# FULLY NON-LINEAR ELLIPTIC EQUATIONS ON COMPACT HERMITIAN MANIFOLDS

Gábor Székelyhidi

## Abstract

We derive a priori estimates for solutions of a general class of fully non-linear equations on compact Hermitian manifolds. Our method is based on ideas that have been used for different specific equations, such as the complex Monge–Ampère, Hessian and inverse Hessian equations. As an application we solve a class of Hessian quotient equations on Kähler manifolds assuming the existence of a suitable subsolution. The method also applies to analogous equations on compact Riemannian manifolds.

## 1. Introduction

Let $(M, \alpha)$ be a compact Hermitian manifold of dimension $n$, and fix a real $(1,1)$-form $\chi$. For any $C^2$ function $u : M \to \mathbf{R}$ we obtain a new real $(1,1)$-form $g = \chi + \sqrt{-1}\partial\bar{\partial}u$, and we can define the endomorphism of $T^{1,0}M$ given by $A^i_j = \alpha^{i\bar{p}}g_{j\bar{p}}$. This is a Hermitian endomorphism with respect to the metric $\alpha$. We consider equations for $u$ that can be written in the form

$$(1) \qquad\qquad F(A) = h,$$

for a given function $h$ on $M$, where

$$(2) \qquad\qquad F(A) = f(\lambda_1, \ldots, \lambda_n)$$

is a smooth symmetric function of the eigenvalues of $A$. Such equations have been studied extensively in the literature, going back to the work of Caffarelli–Nirenberg–Spruck [5] on the Dirichlet problem in the real case, when $\alpha$ is the Euclidean metric and $M$ is a domain in $\mathbf{R}^n$.

We assume that $f$ is defined in an open symmetric cone $\Gamma \subsetneq \mathbf{R}^n$, with vertex at the origin, containing the positive orthant $\Gamma_n$. In addition,

(i) $f_i > 0$ for all $i$, and $f$ is concave,
(ii) $\sup_{\partial\Gamma} f < \inf_M h$,
(iii) For any $\sigma < \sup_\Gamma f$ and $\lambda \in \Gamma$ we have $\lim_{t\to\infty} f(t\lambda) > \sigma$.

Assumption (ii) ensures that the relevant level sets of $f$ do not intersect the boundary of $\Gamma$. Assumption (iii) is satisfied by many natural equations, for instance, if $f$ is homogeneous of degree 1 and $f > 0$ in $\Gamma$.

**Definition 1.** We say that a smooth function $\underline{u}$ is a $\mathcal{C}$-subsolution of (1), if the following condition holds. At each $x \in M$, define the matrix $B_j^i = \alpha^{i\bar{p}}(\chi_{j\bar{p}} + \partial_j \partial_{\bar{p}} \underline{u})$. Then we require that for each $x \in M$ the set

$$(3) \qquad \{\lambda' \in \Gamma \,:\, f(\lambda') = h(x) \text{ and } \lambda' - \lambda(B(x)) \in \Gamma_n\}$$

is bounded, where $\lambda(B(x))$ denotes the $n$-tuple of eigenvalues of $B(x)$.

In Section 2, we will describe the relationship between this notion and that introduced by Guan [20]. Our main result is the following.

**Theorem 2.** *Suppose that $u$ is a solution, and $\underline{u}$ is a $\mathcal{C}$-subsolution of Equation 1. If we normalize $u$ so that $\sup_M u = 0$, then we have an estimate $\|u\|_{C^{2,\alpha}} < C$, where $C$ depends on the given data $M, \alpha, \chi, h$, and the subsolution $\underline{u}$.*

Note that the main result of Guan [20] is a similar estimate on Riemannian manifolds, but there the constant $C$ depends, in addition, on a $C^1$-bound for $u$. In the Riemannian case this $C^1$-bound can be obtained under certain extra assumptions, as shown in [20], using work of Li [29] and Urbas [45]. In addition, the subsolution condition in [20] is more restrictive than ours. As we will discuss in Section 8, our methods apply with almost no change to the Riemannian case as well, resulting in an estimate analogous to Theorem 2.

We first prove a $C^0$-estimate, generalizing the approach of Blocki [2], using the Alexandroff–Bakelman–Pucci maximum principle, in the case of the complex Monge–Ampère equation. For higher order estimates we use the method that was employed in the case of the complex Hessian equations. In other words a $C^1$-bound is derived by combining a second derivative bound of the form

$$(4) \qquad \sup |\partial \overline{\partial} u| \le C(1 + \sup |\nabla u|^2),$$

due to Hou–Ma–Wu [25] in the case of the Hessian equation, with a blowup argument and Liouville-type theorem due to Dinew-Kolodziej [11]. The gradient bound combined with (4) then bounds $|\partial \overline{\partial} u|$, at which point the Evans–Krylov theory [12, 26], adapted to the complex setting (see, for instance, Tosatti–Wang–Weinkove–Yang [38]) can be used to obtain the required $C^{2,\alpha}$-estimate. Note that as a consequence of the blowup argument the constant $C$ is not explicit in Theorem 2.

Perhaps the most important equation of the form (1) is the complex Monge–Ampère equation, where we take $f = \log \lambda_1 \cdot \ldots \cdot \lambda_n$. The Monge–Ampère equation was first solved on compact Kähler manifolds

by Yau [**47**], and on compact Hermitian manifolds by Tosatti-Weinkove [**41**] with some earlier work by Cherrier [**6**], Hanani [**24**] and Guan–Li [**22**]. See also Phong–Song–Sturm [**31**] for a recent survey. Note that in this case $\underline{u}$ being a $\mathcal{C}$-subsolution is equivalent to $\chi + \sqrt{-1}\partial\bar{\partial}\underline{u}$ being positive definite, and so Theorem 2 can be used to recover these existence results.

A related equation, the Monge–Ampère equation for $(n-1)$-plurisubharmonic functions, was introduced by Fu–Wang–Wu [**16**]. In this case we take $f = \log \widetilde{\lambda}_1 \cdot \ldots \cdot \widetilde{\lambda}_n$, where

$$(5) \qquad \widetilde{\lambda}_k = \frac{1}{n-1} \sum_{i \neq k} \lambda_i,$$

for each $k$. In terms of matrices, if we define the map $P(N) = \frac{1}{n-1}(\mathrm{Tr}(N) \cdot I - N)$, then the corresponding operator $F$ is given by $F(A) = \log \det P(A)$. For this equation, Theorem 2 recovers the a priori estimates of Tosatti–Weinkove [**40**, **39**], and more general equations in terms of the $\widetilde{\lambda}_k$ can be considered as well. We will discuss this in more detail in Section 7.

A setting when the subsolution property is more subtle is the inverse $\sigma_k$-equations for $1 \leq k \leq n-1$, where we take

$$(6) \qquad f = \left(\frac{\sigma_n}{\sigma_k}\right)^{\frac{1}{n-k}},$$

for the elementary symmetric functions $\sigma_i$, and the cone $\Gamma = \Gamma_n$. When $h$ is constant, the equation can be written as

$$(7) \qquad \omega^{n-k} \wedge \alpha^k = c\omega^n,$$

for a constant $c$, where $\omega = \chi + \sqrt{-1}\partial\bar{\partial}u$ is the unknown metric. When $\alpha, \chi$ are Kähler, then we can determine $c$ a priori, since

$$(8) \qquad c = \frac{[\omega]^{n-k} \cup [\alpha]^k}{[\omega]^n}.$$

Fixing this value of $c$, if $k = n-1$ then Song–Weinkove [**32**] showed that a solution exists if there is a metric $\chi' = \chi + \sqrt{-1}\partial\bar{\partial}\underline{u}$ satisfying

$$(9) \qquad nc\chi'^{n-1} - (n-1)\chi'^{n-2} \wedge \alpha > 0,$$

in the sense of positivity of $(n-1, n-1)$-forms. This turns out to be the same as $\underline{u}$ being a $\mathcal{C}$-subsolution. This result was later generalized by Fang–Lai–Ma [**14**] to general $k$, and existence results for general $k$ and non-constant $h$ on Hermitian manifolds were obtained by Guan–Sun [**23**], Sun [**35**].

Using the continuity method, Theorem 2 can be used to obtain such existence results for Equation (1), under certain assumptions, however, it seems to be difficult to state a satisfactory general existence result, whenever the subsolution condition is non-trivial in the sense that it

depends on $h$. We give one such result, Proposition 26 in the Riemannian case, and the same proof works in the Hermitian case too. The difficulty is that one needs extra conditions to ensure that we have a subsolution along the whole continuity path (see also Guan–Sun [**23**] for such results in the case of the inverse $\sigma_k$ equation). The source of this difference between Equation (1) on a compact manifold, and the corresponding Dirichlet problem, is that on a compact manifold the constant functions are in the cokernel of the linearization.

As an illustration, we will consider general Hessian quotient equations, of the form

$$(10) \qquad \omega^l \wedge \alpha^{n-l} = c\omega^k \wedge \alpha^{n-k},$$

where $(M, \alpha)$ is Kähler, $1 \le l < k \le n$, the form $\omega = \chi + \sqrt{-1}\partial\overline{\partial}u$ is the unknown, and $c$ is determined by

$$(11) \qquad c = \frac{\int_M \chi^l \wedge \alpha^{n-l}}{\int_M \chi^k \wedge \alpha^{n-k}}.$$

In analogy with the results of Song–Weinkove and Fang–Lai–Ma for the case $k = n$, we will show the following.

**Corollary 3.** *Suppose that there is a form $\chi' = \chi + \sqrt{-1}\partial\overline{\partial}\underline{u}$ which is $k$-positive (i.e. the eigenvalues satisfy $\sigma_1, \ldots, \sigma_k > 0$), and, in addition,*

$$(12) \qquad kc\chi'^{k-1} \wedge \alpha^{n-k} - l\chi'^{l-1} \wedge \alpha^{n-l} > 0,$$

*in the sense of positivity of $(n-1, n-1)$-forms. Then (10) has a solution $\omega = \chi + \sqrt{-1}\partial\overline{\partial}u$.*

There do not seem to be any previous existence results on compact manifolds for these equations in the literature when $k < n$, although a priori $C^0$ bounds for the solution $u$ have been found recently by Sun [**36**, **37**]. The corresponding Dirichlet problem on Euclidean domains does not fit into the framework of Caffarelli–Nirenberg–Spruck [**5**], but was subsequently solved by Trudinger [**44**].

It is an interesting problem to find geometric assumptions under which the existence of a $\mathcal{C}$-subsolution can be ensured. In the case of the Dirichlet problem in Euclidean domains $\Omega$, Caffarelli–Nirenberg–Spruck [**5**] showed that a subsolution exists under a suitable convexity type condition on the boundary $\partial\Omega$ (see also Li [**28**] for analogous results in the complex case). For the complex Monge–Ampère equation on compact Kähler manifolds, the result of Demailly–Paun [**10**], characterizing the Kähler cone, gives such a geometric condition. Indeed, this result shows that a real $(1, 1)$-class $[\chi]$ on a compact Kähler manifold $(M, \alpha)$ contains a Kähler metric, if and only if for all analytic subvarieties $V \subset M$ of dimension $p = 1, \ldots, n$ we have

$$(13) \qquad \int_V \chi^k \wedge \alpha^{p-k} > 0, \text{ for } 1 \le k \le p.$$

In [**27**], Lejmi and the author proposed a similar condition, conjectured to ensure the existence of a metric $\chi' \in [\chi]$ satisfying the positivity condition (9). The condition is that $[\chi]$ admits a Kähler metric, and, in addition,

$$
\text{(14)} \qquad \int_V c\chi^p - p\chi^{p-1} \wedge \alpha > 0,
$$

for all analytic subvarieties $V \subset M$ of dimension $p = 1, \ldots, n-1$. For $V = M$ equality has to hold by (8). Recently, in [**8**], Collins and the author resolved this conjecture on toric manifolds. We expect that analogous results should hold for a large class of equations on Kähler manifolds, and we state a conjecture to this effect for the Hessian quotient equations in Section 7. In addition, it is natural to expect that for the Dirichlet problem on Kähler manifolds with boundary, the appropriate subsolutions can be constructed whenever the boundary satisfies a suitable convexity assumption, and a geometric condition as above is satisfied for all compact subvarieties of the interior. We hope to explore such results in future work.

In Section 2, we give the basic definition and properties of $\mathcal{C}$-subsolutions. We prove $C^0$-estimates in Section 3, generalizing the approach of Blocki [**2**]. We prove a $C^2$-estimate of the form (4) in Section 4, modeled on the work of Hou–Ma–Wu [**25**]. To complete the proof of Theorem 2 we use a blowup argument and Liouville-type theorem analogous to those of Dinew–Kolodziej [**11**] in Sections 5, 6. In Section 7, we give the proof of Corollary 3. Finally, in Section 8, we discuss analogous problems on compact Riemannian manifolds.

## 2. Subsolutions

As in the introduction, let $\Gamma \subsetneq \mathbf{R}^n$ be a symmetric, open, convex cone with vertex at the origin, containing the positive orthant $\Gamma_n$, and let $f : \Gamma \to \mathbf{R}$ be a smooth, concave function, satisfying the monotonicity condition $f_i > 0$ for all $i$. We denote by $\mathcal{F}$ the function $\mathcal{F}(\lambda) = \sum_i f_i(\lambda)$.

Define

$$
\text{(15)} \qquad \sup_{\partial\Gamma} f = \sup_{\lambda' \in \partial\Gamma} \limsup_{\lambda \to \lambda'} f(\lambda).
$$

For any $\sigma > \sup_{\partial\Gamma} f$, the set

$$
\text{(16)} \qquad \Gamma^\sigma = \{\lambda \, : \, f(\lambda) > \sigma\}
$$

is a convex open set. Fix a value of $\sigma$ for which $\Gamma^\sigma \neq \emptyset$. Then the level set $\partial \Gamma^\sigma = f^{-1}(\sigma)$ is a smooth hypersurface. In view of Definition 1 we are interested in those $\mu \in \mathbf{R}^n$, for which the set $(\mu + \Gamma_n) \cap \partial \Gamma^\sigma$ is bounded. These $\mu$ represent the possible eigenvalues of a $\mathcal{C}$-subsolution.

For any $\lambda \in \partial \Gamma^\sigma$ let us write $\mathbf{n}_\lambda$ for the inward pointing unit normal vector, i.e.,

$$
(17) \qquad \mathbf{n}_\lambda = \frac{\nabla f}{|\nabla f|}.
$$

Note that since $f_i > 0$ for all $i$, we have

$$
(18) \qquad \sum_{i=1}^n f_i(\lambda)^2 \leq \left( \sum_{i=1}^n f_i(\lambda) \right)^2 \leq n \sum_{i=1}^n f_i(\lambda)^2,
$$

and so

$$
(19) \qquad |\nabla f| \leq \mathcal{F} \leq \sqrt{n} |\nabla f|.
$$

In particular, the unit normal $\mathbf{n}$ is a bounded multiple of $\mathcal{F}^{-1} \nabla f$.

**Remark 4.** Using this setup, Guan [20] introduced a convex open set $\mathcal{C}_\sigma^+ \subset \Gamma$, which consists of those $\mu$ for which the set

$$
(20) \qquad \partial \Gamma^\sigma(\mu) = \{ \lambda \in \partial \Gamma^\sigma \ : \ (\lambda - \mu) \cdot \mathbf{n}_\lambda > 0 \}
$$

is bounded. In turn this leads to a notion of subsolution for the equation $F(A) = h$ similar to Definition 1, except one requires that $\lambda(B) \in \mathcal{C}_{h(x)}^+$ at each point. Since $\mathbf{n}$ has positive entries, we have

$$
(21) \qquad (\mu + \Gamma_n) \cap \partial \Gamma^\sigma \subset \partial \Gamma^\sigma(\mu),
$$

and so this notion of subsolution is more restrictive than that of a $\mathcal{C}$-subsolution. As a simple example, if $\partial \Gamma^\sigma$ is a hyperplane, as is the case for a linear elliptic equation, then $\mathcal{C}_\sigma^+$ is empty, but $(\mu + \Gamma_n) \cap \Gamma^\sigma$ is bounded for all $\mu \in \mathbf{R}^n$.

The main result that we need is the following, which is a refinement of [20, Theorem 2.16].

