial/\partial x_i)^A = (T\Phi(\partial/\partial x_i))^A$ for $i = 1, \cdots, n$. Hence we get

$$T\Phi^A((\partial/\partial x_i)^A_{\tilde{x}_0}) = (T\Phi^A(\partial/\partial x_i)^A)_{\tilde{x}_0} = (T\Phi(\partial/\partial x_i))^A_{\tilde{x}_0} .$$

Put $X = T\Phi(\partial/\partial x_i) - \alpha(\partial/\partial x_i)$. Then X is a vector field around x_0 on M with $X_{x_0} = 0$. Therefore by Lemma 7.3 we get $(X^A)_{\tilde{x}_0} = 0$, which implies

$$(T\Phi(\partial/\partial x_i))^A_{\tilde{x}_0} = (\alpha(\partial/\partial x_i))^A_{\tilde{x}_0} = \alpha \cdot (\partial/\partial x_i)^A_{\tilde{x}_0} .$$

Take an arbitrary $a \in A$. Then we have

$$T\Phi^A(a(\partial/\partial x_i))^A_{\tilde{x}_0} = a \cdot (T\Phi(\partial/\partial x_i))^A_{\tilde{x}_0} = a \cdot \alpha(\partial/\partial x_i)^A_{\tilde{x}_0} = \alpha \cdot (a(\partial/\partial x_i)^A_{\tilde{x}_0}) .$$

Since $\{a(\partial/\partial x_i)^A \mid a \in A\}$ span the tangent space $T_{\tilde{x}_0}M^A$ (cf. Lemma 3.9), we get $T_{\tilde{x}_0}\Phi^A = \alpha \cdot 1_{T_{\tilde{x}_0}M^A}$.

Corollary 7.5. *Let Φ be the affine symmetry at a point $x_0 \in M$ with respect to an affine connection ∇ on M. Then Φ^A is the affine symmetry of M^A at $\tilde{x}_0$ with respect to ∇^A.*

Proof. Since Φ leaves ∇ invariant, Φ^A leaves ∇^A invariant by Lemma 7.2. Next, since Φ is the affine symmetry we see that $T_{x_0}\Phi = -1_{T_{x_0}M}$. Thus by Lemma 7.4 we get $T_{\tilde{x}_0}\Phi^A = -1_{T_{\tilde{x}_0}M^A}$, which means that Φ^A is the affine symmetry at $\tilde{x}_0$.

Proposition 7.6. *Let ∇ be an affine connection on M and let $X \in T_0^1(M)$ be an infinitesimal affine transformation of ∇. Then, for any $a \in A$, aX^A is also an infinitesimal affine transformation of ∇^A.*

Proof. A necessary and sufficient condition for X to be an infinitesimal affine transformation of M is that

$$L_X \circ \nabla_Y - \nabla_Y \circ L_X = \nabla_{[X,Y]}$$

for every $Y \in \mathscr{T}^1_0(M)$, where L_X (or $L(X)$) denotes the Lie derivation with respect to X. Therefore we have to prove

$$(7.2) \qquad L(a\cdot X^A)(\nabla^A(\tilde{Y}, K)) - \nabla^A(\tilde{Y}, L(aX^A)K) = \nabla^A([aX^A, \tilde{Y}], K)$$

for every $K \in \mathscr{T}(M^A)$ and $\tilde{Y} \in \mathscr{T}^1_0(M^A)$. To prove (7.2) it suffices to prove (7.2) for the special cases, where $\tilde{Y} = bY^A$ with $Y \in \mathscr{T}^1_0(M)$, $b \in A$, and $K = c\cdot Z^A$ or θ^A with $Z \in \mathscr{T}^1_0(M)$, $\theta \in \mathscr{T}^0_1(M)$ and $c \in A$. Moreover, to prove (7.2) for $K = \theta^A$, it suffices to prove it for $\theta = df$ with $f \in \mathscr{T}^0_0(M)$.

If $K = cZ^A$, we calculate as follows:

$$\begin{aligned} L_{aX^A} \tilde{\nabla}_{bY^A} cZ^A - \tilde{\nabla}_{bY^A} L_{aX^A} cZ^A &= [aX^A, bc(\nabla_Y Z)^A] - \tilde{\nabla}_{bY^A}[aX^A, cZ^A] \\ &= abc[X, \nabla_Y Z]^A - bac(\nabla_Y[X, Z])^A \\ &= abc((L_X\nabla_Y - \nabla_Y L_X)Z)^A = abc(\nabla_{[X,Y]}Z)^A \\ &= \tilde{\nabla}_{[aX^A, bY^A]} cZ^A . \end{aligned}$$

If $K = df^A$, we have

$$\begin{aligned} &L_{aX^A} \tilde{\nabla}_{bY^A}(df^A) - \tilde{\nabla}_{bY^A} L_{aX^A}(df^A)(cZ^A) \\ &= (aX^A)(\tilde{\nabla}_{bY^A} df^A)(cZ^A) - (\tilde{\nabla}_{bY^A} df^A)(aX^A, cZ^A) - (\tilde{\nabla}_{bY^A} d(aX^A f^A))(cZ^A) \\ &= (aX^A)\{(bY^A)(cZ^A)(f^A) - (\tilde{\nabla}_{bY^A} cZ^A)f^A\} \\ &\quad - \{bY^A[aX^A, cZ^A]f^A - (\tilde{\nabla}_{bY^A}[aX^A, cZ^A])f^A\} \\ &\quad - \{bY^A(cZ^A)(aX^A)f^A - (\tilde{\nabla}_{bY^A} cZ^A)(aX^A)f^A\} \\ &= abc(\{L_X \circ \nabla_Y - \nabla_Y \circ L_X\}(df)(Z))^A = abc((\nabla_{[X,Y]}(df))Z)^A \\ &= abc([X, Y]Zf - (\nabla_{[X,Y]}Z)f)^A \\ &= [aX^A, bY^A](cZ^A)f^A - (\tilde{\nabla}_{[aX^A, bY^A]} cZ^A)f^A \\ &= (\tilde{\nabla}_{[aX^A, bY^A]}(df)^A)(cZ^A) . \end{aligned}$$

