

MINIMAL SURFACES IN THE ROUND THREE-SPHERE BY DOUBLING THE EQUATORIAL TWO-SPHERE, I

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Abstract

We construct closed embedded minimal surfaces in the round three-sphere $\mathbb{S}^3(1)$, resembling two parallel copies of the equatorial two-sphere \mathbb{S}_{eq}^2 , joined by small catenoidal bridges symmetrically arranged either along two parallel circles of \mathbb{S}_{eq}^2 , or along the equatorial circle and the poles. To carry out these constructions we refine and reorganize the doubling methodology in ways which we expect to apply also to further constructions. In particular, we introduce what we call “linearized doubling”, which is an intermediate step where singular solutions to the linearized equation are constructed subject to appropriate linear and nonlinear conditions. Linearized doubling provides a systematic approach for dealing with the obstructions involved and also understanding in detail the regions further away from the catenoidal bridges.

1. Introduction

The general framework. This article is an important step in the author’s program to develop doubling constructions for minimal surfaces by singular perturbation methods. It is also the first article in a series in which we discuss gluing constructions for closed embedded minimal surfaces in the round three-sphere $\mathbb{S}^3(1)$ by doubling the equatorial two-sphere \mathbb{S}_{eq}^2 . Doublings of the equatorial two-sphere \mathbb{S}_{eq}^2 are important because their area is close to 8π (the area of two equatorial two-spheres), a feature they share with the celebrated surfaces constructed by Lawson in 1970 [21]. The classification of the low area closed embedded minimal surfaces in the round three-sphere $\mathbb{S}^3(1)$, especially of those of area close to 8π or less, is a natural open question. This is further motivated by the recent resolutions of the Lawson conjecture by Brendle [1] and the Willmore conjecture by Marques and Neves [23] where they also characterize the Clifford torus and the equatorial sphere as the only examples of area $\leq 2\pi^2$. We refer to [2] for a survey of existence and uniqueness results for minimal surfaces in the round three-sphere.

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The general idea of doubling constructions by gluing methods was proposed and discussed in [17, 20, 18]. Gluing methods have been applied extensively and with great success in Gauge Theories by Donaldson, Taubes, and others. The particular kind of gluing methods used relates most closely to the methods developed in [24] and [10], especially as they evolved and were systematized in [13, 14, 15]. We refer to [17] for a general discussion of this gluing methodology and to [18] for a detailed general discussion of doubling by gluing methods.

Roughly speaking, in such doubling constructions one starts with an approximately minimal surface consisting of two approximately parallel copies of a given minimal surface Σ with a number of discs removed and replaced by approximately catenoidal bridges. The initial surface is then perturbed to minimality by Partial Differential Equations methods. Understanding such constructions in full generality seems beyond the immediate horizon at the moment. In the first such construction [20], there is so much symmetry imposed that the position of the catenoidal bridges is completely fixed and all bridges are identical modulo the symmetries. Moreover, the bridges are uniformly distributed, that is when their number is large enough, there are bridges located inside any pre-assigned domain of Σ . Wiygul [26, 25] has extended that construction to situations where the symmetries do not determine the vertical (that is perpendicular to Σ) position of the bridges.

In this article for the first time we deal with situations where the horizontal position of the bridges is not determined by the symmetries, that is the bridges can slide along Σ , or there are more than one bridge modulo the symmetries. Equally importantly the bridges are not uniformly distributed on Σ , that is they stay away from certain fixed domains of Σ even when the number of the bridges tends to infinity. To realize such constructions we introduce what we call “linearized doubling”, which is an intermediate step in the construction, where singular solutions to the linearized equation are constructed, subject to appropriate linear and nonlinear conditions. Linearized doubling provides a systematic approach for dealing with the obstructions involved and also provides a detailed understanding of the regions further away from the catenoidal bridges.

We expect that linearized doubling will be indispensable in developing further constructions except in the (rare) cases of exceptionally high symmetry. Since there is an abundance of such potential constructions, linearized doubling will have many further profound applications. Note, for example, the potential doubling constructions of free boundary minimal surfaces, or of self-shrinkers of the mean curvature flow, which we will discuss elsewhere.

Unlike the case of desingularization constructions, doubling constructions generalize to higher dimensions: In another article under preparation [8], we generalize the current results to doubling the equatorial $\mathbb{S}^{n-1}(1)$ in the round $\mathbb{S}^n(1)$ for any $n > 3$. Although the existence of infinitely many closed embedded smooth minimal hypersurfaces of some simple topological types in the round sphere of dimension $n > 3$ was established by Hsiang [6, 7] and of unknown topological type for $3 \leq n \leq 7$ by Marques–Neves [22], our construction in [8] provides for the first time infinitely many topological types of closed embedded smooth minimal hypersurfaces in the round sphere of any dimension $n > 3$. Note that the constructions in [8] like the ones in this article are fairly explicit with the volume of the hypersurfaces constructed uniformly bounded (depending on the dimension).

We return now to the doublings of the equatorial two-sphere \mathbb{S}_{eq}^2 constructed in this article and the rest of the series. All these doublings are symmetric under a group $\mathcal{G}_{\mathbb{S}^3, m}$. $\mathcal{G}_{\mathbb{S}^3, m}$ is defined (see 2.14) as the group of isometries of $\mathbb{S}^3(1)$ which map L_{mer} to itself, where L_{mer} (see 2.13) is the union of m_{mer} meridians arranged with maximal symmetry. The centers of the catenoidal bridges we employ in the construction form a set L which we call the configuration of the construction. L is invariant under $\mathcal{G}_{\mathbb{S}^3, m}$ and, therefore, we can write $L = L_{mer} \cap L_{par}$ where L_{par} is the union of m_{par} parallel circles symmetrically arranged with respect to the equator. The number of bridges used is, therefore, $m_{mer}m_{par}$, or when the poles (as degenerate circles) are included, $m_{mer}(m_{par} - 2) + 2$. The latitude of the circles in L_{par} (except for the equator and poles if included) has to be appropriately chosen for the construction to work. We call this “horizontal balancing”. As discussed in [18] and later in 6.31 the construction fails when L lies on an equatorial circle. We need, therefore, $m_{mer} \geq 3$ and $m_{par} \geq 2$.

The perturbation methods we employ require that the catenoidal bridges are small so that they do not interact with each other too much. To ensure this we need the number of catenoidal bridges to be large. Moreover, our current approach relies on a comparison with and careful analysis of certain rotationally invariant solutions which are controlled by ODEs, and this imposes the extra requirement that m_{mer} is large. We only present the two simplest possible cases in this article in order to emphasize the fundamental ideas and minimize technical issues: In the first case (see theorem 7.1, also announced and discussed in [18]) $m_{par} = 2$ and, therefore, we have two parallel circles and the number of catenoidal bridges is $2m_{mer}$; in the second case (see theorem 7.3) $m_{par} = 3$ with parallel circles the two poles (which we count as degenerate parallel circles) and the equator circle, and, therefore, we have $m_{mer} + 2$ bridges.

The approach here can be extended to apply at least to the case when m_{mer} is large in terms of m_{par} [19]. The exact limitations of the applicability of this approach are currently under investigation although there certainly exist cases where the ODE model is inadequate, as, for example, when m_{par} is large and m_{mer} small. In such cases further ideas will be needed to carry out the construction.

Outline of the approach. The constructions in this article and articles in preparation using the same approach are based on the following two main ideas: The first idea involves the introduction of an intermediate step in the construction, as mentioned earlier, where singular solutions of the linearized equation on the given surface being doubled (the equatorial two-sphere in this article) are constructed and analyzed. These singular solutions have logarithmic singularities at the points where we plan to place the catenoidal bridges. The initial surfaces are constructed by gluing the catenoidal bridges to appropriately modified graphs of these singular solutions with neighborhoods of the singular points excised.

More precisely the simplest singular solutions of the linearized equation we consider satisfy the linearized equation away from the singularities and can be viewed also as multi-Green's functions for the linearized equation. We call them *linearized doubling* (LD) solutions (see 3.1). If we use an LD solution to construct an initial surface as described above, to ensure that the error introduced by the gluing is small, the LD solution has to satisfy certain matching conditions. Unfortunately the supply of LD solutions which satisfy these matching conditions is inadequate for our purposes. This can be remedied, however, by expanding the class of LD solutions under consideration to a larger class of solutions which satisfy the linearized equation only modulo a certain space which we call $\mathcal{K}[L]$ (see 3.2) which plays also the role of the (extended) substitute kernel used in the linear theory in various earlier constructions [18, 17, 20, 5, 4, 16, 15, 14, 13, 11, 12, 10, 9]. We call those of the solutions in the expanded class that satisfy the desired matching conditions *matched linearized doubling* (MLD) solutions (see 3.4). MLD solutions are in sufficient supply because by an easy technical step it is possible to convert any LD solution (even if it does not satisfy the matching conditions) to a corresponding MLD solution. In doing so we trade the failure to satisfy the matching conditions for the failure to satisfy the precise linearized equation.

It is rather difficult to estimate the LD and MLD solutions carefully so that we have satisfactory control of the construction. In particular, we need to construct families of MLD solutions which satisfy the balancing and unbalancing conditions as required by the general approach (see [17, 18] for a discussion of the general approach). The second main idea of this article allows us in certain cases to achieve the required con-

trol by comparing the LD and MLD solutions to certain ODE solutions which can be well understood. In particular, the study of balancing and unbalancing questions is reduced to the ODE framework. The implementation of this idea relies on the rotational invariance of the original surface (the equatorial two-sphere in this article) and the largeness of m_{mer} . If these conditions are not satisfied the questions involving the LD and MLD solutions (and the corresponding doubling constructions) are still open.

At a more technical level we remark that in this article we experimented with constructing the initial surfaces carefully so that they are exactly minimal away from the gluing regions and the support of the functions in $\mathcal{K}[L]$. This reduces the error terms we have to deal with later, at the expense of complicating the construction of the initial surfaces. We also note that we organized the presentation so that the results using standard or earlier methodology (sections 2, 3, 4 and 7) are separated from the more innovative steps of constructing and analyzing the LD and MLD solutions (sections 5 and 6).