**Proposition 5.** *Suppose that $\mu \in \mathbf{R}^n$ is such that for some $\delta, R > 0$*

$$
(22) \qquad (\mu - 2\delta \mathbf{1} + \Gamma_n) \cap \partial \Gamma^\sigma \subset B_R(0),
$$

*where $B_R(0)$ is the ball of radius $R$ around the origin and $\mathbf{1} = (1, 1, \ldots, 1)$.*

*Then there is a constant $\kappa > 0$ depending on $\delta$ and on the set in (22) (more precisely the normal vectors of $\partial \Gamma^\sigma$ on this set), such that if $\lambda \in \partial \Gamma^\sigma$ and $|\lambda| > R$, then either*

$$
(23) \qquad \sum_{i=1}^n f_i(\lambda)(\mu_i - \lambda_i) > \kappa \mathcal{F}(\lambda),
$$

*or $f_i(\lambda) > \kappa \mathcal{F}(\lambda)$ for all $i$.*

*Proof.* Consider the set

$$(24) \qquad A_\delta = \{v \in \Gamma \,:\, f(v) \le \sigma, \text{ and } v - \mu - \delta\mathbf{1} \in \overline{\Gamma_n}\}.$$

Because of (22) this is a compact set. For each $v \in A_\delta$ consider the cone $\mathcal{C}_v$ with vertex at the origin defined by

$$(25) \quad \mathcal{C}_v = \{w \in \mathbf{R}^n \,:\, v + tw \in (\mu - 2\delta\mathbf{1} + \Gamma_n) \cap \partial\Gamma^\sigma \text{ for some } t > 0\}.$$

In other words the cone $v + \mathcal{C}_v$ has vertex $v$ and cross section $(\mu - 2\delta\mathbf{1} + \Gamma_n) \cap \partial\Gamma^\sigma$. Since $f_i > 0$ for all $i$, the set $(\mu - 2\delta\mathbf{1} + \Gamma_n) \cap \partial\Gamma^\sigma$ is strictly larger than $(\mu - \delta\mathbf{1} + \Gamma_n) \cap \partial\Gamma^\sigma$, i.e.,

$$(26) \qquad \overline{(\mu - \delta\mathbf{1} + \Gamma_n) \cap \partial\Gamma^\sigma} \subset (\mu - 2\delta\mathbf{1} + \Gamma_n) \cap \partial\Gamma^\sigma.$$

This implies that the cone $\mathcal{C}_v$ is strictly larger than $\Gamma_n$. Let us denote by $\mathcal{C}_v^*$ the dual cone, i.e.,

$$(27) \qquad \mathcal{C}_v^* = \{x \in \mathbf{R}^n \,:\, \langle x, y \rangle > 0 \text{ for all } y \in \mathcal{C}_v\}.$$

Being strictly larger than $\Gamma_n$ means that there is an $\epsilon > 0$ such that if $x \in \mathcal{C}_v^*$ is a unit vector, then each entry of $x$ satisfies $x_i > \epsilon$. Since $A_\delta$ is compact, we can choose a uniform $\epsilon$ that works for all $v \in A_\delta$.

Suppose that $\lambda \in \partial\Gamma^\sigma$, and $|\lambda| > R$. Let $T_\lambda$ be the tangent plane to $\partial\Gamma^\sigma$ at $\lambda$. There are two possibilities:

- If $T_\lambda$ intersects $A_\delta$, in a point $v$, say, then the cone $v + \mathcal{C}_v$ lies above $T_\lambda$ (i.e. $\Gamma^\sigma$ lies on the same side of $T_\lambda$ as $v + \mathcal{C}_v$). This implies that the normal vector of $T_\lambda$ is in the dual cone, i.e. $\mathbf{n}_\lambda \in \mathcal{C}_v^*$. But then each entry of $\mathbf{n}_\lambda$ is greater than $\epsilon$, i.e., $f_i > \epsilon|\nabla f|$ for each $i$. Because of (19) this implies

$$(28) \qquad\qquad f_i > \frac{\epsilon}{\sqrt{n}}\mathcal{F},$$

  for each $i$.
- If $T_\lambda$ does not intersect $A_\delta$, then $\mu$ must be of distance at least $\delta$ from $T_\lambda$. This means that $(\mu - \lambda) \cdot \mathbf{n}_\lambda > \delta$. Writing this out in components, we have

$$(29) \qquad\qquad \sum_{i=1}^n f_i(\lambda)\,(\mu_i - \lambda_i) > \delta|\nabla f(\lambda)|,$$

  which by (19) implies (23). $\qquad\qquad$ q.e.d.

We need to apply this to the function $F$ defined on the space of Hermitian matrices $A$ by $F(A) = f(\lambda(A))$, where

$$(30) \qquad\qquad\qquad \lambda(A) = (\lambda_1, \ldots, \lambda_n)$$

denotes the eigenvalues of $A$. Let us write $F^{ij}$ for the derivative of $F$ with respect to the $ij$-entry of $A$. Then, similarly to Guan [20, Theorem 2.18], we have the following.

**Proposition 6.** *Let $[a,b] \subset (\sup_{\partial\Gamma} f, \sup_{\Gamma} f)$ and $\delta, R > 0$. There exists $\kappa > 0$ with the following property. Suppose that $\sigma \in [a,b]$ and $B$ is a Hermitian matrix such that*

$$(31) \qquad (\lambda(B) - 2\delta\mathbf{1} + \Gamma_n) \cap \partial\Gamma^\sigma \subset B_R(0).$$

*Then for any Hermitian matrix $A$ with $\lambda(A) \in \partial\Gamma^\sigma$ and $|\lambda(A)| > R$ we either have*

$$(32) \qquad \sum_{p,q} F^{pq}(A)\left[B_{pq} - A_{pq}\right] > \kappa \sum_p F^{pp}(A),$$

*or $F^{ii}(A) > \kappa \sum_p F^{pp}(A)$ for all $i$.*

*Proof.* The proof is essentially the same as that of [**20**, Theorem 2.18], but we give some details for the reader's convenience. Suppose that $A$ is diagonal, and its eigenvalues satisfy $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. This implies that $F^{pq} = 0$ if $p \neq q$, and that $F^{11} \leq F^{22} \leq \ldots \leq F^{nn}$. Let $\mu_1, \ldots, \mu_n$ be the eigenvalues of $B$ ordered so that $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$. The matrix $B$ may not be diagonal, but the Schur–Horn theorem implies that the $n$-tuple of diagonal entries $(B_{11}, \ldots, B_{nn})$ is in the convex hull of the vectors obtained by permuting the entries of $(\mu_1, \ldots, \mu_n)$. In particular, it follows that

$$(33) \qquad \sum_i F^{ii}(A)B_{ii} \geq F^{ii}(A)\mu_i.$$

Since $A$ is diagonal, we have $F^{ii} = f_i(\lambda)$, and (31) implies that we can apply Proposition 5 to obtain the required inequalities. We obtain uniform $\kappa > 0$, since the assumptions on $\sigma$ imply that the sets $(\lambda(B) - 2\delta\mathbf{1} + \Gamma_n) \cap \partial\Gamma^\sigma$ move in a compact family.                          q.e.d.

We recall the definition of a $\mathcal{C}$-subsolution from the introduction.

**Definition 7.** Suppose, as in the introduction that $(M, \alpha)$ is Hermitian and $\chi$ is a real $(1,1)$-form. We say that $\underline{u}$ is a $\mathcal{C}$-subsolution for the equation $F(A) = h$, if at each $x \in M$ the set

$$(34) \qquad \left(\lambda\left[\alpha^{j\bar{p}}(\chi_{i\bar{p}} + \underline{u}_{i\bar{p}})\right] + \Gamma_n\right) \cap \partial\Gamma^{h(x)}$$

is bounded. Let us also say that $\underline{u}$ is admissible, if $\lambda\left[\alpha^{j\bar{p}}(\chi_{i\bar{p}} + \underline{u}_{i\bar{p}})\right] \in \Gamma$. An observation pointed out to the author by Wei Sun is that a $\mathcal{C}$-subsolution need not be admissible.

**Remark 8.** In examples it is useful to have an alternative description of the set of $\mathcal{C}$-subsolutions. Following Trudinger [**44**], let us denote by $\Gamma_\infty$ the projection of $\Gamma$ onto $\mathbf{R}^{n-1}$:

$$(35) \qquad \Gamma_\infty = \{(\lambda_1, \ldots, \lambda_{n-1}) : (\lambda_1, \ldots, \lambda_n) \in \Gamma \text{ for some } \lambda_n\}.$$

For $\mu \in \mathbf{R}^n$, the set $(\mu + \Gamma_n) \cap \partial\Gamma^\sigma$ is bounded, if and only if

$$(36) \qquad \lim_{t\to\infty} f(\mu + t\mathbf{e}_i) > \sigma,$$

for all $i$, where $\mathbf{e}_i$ is the $i^{\text{th}}$ standard basis vector. Note that this limit is defined as long as any $(n-1)$-tuple $\mu'$ in $\mu$ satisfies $\mu' \in \Gamma_\infty$. In other words it is defined for $\mu \in \widetilde{\Gamma}$, where $\widetilde{\Gamma} \subset \mathbf{R}^n$ is given by

$$(37) \quad \widetilde{\Gamma} = \{\mu \in \mathbf{R}^n : \text{ there exists } t > 0 \text{ such that } \mu + t\mathbf{e}_i \in \Gamma \text{ for all } i\}.$$

Note that $\Gamma \subset \widetilde{\Gamma}$.

For any $\lambda' = (\lambda_1, \ldots, \lambda_{n-1}) \in \Gamma_\infty$, consider the limit

$$(38) \qquad\qquad \lim_{\lambda_n \to \infty} f(\lambda_1, \ldots, \lambda_n).$$

Then as in [44] this limit is either finite for all $\lambda'$ or infinite for all $\lambda'$ because of the concavity of $f$. If the limit is infinite, then $(\mu + \Gamma_n) \cap \partial\Gamma^\sigma$ is bounded for all $\sigma$ and $\mu \in \widetilde{\Gamma}$. In particular, any admissible $\underline{u}$ is a $\mathcal{C}$-subsolution in this case.

If the limit is finite, define the function $f_\infty$ on $\Gamma_\infty$ by

$$(39) \qquad f_\infty(\lambda_1, \ldots, \lambda_{n-1}) = \lim_{\lambda_n \to \infty} f(\lambda_1, \ldots, \lambda_n).$$

From the above it is clear that $(\mu + \Gamma_n) \cap \partial\Gamma^\sigma$ for $\mu \in \widetilde{\Gamma}$ is bounded if and only if $f_\infty(\mu') > \sigma$, where $\mu' \in \Gamma_\infty$ denotes any $(n-1)$-tuple of entries of $\mu$.

We will need the following consequences of our structural assumptions for $f$.

**Lemma 9.** *Under the assumptions (i), (ii), (iii) for $f$ in the introduction, we have the following, for any $\sigma \in (\sup_{\partial\Gamma} f, \sup_\Gamma f)$:*

(a) *There is an $N > 0$ depending on $\sigma$, such that $\Gamma + N\mathbf{1} \subset \Gamma^\sigma$,*
(b) *there is a $\tau > 0$, depending on $\sigma$, such that $\mathcal{F}(\lambda) > \tau$ for any $\lambda \in \partial\Gamma^\sigma$.*

*Proof.* To prove (a), let $x \in \partial\Gamma^\sigma$ be the closest point to the origin. By the convexity of $\Gamma^\sigma$ and symmetry under permuting the variables, we must have $x = N\mathbf{1}$ for some $N > 0$. We claim that $\Gamma + N\mathbf{1} \subset \Gamma^\sigma$. Indeed, for any $\lambda \in \Gamma$, assumption (iii) implies that there is some $T > 1$, such that $T\lambda \in \Gamma^\sigma$. The convexity of $\Gamma^\sigma$ implies that then $x + t\lambda \in \Gamma^\sigma$ for all $t \in (0, T]$, and so, in particular, $x + \lambda \in \Gamma^\sigma$. This proves (a).

To prove (b), first choose $\sigma' > \sigma$ such that $\sigma' \in (\sup_{\partial\Gamma} f, \sup_\Gamma f)$ as well. Part (a) implies that if $f(\lambda) = \sigma$, then $f(\lambda + N\mathbf{1}) > \sigma'$. By concavity we have

$$(40) \qquad\qquad f(\lambda + N\mathbf{1}) \leq f(\lambda) + N \sum_{i=1}^{n} f_i(\lambda),$$

which implies $\mathcal{F}(\lambda) \geq N^{-1}(\sigma' - \sigma)$, which is the bound that we wanted.

<div align="right">q.e.d.</div>

## 3. $C^0$-estimates

In this section, we prove a priori $C^0$-estimates for solutions of Equation (1). We will use the following variant of the Alexandroff–Bakelman–Pucci maximum principle, similar to Gilbarg–Trudinger [18, Lemma 9.2].

**Proposition 10.** *Let $v : B(1) \to \mathbf{R}$ be smooth, such that $v(0) + \epsilon \leq \inf_{\partial B(1)} v$, where $B(1)$ denotes the unit ball in $\mathbf{R}^n$. Define the set*
(41)
$$
P = \left\{ x \in B(1) \; : \; \begin{array}{l} |Dv(x)| < \dfrac{\epsilon}{2}, \;\; and \\[6pt] v(y) \geq v(x) + Dv(x) \cdot (y - x) \;\; for \; all \; y \in B(1) \end{array} \right\}.
$$
*Then for a dimensional constant $c_0 > 0$ we have*
(42)
$$
c_0 \epsilon^n \leq \int_P \det(D^2 v).
$$

*Proof.* The proof follows the argument of [18, Lemma 9.2]. Consider the graph of $v$, and let $\xi \in \mathbf{R}^n$ be such that $|\xi| < \frac{\epsilon}{2}$. The graph of the function $l(x) = v(0) + \xi \cdot x$ lies below the graph of $v$ on the boundary $\partial B(1)$ by our assumption on $v$, and it intersects the graph of $v$ at $(0, v(0))$. This implies that for some $k \geq 0$, the graph of $l(x) - k$ is tangent to the graph of $v$ at some point $x \in B(1)$, and considering the largest such $k$ we will have $x \in P$. In particular, the ball $B(\epsilon/2)$ is in the image of $P$ under the gradient of $v$, i.e. $B(\epsilon/2) \subset \nabla v(P)$. The inequality (42) follows by comparing volumes.                q.e.d.

The main result of this section is the following.

**Proposition 11.** *Suppose that $F(A) = h$, where $A^{ij} = \alpha^{j\bar{p}} g_{i\bar{p}}$ and $g = \chi + \sqrt{-1}\partial\bar{\partial}u$ for a fixed background form $\chi$, as in the introduction. Assume that we have a $\mathcal{C}$-subsolution $\underline{u}$, and normalize $u$ so that $\sup(u - \underline{u}) = 0$. There is a constant $C$, depending on the given data, including $\underline{u}$ such that*
(43)
$$
\sup_M |u| < C.
$$

*Proof.* Our proof is based on the method that Blocki [2] (see also [3]) used in the case of the complex Monge–Ampère equation. To simplify notation we can assume $\underline{u} = 0$, by changing $\chi$. We, therefore, have $\sup_M u = 0$, and our goal is to obtain a lower bound for $L = \inf_M u$

Note that our assumptions for $\Gamma$ imply (see [5]) that
(44)
$$
\Gamma \subset \{(\lambda_1, \ldots, \lambda_n) \; : \; \sum_i \lambda_i > 0\},
$$

which in turn implies that $\text{tr}_\alpha g > 0$. It follows that we have a lower bound for $\Delta u$ (the complex Laplacian with respect to $\alpha$), and so using the Green's function of a Gauduchon metric conformal to $\alpha$ as in Tosatti–Weinkove [42], we have a uniform bound for $\|u\|_{L^1}$.