Theorem 7.7. *Let M be an affine symmetric space with connection ∇. Then M^A is also an affine symmetric space with connection ∇^A.*

Proof. Let G be the connected component of the group of all affine transformations of M. Then G operates transitively on M. Let $X_1, \cdots, X_m$ be a basis of the Lie algebra g of G. We denote by $X^* \in \mathscr{T}^1_0(M)$ the vector field induced by the one-parameter group of affine transformations generated by $X \in g$. Now we can show that aX^A is a left invariant vector field on the Lie group G^A and that $(a\cdot X^A)^* = a\cdot(X^*)^A$ holds for $a \in A$ (the detail will be omitted), which implies that $a\cdot(X^*)^A$ is complete, i.e., generates a global one-parameter group of affine transformations of M^A (cf. Proposition 7.6). Hence we see that any element of G^A is an affine transformation of M^A. The transitivity of G shows that dim $(\{X_x^* \mid X \in g\}) = \dim M$ for any $x \in M$, which implies

$$\dim (\{(a \cdot (X^*)^A)_{x'} \mid a \in A,\ X \in g\}) = \dim M^A$$

for any $x' \in M^A$ and hence the transitivity of G^A on M^A follows. On the other hand, by Corollary 7.5 we have an affine symmetry at $\tilde{x}_0$ of M^A for $x_0 \in M$. Hence M^A is affinely symmetric.

Proposition 7.8. *Let ∇ be an affine connection on M. If M^A is affinely symmetric with respect to ∇^A, then M is also so with respect to ∇.*

Proof. Consider the map $\zeta: M \to M^A$ defined by $(\zeta(x))f = f(x)$ for $x \in M$, $f \in C^\infty(M)$. Let $\gamma: I \to M$ be a curve on M, where I is an open interval in R. Put $\tilde{\gamma} = \zeta \circ \gamma$. From (6.4), we see that

$$\tilde{\Gamma}_{(j,0)}{}^{(i,\lambda)}{}_{(k,0)}(\zeta(x)) = \delta_0^\lambda \Gamma_j{}^i{}_k(x)$$

for $i, j, k = 1, \cdots, n$; $\lambda = 0, 1, \cdots, N$, from which we can verify that γ is a geodesic on M if and only if $\tilde{\gamma}$ is so on M^A. Further, we can conclude that the submanifold $\tilde{M} = \zeta(M)$ is a totally geodesic submanifold of M^A with respect to ∇^A and that the induced affine connection ∇' on M is isomorphic with ∇ by the diffeomorphism $\zeta: M \to \tilde{M}$.

Now, take an arbitrary point $x \in M$ and consider $\tilde{x} = \zeta(x) \in M^A$. Since M^A is affinely symmetric, there exists an affine symmetry Φ of M^A at $\tilde{x}$. Since $T_{\tilde{x}}\Phi = -1_{T_{\tilde{x}}M^A}$, and $\tilde{M}$ is totally geodesic, we see that $\Phi(\tilde{M}) = \tilde{M}$ and that $\Phi|_{\tilde{M}}: \tilde{M} \to \tilde{M}$ is an affine transformation of ∇'. Then $\Phi|_{\tilde{M}}$ induces the affine symmetry $\Psi: M \to M$ of M at x.

References

[1] C. Ehresmann, *Les prolongements d'une variété différentiable.* I, *Calcul des jets, prolongement principal,* C. R. Acad. Sci. Paris **233** (1951) 598–600.

[2] ——, *Les prolongements d'une variété différentiable.* II, *L'espace des jets d'ordre r de* V_n *dans* V_m, C. R. Acad. Sci. Paris **233** (1951) 777–779.

[3] S. Helgason, *Differential geometry and symmetric spaces,* Academic Press, New York, 1962.

[4] A. J. Ledger & K. Yano, *The tangent bundle of a locally* symmetric space, J. London Math. Soc. **40** (1965) 487–492.

[5] A. Morimoto, *Liftings of some types of tensor fields and connections to tangent bundles of* p^r*-velocities,* Nagoya Math. J. **40** (1970) 13–31.

[6] ——, *Prolongations of geometric structures,* Lecture Notes, Nagoya University, 1969.

[7] A. Weil, *Théorie des points proches sur les variété différentiables,* Colloq. Topologie Géométrie Différentielle, Strasbourg, 1953, 111–117.

[8] K. Yano & S. Ishihara, *Differential geometry of tangent bundles of order* 2, Kōdai Math. Sem. Rep. **20** (1968) 318–354.

[9] ——, *Tangent and cotangent bundles,* Marcel Dekker, New York, 1973.

[10] K. Yano & S. Kobayashi, *Prolongations of tensor fields and connections to tangent bundles.* I, J. Math. Soc. Japan **18** (1966) 194–210.

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