Organization of the presentation. The main body of this article consists of three parts. The first part consists of sections 2, 3, and 4, where we present a general construction of initial approximate minimal surfaces based on LD solutions and MLD solutions. The second part of the paper consists of sections 5 and 6 where we construct and study in detail the LD and MLD solutions needed for the constructions carried out in this paper. Finally, in the last part which consists of section 7 only we combine the earlier results to prove the main results of this paper.

In more detail now, in section 2, we review the elementary geometry of the geometric objects we are interested in, and we establish the corresponding notation. In particular, we study aspects of the geometry of the round three-sphere and its equator, the symmetries we impose, and the catenoidal bridges we will be using later. In section 3, we discuss in detail linearized doubling, the LD and MLD solutions, and we construct the initial surfaces by gluing MLD solutions and catenoidal bridges. We also discuss geometric aspects of the initial surfaces needed later in understanding their perturbations. In section 4, we develop the perturbation theory on the initial surfaces constructed in section 3: We solve the linearized equation on the initial surfaces and we also estimate the solutions and the corresponding nonlinear terms. Note that the theory in sections 3 and 4 is developed with a general setting in mind (see also 3.21) and is not restricted to the cases we actually pursue in this article.

In section 5, we carefully study and estimate the LD and MLD solutions needed for the construction of doublings where the catenoidal bridges are distributed on two parallel circles. In section 6, we do the

same in the case where the catenoidal bridges are distributed on the equatorial circle with two more bridges, one at each pole. Finally, in section 7, we use the MLD solutions constructed in sections 5 and 6 to construct our minimal surfaces by using the results of sections 3 and 4.

General notation and conventions. In comparing equivalent norms we will find the following notation useful.

Definition 1.1. If $a, b > 0$ and $c > 1$ we write $a \sim_c b$ to mean that the inequalities $a \leq cb$ and $b \leq ca$ hold.

We discuss now the Hölder norms we use. We use the standard notation $\|u : C^{k,\beta}(\Omega, g)\|$ to denote the standard $C^{k,\beta}$ -norm of a function or more generally tensor field u on a domain Ω equipped with a Riemannian metric g . Actually the definition is completely standard only when $\beta = 0$ because then we just use the covariant derivatives and take a supremum norm when they are measured by g . When $\beta \neq 0$ we have to use parallel transport along geodesic segments connecting any two points of small enough distance in order to define the Hölder seminorms and this could lead to complications in some cases. In this paper we take care to avoid situations where such complications may arise and so we will not discuss this issue further.

In this paper we use also weighted Hölder norms. The definition we use is somewhat more flexible than the one used in some earlier work (for example, in [16, 10, 14, 20, 4]):

Definition 1.2. Assuming that Ω is a domain inside a manifold, g is a Riemannian metric on the manifold, $\rho, f : \Omega \rightarrow (0, \infty)$ are given functions, $k \in \mathbb{N}_0$, $\beta \in [0, 1)$, $u \in C^{k,\beta}_{loc}(\Omega)$ or more generally u is a $C^{k,\beta}_{loc}$ tensor field (section of a vector bundle) on Ω , and that the injectivity radius in the manifold around each point x in the metric $\rho^{-2}(x)g$ is at least $1/10$, we define

$$\|u : C^{k,\beta}(\Omega, \rho, g, f)\| := \sup_{x \in \Omega} \frac{\|u : C^{k,\beta}(\Omega \cap B_x, \rho^{-2}(x)g)\|}{f(x)},$$

where B_x is a geodesic ball centered at x and of radius $1/100$ in the metric $\rho^{-2}(x)g$. For simplicity we may omit any of β , ρ , or f , when $\beta = 0$, $\rho \equiv 1$, or $f \equiv 1$, respectively.

f can be thought of as a “weight” function because $f(x)$ controls the size of u in the vicinity of the point x . ρ can be thought of as a function which determines the “natural scale” $\rho(x)$ at the vicinity of each point x . Note that if u scales nontrivially we can modify appropriately f by multiplying by the appropriate power of ρ . Note that from the definition follows that we always have

$$(1.3) \quad \|\nabla u : C^{k-1,\beta}(\Omega, \rho, g, \rho^{-1}f)\| \leq \|u : C^{k,\beta}(\Omega, \rho, g, f)\|,$$

and the multiplicative property

$$(1.4) \quad \|u_1 u_2 : C^{k,\beta}(\Omega, \rho, g, f_1 f_2)\| \leq C(k) \|u_1 : C^{k,\beta}(\Omega, \rho, g, f_1)\| \|u_2 : C^{k,\beta}(\Omega, \rho, g, f_2)\|.$$

We will be using extensively cut-off functions, and for this reason we adopt the following.

Definition 1.5. We fix a smooth function $\Psi : \mathbb{R} \rightarrow [0, 1]$ with the following properties:

- (i). Ψ is nondecreasing.
- (ii). $\Psi \equiv 1$ on $[1, \infty)$ and $\Psi \equiv 0$ on $(-\infty, -1]$.
- (iii). $\Psi - \frac{1}{2}$ is an odd function.

Given now $a, b \in \mathbb{R}$ with $a \neq b$, we define smooth functions $\psi_{cut}[a, b] : \mathbb{R} \rightarrow [0, 1]$ by

$$(1.6) \quad \psi_{cut}[a, b] := \Psi \circ L_{a,b},$$

where $L_{a,b} : \mathbb{R} \rightarrow \mathbb{R}$ is the linear function defined by the requirements $L(a) = -3$ and $L(b) = 3$.

Clearly then $\psi_{cut}[a, b]$ has the following properties:

- (i). $\psi_{cut}[a, b]$ is weakly monotone.
- (ii). $\psi_{cut}[a, b] = 1$ on a neighborhood of b and $\psi_{cut}[a, b] = 0$ on a neighborhood of a .
- (iii). $\psi_{cut}[a, b] + \psi_{cut}[b, a] = 1$ on \mathbb{R} .

Suppose now we have two sections f_0, f_1 of some vector bundle over some domain Ω . (A special case is when the vector bundle is trivial and f_0, f_1 real-valued functions). Suppose we also have some real-valued function d defined on Ω . We define a new section

$$(1.7) \quad \Psi[a, b; d](f_0, f_1) := \psi_{cut}[a, b] \circ d f_1 + \psi_{cut}[b, a] \circ d f_0.$$

Note that $\Psi[a, b; d](f_0, f_1)$ is then a section which depends linearly on the pair (f_0, f_1) and transits from f_0 on Ω_a to f_1 on Ω_b , where Ω_a and Ω_b are subsets of Ω which contain $d^{-1}(a)$ and $d^{-1}(b)$ respectively, and are defined by

$$\Omega_a = d^{-1}((-\infty, a + \frac{1}{3}(b - a))), \quad \Omega_b = d^{-1}((b - \frac{1}{3}(b - a), \infty)),$$

when $a < b$, and

$$\Omega_a = d^{-1}((a - \frac{1}{3}(a - b), \infty)), \quad \Omega_b = d^{-1}((-\infty, b + \frac{1}{3}(a - b))),$$

when $b < a$. Clearly if f_0, f_1, d are smooth then $\Psi[a, b; d](f_0, f_1)$ is also smooth.

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2. Elementary geometry and notation

The parametrization Θ and the coordinates xyz . We consider now the unit three-sphere $\mathbb{S}^3(1) \subset \mathbb{R}^4$. We denote by (x_1, x_2, x_3, x_4) the standard coordinates of \mathbb{R}^4 and we define by

$$(2.1) \quad \mathbb{S}_{eq}^2 := \mathbb{S}^3(1) \cap \{x_4 = 0\},$$

an equatorial two-sphere in $\mathbb{S}^3(1)$. To facilitate the discussion we fix spherical coordinates (x, y, z) on $\mathbb{S}^3(1)$ (see 2.12) by defining a map $\Theta : \mathbb{R}^3 \rightarrow \mathbb{S}^3(1)$ by

$$(2.2) \quad \Theta(x, y, z) = (\cos x \cos y \cos z, \cos x \sin y \cos z, \sin x \cos z, \sin z).$$

Note that in the above notation we can think of x as the geographic latitude on \mathbb{S}_{eq}^2 and of y as the geographic longitude. We will also refer to

$$(2.3) \quad \begin{aligned} \mathbb{P}_0 &:= \mathbb{S}_{eq}^2 \cap \{x_3 = 0\} = \Theta(\{x = z = 0\}), \\ p_N &:= (0, 0, 1, 0) = \Theta(\pi/2, y, 0), \\ p_S &:= (0, 0, -1, 0) = \Theta(-\pi/2, y, 0), \end{aligned}$$

as the equator circle, the North pole, and the South pole of \mathbb{S}_{eq}^2 respectively. More generally to facilitate reference to circles of latitude we introduce the notation for $x \in [-1, 1]$

$$(2.4) \quad \mathbb{P}_x := \mathbb{S}_{eq}^2 \cap \{x_3 = x\}.$$

We have then that $\mathbb{P}_{\sin x}$ is the circle (or pole) of latitude x (which is consistent with the definition of the equator circle \mathbb{P}_0 above), $\mathbb{P}_{-1} = \{p_S\}$, and $\mathbb{P}_1 = \{p_N\}$.

Clearly the standard metric of $\mathbb{S}^3(1)$ is given in the coordinates of 2.2 by

$$(2.5) \quad \Theta^*g = \cos^2 z (dx^2 + \cos^2 x dy^2) + dz^2.$$

Finally, we define a nearest-point projection by

$$(2.6) \quad \begin{aligned} \Pi_{\mathbb{S}_{eq}^2} : \mathbb{S}^3(1) \setminus \{(0, 0, 0, \pm 1)\} &\rightarrow \mathbb{S}_{eq}^2, \\ \Pi_{\mathbb{S}_{eq}^2}(x_1, x_2, x_3, x_4) &= \frac{1}{|(x_1, x_2, x_3, 0)|} (x_1, x_2, x_3, 0). \end{aligned}$$

Clearly we have

$$(2.7) \quad \Pi_{\mathbb{S}_{eq}^2} \circ \Theta(x, y, z) = \Theta(x, y, 0).$$

We introduce now some convenient notation.