An alternative, more local argument for obtaining a bound for $\||u|^p\|_{L^1}$ for some $p > 0$ follows from the weak Harnack inequality, Gilbarg–Trudinger [**18**, Theorem 9.22]. Indeed, suppose that we cover $M$ with a finite number of coordinate balls $2B_i$, so that the balls $B_i$ of half the radius still cover $M$. Using the lower bound $\Delta u \geq -C$, and the fact that $u$ is non-positive, we can apply the weak Harnack inequality in each ball $2B_i$ to obtain an estimate of the form

$$(45) \qquad \left(\int_{B_i} (-u)^p\right)^{1/p} \leq C_1\big[\inf_{B_i}(-u) + 1\big],$$

with $p, C_1 > 0$ depending only on the covering and the metric $\alpha$. Our assumption $\sup_M u = 0$ implies that we have one ball, say $B_0$ in which $\inf_{B_0}(-u) = 0$. Then (45) implies a bound on the integral of $|u|^p$ on $B_0$. But this in turn gives a bound $\inf_{B_i} |u| < C_2$, for all balls $B_i$ with $B_i \cap B_0 \neq \emptyset$, and then (45) can be used again to bound the integrals of $|u|^p$ on each such $B_i$. Continuing in this way, we will end up with bounds on each ball, and so we obtain an a priori bound for $\||u|^p\|_{L^1}$ on $M$.

Being a $\mathcal{C}$-subsolution means that for each $x$ the set

$$(46) \qquad \big(\lambda(\alpha^{j\bar{p}}\chi_{i\bar{p}}) + \Gamma_n\big) \cap \partial\Gamma^{h(x)}$$

is bounded. There is then a $\delta > 0$ and $R > 0$ such that at each $x$ we have

$$(47) \qquad (\lambda(\alpha^{j\bar{p}}\chi_{i\bar{p}}) - \delta\mathbf{1} + \Gamma_n) \cap \partial\Gamma^{h(x)} \subset B_R(0).$$

Let us work in local coordinates $z_i$, for which the infimum $L$ is achieved at the origin, and the coordinates are defined for $|z_i| < 1$, say. We write $B(1) = \{z : |z| < 1\}$. Let $v = u + \epsilon|z|^2$ for a small $\epsilon > 0$. We have $\inf v = L = v(0)$, and $v(z) \geq L + \epsilon$ for $z \in \partial B(1)$. From Proposition 10 we obtain

$$(48) \qquad c_0\epsilon^{2n} \leq \int_P \det(D^2 v),$$

where $P$ is defined as in (41). As in Blocki [**2**], at any point $x \in P$ we have $D^2 v(x) \geq 0$ and so

$$(49) \qquad \det(D^2 v) \leq 2^{2n} \det(v_{i\bar{j}})^2.$$

At the same time, if $x \in P$, then $D^2 v(x) \geq 0$ implies that $u_{i\bar{j}}(x) \geq -\epsilon\delta_{i\bar{j}}$. If $\epsilon$ is sufficiently small (depending on the metric $\alpha$ and the choice of $\delta$), then this implies that at $x \in P$

$$(50) \qquad \lambda\big[\alpha^{j\bar{p}}(\chi_{i\bar{p}} + u_{i\bar{p}})\big] \in \lambda(\alpha^{j\bar{p}}\chi_{i\bar{p}}) - \delta\mathbf{1} + \Gamma_n.$$

According to the equation $F(A) = h$ we also have $\lambda\big[\alpha^{j\bar{p}}(\chi_{i\bar{p}} + u_{i\bar{p}})\big] \in \partial\Gamma^{h(x)}$ at $x$, so from (47) we get an upper bound $|u_{i\bar{j}}| < C$. This gives

a bound for $v_{i\bar{j}}$ at any $x \in P$, so from (49) and (48) we get

$$
(51) \qquad c_0 \epsilon^{2n} \leq C' \text{vol}(P).
$$

By definition, for $x \in P$ we have $v(0) > v(x) - \epsilon/2$, and so $v(x) < L + \epsilon/2$. This implies

$$
(52) \qquad \text{vol}(P) \leq \frac{\||v|^p\|_{L^1}}{|L + \frac{\epsilon}{2}|^p}.
$$

Since we already have a bound for $\||v|^p\|_{L^1}$, this inequality contradicts (51) if $|L|$ is very large. $\hfill$ q.e.d.

**Remark 12.** This method can also be used to obtain $C^0$-estimates for more general types of equations, where the matrix $A$ in the equation $F(A) = h$ depends on the gradient of $u$ as well. We illustrate this with an example taken from the recent work of Tosatti–Weinkove [**39**] on $(n-1)$-plurisubharmonic functions. On a Hermitian manifold $(M, \alpha)$, given another Hermitian metric $\chi$ the equation can be written as

$$
(53) \qquad \det\left(\chi + \frac{1}{n-1}\big[(\Delta u)\alpha - \sqrt{-1}\partial\bar{\partial}u\big] + *E\right) = e^h \det\alpha,
$$

where $*$ is the Hodge star operator of $\alpha$ and

$$
(54) \qquad E = \frac{1}{(n-1)!}\text{Re}\big[\sqrt{-1}\partial u \wedge \bar{\partial}(\alpha^{n-2})\big].
$$

In addition, $\Delta$ is the complex Laplacian with respect to $\alpha$. It is assumed that $\alpha$ is a Gauduchon metric, and the form inside the determinant in (53) is positive definite. Normalizing $u$ so that $\sup_M u = 0$, it is shown in [**39**] that this implies an $L^1$-bound $\|u\|_{L^1} < C$. An argument using the weak Harnack inequality as above can also be used to bound $\||u|^p\|_{L^1}$ for some $p > 0$. In [**39**] this is then used together with a Moser iteration argument to bound $\inf_M u$. We can obtain a different proof of this bound along the lines of the argument above.

As in the proof of Proposition 11, choose coordinates $z_i$ in which the infimum $L = \inf_M u$ is achieved at the origin and for a small $\epsilon > 0$ we let $v = u + \epsilon|z|^2$. We apply Proposition 10 to obtain

$$
(55) \qquad c_0 \epsilon^{2n} \leq \int_P \det(D^2 v),
$$

with the set $P$ in (41). By definition, if $x \in P$, then we have $D^2 v \geq 0$ and so $u_{i\bar{j}}(x) \geq -\epsilon\delta_{ij}$, and, in addition, $|Dv(x)| < \epsilon/2$. If $\epsilon$ is chosen sufficiently small (depending on $\chi$ and $\alpha$), then Equation (53) implies an upper bound for $u_{i\bar{j}}(x)$ at all $x \in P$, using that $\chi$ is positive definite. From (55) we then get

$$
(56) \qquad c_0 \epsilon^{2n} \leq C' \text{vol}(P),
$$

but as before,

$$\text{(57)} \qquad \text{vol}(P) \leq \frac{\||v|^p\|_{L^1}}{|L + \frac{\epsilon}{2}|^p},$$

which is a contradiction if $|L|$ is too large.

It is an interesting problem whether the $C^2$-estimate in Section 4 can also be extended to more general equations $F(A) = h$, where $A$ depends on the gradient of $u$ as well. In particular, this estimate is not known at present for Equation (53). Note that in the Riemannian case $C^2$-estimates for such equations, under certain conditions, are derived in Guan–Jiao [**21**].

## 4. $C^2$-estimates

Our goal in this section is the following estimate for the complex Hessian of $u$ in terms of the gradient. As in the introduction we assume that $u$ satisfies an equation of the form $F(A) = h$, where $A^{ij} = \alpha^{j\bar{p}} g_{i\bar{p}}$ and $g_{i\bar{j}} = \chi_{i\bar{j}} + u_{i\bar{j}}$ for a given form $\chi$. In addition, we assume the existence of a $\mathcal{C}$-subsolution $\underline{u}$.

**Proposition 13.** *We have an estimate*

$$\text{(58)} \qquad |\partial\bar{\partial}u| \leq C(1 + \sup_M |\nabla u|_\alpha^2),$$

*where the constant depends on the background data, in particular, $\|\alpha\|_{C^2}$, $\|h\|_{C^2}$, $\|\chi\|_{C^2}$ and the subsolution $\underline{u}$.*

To simplify notation, we will assume that the subsolution $\underline{u} = 0$, since otherwise we could simply modify the background form $\chi$. By definition this means that for each $x \in M$ the sets $(\lambda(B(x)) + \Gamma_n) \cap \partial\Gamma^{h(x)}$ are bounded, where $B^{ij} = \alpha^{j\bar{p}}\chi_{i\bar{p}}$. We can find $\delta, R > 0$ such that at each $x$,

$$\text{(59)} \qquad (\lambda(B) - 2\delta\mathbf{1} + \Gamma_n) \cap \partial\Gamma^{h(x)} \subset B_R(0).$$

In particular, by Proposition 6 we have a $\kappa > 0$ with the following property: at any $x \in M$, if $|\lambda(A)| > R$ and $A$ is diagonal with eigenvalues $\lambda_1, \ldots, \lambda_n$, then either

$$\text{(60)} \qquad F^{ii}(A) > \kappa \sum_p F^{pp}(A) \text{ for all } i,$$

or

$$\text{(61)} \qquad \sum_p F^{pp}(A)\big[B^{pp} - \lambda_p\big] > \kappa \sum_p F^{pp}(A).$$

Also, Lemma 9 implies that we have a constant $\tau > 0$ such that $\sum_p F^{pp}(A) > \tau$.

Our calculation will mostly follow that of Hou–Ma–Wu [**25**] which in turn is based on ideas in Chou–Wang [**7**]. One key difference is that instead of using $g_{1\bar{1}}$ in suitable coordinates, we use the maximum

eigenvalue of the matrix $A$. This introduces good positive terms which are useful in the Hermitian case, somewhat similar to the extra term $\log \alpha^{p\bar{q}} g_{1\bar{q}} g_{p\bar{1}}$ introduced in Tosatti–Weinkove [**39**]. The idea of exploiting the inequality (61) is from Guan [**20**]. A refinement of this also appears in Guan [**19**] where the two possibilities (60) and (61) are exploited, although the setup is not the same as ours.

We first review some basic formulas for the derivatives of eigenvalues which can be found in Spruck [**33**], for instance. The derivatives of the eigenvalue $\lambda_i$ at a diagonal matrix with distinct eigenvalues are

$$(62) \qquad \qquad \lambda_i^{pq} = \delta_{pi} \delta_{qi},$$

$$(63) \qquad \lambda_i^{pq,rs} = (1 - \delta_{ip}) \frac{\delta_{iq} \delta_{ir} \delta_{ps}}{\lambda_i - \lambda_p} + (1 - \delta_{ir}) \frac{\delta_{is} \delta_{ip} \delta_{rq}}{\lambda_i - \lambda_r},$$

where $\lambda_i^{pq}$ denotes the derivative with respect to the $pq$-entry.

It follows from this that for any symbols $A_k^{pq}$ we have

$$(64) \qquad \lambda_1^{pq,rs} A_k^{pq} A_{\bar{k}}^{rs} = \sum_{p>1} \frac{A_k^{p1} A_{\bar{k}}^{1p} + A_k^{1p} A_{\bar{k}}^{p1}}{\lambda_1 - \lambda_p}.$$

If $F(A) = f(\lambda_1, \ldots, \lambda_n)$ in terms of a symmetric function of the eigenvalues, then at a diagonal matrix $A$ with distinct eigenvalues we have (see Andrews [**1**], Gerhardt [**17**])

$$(65) \qquad \qquad F^{ij} = \delta_{ij} f_i,$$

$$(66) \qquad F^{ij,rs} = f_{ir} \delta_{ij} \delta_{rs} + \frac{f_i - f_j}{\lambda_i - \lambda_j} (1 - \delta_{ij}) \delta_{is} \delta_{jr}.$$

Note that these formulas make sense even when the eigenvalues are not distinct, since $F$ is a smooth function on the space of matrices if $f$ is symmetric. In particular, as $\lambda_i \to \lambda_j$ we also have $f_i \to f_j$. It follows that

$$(67) \qquad \begin{aligned} F^{ij,rs} u_{i\bar{j}k} u_{r\bar{s}k} &= f_{ij} u_{i\bar{i}k} u_{j\bar{j}k} + \sum_{p \neq q} \frac{f_p - f_q}{\lambda_p - \lambda_q} |u_{p\bar{q}k}|^2 \\ &\leq f_{ij} u_{i\bar{i}k} u_{j\bar{j}k} + \sum_{i>1} \frac{f_1 - f_i}{\lambda_1 - \lambda_i} |u_{i\bar{1}k}|^2, \end{aligned}$$

since if $f$ is concave and symmetric, one can show (see Spruck [**33**]) that $\frac{f_i - f_j}{\lambda_i - \lambda_j} \leq 0$. In particular, $f_i \leq f_j$ if $\lambda_i \geq \lambda_j$.

We want to apply the maximum principle to a function $G$ of the form

$$(68) \qquad \qquad G = \log \lambda_1 + \phi(|\nabla u|^2) + \psi(u),$$

where $\lambda_1 : M \to \mathbf{R}$ is the largest eigenvalue of the matrix $A$ at each point. Since the eigenvalues of $A$ need not be distinct at the point where $G$ achieves its maximum, we will perturb $A$ slightly.

To do this, choose local coordinates $z_i$, such that $G$ achieves its maximum at the origin, and at the origin $A$ is diagonal with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Let $B$ be a diagonal matrix such that $B^{11} = 0$ and $0 < B^{22} < \ldots < B^{nn}$ are small, satisfying $B^{nn} < 2B^{22}$. Define the matrix $\widetilde{A} = A - B$. At the origin, $\widetilde{A}$ has eigenvalues

$$(69) \qquad \widetilde{\lambda_1} = \lambda_1, \quad \widetilde{\lambda_i} = \lambda_i - B^{ii} \text{ if } i > 1.$$

Since these are distinct, the eigenvalues of $\widetilde{A}$ define smooth functions near the origin.

In the calculations below we use derivatives with respect to the Chern connection of $\alpha$. From the formulas for the derivatives of the $\widetilde{\lambda}_i$, we have

$$
\begin{aligned}
\widetilde{\lambda}_{1,k} &= \widetilde{\lambda}_1^{pq}(\widetilde{A}^{pq})_k = g_{1\bar{1}k} - (B^{11})_k \\
\widetilde{\lambda}_{1,k\bar{k}} &= \widetilde{\lambda}_1^{pq,rs}(\widetilde{A}^{pq})_k(\widetilde{A}^{rs})_{\bar{k}} + \widetilde{\lambda}_1^{pq}(\widetilde{A}^{pq})_{k\bar{k}} \\
&= g_{1\bar{1}k\bar{k}} + \sum_{p>1} \frac{|g_{p\bar{1}k}|^2 + |g_{1\bar{p}k}|^2}{\lambda_1 - \widetilde{\lambda}_p} \\
&\quad + (B^{11})_{k\bar{k}} - 2\mathrm{Re} \sum_{p>1} \frac{g_{p\bar{1}k}(B^{1\bar{p}})_{\bar{k}} + g_{1\bar{p}k}(B^{p\bar{1}})_{\bar{k}}}{\lambda_1 - \widetilde{\lambda}_p} \\
&\quad + \widetilde{\lambda}_1^{pq,rs}(B^{pq})_k(B^{rs})_{\bar{k}},
\end{aligned}
$$
(70)

where we used Equation (64). Note that $B$ is a constant matrix in our local coordinates, but its covariant derivatives may not vanish. The assumption $\sum_i \lambda_i > 0$ implies that $\sum_i \widetilde{\lambda}_i > 0$ if the matrix $B$ is sufficiently small, and so $|\widetilde{\lambda}_i| < (n-1)\lambda_1$ for all $i$, which implies $(\lambda_1 - \widetilde{\lambda}_p)^{-1} \geq (n\lambda_1)^{-1}$. Since we are trying to bound $\lambda_1$ from above, we can assume $\lambda_1 > 1$. We can also absorb the terms $g_{p\bar{1}k}(B^{1\bar{p}})_{\bar{k}}$ using

$$(71) \qquad \left| g_{p\bar{1}k}(B^{1\bar{p}})_{\bar{k}} \right| \leq \frac{1}{4}|g_{p\bar{1}k}|^2 + C|B^{nn}|^2,$$

using that the derivative of $B$ is controlled by $B^{nn}$. In addition, for $p > 1$ we have

$$(72) \qquad \frac{1}{\lambda_1 - \widetilde{\lambda}_p} \leq \frac{1}{B^{22}} < \frac{2}{B^{nn}},$$

and so

$$(73) \qquad \frac{|B^{nn}|^2}{\lambda_1 - \widetilde{\lambda}_p} + \widetilde{\lambda}_1^{pq,rs}(B^{pq})_k(B^{rs})_{\bar{k}} = O(B^{nn}).$$

It follows that

$$(74) \qquad \widetilde{\lambda}_{1,k\bar{k}} \geq g_{1\bar{1}k\bar{k}} + \frac{1}{2n\lambda_1} \sum_{p>1}(|g_{p\bar{1}k}|^2 + |g_{1\bar{p}k}|^2) - 1,$$

if $B$ is chosen sufficiently small.

Using that $g_{i\bar{j}} = \chi_{i\bar{j}} + u_{i\bar{j}}$, we get

(75)
$$\widetilde{\lambda}_{1,k\bar{k}} \geq \chi_{1\bar{1}k\bar{k}} + u_{1\bar{1}k\bar{k}} + \frac{1}{2n\lambda_1} \sum_{p>1} (|\chi_{p\bar{1}k} + u_{p\bar{1}k}|^2 + |\chi_{1\bar{p}k} + u_{1\bar{p}k}|^2) - 1$$

$$\geq u_{1\bar{1}k\bar{k}} + \frac{1}{3n\lambda_1} \sum_{p>1} (|u_{p\bar{1}k}|^2 + |u_{1\bar{p}k}|^2) - C_0,$$

where $C_0$ is a constant depending only on the background data (including $\chi$). From here on out $C_0$ will always denote such a constant which may vary from line to line, but does not depend on other parameters that we choose later on.

Commuting derivatives, we obtain

(76)     $$u_{1\bar{1}k\bar{k}} = u_{k\bar{k}1\bar{1}} - 2\mathrm{Re}(u_{k\bar{p}1}\overline{T_{k1}^p}) + u_{i\bar{j}} * R + u_{i\bar{j}} * T * T,$$

where $R, T$ are the curvature and torsion of $\alpha$, and $*$ denotes a contraction. Using this in (75) we get

(77) $$\widetilde{\lambda}_{1,k\bar{k}} \geq u_{k\bar{k}1\bar{1}} + \frac{1}{3n\lambda_1} \sum_{p>1} (|u_{p\bar{1}k}|^2 + |u_{1\bar{p}k}|^2) - 2\mathrm{Re}(u_{k\bar{p}1}\overline{T_{k1}^p}) - C_0\lambda_1,$$

since $u_{i\bar{j}}$ is controlled by $\lambda_1$, and we assumed $\lambda_1 > 1$.

We have $u_{k\bar{p}1} = u_{1\bar{p}k} + u_{i\bar{j}} * T$. This means that we can absorb almost all of the terms $u_{k\bar{p}1}\overline{T_{k1}^p}$ using the good positive sum, except for $u_{k\bar{1}1}\overline{T_{k1}^1}$. We also rewrite $u$ in terms of $g$, to finally obtain

(78)                 $$\widetilde{\lambda}_{1,k\bar{k}} \geq g_{k\bar{k}1\bar{1}} - 2\mathrm{Re}(g_{k\bar{1}1}\overline{T_{k1}^1}) - C_0\lambda_1.$$

Differentiating the equation $F(A) = h$, we have

(79)                         $$h_1 = F^{ij}g_{i\bar{j}1} = F^{kk}g_{k\bar{k}1},$$

(80)                 $$h_{1\bar{1}} = F^{pq,rs}g_{p\bar{q}1}g_{r\bar{s}\bar{1}} + F^{kk}g_{k\bar{k}1\bar{1}},$$

using that $F^{ij}$ is diagonal at the origin (since $A$ is diagonal). Using this in Equation (78) we get

(81)     $$F^{kk}\widetilde{\lambda}_{1,k\bar{k}} \geq -F^{pq,rs}g_{p\bar{q}1}g_{r\bar{s}\bar{1}} - 2F^{kk}\mathrm{Re}(g_{k\bar{1}1}\overline{T_{k1}^1}) - C_0\lambda_1\mathcal{F},$$

where we wrote $\mathcal{F} = \sum_k F^{kk}$ and used that $\mathcal{F} > \tau$ to absorb a constant into $\lambda_1\mathcal{F}$. Defining the linearized operator $Lw = F^{ij}w_{i\bar{j}}$, we have

(82)
$$L(\log\widetilde{\lambda}_1) \geq \frac{-F^{pq,rs}g_{p\bar{q}1}g_{r\bar{s}\bar{1}}}{\lambda_1} - \frac{F^{kk}|g_{1\bar{1}k}|^2}{\lambda_1^2}$$
$$- \frac{F^{kk}}{\lambda_1}2\mathrm{Re}(g_{k\bar{1}1}\overline{T_{k1}^1}) - C_0\mathcal{F}.$$

We have

$$g_{1\bar{1}k} = \chi_{1\bar{1}k} + u_{1\bar{1}k}$$

(83)
$$= \chi_{1\bar{1}k} + u_{k\bar{1}1} - T_{k1}^1\lambda_1$$

$$= (\chi_{1\bar{1}k} - \chi_{k\bar{1}1}) + g_{k\bar{1}1} - T_{k1}^1\lambda_1,$$

and so

(84) $\qquad |g_{1\bar{1}k}|^2 \leq |g_{k\bar{1}1}|^2 - 2\lambda_1\mathrm{Re}(g_{k\bar{1}1}\overline{T_{k1}^1}) + C_0(\lambda_1^2 + |g_{1\bar{1}k}|).$

Using this and (83) again in Equation (82), we get
(85)
$$L(\log \widetilde{\lambda}_1) \geq \frac{-F^{pq,rs}g_{p\bar{q}1}g_{r\bar{s}\bar{1}}}{\lambda_1} - \frac{F^{kk}|g_{k\bar{1}1}|^2}{\lambda_1^2} - C_0(\mathcal{F} + \lambda_1^{-2}|F^{kk}g_{1\bar{1}k}|).$$

As a reminder, we note that in this calculation $\widetilde{\lambda}_1$ denotes the largest eigenvalue of the perturbed endomorphism $\widetilde{A} = A - B$. At the point where we are calculating, this coincides with the largest eigenvalue of $A$, but at nearby points it is a small perturbation. We could take $B \to 0$, and obtain the same differential inequality (85) for the largest eigenvalue of $A$ as well, but this would only hold in a viscosity sense because the largest eigenvalue of $A$ may not be $C^2$ at the origin, if some eigenvalues coincide.

We now begin the main calculation for proving Proposition 13.

*Proof of Proposition 13.* Set $K = \sup|\nabla u|^2 + 1$, and consider the function

(86) $$G = \log \widetilde{\lambda}_1 + \phi(|\nabla u|^2) + \psi(u),$$

where $\phi$ is the same as the function used in [25]:

(87) $$\phi(t) = -\frac{1}{2}\log\left(1 - \frac{t}{2K}\right),$$

and it satisfies

(88) $$(4K)^{-1} < \phi' < (2K)^{-1}, \quad \phi'' = 2\phi'^2 > 0.$$

We normalize $u$ so that $\inf u = 0$, so from Proposition 11 we already have a bound on $\sup u$. We then let $\psi : [0, \sup u] \to \mathbf{R}$ be defined by

(89) $$\psi(t) = -2At + \frac{A\tau}{2}t^2,$$

where $\tau$ is chosen sufficiently small depending on $\sup u$ (we decrease the $\tau$ from before if necessary), so that $\psi$ satisfies the bounds

(90) $$A \leq -\psi' \leq 2A, \quad \psi'' = A\tau.$$

Here $A$ is a large constant that we will choose later.

Let us write $w_k = -(B^{11})_k/\lambda_1$, which appears in the derivative of $\log \widetilde{\lambda}_1$. We assume $\lambda_1 > 1$, so this is a bounded quantity. We have

(91)
$$G_k = \frac{g_{1\bar{1}k}}{\lambda_1} + \phi'(u_{pk}u_{\bar{p}} + u_p u_{\bar{p}k}) + \psi' u_k + w_k,$$

$$G_{kk} = (\log \widetilde{\lambda}_1)_{k\bar{k}} + \phi'' \Big| u_{pk}u_{\bar{p}} + u_p u_{\bar{p}k} \Big|^2$$
$$+ \phi'\Big(u_{pk\bar{k}}u_{\bar{p}} + u_p u_{\bar{p}k\bar{k}} + \sum_p (|u_{pk}|^2 + |u_{\bar{p}k}|^2)\Big) + \psi'' u_k u_{\bar{k}} + \psi' u_{k\bar{k}}.$$

Commuting derivatives, we have the identities

(92)
$$u_{pk\bar{k}} = u_{k\bar{k}p} - T^q_{kp}u_{q\bar{k}} + R_{k\bar{k}p}{}^q u_q$$
$$= u_{k\bar{k}p} - T^k_{kp}\lambda_k + T^q_{kp}\chi_{q\bar{k}} + R_{k\bar{k}p}{}^q u_q,$$

and

(93)
$$u_{\bar{p}k\bar{k}} = u_{k\bar{k}\bar{p}} - \overline{T^q_{kp}}u_{k\bar{q}}$$
$$= u_{k\bar{k}\bar{p}} - \overline{T^k_{kp}}\lambda_k + \overline{T^q_{kp}}\chi_{k\bar{q}}.$$

Differentiating the equation $F(A) = h$ once, we have

(94)
$$F^{kk}u_{k\bar{k}p} = F^{kk}(g_{k\bar{k}p} - \chi_{k\bar{k}p}) = h_p - F^{kk}\chi_{k\bar{k}p},$$

and so

(95)
$$F^{kk}u_{pk\bar{k}}u_{\bar{p}}$$
$$= h_p u_{\bar{p}} - F^{kk}\chi_{k\bar{k}p}u_{\bar{p}} - T^k_{kp}F^{kk}\lambda_k u_{\bar{p}} + T^q_{kp}F^{kk}\chi_{q\bar{k}}u_{\bar{p}} + F^{kk}R_{k\bar{k}p}{}^q u_q u_{\bar{p}}$$
$$\geq -C_0(K^{1/2} + K^{1/2}\mathcal{F} + K^{1/2}\mathcal{F} + K\mathcal{F}) - \epsilon_1 F^{kk}\lambda_k^2 - C_{\epsilon_1}\mathcal{F}K$$
$$\geq -C_0 K\mathcal{F} - \epsilon_1 F^{kk}\lambda_k^2 - C_{\epsilon_1}\mathcal{F}K.$$

We have used the inequality (valid for each $k, p$)

(96)
$$|F^{kk}\lambda_k u_{\bar{p}}| \leq \epsilon_1 F^{kk}\lambda_k^2 + C_{\epsilon_1}F^{kk}K,$$

for any $\epsilon_1 > 0$ and corresponding $C_{\epsilon_1} > 0$, which implies that if we sum over $k$ then

(97)
$$|F^{kk}T^k_{kp}\lambda_k u_{\bar{p}}| \leq \epsilon_1 F^{kk}\lambda_k^2 + C_{\epsilon_1}\mathcal{F}K.$$

The same estimate also holds for $F^{kk}u_{\bar{p}k\bar{k}}u_p$ (which has fewer terms). We have $\phi' < (2K)^{-1}$, so combining these estimates we obtain

(98)
$$\phi' F^{kk}(u_{pk\bar{k}}u_{\bar{p}} + u_p u_{\bar{p}k\bar{k}}) \geq -C_0\mathcal{F} - \epsilon_1 K^{-1}F^{kk}\lambda_k^2 - C_{\epsilon_1}\mathcal{F}.$$

This implies that at the maximum of $G$

$$
\begin{aligned}
0 \geq LG \geq{}& L(\log \widetilde{\lambda}_1) + F^{kk}\phi'' \left| u_{pk}u_{\bar{p}} + u_p u_{\bar{p}k} \right|^2 \\
& + F^{kk}\phi' \sum_p (|u_{pk}|^2 + |u_{\bar{p}k}|^2) \\
& + \psi'' F^{kk} u_k u_{\bar{k}} + \psi' F^{kk} u_{k\bar{k}} - C_0 \mathcal{F} \\
\geq{}& \frac{-F^{ij,rs} g_{i\bar{j}1} g_{r\bar{s}1}}{\lambda_1} - \frac{F^{kk}|g_{k\bar{1}1}|^2}{\lambda_1^2} + F^{kk}\phi'' \left| u_{pk}u_{\bar{p}} + u_p u_{\bar{p}k} \right|^2 \\
& + F^{kk}\phi' \sum_p (|u_{pk}|^2 + |u_{\bar{p}k}|^2) + \psi'' F^{kk} u_k u_{\bar{k}} + F^{kk}\psi' u_{k\bar{k}} \\
& - C_0(\mathcal{F} + \lambda_1^{-2}|F^{kk} g_{1\bar{1}k}|) - C_{\epsilon_1}\mathcal{F} - \epsilon_1 K^{-1} F^{kk}\lambda_k^2.
\end{aligned}
\tag{99}
$$

We have

$$
\begin{aligned}
F^{kk}|u_{k\bar{k}}|^2 &= F^{kk}(\lambda_k - \chi_{k\bar{k}})^2 \\
&\geq \frac{1}{2} F^{kk}\lambda_k^2 - C_0\mathcal{F}.
\end{aligned}
\tag{100}
$$

Note that $\phi' > (4K)^{-1}$, so if we choose $\epsilon_1 = \frac{1}{16}$, we can use half of the $\phi' F^{kk}|u_{k\bar{k}}|^2$ term to cancel the negative $\epsilon_1$ term. Since this fixes $\epsilon_1$, we can absorb the $C_{\epsilon_1}$ term into $C_0$. It follows that

$$
\begin{aligned}
0 \geq{}& \frac{-F^{ij,rs} g_{i\bar{j}1} g_{r\bar{s}1}}{\lambda_1} - \frac{F^{kk}|g_{k\bar{1}1}|^2}{\lambda_1^2} + F^{kk}\phi'' \left| u_{pk}u_{\bar{p}} + u_p u_{\bar{p}k} \right|^2 \\
& + \frac{1}{32K} F^{kk}\lambda_k^2 + \frac{1}{16K} F^{kk} \sum_p (|u_{pk}|^2 + |u_{\bar{p}k}|^2) + \psi'' F^{kk} u_k u_{\bar{k}} \\
& + \psi' F^{kk} u_{k\bar{k}} - C_0(\mathcal{F} + \lambda_1^{-2}|F^{kk} g_{1\bar{1}k}|).
\end{aligned}
\tag{101}
$$

To deal with the final term, we use the equation $G_k = 0$. This implies that

$$
\frac{g_{1\bar{1}k}}{\lambda_1} = -\phi'(u_{pk}u_{\bar{p}} + u_p u_{\bar{p}k}) - \psi' u_k - w_k,
\tag{102}
$$

so

$$
\begin{aligned}
\lambda_1^{-2} F^{kk}|g_{1\bar{1}k}| \leq{}& \frac{1}{2K}\lambda_1^{-1} F^{kk}\left( |u_{pk}| + |u_{\bar{p}k}| \right) K^{1/2} \\
& + 2A\lambda_1^{-1} F^{kk}|u_k| + C_0\mathcal{F}.
\end{aligned}
\tag{103}
$$

It is clear that the term involving $|u_{pk}| + |u_{\bar{p}k}|$ can be absorbed by the fifth term in (101), up to adding a further multiple of $\mathcal{F}$. We, therefore,

have

$$LG \geq \frac{-F^{ij,rs}g_{i\bar{j}1}g_{r\bar{s}1}}{\lambda_1} - \frac{F^{kk}|g_{k\bar{1}1}|^2}{\lambda_1^2} + F^{kk}\phi''\Big|u_{pk}u_{\bar{p}} + u_p u_{\bar{p}k}\Big|^2 +$$

(104)
$$+ \frac{1}{20K}F^{kk}\left(\lambda_k^2 + \sum_p |u_{pk}|^2\right)$$

$$+ \psi''F^{kk}u_k u_{\bar{k}} + \psi'F^{kk}u_{k\bar{k}} - C_0(\mathcal{F} + A\lambda_1^{-1}F^{kk}|u_k|),$$

Following Hou–Ma–Wu [25] we deal with two cases separately, depending on whether $-\lambda_n > \delta\lambda_1$ or not, for a small $\delta > 0$ to be chosen later.