NOTATION 2.8. For X a subset of \mathbb{S}_{eq}^2 we will write \mathbf{d}_X for the distance function from X , that is $\mathbf{d}_X(p)$ denotes the distance in \mathbb{S}_{eq}^2 of some $p \in \mathbb{S}_{eq}^2$ from X with respect to the standard metric. Moreover, for $\delta > 0$ we define a tubular neighborhood of X by

$$D_X(\delta) := \{p \in \mathbb{S}_{eq}^2 : \mathbf{d}_X(p) < \delta\}.$$

If X is finite we just enumerate its points in both cases, for example, $\mathbf{d}_q(p)$ is the geodesic distance between p and q and $D_q(\delta)$ is the geodesic disc in \mathbb{S}_{eq}^2 of center q and radius δ . □

Symmetries of Θ and symmetries of the construction. We first define reflections $\widehat{\underline{X}}, \widehat{\underline{Y}}_c, \widehat{\underline{Y}} := \widehat{\underline{Y}}_0$, and $\widehat{\underline{Z}}$ in \mathbb{R}^3 , and translations $\widehat{\underline{Y}}_c$ in \mathbb{R}^3 , where $c \in \mathbb{R}$, by

$$(2.9) \quad \begin{aligned} \widehat{\underline{X}}(x, y, z) &:= (-x, y, z), & \widehat{\underline{Y}}_c(x, y, z) &:= (x, 2c - y, z), \\ \widehat{\underline{Z}}(x, y, z) &:= (x, y, -z), & \widehat{\underline{Y}}_c(x, y, z) &:= (x, y + c, z). \end{aligned}$$

All these clearly preserve

$$(2.10) \quad \text{Dom}_\Theta := \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

We also define corresponding reflections $\underline{X}, \underline{Y}_c, \underline{Y} := \underline{Y}_0$, and \underline{Z} in \mathbb{R}^4 , and rotations \underline{Y}_c in \mathbb{R}^4 , all of which preserve $\mathbb{S}^3(1) \subset \mathbb{R}^4$, by

$$(2.11) \quad \begin{aligned} \underline{X}(x_1, x_2, x_3, x_4) &:= (x_1, x_2, -x_3, x_4), \\ \underline{Y}(x_1, x_2, x_3, x_4) &:= (x_1, -x_2, x_3, x_4), \\ \underline{Z}(x_1, x_2, x_3, x_4) &:= (x_1, x_2, x_3, -x_4), \\ \underline{Y}_c(x_1, x_2, x_3, x_4) &:= (x_1 \cos 2c + x_2 \sin 2c, x_1 \sin 2c - x_2 \cos 2c, x_3, x_4), \\ \underline{Y}_c(x_1, x_2, x_3, x_4) &:= (x_1 \cos c - x_2 \sin c, x_1 \sin c + x_2 \cos c, x_3, x_4). \end{aligned}$$

Note that $\underline{X}, \underline{Y}, \underline{Z}$, and \underline{Y}_c are reflections with respect to the 3-planes $\{x_3 = 0\}, \{x_2 = 0\}, \{x_4 = 0\}$, and $\underline{Y}_c(\{x_2 = 0\})$, respectively. \underline{Z} fixes \mathbb{S}_{eq}^2 pointwise and exchanges its two sides in $\mathbb{S}^3(1)$. Clearly $\underline{Y}_{2\pi}$ is the identity map. We record the symmetries of Θ in the following lemma:

Lemma 2.12. Θ restricted to Dom_Θ is a covering map onto $\mathbb{S}^3(1) \setminus \{x_1 = x_2 = 0\}$. Moreover, the following hold:

- (i). The group of covering transformations is generated by $\widehat{\underline{Y}}_{2\pi}$.
- (ii). $\underline{X} \circ \Theta = \Theta \circ \widehat{\underline{X}}, \underline{Y}_c \circ \Theta = \Theta \circ \widehat{\underline{Y}}_c, \underline{Z} \circ \Theta = \Theta \circ \widehat{\underline{Z}}$, and $\underline{Y}_c \circ \Theta = \Theta \circ \widehat{\underline{Y}}_c$.

The symmetry group of our constructions depends on a large number $m \in \mathbb{N}$ which we assume now fixed. We define $L_{mer} = L_{mer}[m] \subset \mathbb{S}_{eq}^2$ to be the union of m meridians symmetrically arranged:

$$(2.13) \quad L_{mer} = L_{mer}[m] := \Theta(\{(x, y, 0) : x \in [-\pi/2, \pi/2], y = 2\pi i/m, i \in \mathbb{Z}\}).$$

Definition 2.14. We denote by $\mathcal{G}_{\mathbb{S}^3,m}$ and $\mathcal{G}_{\mathbb{S}_{eq}^2,m}$ the groups of isometries of $\mathbb{S}^3(1)$ and \mathbb{S}_{eq}^2 respectively which fix $L_{mer}[m]$ as a set.

Clearly $\mathcal{G}_{\mathbb{S}^3,m}$ is a finite group and is generated by the reflections \underline{X} , \underline{Y} , \underline{Z} and $\underline{Y}_{\pi/m}$. $\mathcal{G}_{\mathbb{S}_{eq}^2,m}$ can be identified with the subgroup of $\mathcal{G}_{\mathbb{S}^3,m}$ which is generated by \underline{X} , \underline{Y} , and $\underline{Y}_{\pi/m}$.

The linearized equation and rotationally invariant solutions. It will be easier later to state some of our estimates if we use a scaled metric on \mathbb{S}_{eq}^2 and scaled coordinates (\tilde{x}, \tilde{y}) defined by

$$(2.15) \quad \tilde{g} := m^2 g_{\mathbb{S}_{eq}^2}, \quad \tilde{x} = mx, \quad \tilde{y} = my.$$

To simplify the notation we also define linear operators acting on twice differentiable functions on domains of \mathbb{S}_{eq}^2 by

$$(2.16) \quad \mathcal{L}' := \Delta + 2, \quad \mathcal{L}'_{\tilde{g}} := \Delta_{\tilde{g}} + 2m^{-2} = m^{-2}\mathcal{L}'.$$

\mathcal{L}' is of course the linearized operator for the mean curvature on \mathbb{S}_{eq}^2 .

By a rotationally invariant function we mean a function on a domain of \mathbb{S}_{eq}^2 which depends only on the latitude x . The linearized equation $\mathcal{L}'\phi = 0$ amounts to an ODE when the solution ϕ is rotationally invariant. Motivated by this we introduce some notation to simplify the presentation.

NOTATION 2.17. Consider a function space X consisting of functions defined on a domain $\Omega \subset \mathbb{S}_{eq}^2$. If Ω is invariant under the action of $\mathcal{G}_{\mathbb{S}_{eq}^2,m}$ we use a subscript “*sym*” to denote the subspace $X_{sym} \subset X$ consisting of those functions in X which are invariant under the action of $\mathcal{G}_{\mathbb{S}_{eq}^2,m}$. If Ω is a union of parallel circles we use a subscript “*x*” to denote the subspace of functions X_x consisting of rotationally invariant functions which, therefore, depend only on x . If, moreover, Ω is invariant under reflection with respect to the equator of \mathbb{S}_{eq}^2 we use a subscript “ $|x|$ ” to denote the subspace of functions $X_{|x|} = X_x \cap X_{sym}$ consisting of those functions which depend only on $|x|$. \square

For example, we have $C_{|x|}^0(\mathbb{S}_{eq}^2) \subset C_{sym}^0(\mathbb{S}_{eq}^2) \subset C^0(\mathbb{S}_{eq}^2)$ and $C_{|x|}^0(\mathbb{S}_{eq}^2) \subset C_x^0(\mathbb{S}_{eq}^2)$, but $C_x^0(\mathbb{S}_{eq}^2)$ is not a subset of $C_{sym}^0(\mathbb{S}_{eq}^2)$.

Definition 2.18. We define rotationally invariant functions $\phi_{odd} \in C_x^\infty(\mathbb{S}_{eq}^2)$ and $\phi_{even} \in C_{|x|}^\infty(\mathbb{S}_{eq}^2 \setminus \{p_N, p_S\})$ by

$$\phi_{odd} = \sin x, \quad \phi_{even} = 1 - \sin x \log \frac{1 + \sin x}{\cos x} = 1 + \sin x \log \frac{1 - \sin x}{\cos x}.$$

Lemma 2.19. ϕ_{even} and ϕ_{odd} are even and odd in x respectively. They satisfy $\mathcal{L}'\phi_{even} = 0$ and $\mathcal{L}'\phi_{odd} = 0$. Moreover, ϕ_{even} is strictly decreasing on $[0, \pi/2)$ where it has a unique root we will denote by x_{root} .

Proof. ϕ_{even} corresponds to a translation and is a first harmonic of the Laplacian on \mathbb{S}_{eq}^2 . ϕ_{odd} is the pushforward of the scaling of the catenoid by the Gauss map and we can finish the proof using this. Alternatively it is straightforward to check by direct calculation. q.e.d.

We discuss now the Green's function for \mathcal{L}' on \mathbb{S}_{eq}^2 :

Lemma 2.20. *There is a function $G \in C^\infty((0, \pi))$ uniquely characterized by (i) and (ii) and, moreover, satisfying (iii–vii) below. We denote by r the standard coordinate of \mathbb{R}^+ :*

- (i). For small r we have $G(r) = (1 + O(r^2)) \log r$.
- (ii). For each $p \in \mathbb{S}_{eq}^2$ we have $\mathcal{L}'G_p = 0$ where $G_p := G \circ \mathbf{d}_p \in C^\infty(\mathbb{S}_{eq}^2 \setminus \{p, -p\})$ (recall 2.8).
- (iii). $G_{p_N} = (\log 2 - 1) \phi_{odd} + \phi_{even}$ (recall 2.3).
- (iv). $G(r) = 1 + \cos r (-1 + \log \frac{2 \sin r}{1 + \cos r})$.
- (v). $\frac{\partial G}{\partial r}(r) = -\sin r \log \frac{2 \sin r}{1 + \cos r} + \frac{1}{\sin r} + \frac{\sin r \cos r}{1 + \cos r}$.
- (vi). $\|G - \cos r \log r : C^k((0, 1), r, dr^2, r^2)\| \leq C(k)$.
- (vii). $\|G : C^k((0, 1), r, dr^2, |\log r|)\| \leq C(k)$.

Proof. Since $\mathbf{d}_{p_N} = \frac{\pi}{2} - x$ we have by direct calculation using 2.18 that

$$\begin{aligned} (\log 2 - 1)\phi_{odd} + \phi_{even} &= 1 - \cos \circ \mathbf{d}_{p_N} + \cos \circ \mathbf{d}_{p_N} \log \frac{2 \sin \circ \mathbf{d}_{p_N}}{1 + \cos \circ \mathbf{d}_{p_N}} = \\ &= (1 + O(\mathbf{d}_{p_N}^2)) \log \circ \mathbf{d}_{p_N}. \end{aligned}$$

This clearly implies (i–iv). (v) follows from (iv) by direct calculation. (vi) follows from (iv) and (v). (vii) follows from (vi). q.e.d.

For future reference we define a decomposition of functions on domains of \mathbb{S}_{eq}^2 as follows.

Definition 2.21. Given a function φ on some domain $\Omega \subset \mathbb{S}_{eq}^2$ we define a rotationally invariant function φ_{avg} on the union Ω' of the parallel circles on which φ is integrable (whether contained in Ω or not), by requesting that on each such circle C

$$\varphi_{avg}|_C := \text{avg}_C \varphi.$$

We also define φ_{osc} on $\Omega \cap \Omega'$ by $\varphi_{osc} := \varphi - \varphi_{avg}$.

Catenoidal bridges. Recall now that a catenoid of size τ in Euclidean three-space can be parametrized conformally on a cylinder $\mathbb{R} \times \mathbb{S}^1(1)$ by (2.22)

$$\begin{aligned} \widehat{X}_{cat}(t, \theta) &:= (\tau \cosh t \cos \theta, \tau \cosh t \sin \theta, \tau t) = (r(t) \cos \theta, r(t) \sin \theta, z(t)), \\ &\text{where } r(t) := \tau \cosh t, \quad z(t) := \tau t. \end{aligned}$$

Alternatively the part above the waist can be given as a radial graph of a function $\varphi_{cat} : [\tau, \infty) \rightarrow \mathbb{R}$ defined by

$$\begin{aligned}
 (2.23) \quad \varphi_{cat}(r) &:= \tau \operatorname{arccosh} \frac{r}{\tau} = \tau \left(\log r - \log \tau + \log \left(1 + \sqrt{1 - \tau^2 r^{-2}} \right) \right) = \\
 &= \tau \left(\log \frac{2r}{\tau} + \log \left(\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{\tau^2}{r^2}} \right) \right),
 \end{aligned}$$

where we denote by (x^1, x^2, x^3) the standard Cartesian coordinates of \mathbb{R}^3 and r is the polar coordinate on the $x^1 x^2$ -plane defined by $r := \sqrt{(x^1)^2 + (x^2)^2}$. By direct calculation or balancing considerations we have for future reference that

$$(2.24) \quad \frac{\partial \varphi_{cat}}{\partial r}(r) = \frac{\tau}{\sqrt{r^2 - \tau^2}}.$$

Lemma 2.25. $\| \varphi_{cat}(r) - \tau \log \frac{2r}{\tau} : C^k((9\tau, \infty), r, dr^2, r^{-2}) \| \leq C(k) \tau^3$.

Proof. This follows easily from 2.23 and 2.24. q.e.d.

Because of the rotational invariance it simplifies the presentation to use exactly minimal catenoidal bridges in the construction of the minimal surfaces, unlike in [20, 26, 25] where the catenoidal bridges used are only approximately minimal:

Lemma 2.26 (\underline{G}_τ and $\underline{G}_{p,\tau}$). *For $\tau > 0$ small enough there is a function $\underline{G}_\tau \in C^0([\tau, 9^{-2}]) \cap C^\infty((\tau, 9^{-2}))$ uniquely characterized by (i) and (ii) and, moreover, satisfying (iii):*

- (i). *The initial conditions $\underline{G}_\tau(\tau) = 0$ and $\frac{\partial \underline{G}_\tau}{\partial r}(r) \rightarrow \infty$ as $r \rightarrow \tau+$ hold.*
- (ii). *For each $p \in \mathbb{S}_{eq}^2$ the graph of $\underline{G}_{p,\tau} := \underline{G}_\tau \circ \mathbf{d}_p$ is minimal in $\mathbb{S}^3(1)$ (recall 2.8).*
- (iii). *If τ is small enough in terms of given $k \in \mathbb{N}$ and $\alpha \in (0, 1/2)$, then there is a constant b_{cat} such that*

$$\begin{aligned}
 \| \underline{G}_\tau - \varphi_{cat} - b_{cat} : C^k((9\tau, 9\tau^\alpha), r, dr^2, r^2) \| &\leq C(k, \alpha) \tau |\log \tau|, \\
 \text{where } b_{cat} \text{ depends only on } \tau \text{ and satisfies } |b_{cat}| &< C\tau^2.
 \end{aligned}$$

Proof. We can assume without loss of generality that $p = p_N$. The graph of $\underline{G}_{p,\tau}$ can be parametrized on the portion of a cylinder by

$$\begin{aligned}
 Y(r, \theta) &= (\sin r \cos \theta \cos \underline{G}_\tau(r), \sin r \sin \theta \cos \underline{G}_\tau(r), \\
 &\quad \cos r \cos \underline{G}_\tau(r), \sin \underline{G}_\tau(r)) \in \mathbb{S}^3(1) \subset \mathbb{R}^4,
 \end{aligned}$$

where clearly $\Pi_{\mathbb{S}_{eq}^2} \circ Y(r, \theta) = (\sin r \cos \theta, \sin r \sin \theta, \cos r) \in \mathbb{S}_{eq}^2$ (recall 2.6). Straightforward calculation implies then that

$$\begin{aligned} \frac{\partial Y}{\partial r}(r, \theta) &= (\cos r \cos \underline{G}_\tau(r) - \frac{\partial \underline{G}_\tau}{\partial r}(r) \sin r \sin \underline{G}_\tau(r))(\cos \theta, \sin \theta, 0, 0) \\ &\quad + (0, 0, -\sin r \cos \underline{G}_\tau(r) - \frac{\partial \underline{G}_\tau}{\partial r}(r) \cos r \sin \underline{G}_\tau(r), \frac{\partial \underline{G}_\tau}{\partial r}(r) \cos \underline{G}_\tau(r)), \end{aligned}$$

which implies further that

$$\left| \frac{\partial Y}{\partial r}(r, \theta) \right|^2 = \cos^2 \underline{G}_\tau(r) + \left(\frac{\partial \underline{G}_\tau}{\partial r}(r) \right)^2.$$

We will apply the standard balancing formula (see, for example, [17, 18]) with Killing field \vec{K} given by

$$\vec{K} \Big|_{(x_1, x_2, x_3, x_4)} := (0, 0, -x_4, x_3).$$

Using 2.5 we calculate that the length of the circle $Y(\{r\} \times \mathbb{S}^1)$ is $2\pi \sin r \cos \underline{G}_\tau(r)$ and then we have

$$\begin{aligned} \int_{Y(\{r\} \times \mathbb{S}^1)} \vec{\eta} \cdot \vec{K} &= 2\pi \sin r \cos \underline{G}_\tau(r) \left(\cos^2 \underline{G}_\tau(r) + \left(\frac{\partial \underline{G}_\tau}{\partial r}(r) \right)^2 \right)^{-1/2} \\ &\quad \cdot \left(\sin r \sin \underline{G}_\tau(r) \cos \underline{G}_\tau(r) + \frac{\partial \underline{G}_\tau}{\partial r}(r) \cos r \right). \end{aligned}$$

By the balancing formula this is independent of r and so equals the value at $r = \tau$. We conclude then

$$\begin{aligned} \sin r \cos \underline{G}_\tau(r) \left(\sin r \sin \underline{G}_\tau(r) \cos \underline{G}_\tau(r) + \frac{\partial \underline{G}_\tau}{\partial r}(r) \cos r \right) &= \\ = \left(\cos^2 \underline{G}_\tau(r) + \left(\frac{\partial \underline{G}_\tau}{\partial r}(r) \right)^2 \right)^{1/2} \sin \tau \cos \tau. \end{aligned}$$

By squaring both sides, calculating, and solving for $\frac{\partial}{\partial r} \underline{G}_\tau(r)$, we obtain (2.27)

$$\begin{aligned} A(r) \left(\frac{\partial \underline{G}_\tau}{\partial r}(r) \right)^2 + 2B(r) \frac{\partial \underline{G}_\tau}{\partial r}(r) &= C(r), \\ \frac{\partial \underline{G}_\tau}{\partial r} &= -\frac{B}{A} + \sqrt{\frac{C}{A} + \frac{B^2}{A^2}}, \quad \text{where} \\ A(r) &:= \sin^2 r \cos^2 r \cos^2 \underline{G}_\tau(r) - \sin^2 \tau \cos^2 \tau, \\ B(r) &:= \sin^3 r \cos r \sin \underline{G}_\tau(r) \cos^3 \underline{G}_\tau(r), \\ C(r) &:= \sin^2 \tau \cos^2 \tau \cos^2 \underline{G}_\tau(r) - \sin^4 r \sin^2 \underline{G}_\tau(r) \cos^4 \underline{G}_\tau(r). \end{aligned}$$

Let $b_{cat} = \underline{G}_\tau(9\tau) - \varphi_{cat}(9\tau)$. Using the smooth dependence of ODE solutions on the coefficients it is easy to confirm that $|b_{cat}| < C\tau^2$.