**Case 1:** $-\lambda_n > \delta\lambda_1$, so, in particular, $\lambda_n < 0$. We use the equation $G_k = 0$ to write

(105)
$$-\frac{F^{kk}|g_{1\bar{1}k}|^2}{\lambda_1^2} = -F^{kk}\Big|\phi'(u_{pk}u_{\bar{p}} + u_p u_{\bar{p}k}) + \psi'u_k + w_k\Big|^2$$

$$\geq -2\phi'^2 F^{kk}\Big|u_{pk}u_{\bar{p}} + u_p u_{\bar{p}k}\Big|^2 - 3(2A)^2\mathcal{F}K - C_0.$$

From (83) we also have

(106)
$$\frac{|g_{k\bar{1}1}|^2}{\lambda_1^2} \leq \frac{|g_{1\bar{1}k}|^2}{\lambda_1^2} + C_0(1 + \lambda_1^{-1}|g_{1\bar{1}k}|),$$

and using the inequality (103) absorbing the $|u_{pk}|, |u_{\bar{p}k}|$ terms in the same way, we obtain

(107)
$$\frac{F^{kk}|g_{k\bar{1}1}|^2}{\lambda_1^2} \leq \frac{F^{kk}|g_{1\bar{1}k}|^2}{\lambda_1^2} + 2AF^{kk}|u_k| + C_0\mathcal{F}$$

$$\leq \frac{F^{kk}|g_{1\bar{1}k}|^2}{\lambda_1^2} + 2A\mathcal{F}K^{1/2} + C_0\mathcal{F}.$$

Using this and (105) in our inequality (104) for $LG$, together with concavity of $F$ and $\phi'' = 2\phi'^2$, we find that at the origin

(108)
$$0 \geq \frac{1}{32K}F^{kk}\lambda_k^2 + \psi'F^{kk}u_{k\bar{k}} - C_0(\mathcal{F} + A\lambda_1^{-1}\mathcal{F}K^{1/2})$$

$$- 2A\mathcal{F}K^{1/2} - 12A^2\mathcal{F}K$$

$$\geq \frac{1}{32K}F^{kk}\lambda_k^2 + \psi'F^{kk}u_{k\bar{k}} - C_0\mathcal{F} - 14A^2\mathcal{F}K,$$

where we have assumed that $\lambda_1 > C_0$ and $A > 1$. Note that $F^{nn} \geq \frac{1}{n}\mathcal{F}$, and $\lambda_n^2 > \delta^2\lambda_1$ by our assumption. In addition,

(109)
$$\psi'F^{kk}u_{k\bar{k}} = \psi'F^{kk}g_{k\bar{k}} - \phi'F^{kk}\chi_{k\bar{k}}$$

$$\geq -2A\mathcal{F}\lambda_1 - C_0\mathcal{F},$$

and so from (108) we have

$$(110) \qquad 0 \geq \frac{1}{32nK} \mathcal{F} \delta^2 \lambda_1^2 - 2A\mathcal{F}\lambda_1 - C_0\mathcal{F} - 14A^2\mathcal{F}K.$$

Dividing by $K\mathcal{F}$ and rearranging we get the required bound for $K^{-1}\lambda_1$, with the bound depending on $\delta, A$ that will be chosen in Case 2 below.

**Case 2:** $-\lambda_n \leq \delta\lambda_1$. Define the set

$$(111) \qquad I = \{i \,:\, F^{ii} > \delta^{-1}F^{11}\}.$$

Using that $G_k = 0$ as above, we have

$$(112) \qquad \begin{aligned} &-\sum_{k \notin I} \frac{F^{kk}|g_{1\bar{1}k}|^2}{\lambda_1^2} \\ &= -2\phi'^2 \sum_{k \notin I} F^{kk}\left|u_{pk}u_{\bar{p}} + u_p u_{\bar{p}k}\right|^2 - 3\psi'^2 \sum_{k \notin I} F^{kk}|u_k|^2 - C_0 \\ &\geq -\phi'' \sum_{k \notin I} F^{kk}\left|u_{pk}u_{\bar{p}} + u_p u_{\bar{p}k}\right|^2 - 3(2A)^2\delta^{-1}F^{11}K - C_0. \end{aligned}$$

In addition, for $k \notin I$, from (107) we have

$$(113) \qquad \frac{F^{kk}|g_{k\bar{1}1}|^2}{\lambda_1^2} \leq \frac{F^{kk}|g_{1\bar{1}k}|^2}{\lambda_1^2} + 2A\delta^{-1}F^{11}K^{1/2} + C_0\mathcal{F}.$$

Our inequality (104) for $LG$ then implies
$$(114)$$
$$\begin{aligned} 0 \geq\;& \frac{-F^{ij,rs}g_{i\bar{j}1}g_{r\bar{s}1}}{\lambda_1} - \sum_{k \in I} \frac{F^{kk}|g_{k\bar{1}1}|^2}{\lambda_1^2} + \phi'' \sum_{k \in I} F^{kk}\left|u_{pk}u_{\bar{p}} + u_p u_{\bar{p}k}\right|^2 \\ &+ \psi'' F^{kk}|u_k|^2 + \frac{1}{32K}F^{kk}\lambda_k^2 + \psi' F^{kk}u_{k\bar{k}} \\ &- C_0(\mathcal{F} + A\lambda_1^{-1}F^{kk}|u_k|) - 13A^2\delta^{-1}F^{11}K. \end{aligned}$$

We want to choose $\delta$ so small that

$$(115) \qquad \frac{4\psi'^2\delta}{1-\delta} \leq \frac{1}{2}\psi''.$$

Note that $|\psi'| \leq 2A$, and $\psi'' = \tau A$ for a fixed $\tau > 0$, so we can choose $\delta \leq \delta_0 A^{-1}$, for some fixed number $\delta_0$ (depending on $\tau$).

To deal with the first four terms in (114) we use that $G_k = 0$ to obtain

$$(116) \qquad \begin{aligned} 2\phi'^2 \sum_{k \in I} F^{kk}\left|u_{pk}u_{\bar{p}} + u_p u_{\bar{p}k}\right|^2 &= 2\sum_{k \in I} F^{kk}\left|\frac{g_{1\bar{1}k}}{\lambda_1} + \psi'u_k + w_k\right|^2 \\ &\geq 2\delta \sum_{k \in I} \frac{F^{kk}|g_{1\bar{1}k}|^2}{\lambda_1^2} - \frac{4\delta\psi'^2}{1-\delta} \sum_{k \in I} F^{kk}|u_k|^2 - C_0, \end{aligned}$$

just as in [**25**], using the elementary inequality $|a+b|^2 \geq \delta|a|^2 - \frac{\delta}{1-\delta}|b|^2$.
More precisely we used

(117) $\quad |a+b+c|^2 \geq \delta|a|^2 - \frac{\delta}{1-\delta}|b+c|^2 \geq \delta|a|^2 - \frac{2\delta}{1-\delta}|b|^2 - \frac{2\delta}{1-\delta}|c|^2.$

Using (107) we have

(118)
$$2\delta \sum_{k \in I} \frac{F^{kk}|g_{1\bar{1}k}|^2}{\lambda_1^2} \geq 2\delta \sum_{k \in I} \frac{F^{kk}|g_{k\bar{1}1}|^2}{\lambda_1^2} - 4\delta A F^{kk}|u_k|^2 - \delta C_0 \mathcal{F}$$
$$\geq 2\delta \sum_{k \in I} \frac{F^{kk}|g_{k\bar{1}1}|^2}{\lambda_1^2} - \delta^2 A^2 F^{kk}|u_k|^2 - C_0 \mathcal{F}.$$

We also have

(119) $\qquad A\lambda_1^{-1} F^{kk}|u_k| \leq \frac{1}{2}A^2 \lambda_1^{-1} F^{kk}|u_k|^2 + \frac{1}{2}\lambda_1^{-1}\mathcal{F}.$

Using this, (118) and (116) in (114) we have
(120)
$$0 \geq \frac{-F^{ij,rs}g_{i\bar{j}1}g_{r\bar{s}\bar{1}}}{\lambda_1} - (1-2\delta)\sum_{k \in I} \frac{F^{kk}|g_{k\bar{1}1}|^2}{\lambda_1^2}$$
$$+ \left(\frac{1}{2}\psi'' - \delta^2 A^2 - A^2\lambda_1^{-1}\right)F^{kk}|u_k|^2 + \frac{1}{32K}F^{kk}\lambda_k^2 + \psi' F^{kk}u_{k\bar{k}}$$
$$- C_0\mathcal{F} - 13A^2\delta^{-1}F^{11}K.$$

In addition, we claim that

(121) $\qquad \frac{-F^{ij,rs}g_{i\bar{j}1}g_{r\bar{s}\bar{1}}}{\lambda_1} - (1-2\delta)\sum_{k \in I} \frac{F^{kk}|g_{k\bar{1}1}|^2}{\lambda_1^2} \geq 0.$

Indeed, from Equation (67) we have

(122) $\qquad F^{ij,rs}g_{i\bar{j}1}g_{r\bar{s}\bar{1}} \leq \sum_{k \in I} \frac{F^{11} - F^{kk}}{\lambda_1 - \lambda_k}|g_{k\bar{1}1}|^2,$

using that $\frac{F^{11}-F^{kk}}{\lambda_1-\lambda_k} \leq 0$. For $k \in I$ we have $F^{11} < \delta F^{kk}$, and so

(123) $\qquad \frac{F^{11} - F^{kk}}{\lambda_1 - \lambda_k} \leq \frac{(\delta-1)F^{kk}}{\lambda_1 - \lambda_k}.$

It is, therefore, enough to show

(124) $\qquad \frac{\delta-1}{\lambda_1 - \lambda_k} \leq -\frac{1-2\delta}{\lambda_1}.$

Rearranging, this is equivalent to $(2\delta-1)\lambda_k \leq \delta\lambda_1$. If $\lambda_k \geq 0$, this is
clear, while if $\lambda_k < 0$, then

(125) $\qquad (2\delta-1)\lambda_k \leq -\lambda_k \leq -\lambda_n \leq \delta\lambda_1,$

where we used our assumption for Case 2.

Using (121) in (120) we have

$$(126)$$

$$0 \geq \left( \frac{1}{2} \psi'' - \delta^2 A^2 - A^2 \lambda_1^{-1} \right) F^{kk} |u_k|^2 + \frac{1}{32K} F^{kk} \lambda_k^2 + \psi' F^{kk} u_{k\bar{k}}$$
$$- C_0 \mathcal{F} - 13 A^2 \delta^{-1} F^{11} K.$$

Suppose that $\lambda_1 > R$ with the $R$ in (59). There are two cases to consider:

- If (61) holds, then we have $\psi' F^{kk} u_{k\bar{k}} > A\kappa\mathcal{F}$. Choosing $A$ so that $A\kappa > C_0$, Equation (126) implies

$$(127)$$

$$0 \geq \left( \frac{1}{2} \psi'' - \delta^2 A^2 - A^2 \lambda_1^{-1} \right) F^{kk} |u_k|^2 + \frac{1}{32K} F^{kk} \lambda_k^2 - 13 A^2 \delta^{-1} F^{11} K.$$

At this point $\psi'' = \tau A$ is fixed, so if we choose $\delta$ sufficiently small, and $\lambda_1$ is sufficiently large, then first term is positive. We then obtain

$$(128) \qquad 0 \geq \frac{1}{32K} F^{11} \lambda_1^2 - 13 A^2 \delta^{-1} F^{11} K,$$

from which the required bound for $K^{-1} \lambda_1$ follows.

- If (61) does not hold, then we must have $F^{11} > \kappa\mathcal{F}$. We have

$$(129)$$

$$F^{kk} u_{k\bar{k}} = F^{kk} \lambda_k - F^{kk} \chi_{k\bar{k}}$$
$$\leq \frac{1}{128 AK} F^{kk} \lambda_k^2 + C_0 AK\mathcal{F},$$

which together with (126) and $|\psi'| < 2A$ implies

$$(130) \qquad 0 \geq \frac{1}{64K} \kappa\mathcal{F}\lambda_1^2 - C_0 A^2 K\mathcal{F} - 13 A^2 \delta^{-1} \mathcal{F} K,$$

where we also used the bounds $\kappa\mathcal{F} < F^{11} < \mathcal{F}$. This inequality again implies the bound of the form $\lambda_1 < CK$, which we are after.

q.e.d.

**Remark 14.** Under an extra concavity condition for $f$, one can obtain $C^2$-estimates directly from $C^0$-estimates, as is well-known in the case of the Monge–Ampère equation. See, e.g., Yau [**47**], Tosatti–Weinkove [**42**] and also Weinkove [**46**], Fang–Lai–Ma [**14**], Sun [**35**] for the inverse $\sigma_k$-equations. On Hermitian manifolds this technique was used to obtain the $C^2$-estimate by Tosatti–Weinkove [**43**].

Suppose that in addition to the structural conditions (i), (ii), (iii) in the introduction, $f$ satisfies the extra conditions

$$(131) \qquad f_{11} + \frac{f_1}{\lambda_1} \leq 0, \quad \text{and} \quad \lambda_1 f_1 \leq \lambda_i f_i \quad \text{for all } i > 1,$$

where $\lambda_1$ is the largest eigenvalue, and we work at a diagonal matrix $A$, as before. See Fang–Lai [**13**] for related conditions. We assume also

that the Hessian $f_{ij}$ is diagonal (we thank Rirong Yuan for pointing out the need for such an additional assumption). In particular, it is easy to check that these conditions hold for the complex Monge–Ampère equation written in the form $f(\lambda) = \log \lambda_1 \cdot \ldots \lambda_n$.

These conditions imply a stronger concavity property of $F$, which allows us to get rid of the first two terms in (85). Indeed, from (67) we have

$$(132) \qquad F^{pq,rs} g_{p\bar{q}1} g_{r\bar{s}\bar{1}} \leq f_{kk} |g_{k\bar{k}1}|^2 + \sum_{k>1} \frac{f_1 - f_k}{\lambda_1 - \lambda_k} |g_{k\bar{1}1}|^2.$$

For $k > 1$ the condition $\lambda_1 f_1 \leq \lambda_k f_k$ implies

$$(133) \qquad -\frac{1}{\lambda_1} \frac{f_1 - f_k}{\lambda_1 - \lambda_k} - \frac{f_k}{\lambda_1^2} \geq 0.$$

From this we have

$$(134) \quad -\frac{F^{pq,rs} g_{p\bar{q}1} g_{r\bar{s}\bar{1}}}{\lambda_1} - \frac{F^{kk} |g_{k\bar{1}1}|^2}{\lambda_1^2} \geq -\frac{f_{11} |g_{1\bar{1}1}|^2}{\lambda_1} - \frac{F^{11} |g_{1\bar{1}1}|^2}{\lambda_1^2} \geq 0,$$

using the assumption $\lambda_1 f_{11} + f_1 \leq 0$ as well. Using (85) we have

$$(135) \qquad L(\log \widetilde{\lambda}_1) \geq -C_0 (\mathcal{F} + \lambda_1^{-2} F^{kk} |g_{1\bar{1}k}|).$$

Note that (131) implies that $\lambda_i > 0$ for all $i$, and so, in particular, we must have $\Gamma = \Gamma_n$. Suppose that $\sigma > \sup_{\partial \Gamma} f$ and $\partial \Gamma^\sigma$ is non-empty, and let $x_0 \in \partial \Gamma^\sigma$ be the closest point to the origin. For any $\epsilon > 0$ we can find an $R > 0$ such that if $\lambda \in \partial \Gamma^\sigma$ and $\lambda_1 > R$, then $f_1 < \epsilon \mathcal{F}$ at $\lambda$. To see this, let $\mathbf{n}_\lambda$ be the inward pointing unit normal to $\partial \Gamma^\sigma$ at $\lambda$. By convexity we have

$$(136) \qquad \mathbf{n}_\lambda \cdot (\lambda - x_0) < 0,$$

which implies $\mathbf{n}_\lambda \cdot \lambda < \mathbf{n}_\lambda \cdot x_0$. In particular, we have

$$(137) \qquad \lambda_1 \frac{f_1(\lambda)}{|\nabla f(\lambda)|} < |x_0|.$$

In view of the equivalence between $|\nabla f|$ and $\mathcal{F}$ (see (19)), this implies our claim. In particular, in Proposition 5 only the alternative (23) occurs. It follows that if the largest eigenvalue $\lambda_1$ is sufficiently large, then Equation (61) will be satisfied, if we assume the existence of a $\mathcal{C}$-subsolution.