Using also 2.23 we conclude $(9\tau, 10\tau) \subset S'$, where $S' := \{r \in (9\tau, 9\tau^\alpha) : \underline{G}_\tau \leq 10\tau \log \frac{r}{\tau}\}$. Let S be the connected component of S' containing $(9\tau, 10\tau)$. We have then on S that $\underline{G}_\tau \leq C\tau |\log \tau|$, and, therefore,

$$\begin{aligned} A(r) &= (r^2 - \tau^2) (1 + O(r^2 + \tau^2 \log^2 \tau)), \\ B(r) &= O(r^3 \tau |\log \tau|), \\ C(r) &= \tau^2 (1 + O(r^2 + \tau^2 \log^2 \tau)). \end{aligned}$$

Using 2.27 and 2.24 we obtain that on S

$$(2.28) \quad \frac{\partial \underline{G}_\tau}{\partial r}(r) = \frac{\partial \varphi_{cat}}{\partial r}(r) + O(\tau r |\log \tau|).$$

By integrating we conclude that on S

$$\underline{G}_\tau(r) = \varphi_{cat}(r) + b_{cat} + O(\tau r^2 |\log \tau|).$$

We conclude then that on S we have $\underline{G}_\tau \leq 8\tau \log \frac{r}{\tau}$ and hence $S = (9\tau, 9\tau^\alpha)$. Finally, using 2.27 we can estimate the higher order derivatives and conclude (iii). q.e.d.

Corollary 2.29. *For τ small enough in terms of given $k \in \mathbb{N}$ and $\alpha \in (0, 1/2)$ the following holds.*

$$\begin{aligned} \|\underline{G}_\tau - \tau \log(2r/\tau) : C^k((9\tau, 9\tau^\alpha), r, dr^2, \tau^{2\alpha} |\log \tau| + \tau^2 r^{-2})\| \\ \leq C(k, \alpha) \tau. \end{aligned}$$

Proof. This follows by combining 2.25 and 2.26.iii and using that $\tau + r^2 |\log \tau| \leq 2\tau^{2\alpha} |\log \tau|$ on the interval under consideration. q.e.d.

Corollary 2.30. *For τ small enough in terms of given $k \in \mathbb{N}$ and $\alpha \in (0, 1/2)$ the following holds.*

$$\begin{aligned} \|\underline{G}_\tau - \tau G + \tau \log(\tau/2) \cos r : C^k((\tau^\alpha, 9\tau^\alpha), \tau^{-2\alpha} dr^2)\| \\ \leq C(k, \alpha) \tau^{1+2\alpha} |\log \tau|. \end{aligned}$$

Proof. We have $G - \cos r \log \frac{\tau}{2} - \log \frac{2r}{\tau} = (1 - \cos r) (1 + \log \frac{\tau}{2} - \log r) + \cos r \log \frac{2 \sin r}{r(1 + \cos r)}$ by an easy calculation based on 2.20.iv. This implies that

$$\begin{aligned} \|G - \cos r \log(\tau/2) - \log(2r/\tau) : C^k((9\tau, 9\tau^\alpha), r, dr^2, r^2)\| \\ \leq C(k, \alpha) |\log \tau|. \end{aligned}$$

Combining this with 2.29 we conclude the proof. q.e.d.

CONVENTION 2.31. We fix now some small $\alpha > 0$ which we will assume as small in absolute terms as needed. □

Definition 2.32. For $\tau \in (0, 1)$ and $p \in \mathbb{S}^3(1)$ we define $\text{Exp}_{p,\tau} := \mathbb{R}_{p,\tau} \circ \exp_{g,p}$ where $\mathbb{R}_{p,\tau} : T_p \mathbb{S}^3(1) \rightarrow T_p \mathbb{S}^3(1)$ is defined by $\mathbb{R}_{p,\tau}(v) = \tau v$ and $\exp_{g,p}$ denotes the exponential map of $(\mathbb{S}^3(1), g)$ at p . Let

$B_{p,\tau} \subset T_p\mathbb{S}^3(1)$ be the ball such that the restriction of $\text{Exp}_{p,\tau}$ to $B_{p,\tau}$ is a diffeomorphism onto $\mathbb{S}^3(1) \setminus \{-p\}$. We define a metric on $B_{p,\tau}$ by $\tilde{g}_{p,\tau} := \text{Exp}_{p,\tau}^*(\tau^{-2}g)$.

Lemma 2.33. *The estimate*

$$\|\tilde{g}_{p,\tau} - h_p : C^k(B_0(9^2\tau^{\alpha-1}) \setminus \{0\}, \tilde{R}, h_p, \tilde{R}^4)\| \leq C(k)\tau^2$$

holds, where $B_0(9^2\tau^{\alpha-1}) \subset T_p\mathbb{S}^3(1)$ is the ball centered at the origin and of radius $9^2\tau^{\alpha-1}$ with respect to h_p , \tilde{R} denotes the distance from the origin in the h_p metric, and h_p is the Euclidean metric on $T_p\mathbb{S}^3(1)$ defined by $h_p := g|_p = \tilde{g}_{p,\tau}|_p$.

Proof. Clearly $h_p = d\tilde{R}^2 + \tilde{R}^2 g_{\mathbb{S}^2(1)}$ and $\tilde{g}_{p,\tau} = d\tilde{R}^2 + \tau^{-2} \sin^2(\tau\tilde{R}) g_{\mathbb{S}^2(1)}$. By calculating and using the definitions we obtain

$$\|\tilde{R}^2 g_{\mathbb{S}^2(1)} : C^k(B_{p,\tau} \setminus \{0\}, \tilde{R}, h_p, \tilde{R}^2)\| \leq C(k),$$

and

$$\begin{aligned} \|\tau^{-2} \tilde{R}^{-2} \sin^2(\tau\tilde{R}) - 1 : C^k(B_0(9^2\tau^{\alpha-1}) \setminus \{0\}, \tilde{R}, h_p, \tilde{R}^2)\| \\ \leq C(k)\tau^2. \end{aligned}$$

Using 1.4 we complete the proof. q.e.d.

Definition 2.34. We define the catenoidal bridge $\mathbb{K}_{p,\tau}$ centered at $p \in \mathbb{S}_{eq}^2$ and of waist size τ to be the union of the graphs of $\pm \underline{G}_{p,\tau}$ restricted to $D_p(9\tau^\alpha) \setminus D_p(\tau)$ where α is as in 2.31. For $\tau > 0$ small enough we define (recall 2.32) $\tilde{\mathbb{K}}_{p,\tau} := \tilde{\mathbb{K}}_{p,-\tau} := \text{Exp}_{p,\tau}^{-1}(\mathbb{K}_{p,\tau})$. Finally, we define $\tilde{\mathbb{K}}_{p,0}$ to be the standard catenoid in the Euclidean space $(T_p\mathbb{S}^3(1), h_p)$, appropriately placed so that $\tilde{\mathbb{K}}_{p,\tau}$ depends smoothly on τ for $|\tau|$ small enough.

Note that the last statement above applies since $\tilde{\mathbb{K}}_{p,\tau}$ is controlled by an ODE with initial conditions at the waist independent of τ and coefficients smoothly depending on τ .

Definition 2.35. For $p \in \mathbb{S}^2$ we define $\tilde{\Pi}_{\mathbb{K},p}$ to be the nearest point projection from an appropriate neighborhood of $\tilde{\mathbb{K}}_{p,0}$ in $(T_p\mathbb{S}^3(1), h_p)$ to $\tilde{\mathbb{K}}_{p,0}$. We also define $\tilde{r} : T_p\mathbb{S}^3(1) \rightarrow \mathbb{R}$ to be the distance from the axis of $\tilde{\mathbb{K}}_{p,0}$ in $T_p\mathbb{S}^3(1)$ with respect to h_p .

Lemma 2.36. *For τ small enough the restriction of $\tilde{\Pi}_{\mathbb{K},p}$ to $\tilde{\mathbb{K}}_{p,\tau}$ is well defined and is, moreover, a smooth diffeomorphism onto a domain $\tilde{\Omega}_\tau \subset \tilde{\mathbb{K}}_{p,0}$. Moreover, $\tilde{\mathbb{K}}_{p,\tau}$ is the graph in the Euclidean space $(T_p\mathbb{S}^3(1), h_p)$ over $\tilde{\Omega}_\tau \subset \tilde{\mathbb{K}}_{p,0}$ of a function $\tilde{\varphi}_\tau$ which satisfies*

$$\|\tilde{\varphi}_\tau : C^k(\tilde{\Omega}_\tau, \tilde{r}, \tilde{g}_0, \tau + \tau^2 \tilde{r}^2 |\log \tau|)\| \leq C(k, \alpha),$$

where \tilde{g}_0 is the metric on $\tilde{\mathbb{K}}_{p,0}$ induced by the Euclidean metric h_p on $T_p\mathbb{S}^3(1)$ and \tilde{r} is as in 2.35.