Recall that $g_{i\bar{j}} = \chi_{i\bar{j}} + u_{i\bar{j}}$. Let us normalize $u$ so that $\inf_M u = 0$, and consider the function

$$(138) \qquad G = \log(\lambda_1) + \psi(u),$$

where $\psi : [0, \infty) \to \mathbf{R}$ is defined by

$$(139) \qquad \psi(x) = -Ax + \frac{1}{1+x} - 1,$$

for a constant $A > 1$ to be chosen later (this is the function used by Phong–Sturm [**30**]). Then for $x \geq 0$ we have

(140)
$$-Ax - 1 < \psi \leq -Ax,$$
$$A < -\psi' < 2A,$$
$$\psi'' = \frac{2}{(1+x)^3}.$$

At the maximum of $G$ we compute in coordinates as before, i.e., $\alpha$ is the identity and $g$ is diagonal, and we make a small perturbation to the matrix $\alpha^{i\bar{p}} g_{j\bar{p}}$ by a diagonal matrix, so that the eigenvalues are distinct at this point (this does not change $\lambda_1$ at the origin). Writing $w_k = -(B^{11})_k/\lambda_1$ as before, at the origin we have

(141)
$$0 = G_k = \frac{g_{1\bar{1}k}}{\lambda_1} + \psi' u_k + w_k,$$
$$0 \geq L(G) = L(\log \widetilde{\lambda}_1) + \psi'' F^{kk} u_k u_{\bar{k}} + F^{kk} \psi' u_{k\bar{k}},$$

and so using (135)

(142) $\quad 0 \geq \psi'' F^{kk} u_k u_{\bar{k}} + \psi' F^{kk}(g_{k\bar{k}} - \chi_{k\bar{k}}) - C_0(\mathcal{F} + \lambda_1^{-2} F^{kk}|g_{1\bar{1}k}|).$

Using the properties of $\psi$, if we assume that $\lambda_1 > R$ for sufficiently large $R$, we will have

(143) $\quad 0 \geq 2(1 + u(0))^{-3} F^{kk}|u_k|^2 + A\kappa\mathcal{F} - C_0\mathcal{F} - C_0\lambda_1^{-2} F^{kk}|g_{1\bar{1}k}|.$

Choose $A$ so that $A\kappa = C_0 + 1$. This implies

(144) $\quad 0 \geq 2(1 + u(0))^{-3} F^{kk}|u_k|^2 + \mathcal{F} - C_0\lambda_1^{-2} F^{kk}|g_{1\bar{1}k}|.$

Using the equation $G_k = 0$ we have (we can assume that $|w_k| < 1$)

(145)
$$C_0\lambda_1^{-2} F^{kk}|g_{1\bar{1}k}| = C_0\lambda_1^{-1} F^{kk}(|\psi' u_k| + 1)$$
$$\leq \lambda_1^{-1} F^{kk}|u_k|^2 + \left[(2C_0 A)^2 + C_0\right]\lambda_1^{-1}\mathcal{F}.$$

Using this in (144), we have

(146) $\quad 0 \geq \left(2(1 + u(0))^{-3} - \lambda_1^{-1}\right) F^{kk}|u_k|^2 + \left(1 - \left[(2C_0 A)^2 + C_0\right]\lambda_1^{-1}\right)\mathcal{F}.$

It follows that at the maximum of $G$ we must have

(147)
$$\lambda_1 < \frac{1}{2}(1 + u(0))^3 + C_1,$$

where $C_1$ only depends on the background data (we also choose $C_1 > R$). Therefore,

(148)
$$G(0) < 3\log(1 + u(0)) + C_2 - Au(0).$$

Since $\log(1 + x) \leq x$, we obtain

(149)
$$G(0) < C_2 - (A - 3)u(0) \leq C_2.$$

Since $G$ achieves its maximum at 0, we have $G(p) \leq C_2$ for all $p$, from which we, finally, obtain the global estimate

$$(150) \qquad \qquad \mathrm{tr}_\alpha g \leq C e^{A(u - \inf_M u)}.$$

Note that we assumed $\inf_M u = 0$ earlier but wrote this last estimate in the usual form that is independent of the normalization of $u$. At this point we can either use the $C^0$-estimate Proposition 11 to obtain an estimate for $\mathrm{tr}_\alpha g$, or this $C^0$-estimate can be directly derived from (150) as in Weinkove [**46**] or Tosatti–Weinkove [**41**].

## 5. Liouville theorem

Suppose as before, that $\Gamma \subset \mathbf{R}^n$ is an open convex cone, containing the positive orthant $\Gamma_n$ and not equal to all of $\mathbf{R}^n$. In addition, assume that $\Gamma$ is preserved under permuting the coordinates. It follows that

$$(151) \qquad \qquad \Gamma \subset \{(x_1, \ldots, x_n) : \sum x_i > 0\}.$$

**Definition 15.** Suppose $u : \mathbf{C}^n \to \mathbf{R}$ is continuous. We say that $u$ is a (viscosity) $\Gamma$-subsolution if for all $h \in C^2$ such that $u - h$ has a local maximum at $z$, we have $\lambda(h_{i\bar{j}}) \in \overline{\Gamma}$, where $\lambda(A)$ denotes the eigenvalues of the Hermitian matrix $A$.

We say that $u$ is a $\Gamma$-solution, if it is a $\Gamma$-subsolution and, in addition, for all $z \in \mathbf{C}^n$, if $h \in C^2$ and $u - h$ has a local minimum at $z$, then $\lambda(h_{i\bar{j}}(z)) \in \mathbf{R}^n \setminus \Gamma$.

Note that (151) implies that every $\Gamma$-subsolution is subharmonic. Suppose that we define the function $F_0$ on the space of Hermitian matrices by the property that

$$(152) \qquad \qquad \lambda(A) - F_0(A)(1, 1, \ldots, 1) \in \overline{\Gamma},$$

and define $F$ on the space of symmetric $2n \times 2n$ matrices by $F(M) = F_0(p(M))$, where

$$(153) \qquad \qquad p(M) = \frac{M + J^T M J}{2},$$

and $J$ is the standard complex structure. Then a $\Gamma$-subsolution (resp. solution) $u$ is the same as a viscosity subsolution (resp. solution) of the nonlinear equation $F(D^2 u) = 0$. Note that $F$ is concave and elliptic, but in general not uniformly elliptic. Many of the basic results about viscosity subsolutions and solutions found in Caffarelli–Cabré [**4**] can still be applied with the same proofs. In particular, we have the following.

**Proposition 16.**     1) *If $u_k$ are $\Gamma$-solutions (resp. subsolutions) converging locally uniformly to $u$, then $u$ is also a $\Gamma$-solution (resp. subsolution).*

2) *If $u, v$ are $\Gamma$-subsolutions, then $\frac{1}{2}(u + v)$ is also a $\Gamma$-subsolution, using that $\Gamma$ is convex.*

An important consequence is that mollifications of $\Gamma$-subsolutions are again $\Gamma$-subsolutions. Indeed, a mollification can be written as a uniform limit of averages of a larger and larger number of translates.

We will use the following simple comparison result.

**Lemma 17.** *Suppose that $u$ is a smooth $\Gamma$-subsolution and $v$ is a $\Gamma$-solution on a bounded open set $\Omega \subset \mathbf{C}^n$, and, in addition, $u = v$ on $\partial\Omega$. Then $u \le v$ in $\Omega$.*

*Proof.* If $v < u$ at some point in $\Omega$, then $v - u$ achieves a negative minimum at a point in $\Omega$, so for small $\epsilon > 0$, the function $v - u - \epsilon|z|^2$ also has a minimum, at a point $p \in \Omega$. Since $v$ is a $\Gamma$-solution and $u$ is smooth, this implies that

$$(154) \qquad \lambda(u_{i\bar{j}} + \epsilon\delta_{i\bar{j}}) \in \mathbf{R}^n \setminus \Gamma.$$

This contradicts that $u$ is a $\Gamma$-subsolution. Indeed, we have $\lambda(u_{i\bar{j}}) \in \overline{\Gamma}$, and since $\Gamma_n \subset \Gamma$ and $\Gamma$ is convex, this implies $\lambda(u_{i\bar{j}} + \epsilon\delta_{i\bar{j}}) \in \Gamma$.  q.e.d.

Note that since the $F$ defined by (152) is not uniformly elliptic, the comparison result might not hold in full generality if $u$ is only a continuous $\Gamma$-subsolution. The following lemmas will be used in an inductive argument.

**Lemma 18.** *Suppose that $\Gamma \ne \Gamma_n$. Then $\Gamma' \subset \mathbf{R}^{n-1}$ given by*

$$(155) \qquad \Gamma' = \{(x_1, \ldots, x_{n-1}) : (x_1, \ldots, x_{n-1}, 0) \in \Gamma\}$$

*satisfies the same conditions as $\Gamma$. I.e. $\Gamma'$ is a symmetric, open convex cone, containing $\Gamma_{n-1}$, and $\Gamma' \ne \mathbf{R}^{n-1}$. In addition, $\overline{\Gamma} \cap \{x_n = 0\} = \overline{\Gamma'}$.*

*Proof.* It is clear that $\Gamma'$ is a symmetric open cone, and $\Gamma' \ne \mathbf{R}^{n-1}$. The assumption $\Gamma \ne \Gamma_n$, and openness of $\Gamma$, means that there is at least one vector in $\Gamma$ with a negative entry, and, in particular, $\Gamma'$ is non-empty. Using that $\Gamma_n \subset \Gamma$ and scaling, we can then obtain that for a small $\epsilon > 0$, we have $\mathbf{e} = (1, 1, \ldots, 1, -\epsilon) \in \Gamma$.

If $(x_1, \ldots, x_{n-1}) \in \Gamma_{n-1}$, then $(x_1 + 1, \ldots, x_{n-1} + 1, \epsilon) \in \Gamma$, and adding $\mathbf{e}$ to this vector we have $(x_1, \ldots, x_{n-1}, 0) \in \Gamma$. This implies $(x_1, \ldots, x_{n-1}) \in \Gamma'$.

It remains to show the claim about $\overline{\Gamma'}$. The inclusion $\overline{\Gamma'} \subset \overline{\Gamma} \cap \{x_n = 0\}$ is clear (we are thinking of $\Gamma'$ as a subset of the hyperplane $\{x_n = 0\}$ in $\mathbf{R}^n$). For the reverse inclusion, suppose that $x = (x_1, \ldots, x_{n-1}, 0) \in \overline{\Gamma} \cap \{x_n = 0\}$. This implies that we have $y^{(k)} = (y_1^{(k)}, \ldots, y_n^{(k)}) \in \Gamma$ converging to $x$. If $y_n^{(k_i)} \le 0$ along a subsequence, then

$$(156) \qquad y - y_n^{(k_i)}(1, 1, \ldots, 1) \in \Gamma \cap \{x_n = 0\}$$

also converges to $x$, and so $(x_1, \ldots, x_{n-1}) \in \overline{\Gamma'}$. Otherwise $y_n^{(k_i)} > 0$ along a subsequence, in which case

$$(157) \qquad\qquad y + \epsilon^{-1} y_n^{(k_i)} \mathbf{e} \in \Gamma$$

converges to $x$, and again $(x_1, \ldots, x_{n-1}) \in \overline{\Gamma'}$.                 q.e.d.

**Lemma 19.** *Suppose that* $v : \mathbf{C}^n \to \mathbf{R}$ *is a* $\Gamma$*-solution,* $\Gamma \ne \Gamma_n$, *and* $v$ *is independent of the variable* $z_n$. *Then letting* $\Gamma' = \Gamma \cap \{x_n = 0\} \subset \mathbf{R}^{n-1}$, *the function* $w(z_1, \ldots, z_{n-1}) = v(z_1, \ldots, z_{n-1}, 0)$ *is a* $\Gamma'$*-solution on* $\mathbf{C}^{n-1}$.

*Proof.* Suppose that $h$ is smooth and $w - h$ has a local maximum at a point $z = (z_1, \ldots, z_{n-1})$. Then $v - H$ has a local maximum at $Z = (z_1, \ldots, z_{n-1}, 0)$, where $H(z_1, \ldots, z_n) = h(z_1, \ldots, z_{n-1})$. Since $v$ is a $\Gamma$-subsolution, we have $\lambda(H_{i\bar{j}}(Z)) \in \overline{\Gamma}$, and one eigenvalue is zero. Using Lemma 18 this implies that $\lambda(h_{i\bar{j}}(z)) \in \overline{\Gamma'}$, so $w$ is a $\Gamma'$-subsolution.

Similarly, if $h$ is smooth and $w - h$ has a local minimum at $z$, then $v - H$ has a local minimum at $Z$, which implies $\lambda(H_{i\bar{j}}(z)) \in \mathbf{R}^n \setminus \Gamma$, and one eigenvalue is zero. So $\lambda(h_{i\bar{j}}(z)) \in \mathbf{R}^{n-1} \setminus \Gamma'$.                 q.e.d.

Our goal in this section is the following result, generalizing the Liouville theorem of Dinew–Kolodziej [**11**] (see also the analogous result of Tosatti–Weinkove [**40**] for $(n-1)$-plurisubharmonic functions). The proof follows their arguments closely.

**Theorem 20.** *Let* $u : \mathbf{C}^n \to \mathbf{R}$ *be a Lipschitz* $\Gamma$*-solution such that* $|u| < C$ *and* $u$ *has Lipschitz constant bounded by* $C$. *Then* $u$ *is constant.*

*Proof.* We use induction over $n$. If $n = 1$, then $u$ is harmonic, while if $\Gamma = \Gamma_n$, then $u$ is plurisubharmonic. In both cases the result follows from the fact that a bounded subharmonic function on $\mathbf{C}$ is constant. We, therefore, assume that $n > 1$ and $\Gamma \ne \Gamma_n$.

Suppose that $u$ is non-constant, $|\nabla u| < c_0$, and $\inf u = 0, \sup u = 1$. For any function $v$ on $\mathbf{C}^n$, let

$$(158) \qquad\qquad [v]_r(z) = \int_{\mathbf{C}^n} v(z + rz') \eta(z') \beta^n(z'),$$

where $\beta = \sum_i dz_i \wedge d\bar{z}_i$ and $\eta : \mathbf{C}^n \to \mathbf{R}$ is a smooth mollifier satisfying $\eta > 0$ in $B(0,1)$, $\eta = 0$ outside $B(0,1)$ and $\int_{\mathbf{C}^n} \eta \beta^n = 1$. We do this instead of taking averages over balls in order to obtain a smooth function. This is used in the comparison result Lemma 17. As in [**11**], Cartan's Lemma implies that

$$(159) \qquad\qquad \lim_{r \to \infty} [u^2]_r(z) = \lim_{r \to \infty} [u]_r(z) = 1,$$

using that $u$ and $u^2$ are subharmonic.

It will be helpful to regularize $u$ slightly, letting $u^\epsilon = [u]_\epsilon$ for $\epsilon > 0$. Just as in [**11**], there are two cases to consider.

**Case 1.** In this case we assume that there is a $\rho > 0$, and sequences $\epsilon_k \to 0$, $x_k \in \mathbf{C}^n$, $r_k \to \infty$ and unit vectors $\xi_k$ (of type $(1,0)$), such that

$$(160) \qquad [u^2]_{r_k}(x_k) + [u]_\rho(x_k) - 2u(x_k) \geq 4/3,$$

and

$$(161) \qquad \lim_{k \to \infty} \int_{B(x_k, r_k)} |\partial_{\xi_k} u^{\epsilon_k}|^2 \beta^n = 0.$$

In this case, translating and rotating $u$ to make $x_k$ the origin, and $\partial_{z^n} = \partial_{\xi_k}$, we obtain a sequence of $\Gamma$-solutions $u_k$, such that

$$(162) \qquad \begin{aligned} [u_k^2]_{r_k}(0) + [u_k]_\rho(0) - 2u_k(0) &\geq 4/3, \\ \lim_{k \to \infty} \int_{B(0, r_k)} |\partial_{z^n} u_k^{\epsilon_k}|^2 \beta^n &= 0. \end{aligned}$$

The uniform Lipschitz bound implies that we can replace $u_k$ by a subsequence, converging locally uniformly to $v : \mathbf{C}^n \to \mathbf{R}$, which by Lemma 16 is also a $\Gamma$-solution with the same Lipschitz bound. In addition, we also have that $u_k^{\epsilon_k} \to v$ locally uniformly, since $\epsilon_k \to 0$. Just as in [**11**], we find that $v$ is independent of $z_n$, and so we can define a function $w : \mathbf{C}^{n-1} \to \mathbf{R}$ by $w(z_1, \ldots, z_{n-1}) = v(z_1, \ldots, z_{n-1}, 0)$, and by Lemma 19, $w$ is a $\Gamma'$-solution with $\Gamma' = \Gamma \cap \{x_n = 0\}$. The induction hypothesis implies that $w$ is constant, and so $v$ is constant, but this contradicts (160), using that $0 \leq u \leq 1$.