Proof. We assume without loss of generality that $p = p_N$. We identify then $T_p\mathbb{S}^3(1)$ with \mathbb{R}^3 so that for $\vec{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ we have

$$\text{Exp}_{p,\tau}(u_1, u_2, u_3) = \cos \tau |\vec{u}| (0, 0, 1, 0) + \frac{\sin \tau |\vec{u}|}{|\vec{u}|} (u_1, u_2, 0, u_3).$$

where $|\vec{u}| = (u_1^2 + u_2^2 + u_3^2)^{1/2}$. The upper half of $\mathbb{K}_{p,\tau}$ can be parametrized by $X_{p,\tau} : [\tau, 9\tau^\alpha) \times \mathbb{S}^1 \rightarrow \mathbb{S}^3(1)$ defined by (recall 2.2)

$$X_{p,\tau}(r, \theta) = \cos \underline{G}_\tau(r) (\sin r \cos \theta, \sin r \sin \theta, \cos r, 0) + \sin \underline{G}_\tau(r) (0, 0, 0, 1).$$

The upper half of $\tilde{\mathbb{K}}_{p,0}$ can be parametrized by $\tilde{X}_{p,0} : [\tau, \infty) \times \mathbb{S}^1 \rightarrow T_p\mathbb{S}^3(1)$ defined by (recall 2.23)

$$\tilde{X}_{p,0}(r, \theta) = \tau^{-1} (r \cos \theta, r \sin \theta, \varphi_{cat}(r)),$$

and, therefore,

$$(2.37) \quad \text{Exp}_{p,\tau} \circ \tilde{X}_{p,0}(r, \theta) = \cos \sqrt{r^2 + \varphi_{cat}^2(r)} (0, 0, 1, 0) + \frac{\sin \sqrt{r^2 + \varphi_{cat}^2(r)}}{\sqrt{r^2 + \varphi_{cat}^2(r)}} (r \cos \theta, r \sin \theta, 0, \varphi_{cat}(r)).$$

Using 2.26.iii and 2.25 we conclude

$$\left\| X_{p,\tau} - \text{Exp}_{p,\tau} \circ \tilde{X}_{p,0} : C^k \left((9\tau, 9\tau^\alpha) \times \mathbb{S}^1, r, (\text{Exp}_{p,\tau} \circ \tilde{X}_{p,0})^* g, \tau + r^2 |\log \tau| \right) \right\| \leq C(k, \alpha) \tau.$$

This implies that the points of $\tilde{\mathbb{K}}_{p,\tau} = \text{Exp}_{p,\tau}^{-1}(\mathbb{K}_{p,\tau})$ are within distance $C \tau^{2\alpha} |\log \tau|$ from $\tilde{\mathbb{K}}_{p,0}$, where for the region $\{\tilde{r} < 10\}$ we use the smooth dependence on τ . The restriction hence of $\tilde{\Pi}_{\mathbb{K},p}$ to $\tilde{\mathbb{K}}_{p,\tau}$ is well defined. Magnifying and using the implicit function theorem we conclude the proof. q.e.d.

Since by 2.36 $\tilde{\mathbb{K}}_{p,\tau}$ is a small perturbation of a domain $\tilde{\Omega}_\tau \subset \tilde{\mathbb{K}}_{p,0}$, its first and second fundamental forms induced by h_p are a small perturbation of those of $\tilde{\Omega}_\tau$. However, we are really interested in the first and second fundamental forms \tilde{g} and \tilde{A} of $\tilde{\mathbb{K}}_{p,\tau}$ induced by $\tilde{g}_{p,\tau}$ (defined in 2.32), or equivalently

$$(2.38) \quad \tilde{g} = \tau^{-2} \text{Exp}_{p,\tau}^* g, \quad \tilde{A} = \tau^{-1} \text{Exp}_{p,\tau}^* A,$$

where g and A denote the first and second fundamental forms of $\mathbb{K}_{p,\tau} \subset \mathbb{S}^3(1)$ induced by the standard metric of $\mathbb{S}^3(1)$ and $\text{Exp}_{p,\tau}^*$ denotes pull-

back by the restriction of $\text{Exp}_{p,\tau}$ to $\tilde{\mathbb{K}}_{p,\tau}$. The next corollary provides the estimates we will need.

Corollary 2.39. *For $\tilde{\Omega}_\tau$, \tilde{g}_0 , and \tilde{r} as in 2.36 we have that*

$$\begin{aligned} & \| (\tilde{\Pi}_{\mathbb{K},p})_* \tilde{g} - \tilde{g}_0 : C^k(\tilde{\Omega}_\tau, \tilde{r}, \tilde{g}_0, \tau\tilde{r} + \tau^2\tilde{r}^4) \| \leq C(k, \alpha), \\ & \| (\tilde{\Pi}_{\mathbb{K},p})_* \tilde{A} - \tilde{A}_0 : C^k(\tilde{\Omega}_\tau, \tilde{r}, \tilde{g}_0, \tau + \tau^2\tilde{r}^3) \| \leq C(k, \alpha), \end{aligned}$$

where $(\tilde{\Pi}_{\mathbb{K},p})_*$ denotes the pushforward by $\tilde{\Pi}_{\mathbb{K},p}$ restricted to $\tilde{\mathbb{K}}_{p,\tau}$, that is the pullback by its inverse, and \tilde{g} and \tilde{A} are as above.

Proof. Let $q \in \tilde{\Omega}_\tau \subset \tilde{\mathbb{K}}_{p,0} \subset T_p\mathbb{S}^3(1)$ and consider the Euclidean metric $\hat{h}_q := \tilde{r}^{-2}(q)h_p$ on $T_p\mathbb{S}^3(1)$. Consider Cartesian orthonormal coordinates on $(T_p\mathbb{S}^3(1), \hat{h}_q)$ so that \hat{X}_{cat} defined as in 2.22 with $\tau = 1$ provides a conformal parametrization of $\tilde{\mathbb{K}}_{p,0}$. We consider the geodesic disc $B'_q \subset \tilde{\mathbb{K}}_{p,0}$ with center q , radius $1/10$, and defined with respect to the metric induced \hat{h}_q . It is then easy to check that there is a constant $C(k)$ which depends only on k such that on $\hat{X}_{cat}^{-1}(B'_q)$ we have that the C^k norms of the coordinates of \hat{X}_{cat} are bounded by $C(k)$ and also $g_{cyl} \leq C(k)\hat{X}_{cat}^*\hat{h}_q$, where g_{cyl} is the standard metric on the cylinder $\mathbb{R} \times \mathbb{S}^1(1)$ and $\hat{X}_{cat}^*\hat{h}_q$ is the pullback by \hat{X}_{cat} of the metric induced by \hat{h}_q .

Note that by 2.36 $\tilde{\mathbb{K}}_{p,\tau}$ is the graph in the Euclidean space $(T_p\mathbb{S}^3(1), \hat{h}_q)$ over $\tilde{\Omega}_\tau$ of the function $\frac{1}{\tilde{r}(q)}\tilde{\varphi}_\tau$. Since we have uniform bounds for the coordinate functions of \hat{X}_{cat} we can combine the estimates in 2.33 and 2.36 to conclude that on B'_q the norms of the differences of the fundamental forms induced by $\tilde{r}^{-2}(q)\tilde{g}_{p,\tau}$ on the graph versus the ones induced by \hat{h}_q on $\tilde{\mathbb{K}}_{p,0}$, are bounded by a constant depending only on k and α times

$$(2.40) \quad \frac{\tau + \tau^2\tilde{r}^2(q)|\log \tau|}{\tilde{r}(q)} + \tau^2\tilde{R}^2(q) \leq C\left(\frac{\tau}{\tilde{r}(q)} + \tau^2\tilde{r}^2(q)\right) \leq C\tau^{2\alpha},$$

where for the last inequality we used that $\tilde{r} \leq 9\tau^{\alpha-1}$ by definition, and we also used that linear terms dominate because of the smallness of the last term in 2.40. By scaling and applying 1.2 we conclude the proof. q.e.d.

3. Linearized doubling and initial surfaces

We expect that the approach developed in this paper, which consists of finding appropriate linearized doubling (LD) solutions first, and using them to “build” the desired minimal surfaces afterward, can be modified to apply to general situations with little or no symmetry (see 3.21). Under this approach the difficulty is shifted to finding and understanding the appropriate LD solutions.

LD and MLD solutions. We proceed now to describe the LD solutions for doubling constructions of \mathbb{S}_{eq}^2 . Note that our definitions although stated for \mathbb{S}_{eq}^2 can easily be modified to apply to any minimal surface. Note also that we can think of an LD solution φ as a multi-Green's function, since clearly in the distributional sense $\mathcal{L}'\varphi$ is a linear combination of delta functions:

Definition 3.1 (LD solutions). Given a finite set $L \subset \mathbb{S}_{eq}^2$ and a function $\tau : L \rightarrow \mathbb{R}$, we define a *linearized doubling (LD) solution of configuration* (L, τ) to be a function $\varphi \in C^\infty(\mathbb{S}_{eq}^2 \setminus L)$ which satisfies the following conditions where τ_p denotes the value of τ at p :

- (i). $\mathcal{L}'\varphi = 0$ on $\mathbb{S}_{eq}^2 \setminus L$.
- (ii). $\forall p \in L$ there is $\widehat{\varphi}_p \in C^\infty(\{p\} \cup (\mathbb{S}_{eq}^2 \setminus (L \cup \{-p\})))$, a smooth extension across p , such that $\widehat{\varphi}_p = \varphi - \tau_p G_p$ on $\mathbb{S}_{eq}^2 \setminus (L \cup \{-p\})$.

The main idea of our current approach is to construct initial surfaces by gluing catenoidal bridges centered at the points of L to (appropriately modified) graphs of the LD solutions. This step requires a satisfactory matching of each LD solution to the catenoidal bridge at the annulus where the gluing occurs. The matching can be controlled by the first terms of the Taylor expansion of each $\widehat{\varphi}_p$ at p . It turns out, however, that well matched LD solutions are not in sufficient supply for our purposes. For this reason we have to employ also LD solutions which are not well matched. Such solutions need to be modified so that they satisfy the matching conditions at the expense of not satisfying the exact linearized equation anymore. The solutions in this new class will only satisfy the linearized equation modulo a space $\mathcal{K}[L]$ which will be defined later in 3.7. $\mathcal{K}[L]$ depends smoothly on L and plays also the role of the (extended) substitute kernel in the linear theory (see 4.17).

Definition 3.2 (LD solutions modulo $\mathcal{K}[L]$). Given L and τ as in 3.1, and also a finite dimensional space $\mathcal{K}[L] \subset C^\infty(\mathbb{S}_{eq}^2)$, we define a *linearized doubling (LD) solution modulo $\mathcal{K}[L]$ of configuration* (L, τ, w) to be a function $\varphi \in C^\infty(\mathbb{S}_{eq}^2 \setminus L)$ which satisfies the same conditions as in 3.1, except that condition (i) is replaced by the following:

- (i'). $\mathcal{L}'\varphi = w \in \mathcal{K}[L] \subset C^\infty(\mathbb{S}_{eq}^2)$ on $\mathbb{S}_{eq}^2 \setminus L$.

Note that LD solutions in the sense of 3.1 are LD solutions in the sense of 3.2 with $w = 0$. We describe now the matching conditions.

Definition 3.3 (Mismatch of LD solutions). Given φ as in 3.1 or 3.2 we define $\mathcal{V}[L] := \bigoplus_{p \in L} \mathcal{V}[p]$, where $\mathcal{V}[p] := \mathbb{R} \oplus T_p^* \mathbb{S}_{eq}^2$, and the mismatch of φ by

$$\mathcal{B}_L \varphi := \bigoplus_{p \in L} (\widehat{\varphi}_p(p) + \tau_p \log(\tau_p/2), d_p \widehat{\varphi}_p) \in \mathcal{V}[L].$$

Among all the LD solutions modulo $\mathcal{K}[L]$ we will be mainly interested in the ones which are well matched:

Definition 3.4 (MLD solutions). We define a *matched linearized doubling (MLD) solution modulo $\mathcal{K}[L]$* of configuration (L, τ, w) to be some φ as in 3.2 which, moreover, satisfies the conditions $\mathcal{B}_L\varphi = 0$ and $\tau_p > 0 \forall p \in L$.

REMARK 3.5. Note that given φ and L as in 3.2, τ, w , each $\widehat{\varphi}_p$, and the second components of $\mathcal{B}_L\varphi$, are uniquely determined and depend linearly on φ . The first components of $\mathcal{B}_L\varphi$ are not linear in φ , however, and this makes the construction harder. \square

The definition of $\mathcal{K}[L]$. In order to describe the support of the functions in $\mathcal{K}[L]$ we have first the following.

CONVENTION 3.6 (The constants δ_p). Given L as in 3.1 we assume that for each $p \in L$ we have chosen a constant $\delta_p > 0$, where each δ_p is small enough so that any two $D_p(9\delta_p)$'s are disjoint for two different points $p \in L$. \square

Definition 3.7 (The obstruction space $\mathcal{K}[L]$). Given L and δ_p 's as in 3.6 we define $\mathcal{K}[L] \subset C^\infty(\mathbb{S}_{eq}^2)$ by $\mathcal{K}[L] := \bigoplus_{p \in L} \mathcal{K}[p]$, where $\mathcal{K}[p]$ is spanned by the following.

- (i). $\mathcal{L}'\Psi [2\delta_p, \delta_p; \mathbf{d}_p] (G_p, \log \delta_p \cos \circ \mathbf{d}_p) = -\mathcal{L}'\Psi [2\delta_p, \delta_p; \mathbf{d}_p] (\log \delta_p \cos \circ \mathbf{d}_p, G_p)$.
- (ii). $\mathcal{L}'\Psi [2\delta_p, \delta_p; \mathbf{d}_p] (0, u_p)$, where u_p is any first harmonic of \mathbb{S}_{eq}^2 vanishing at p .

Note that the functions in $\mathcal{K}[L]$ are supported on $\bigsqcup_{p \in L} (D_p(2\delta_p) \setminus D_p(\delta_p))$. Clearly $\forall p \in L$ we have $\dim \mathcal{K}[p] = 3$ and hence $\dim \mathcal{K}[L] = 3|L|$ where $|L|$ is the number of points in L .

Symmetric LD solutions. Because of the symmetries imposed on our constructions we concentrate now on LD solutions which are invariant under the action of $\mathcal{G}_{\mathbb{S}_{eq}^2, m}$ (recall 2.14). In such a case we can write

$$(3.8) \quad L = L_{mer} \cap L_{par},$$

where L_{par} is the union of a finite number of parallel circles and perhaps $\{p_N, p_S\}$, symmetrically arranged around the equator so that $\mathcal{G}_{\mathbb{S}_{eq}^2, m}L_{par} = L_{par}$. We assume that δ_p 's have been chosen as in 3.6 and so that they are $\mathcal{G}_{\mathbb{S}_{eq}^2, m}$ -invariant. We also define (recall 3.2 and 2.17)

$$(3.9) \quad \begin{aligned} \mathcal{K}_{sym}[L] &:= \mathcal{K}[L] \cap C_{sym}^\infty(\mathbb{S}_{eq}^2), \\ \widehat{\mathcal{K}}_{sym}[L] &:= \{v \in C_{sym}^\infty(\mathbb{S}_{eq}^2) : \mathcal{L}'v \in \mathcal{K}_{sym}[L]\}. \end{aligned}$$

Note that because of the symmetries \mathcal{L}' has no kernel and, therefore, \mathcal{L}' restricted to $\widehat{\mathcal{K}}_{sym}[L]$ provides an isomorphism onto $\mathcal{K}_{sym}[L]$. The dimension of $\mathcal{K}_{sym}[L]$ and $\widehat{\mathcal{K}}_{sym}[L]$ is clearly $k_{eq} + k_{poles} + 2k_{other}$ where $k_{eq} = 1$ if the equatorial circle is included in L_{par} and 0 otherwise,

$k_{poles} = 1$ if the poles are included and 0 otherwise, and $m_{mer} = k_{eq} + k_{poles} + 2k_{other}$. Note now that the symmetries imposed ensure that the configuration of an LD solution uniquely determines the LD solution as in the next lemma:

Lemma 3.10 (Symmetric LD solutions). *Given a finite $\mathcal{G}_{\mathbb{S}_{eq}^2, m}$ -invariant set $L \subset \mathbb{S}_{eq}^2$, a $\mathcal{G}_{\mathbb{S}_{eq}^2, m}$ -invariant function $\tau : L \rightarrow \mathbb{R}$, and $w \in \mathcal{K}_{sym}[L]$, there is a unique $\mathcal{G}_{\mathbb{S}_{eq}^2, m}$ -invariant LD solution modulo $\mathcal{K}[L]$ $\varphi = \varphi[L, \tau, w]$ of configuration (L, τ, w) (recall 3.2). Moreover, the following hold.*

- (i). φ and each $\widehat{\varphi}_p$ depend linearly on (τ, w) .
- (ii). $\varphi_{avg} \in C^0(\mathbb{S}_{eq}^2 \setminus (L \cap \{p_N, p_S\}))$ (recall 2.21) and φ_{avg} is smooth on $\mathbb{S}_{eq}^2 \setminus L_{par}$ where it satisfies the ODE $\mathcal{L}'\varphi_{avg} = w_{avg}$.

If $w = 0$ then we also write $\varphi = \varphi[L, \tau]$ and φ is the unique $\mathcal{G}_{\mathbb{S}_{eq}^2, m}$ -invariant LD solution of configuration (L, τ) as in 3.1.

Proof. We define $\varphi_1 \in C_{sym}^\infty(\mathbb{S}_{eq}^2 \setminus L)$ by requesting that it is supported on $\bigsqcup_{p \in L} (D_p(2\delta_p))$ and $\varphi_1 = \Psi[\delta_p, 2\delta_p; \mathbf{d}_p](G_p, 0)$ on $D_p(2\delta_p)$ for each $p \in L$. Note that $\mathcal{L}'\varphi_1 \in C_{sym}^\infty(\mathbb{S}_{eq}^2)$ (by assigning 0 values on L) and it is supported on $\bigsqcup_{p \in L} (D_p(2\delta_p) \setminus D_p(\delta_p))$. Because the symmetries do not allow the first harmonics of the Laplacian on \mathbb{S}_{eq}^2 , there is $\varphi_2 \in C_{sym}^\infty(\mathbb{S}_{eq}^2)$ such that $\mathcal{L}'\varphi_2 = -\mathcal{L}'\varphi_1 + w$. We can define then $\varphi := \varphi_1 + \varphi_2$. Uniqueness and (i) follow then immediately. To prove (ii) we need to check that φ is integrable on each circle contained in L_{par} and that φ_{avg} is continuous there also. But these follow easily by the logarithmic behavior of G_p (recall 2.20). Since the case $w = 0$ is clearly a special case of the general case the proof is complete. q.e.d.

Next we will need the following.

Definition 3.11 (The map \mathcal{E}_L). We define $\mathcal{V}_{sym}[L]$ to be the subspace of $\mathcal{V}[L]$ (recall 3.3) consisting of those elements which are invariant under the obvious action of $\mathcal{G}_{\mathbb{S}_{eq}^2, m}$. We define then a linear map $\mathcal{E}_L : \widehat{\mathcal{K}}_{sym}[L] \rightarrow \mathcal{V}_{sym}[L]$ by $\mathcal{E}_L(v) := (v(p), d_p v)_{p \in L} \in \mathcal{V}_{sym}[L]$ for $v \in \widehat{\mathcal{K}}_{sym}[L]$ (recall 3.9).

The following assumption is crucial for the construction and will be checked later. Note that besides being used in the linear theory later it also allows us to convert any LD solution φ in the sense of 3.1 to an MLD in the sense of 3.4 by subtracting from it $\mathcal{E}_L^{-1} \mathcal{B}_L \varphi$:

ASSUMPTION 3.12. We assume that the map $\mathcal{E}_L : \widehat{\mathcal{K}}_{sym}[L] \rightarrow \mathcal{V}_{sym}[L]$ is a linear isomorphism.

Definition 3.13. We denote by $\|\mathcal{E}_L^{-1}\|$ the operator norm of $\mathcal{E}_L^{-1} : \mathcal{V}_{sym}[L] \rightarrow \widehat{\mathcal{K}}_{sym}[L]$ with respect to the $C^{2, \beta}(\mathbb{S}_{eq}^2, g)$ norm on the target

and the maximum norm on the domain subject to the standard metric g of \mathbb{S}_{eq}^2 .

Initial surfaces from $\mathcal{G}_{\mathbb{S}_{eq}^2, m}$ -symmetric MLD solutions. In this subsection we construct the initial surfaces by gluing catenoidal bridges to appropriately modified graphs of MLD solutions. More precisely we start by assuming given a $\mathcal{G}_{\mathbb{S}_{eq}^2, m}$ -symmetric MLD solution, $\varphi = \varphi[L, \tau, w]$ in the notation of 3.10. The first step in the construction is to modify the MLD solution so that its graph on an appropriate domain (corresponding to the complement of the catenoidal bridges) is minimal except on the support of the elements of $\mathcal{K}_{sym}[L]$. We then attach the catenoidal bridges and this way we obtain an initial surface where the unwelcome mean curvature is supported on small annuli where the gluing occurs. We choose now the scale of the gluing annuli:

Definition 3.14. For each $p \in L$ we define $\delta'_p := \tau_p^\alpha$ where α is as in 2.31. We will also use the notation $\delta_{min} := \min_{p \in L} \delta_p$, $\tau_{min} := \min_{p \in L} \tau_p$, $\tau_{max} := \max_{p \in L} \tau_p$, and $\delta'_{min} := \min_{p \in L} \delta'_p = \tau_{min}^\alpha$.