**Case 2.** In this case, the assumption in Case 1 does not hold, so for all $\rho > 0$, there is a constant $C_\rho > 0$ such that if $\epsilon < C_\rho^{-1}$, $r > C_\rho$, $x \in \mathbf{C}^n$ and $\xi$ is a unit vector, we have

$$(163) \qquad \int_{B(x,r)} |\partial_\xi u^\epsilon|^2 \, dz \geq C_\rho^{-1},$$

as long as

$$(164) \qquad [u^2]_r(x) + [u]_\rho(x) - 2u(x) \geq 4/3.$$

We choose our origin so that $u(0) < 1/9$, and fix $\rho > 0$ such that $[u]_\rho(0) > 3/4$. Then choose $r > C_\rho$ such that $[u^2]_r(0) > 3/4$ as well. Define the set

$$(165) \qquad U = \{z \ : \ 2u(z) < [u^2]_r(z) + [u]_\rho(z) - 4/3\},$$

so that $0 \in U$.

**Claim.** *There is a constant $c > 0$ such that $[(u^\epsilon)^2]_r - c|z|^2$ is a $\Gamma$-subsolution on $U$ for all $\epsilon < C_\rho^{-1}$. We have*

$$(166) \qquad (u^\epsilon)^2_{i\bar{j}} = 2u^\epsilon u^\epsilon_{i\bar{j}} + 2u^\epsilon_i u^\epsilon_{\bar{j}},$$

*and so*

$$
\begin{aligned}
\left[(u^\epsilon)^2\right]_{r,i\bar{j}}(z) &= \int_{B(0,1)} (u^\epsilon)^2_{i\bar{j}}(z + rz')\,\eta(z')\,\beta^n(z') \\
&= \int_{B(0,1)} 2u^\epsilon u^\epsilon_{i\bar{j}}(z + rz')\,\eta(z')\,\beta^n(z') \\
&\quad + \int_{B(0,1)} 2u^\epsilon_i u^\epsilon_{\bar{j}}(z + rz')\,\eta(z')\,\beta^n(z') \\
&= A_{i\bar{j}} + B_{i\bar{j}}.
\end{aligned}
$$

(167)

*The subsolution property of $u^\epsilon$ together with the fact that $\Gamma$ is convex (i.e. the equation $F(D^2 u) = 0$ is concave) implies that the matrix $A$ satisfies $\lambda(A_{i\bar{j}}) \in \overline{\Gamma}$. Using (163), we find that $B_{i\bar{j}} \geq c_2\beta$ for some $c_2 > 0$, and can choose $c$ such that $(c|z|^2)_{i\bar{j}} \leq c_2\beta$. It then follows that*

$$
\text{(168)} \qquad \left(\left[(u^\epsilon)^2\right]_r - c|z|^2\right)_{i\bar{j}} \geq A_{i\bar{j}},
$$

*in the sense of inequalities for Hermitian matrices, and so $[(u^\epsilon)^2]_r - c|z|^2$ is a $\Gamma$-subsolution. But this converges locally uniformly to $[u^2]_r - c|z|^2$, which is, therefore, also a $\Gamma$-subsolution.*

Consider now the set

$$
\text{(169)} \qquad U' = \{z \,:\, 2u(z) < [u^2]_r(z) - c|z|^2 + [u]_\rho(z) - 4/3\},
$$

which satisfies $U' \subset U$, and, in addition, $U'$ is bounded, by the assumption that $|u| \leq 1$. This contradicts the comparison result Lemma 17, since $u$ is a $\Gamma$-solution and as we have seen, $[u^2]_r - c|z|^2 + [u]_p$ is a $\Gamma$-subsolution on the set $U$. $\qquad$ q.e.d.

## 6. Blowup argument

We now complete the proof of Theorem 2 using a blowup argument analogous to that in [11], using the Liouville-type theorem, Theorem 20.

*Proof of Theorem 2.* Suppose that as in the introduction, $(M, \alpha)$ is Hermitian, $\chi$ is a real $(1,1)$-form, and $g = \chi + \sqrt{-1}\partial\bar{\partial}u$ satisfies $F(A) = h$, where $A^{ij} = \alpha^{j\bar{p}}g_{i\bar{p}}$. We use Proposition 13, together with a contradiction argument to obtain an estimate for $|\nabla u|$, depending on the $C^2$-norms of $\alpha, \chi, h$ and the subsolution $\underline{u}$, which in turn will imply an estimate for $\Delta u$. The $C^{2,\alpha}$-estimate follows from this by the Evans–Krylov theory [12, 26, 38].

To argue by contradiction, suppose that $F(A) = h$, but

$$
\text{(170)} \qquad \sup_M |\nabla u|^2 = |\nabla u(p)|^2 = N,
$$

for some large $N$. Proposition 13 implies that we have

$$
\text{(171)} \qquad |\partial\bar{\partial}u|_\alpha \leq CN,
$$

for a fixed constant $C$. Let $\widetilde{\alpha} = N\alpha$. We can choose coordinates $z_1, \ldots, z_n$ centered at $p$, such that in these coordinates $\widetilde{\alpha}, \chi, h$ satisfy

$$
\begin{aligned}
(172) \qquad \widetilde{\alpha}_{i\bar{j}} &= \delta_{i\bar{j}} + O(N^{-1}|z|), \\
\chi_{i\bar{j}} &= O(N^{-1}), \\
h &= h(p) + O(N^{-1}|z|),
\end{aligned}
$$

and the $z_i$ are defined for $|z_i| < O(N^{1/2})$. The inequality (171) implies that $|\partial\bar{\partial}u|_{\widetilde{\alpha}} \leq C$, and since $\widetilde{\alpha}$ is approximately Euclidean on the ball of radius $O(N^{1/2})$, we obtain a uniform bound

$$
(173) \qquad \|u\|_{C^{1,\alpha}} < C',
$$

on this ball, in these coordinates. The equation $F(A) = h$ implies that

$$
(174) \qquad f\left(N\lambda\big[\widetilde{\alpha}^{j\bar{p}}(\chi_{i\bar{p}} + u_{i\bar{p}})\big]\right) = h(z),
$$

where $f : \Gamma \to \mathbf{R}$ defines the operator $F$. Since we have a fixed bound on $u_{i\bar{p}}$, while $\chi_{i\bar{p}}$ is going to zero and $\widetilde{\alpha}^{j\bar{p}}$ is approaching the identity matrix, we obtain

$$
(175) \qquad \lambda\big[\widetilde{\alpha}^{j\bar{p}}(\chi_{i\bar{p}} + u_{i\bar{p}})\big] = \lambda(u_{i\bar{j}}) + O(N^{-1}|z|).
$$

Suppose now that we have a sequence of such $\alpha, \chi, h$ and the subsolution $\underline{u}$ all bounded in $C^2$, and with $|u|$ uniformly bounded so that Proposition 13 can be applied uniformly, and the constant $N$ in (170) gets larger and larger. The coordinates $z_i$ will then be defined on larger and larger balls, and using the estimate (173) we can choose a subsequence converging uniformly in $C^{1,\alpha}$ to a limit $v : \mathbf{C}^n \to \mathbf{R}$. By the construction we will have global bounds $|v|, |\nabla v| < C$, and $|\nabla v(0)| = 1$.

The proof will be completed by showing that $v$ is a $\Gamma$-solution, in the sense of Definition 15, since that will contradict Theorem 20. To see this, suppose first that we have a $C^2$-function $\psi$, such that $\psi \geq v$, and $\psi(z_0) = v(z_0)$ for some point $z_0$. We need to show that $\lambda(\psi_{i\bar{j}}(z_0)) \in \overline{\Gamma}$. By the construction of $v$, for any $\epsilon > 0$ we can find a $u$ as above, corresponding to a sufficiently large $N$, a number $a$ with $|a| < \epsilon$, and a point $z_1$ with $|z_1 - z_0| < \epsilon$, such that

$$
(176) \qquad \psi + \epsilon|z - z_0|^2 + a \geq u \text{ on } B_1(z_0), \text{ with equality at } z_1.
$$

This implies that $\psi_{i\bar{j}}(z_1) + \epsilon\delta_{i\bar{j}} \geq u_{i\bar{j}}(z_1)$. From (175), and the fact that $\Gamma_n \subset \Gamma$ we obtain that for large $N$, $\lambda\big[\psi_{i\bar{j}}(z_1)\big]$ will be within $2\epsilon$ of $\Gamma$. Letting $\epsilon \to 0$ we find that $\lambda\big[\psi_{i\bar{j}}(z_0)\big] \in \overline{\Gamma}$.

Suppose now that we have a $C^2$-function $\psi$ such that $\psi \leq v$ and $\psi(z_0) = v(z_0)$. We need to show $\lambda\big[\psi_{i\bar{j}}(z_0)\big] \in \mathbf{R}^n \setminus \Gamma$. As above, for any $\epsilon > 0$ we can find a $u$ corresponding to large $N$, and $a, z_1$, such that

$$
(177) \qquad \psi - \epsilon|z - z_0|^2 + a \leq u \text{ on } B_1(z_0), \text{ with equality at } z_1.
$$

This implies that $\psi_{i\bar{j}}(z_1) - \epsilon\delta_{i\bar{j}} \leq u_{i\bar{j}}(z_1)$. This implies that if $\lambda(\psi_{i\bar{j}}(z_1) - 3\epsilon\delta_{i\bar{j}}) \in \Gamma$, then we will have $\lambda(u_{i\bar{j}}) \in \Gamma + 2\epsilon\mathbf{1}$. Using (175), if $N$ is sufficiently large, we will have

$$(178) \qquad \lambda\big[\widetilde{\alpha}^{j\bar{p}}(\chi_{i\bar{p}} + u_{i\bar{p}})\big] \in \Gamma + \epsilon\mathbf{1}.$$

Finally, by Lemma 9 part (a), our assumptions for $f$ in the introduction imply that if $N$ is sufficiently large, then we cannot have (174), since we have a fixed bound for $h$, which must be less than $\sup_\Gamma f$. It follows that we cannot have $\lambda(\psi_{i\bar{j}}(z_1)) \in \Gamma + 3\epsilon\mathbf{1}$. Letting $\epsilon \to 0$ we will have $z_1 \to z_0$, and so $\lambda(\psi_{i\bar{j}}(z_0)) \in \mathbf{R}^n \setminus \Gamma$. This completes the proof that $v$ is a $\Gamma$-solution. q.e.d.

## 7. Hessian quotient equations

In this section, we prove Corollary 3 as an application of Theorem 2, and we also discuss equations for $(n-1)$-plurisubharmonic functions. As we mentioned in the introduction it is somewhat difficult to formulate very general existence results, in contrast to the case of the Dirichlet problem in [5], because on a compact manifold the constant functions are not in the image of the linearized operator of Equation (1). In particular, if we consider only equations with constant right hand side, $F(A) = c$, then a solution can only exist for a unique constant. If we do not know a priori what the right constant is, then we cannot ensure that along a suitable continuity path we have a $\mathcal{C}$-subsolution for the whole path. This issue does not arise when any admissible function is a $\mathcal{C}$-subsolution, which is the case for the complex Monge–Ampère and Hessian equations, for instance. We, therefore, have the following.

**Proposition 21.** *Let $(M, \alpha)$ be compact, Hermitian, let $\chi$ be a $k$-positive real $(1,1)$-form on $M$ and let $1 \leq k \leq n$. Given any smooth function $H$ on $M$, we can find a constant $c$ and a function $u$, such that the form $\omega = \chi + \sqrt{-1}\partial\overline{\partial}u$ satisfies the equation*

$$(179) \qquad \omega^k \wedge \alpha^{n-k} = e^{H+c}\alpha^n.$$

Note that for $k = 1$ this is the Poisson equation whose solution is standard, while for $k = n$ it is the complex Monge–Ampère equation, which was solved on Kähler manifolds by Yau [47] and by Tosatti–Weinkove [41] on Hermitian manifolds. For $1 < k < n$ it was solved by Dinew–Kolodziej [11] on Kähler manifolds, and by Sun [34] on Hermitian manifolds (see also Zhang [48]). For the reader's convenience we present the proof here.

*Proof.* We can write the equation in the form $F(A) = h$, for a positive function $H$ depending on $h$, where $F$ is defined by the function

$$(180) \qquad f = \log \sigma_k,$$

on the $k$-positive cone $\Gamma_k$:

$$(181) \qquad \Gamma_k = \{\lambda : \sigma_1(\lambda), \ldots, \sigma_k(\lambda) > 0\}.$$

This satisfies the structural conditions that we use (see Spruck [**33**]). In addition, $\underline{u} = 0$ is a subsolution if $\chi$ is any $k$-positive form. We can see this using Remark 8 together with the fact that for any $\mu = (\mu_1, \ldots, \mu_n) \in \Gamma_k$ we have

$$(182) \qquad \lim_{t \to \infty} \sigma_k(\mu_1, \ldots, \mu_{n-1}, t) = \infty.$$

We, therefore, have great flexibility in setting up a continuity method. For instance, we can let $H_0$ be the function such that

$$(183) \qquad \chi^k \wedge \alpha^{n-k} = e^{H_0} \alpha^n,$$

and then solve the family of equations

$$(184) \qquad \log \frac{(\chi + \sqrt{-1}\partial\overline{\partial}u_t)^k \wedge \alpha^{n-k}}{\alpha^n} = tH + (1-t)H_0 + c_t,$$

for $t \in [0, 1]$, where $c_t$ are constants. For $t = 0$ a solution is $u_0 = 0, c_0 = 0$. Openness follows from the fact that if $L$ denotes the linearized operator at any $t \in [0, 1]$, then the operator

$$(185) \qquad \begin{aligned} C^{k,\beta} \times \mathbf{R} &\to C^{k-2,\beta}, \\ (v, c) &\mapsto Lv + c \end{aligned}$$

is surjective. To obtain a priori estimates for the solutions we can first obtain bounds for $c_t$ from above and below by looking at the points where $u_t$ achieves its maximum and minimum in Equation (184). Given this, Theorem 2 gives higher order estimates. q.e.d.

We next focus on the Hessian quotient equation

$$(186) \qquad \omega^l \wedge \alpha^{n-l} = c\omega^k \wedge \alpha^{n-k},$$

where $(M, \alpha)$ is Kähler, $1 \le l < k \le n$, and $\omega = \chi + \sqrt{-1}\partial\overline{\partial}u$ with a fixed, closed background form $\chi$. We assume that the constant $c$ is chosen so that

$$(187) \qquad c = \frac{\int_M \chi^l \wedge \alpha^{n-l}}{\int_M \chi^k \wedge \alpha^{n-k}}.$$

The standard way of writing our equation would be to use the function $g = (\sigma_k/\sigma_l)^{1/(k-l)}$ on $\Gamma_k$. This function satisfies the required conditions (see Spruck [**33**]), however, it seems not to be well adapted to setting up a continuity method. Instead we will write the equation in the form

$$(188) \qquad -\frac{\omega^l \wedge \alpha^{n-l}}{\omega^k \wedge \alpha^{n-k}} = -c,$$

which is the same as $F(A) = -c$ with $F$ defined by the function

$$(189) \qquad f = -\frac{\binom{n}{l}^{-1}\sigma_l}{\binom{n}{k}^{-1}\sigma_k}.$$

Note that again $f$ is concave, since $f = -g^{-(k-l)}$. We will use a continuity method interpolating between this, and the Hessian equation, given by the function

$$(190) \qquad f_0 = -\frac{1}{\binom{n}{k}^{-1}\sigma_k}.$$

In other words, we will try to solve the equation

$$(191) \qquad t\frac{\omega^l \wedge \alpha^{n-l}}{\omega^k \wedge \alpha^{n-k}} + (1-t)\frac{\alpha^n}{\omega^k \wedge \alpha^{n-k}} = c_t,$$

for $t \in [0,1]$.

Corollary 3 follows from the following.

**Proposition 22.** *Suppose that $\chi$ is a closed $k$-positive form, satisfying*

$$(192) \qquad kc\chi^{k-1} \wedge \alpha^{n-k} - l\chi^{l-1} \wedge \alpha^{n-l} > 0,$$

*in the sense of positivity of $(n-1, n-1)$-forms, where $c$ is defined by* (187). *Equation* (191) *has a solution for all $t \in [0,1]$, for suitable $c_t$, such that $c_1 = c$.*

*Proof.* For $t = 0$ we can solve the equation using Proposition 21, and openness follows in the same way as in the proof of that proposition. It remains to obtain a priori estimates.

Note first of all, that by integrating (191) on $M$ with respect to $\omega^k \wedge \alpha^{n-k}$, we find that $c_t \geq tc$ for $t \in [0,1]$. Writing Equation (191) in the form

$$(193) \qquad f_t(\lambda) = -t\frac{\binom{n}{l}^{-1}\sigma_l}{\binom{n}{k}^{-1}\sigma_k} - (1-t)\frac{1}{\binom{n}{k}^{-1}\sigma_k} = -c_t,$$

the equation satisfies our structural assumptions, and we claim that $\underline{u} = 0$ is a $\mathcal{C}$-subsolution for it. For this, let $\mu_i$ denote the eigenvalues of $\alpha^{j\bar{p}}\chi_{j\bar{p}}$. By Remark 8 we just need to check that if $\mu'$ denotes any $(n-1)$-tuple from the $\mu_i$, then

$$(194) \qquad \lim_{R \to \infty} f_t(\mu', R) > -c_t,$$

which by the formula for $f_t$ means

$$(195) \qquad -t\frac{\binom{n}{l}^{-1}\sigma_{l-1}(\mu')}{\binom{n}{k}^{-1}\sigma_{k-1}(\mu')} > -c_t.$$

To rewrite this in terms of the forms $\chi, \alpha$, note that at any given point, if we restrict ourselves to the subspace of the tangent space of $M$ spanned by the eigenvectors corresponding to $\mu'$, then on this subspace

$$(196) \qquad \sigma_{i-1}(\mu') = \binom{n-1}{i-1} \frac{\chi^{i-1} \wedge \alpha^{n-i}}{\alpha^{n-1}},$$

for all $i$. Applying this to $i = l, k$ we find that (195) is equivalent to the inequality

$$(197) \qquad k c_t \chi^{k-1} \wedge \alpha^{n-k} - l t \chi^{l-1} \wedge \alpha^{n-l} > 0,$$

for $(n-1, n-1)$-forms. Since $\chi$ is $k$-positive and $c_t \geq tc$, this follows from (192). Theorem 2 will then give uniform estimates for $t$ in any compact interval $[c, 1]$ for $c > 0$. q.e.d.

It is an interesting problem to find geometric conditions which ensure the existence of a $\mathcal{C}$-subsolution. In analogy with the conjecture in [**27**] regarding the case when $k = n, l = n - 1$, it is natural to conjecture the following.

**Conjecture 23.** *Suppose that $\chi$ is a closed $k$-positive form. Then we can find a $k$-positive $\chi' \in [\chi]$ satisfying the inequality (192) with $\chi'$ instead of $\chi$ if and only if for all subvarieties $V \subset M$ of dimension $p = n - l, \ldots, n - 1$ we have*

$$(198) \quad \int_V c \frac{k!}{(k-n+p)!} \chi^{k-n+p} \wedge \alpha^{n-k} - \frac{l!}{(l-n+p)!} \chi^{l-n+p} \wedge \alpha^{n-l} > 0.$$

As we mentioned in the introduction, this conjecture has recently been resolved in [**8**] in the case when $M$ is a toric manifold, and $k = n, l = n - 1$, but cases beyond this are mostly open. Another related problem is to characterize real $(1,1)$-classes which admit $k$-positive representatives, in analogy with the result of Demailly–Paun [**10**] in the case $k = n$. Such $k$-positive representatives provide $\mathcal{C}$-subsolutions for the Hessian equation (179).

An equation related to the complex Monge–Ampère equation was introduced by Fu–Wang–Wu [**16**]. Given Hermitian metrics $\alpha, \eta$ and a function $h$, the equation can be written as

$$(199) \qquad \det\left(\eta_{i\bar{j}} + \frac{1}{n-1}\big[(\Delta u)\alpha_{i\bar{j}} - u_{i\bar{j}}\big]\right) = e^h \det \alpha,$$

where we require that the form inside the determinant on the left hand side is positive definite, and $\Delta$ denotes the (complex) Laplacian with respect to $\alpha$. Let us define the form

$$(200) \qquad \chi_{i\bar{j}} = (\mathrm{tr}_\alpha \eta)\alpha_{i\bar{j}} - (n-1)\eta_{i\bar{j}},$$

and denote by $T$ the map $T(A) = \frac{1}{n-1}(\mathrm{Tr}(A)I - A)$ on matrices, where $I$ is the identity. Write $g_{i\bar{j}} = \chi_{i\bar{j}} + u_{i\bar{j}}$, and $A^i_j = \alpha^{i\bar{p}} g_{j\bar{p}}$ as before. Then

Equation (199) is equivalent to the equation

$$(201) \qquad\qquad \log \det(T(A)) = h.$$

In other words, we can write our equation as $F(A) = h$, where $F$ is determined by the symmetric function

$$(202) \qquad\qquad f(\lambda) = \log \prod_{k=1}^{n} T(\lambda)_k,$$

where abusing notation we denote by $T : \mathbf{R}^n \to \mathbf{R}^n$ the map with components

$$(203) \qquad\qquad T(\lambda)_k = \frac{1}{n-1} \sum_{i \neq k} \lambda_i,$$

giving the map on eigenvalues corresponding to the matrix map $T$ above. The conditions $(i), (ii), (iii)$ in the introduction hold for this function $f$ on the pre-image $T^{-1}(\Gamma_n)$ of the positive orthant under $T$, and so using Theorem 2 we can obtain a priori estimates. Moreover, from Remark 8, we see that $\underline{u} = 0$ is a $\mathcal{C}$-subsolution, whenever $\eta$ is positive definite in Equation (199), since then $f(\lambda) \to \infty$ if we let any one component of $\lambda$ go to infinity. In particular, the same argument as in the proof of Proposition 21 can be used to show that if $\eta$ is positive definite, then Equation (199) admits a smooth solution $u$ for any smooth $h$, up to adding a constant to $h$. This recovers the result of Tosatti–Weinkove [**40**] in the case when $\alpha$ is Kähler, and Tosatti–Weinkove [**39**] when $\alpha$ is Hermitian (see also Fu–Wang–Wu [**15**] for an earlier result under a non-negative curvature assumption for $\alpha$).

One can also consider more general equations in this vein, for instance, those given by the functions

$$(204) \qquad\qquad f(\lambda) = \log \frac{\sigma_k(T(\lambda))}{\sigma_l(T(\lambda))},$$

for $0 \leq l < k \leq n$, where $T : \mathbf{R}^n \to \mathbf{R}^n$ is the map above. Using that $T$ maps $\Gamma_n$ into itself, one sees that these $f$ satisfy the conditions $(i), (ii), (iii)$ in the introduction on the preimage $T^{-1}(\Gamma_k)$ of the $k$-positive cone $\Gamma_k$. When $l > 0$, the subsolution condition is non-trivial (i.e. it depends on $h$), so we do not expect to be able to solve $F(A) = h$ for all $h$, even up to adding a constant to $h$. However, when $l = 0$, and the background form $\chi$ is such that $\underline{u} = 0$ is a $\mathcal{C}$-subsolution, then $0$ will be a $\mathcal{C}$-subsolution for all $h$, since in this case we have $\lim_{t \to \infty} f(\lambda + t\mathbf{e}_i) = \infty$ for all $i$. In particular, the same argument as the proof of Proposition 21 implies the following, which answers a question raised in Tosatti–Wang–Weinkove–Yang [**38**].

**Proposition 24.** *Let $(M, \alpha)$ be a compact Hermitian manifold, $\eta$ a k-positive real $(1,1)$-form, and $1 \leq k \leq n$. For any smooth function $H$*

*on $M$ we can find a constant $c$ and a function $u$ satisfying the equation*

$$(205) \qquad \left(\eta + \frac{1}{n-1}\left[(\Delta u)\alpha - \sqrt{-1}\partial\overline{\partial}u\right]\right)^k \wedge \alpha^{n-k} = e^{H+c}\alpha^n.$$

*Here $\Delta$ denotes the complex Laplacian with respect to $\alpha$, i.e. $\Delta u = \alpha^{p\bar{q}}u_{p\bar{q}}$.*

To see this, note that as for Equation (199), we write the equation in the form $F(A) = h$ where $A^i_j = \alpha^{i\bar{p}}(\chi_{j\bar{p}} + u_{j\bar{p}})$ in terms of the background form $\chi$ given by (200). If $\eta$ is $k$-positive, then the eigenvalues of $\alpha^{i\bar{p}}\chi_{j\bar{p}} = T^{-1}(\alpha^{i\bar{p}}\eta_{j\bar{p}})$ lie in the preimage $T^{-1}(\Gamma_k)$ of the $k$-positive cone, and so $0$ is a $\mathcal{C}$-subsolution.

## 8. Equations on Riemannian manifolds

In this section, we will describe how the methods in this paper apply to equations analogous to (1) on Riemannian manifolds as well. So, in this section, we let $(M, \alpha)$ be a compact Riemannian manifold and $\chi$ a fixed tensor of type (0,2). Suppose we are interested in solving the equation

$$(206) \qquad\qquad\qquad F(A) = h,$$

where analogously to before, $A$ is the endomorphism of the tangent bundle given by $A^i_j = \alpha^{ip}(\chi_{jp} + u_{jp})$ for the unknown function $u$, and $u_{jp}$ denote covariant derivatives with respect to $\alpha$. This endomorphism is symmetric with respect to the inner product defined by $\alpha$ at each point, and as before $F(A) = f(\lambda(A))$ in terms of the eigenvalues $\lambda(A)$ of $A$. We assume that $f$ satisfies the structural conditions (i), (ii), (iii) from the introduction.

Everything that we have done in the Hermitian case applies in this Riemannian setting as well, with almost exactly the same proof. There are no torsion terms, but instead there are some extra curvature terms obtained when commuting derivatives. We briefly note some of the differences. In (76) we obtain

$$(207) \qquad\qquad\qquad u_{11kk} = u_{kk11} + O(\lambda_1),$$

while in (83) we obtain

$$(208) \qquad\qquad\qquad g_{11k} = g_{k11} + O(K^{1/2}),$$

and so

$$(209) \qquad\qquad |g_{11k}|^2 \leq |g_{k11}|^2 + C_0(K^{1/2}|g_{1\bar{1}k}| + K).$$

The upshot is that we have the following analogous inequality to (85):

$$(210) \quad L(\log \widetilde{\lambda}_1) \geq \frac{-F^{pq,rs}g_{pq1}g_{rs1}}{\lambda_1} - \frac{F^{kk}|g_{k11}|^2}{\lambda_1^2} - C_0(\mathcal{F} + K^{1/2}|g_{1\bar{1}k}|).$$

Using this the rest of the argument in the $C^2$-estimate is essentially identical.

The Riemannian case has the distinct advantage in that when using the concavity of the operator as in (67), we get twice as many useful terms as in the complex case:

$$(211) \qquad F^{ij,rs} u_{ij1} u_{rs1} \leq 2 \sum_{k>1} \frac{f_1 - f_k}{\lambda_1 - \lambda_k} |u_{k11}|^2.$$

In the complex case the corresponding extra terms are $|u_{1\bar{k}1}|^2$, which do not appear to be useful in the estimates. A consequence of this is that in the real case it is more straight forward to control the bad negative term involving $|g_{11k}|^2$. See, for instance, Guan–Jiao [21] for such estimates.

In the proof of Theorem 20, when we start the induction argument, the case $n = 1$ corresponds to bounded linear functions on $\mathbf{R}$ being constant, while the case $\Gamma = \Gamma_n$ corresponds to bounded convex functions on $\mathbf{R}^n$ being constant.

Just as before, a function $\underline{u}$ is a $\mathcal{C}$-subsolution for the equation $F(A) = h$, if the matrix $B^i_j = \alpha^{ip}(\chi_{jp} + \underline{u}_{jp})$ is such that the set $(\lambda(B) + \Gamma_n) \cap \partial \Gamma^{h(x)}$ is bounded at each $x \in M$. We then have the following.

**Proposition 25.** *Suppose that there exists a $\mathcal{C}$-subsolution $\underline{u}$ for the equation $F(A) = h$ as above. Normalizing $u$ so that $\sup_M u = 0$, we have a priori estimates $\|u\|_{C^{2,\alpha}} < C$, with constant depending on the background data as well as the subsolution $\underline{u}$.*

This result generalizes several earlier results on these types of equations on compact Riemannian manifolds, such as Li [29], Delanoë [9] who made non-negative curvature assumptions, and Urbas [45], Guan [20], who have stronger structural assumptions. In particular, in Urbas [45] the question of solving the Hessian quotient equations analogous to (186) on compact Riemannian manifolds is raised. This is formulated as the equation

$$(212) \qquad \log F(A) = h + c,$$

where $h$ is a given function, the function $u$ and constant $c$ are the unknowns, and $F$ is given by the function

$$(213) \qquad f = \left( \frac{\sigma_k}{\sigma_l} \right)^{\frac{1}{k-l}},$$

for some $1 \leq l < k \leq n$.

In analogy with the Kähler case, it is natural to expect that these equations do not always have a solution, but it seems to be difficult to formulate a condition as precise as that in Conjecture 23. Instead we formulate a general existence result, focusing for simplicity on equations of the form $F(A) = c$ with constant $c$. Note that if $F$ is homogeneous

and positive, then the restriction to constant right hand side can be removed by scaling the metric $\alpha$.

**Proposition 26.** *Suppose that* $\sup_{\partial\Gamma} f = -\infty$*, and* $\sup_\Gamma f = \infty$*. Let* $h_0 = F(\alpha^{ip}\chi_{jp})$*. If the equation* $F(A) = \sup_M h_0$ *admits a* $\mathcal{C}$*-subsolution* $\underline{u}$*, then the equation* $F(A) = c$ *has a solution for some constant* $c$*.*

*Proof.* We want to use the continuity method to solve the equations

$$(214) \qquad F(A) = c_t + (1-t)h_0,$$

for $t \in [0,1]$ with constants $c_t$. For $t = 0$ the solution is $u = c_0 = 0$, and openness follows using the implicit function theorem as before.

To find a priori estimates, the only thing we need is $\mathcal{C}$-subsolutions for each $t$, and we need to make sure that the range of the right hand side $c_t + (1-t)h_0$ is contained in a compact subset of the range of $f$ in order to obtain uniform constants. Suppose that $u$ is a solution of (214) and $u$ achieves its minimum and maximum at $p \in M$ and $q \in M$, respectively. We then have $F(A) \geq F(\alpha^{ip}\chi_{jp})$ at $p$ and $F(A) \leq F(\alpha^{ip}\chi_{jp})$ at $q$. It follows that

$$(215) \qquad h_0(p) \leq c_t + (1-t)h_0(p),$$

i.e., $c_t \geq th_0(p)$, and, similarly, $c_t \leq th_0(q)$. In particular, we obtain upper and lower bounds for $c_t + (1-t)h_0$, whose range is then in a compact subset of the range of $f$ by our assumption for $f$. More precisely at any $x \in M$ we have

$$(216) \qquad c_t + (1-t)h_0(x) \leq th_0(q) + (1-t)h_0(x) \leq \sup_M h_0,$$

which implies that $\underline{u}$ is a $\mathcal{C}$-subsolution for Equation (214) for each $t$. Proposition 25 then implies the required estimates.                q.e.d.

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Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556
USA
*E-mail address*: gszekely@nd.edu