To simplify the presentation and the construction it is convenient to assume the following which we will confirm later for the actual constructions we carry out (see 5.32.vii and 6.24.v).

CONVENTION 3.15. We assume from now on that the following hold.

- (i). 3.6 holds and τ_{max} is small enough in absolute terms as needed.
- (ii). $\forall p \in L$ we have $9\delta'_p < \tau_p^{\alpha/9} < \delta_p$.
- (iii). $\tau_{max} \leq \tau_{min}^{1-\alpha/9}$.
- (iv). $\forall p \in L$ we have $(\delta_p)^{-2} \|\widehat{\varphi}_p : C^{2,\beta}(\partial D_p(\delta_p), (\delta_p)^{-2}g)\| \leq \tau_p^{1-\alpha/9}$.
- (v). $\|\varphi : C^{3,\beta}_{sym}(\mathbb{S}_{eq}^2 \setminus \bigsqcup_{q \in L} D_q(\delta'_q), g)\| \leq \tau_{min}^{8/9}$.
- (vi). On $\mathbb{S}_{eq}^2 \setminus \bigsqcup_{q \in L} D_q(\delta'_q)$ we have $\tau_{max}^{1+\alpha/5} \leq \varphi$. □

REMARK 3.16. Note that condition 3.15.vi is only needed to ensure embeddedness. For constructions of immersed surfaces which may not be embedded we could drop 3.15.vi. For such constructions we could also allow negative τ_p 's by replacing $\tau_p > 0$ in 3.4 with “ $\tau_p \neq 0$ ” and the first term on the right in 3.3 with “ $\widehat{\varphi}_p(p) + \tau_p \log |\tau_p/2| = 0$ ”. Note that in such a case if 3.15.vi holds the positivity of τ_p is implied anyway. □

In order now to modify φ which by definition satisfies the linearized condition 3.2.i', to another function φ_{nl} which satisfies the nonlinear condition 3.18.i, we first define a cutoff function $\psi'' \in C^\infty_{sym}(\mathbb{S}_{eq}^2)$ by $\psi'' = 1$ on $\mathbb{S}_{eq}^2 \setminus \bigsqcup_{p \in L} D_p(2\delta'_p)$, and on $D_p(2\delta'_p)$ (for each $p \in L$)

$$\psi'' = \Psi[\delta'_p, 2\delta'_p; \mathbf{d}_p](0, 1).$$

We then define inductively sequences $\{u_n\}_{n=1}^\infty \subset C_{sym}^{3,\beta}(\mathbb{S}_{eq}^2)$ and $\{\phi_n\}_{n=-1}^\infty \subset C_{sym}^{3,\beta}(\mathbb{S}_{eq}^2)$ by $\phi_{-1} = 0$, $\phi_0 := \varphi$, and for $n > 0$

$$(3.17) \quad \phi_n = \phi_{n-1} + u_n, \quad \mathcal{L}' u_n = \psi''(Q_{\phi_{n-2}} - Q_{\phi_{n-1}}),$$

where we define Q_{ϕ_k} to vanish on $\bigsqcup_{p \in L} D_p(\delta'_p)$ and to satisfy $H_{\phi_k} = \mathcal{L}'\phi_k + Q_{\phi_k}$ on $\mathbb{S}_{eq}^2 \setminus \bigsqcup_{p \in L} D_p(\delta'_p)$, where H_{ϕ_k} is the mean curvature of the graph of ϕ_k in \mathbb{S}^3 pushed forward to \mathbb{S}_{eq}^2 by the projection $\Pi_{\mathbb{S}_{eq}^2}$ (recall 2.6).

Lemma 3.18. *Given a $\mathcal{G}_{\mathbb{S}_{eq}^2, m}$ -symmetric MLD solution $\varphi = \varphi[L, \tau, w]$ which is as in 3.10 and 3.4 and where 3.15 is satisfied, we can define $\varphi_{nl} = \varphi_{nl}[L, \tau, w] \in C_{sym}^\infty(\mathbb{S}_{eq}^2 \setminus L)$ as the limit of the sequence ϕ_n defined above. Moreover, the following hold.*

- (i). $H_{\varphi_{nl}} = \mathcal{L}'\varphi = w$ on $\mathbb{S}_{eq}^2 \setminus \bigsqcup_{p \in L} D_p(2\delta'_p)$, where $H_{\varphi_{nl}}$ is the mean curvature of the graph of φ_{nl} in \mathbb{S}^3 pushed forward to $\mathbb{S}_{eq}^2 \setminus L$ by the projection $\Pi_{\mathbb{S}_{eq}^2}$ (recall 2.6).
- (ii). $\varphi_{nl} - \varphi$ can be extended to a smooth function on \mathbb{S}_{eq}^2 which satisfies

$$\begin{aligned} \|\varphi_{nl} - \varphi : C_{sym}^{3,\beta}(\mathbb{S}_{eq}^2, g)\| &\leq \\ &\leq C(\delta'_{min})^{-2} \|\varphi : C_{sym}^{3,\beta}(\mathbb{S}_{eq}^2 \setminus \bigsqcup_{p \in L} D_p(\delta'_p), g)\|^2 \leq \tau_{min}^{3/2}. \end{aligned}$$

Proof. By standard linear theory, 3.17, and the triviality of the kernel of \mathcal{L}' on \mathbb{S}_{eq}^2 modulo the symmetries, we conclude that for $n \geq 1$ we have

$$\|u_n : C^{3,\beta}(\mathbb{S}_{eq}^2, g)\| \leq C\|\psi'' : C^{1,\beta}(\mathbb{S}_{eq}^2, g)\| \|Q_{\phi_{n-1}} - Q_{\phi_{n-2}} : C^{1,\beta}(\Omega, g)\|,$$

where $\Omega := \mathbb{S}_{eq}^2 \setminus \bigsqcup_{p \in L} D_p(\delta'_p) \supset \text{supp } \psi''$. Since the quadratic (and higher) terms Q_{ϕ_k} can be expressed as an algebraic expression involving geometric invariants of \mathbb{S}_{eq}^2 , ϕ_k , and the derivatives of ϕ_k , we have

$$\begin{aligned} \|Q_{\phi_{n-1}} - Q_{\phi_{n-2}} : C^{1,\beta}(\Omega, g)\| &\leq \\ &\leq \begin{cases} C \|\varphi : C^{3,\beta}(\Omega, g)\|^2 & (n = 1), \\ C \|\phi_{n-1} - \phi_{n-2} : C^{3,\beta}(\Omega, g)\| \|\phi_{n-2} : C^{3,\beta}(\Omega, g)\| & (n \geq 2). \end{cases} \end{aligned}$$

Combining the last two estimates and substituting u_{n-1} for $\phi_{n-1} - \phi_{n-2}$ we conclude that

$$\begin{aligned} \|u_n : C^{3,\beta}(\mathbb{S}_{eq}^2, g)\| &\leq \\ &\leq \begin{cases} C(\delta'_{min})^{-2} \|\varphi : C^{3,\beta}(\Omega, g)\|^2 & (n = 1), \\ C(\delta'_{min})^{-2} \|u_{n-1} : C^{3,\beta}(\Omega, g)\| \|\phi_{n-2} : C^{3,\beta}(\Omega, g)\| & (n \geq 2). \end{cases} \end{aligned}$$

Since $2\frac{8}{9} - 2\alpha > \frac{3}{2}$ by 2.31, we conclude inductively using 3.15.v that for $n \geq 1$

$$\|u_n : C^{3,\beta}(\mathbb{S}_{eq}^2, g)\| \leq 2^{-n} C(\delta'_{min})^{-2} \|\varphi : C^{3,\beta}(\Omega, g)\|^2 \leq 2^{-n} \tau_{min}^{3/2}.$$

Taking limits and sums and using standard regularity theory for the smoothness we conclude the proof. q.e.d.

Definition 3.19. Given $\varphi = \varphi[L, \tau, w]$ as above we define a function

$$\varphi_{init} = \varphi_{init}[L, \tau, w] : \mathbb{S}_{eq}^2 \setminus \bigsqcup_{p \in L} D_p(\tau_p) \rightarrow [0, \infty),$$

as follows:

- (i). On $\mathbb{S}_{eq}^2 \setminus \bigsqcup_{p \in L} D_p(3\delta'_p)$ we have $\varphi_{init} := \varphi_{nl}[L, \tau, w]$.
- (ii). For each $p \in L$ we have on $D_p(3\delta'_p) \setminus D_p(\tau_p)$ (recall 1.7)

$$\varphi_{init} := \Psi [2\delta'_p, 3\delta'_p; \mathbf{d}_p] (\underline{G}_{p, \tau_p}, \varphi_{nl}[L, \tau, w]).$$

Definition 3.20. Given an LD solution φ as above we define the initial smooth surface $M[L, \tau, w]$ to be the union over $\mathbb{S}_{eq}^2 \setminus \bigsqcup_{p \in L} D_p(\tau_p)$ of the graphs of $\pm \varphi_{init}[L, \tau, w]$.

REMARK 3.21. The approach developed so far is quite general and can be easily modified to apply to doublings of minimal surfaces where the following hold:

- (i). A reflection exists exchanging the two sides of the given surface.
- (ii). The linearized operator has no kernel on the given surface.

If those conditions are not satisfied the approach still applies with further modifications we will describe elsewhere. □

The regions of the initial surfaces.

Lemma 3.22 (The gluing region). *For $M = M[L, \tau, w]$ defined as in 3.20 and $\forall p \in L$ the following hold.*

- (i). $\| \varphi_{init} - \underline{G}_{p, \tau_p} : C^{3, \beta}(D_p(4\delta'_p) \setminus D_p(\delta'_p), (\delta'_p)^{-2} g) \| \leq \tau_p^{1 + \frac{15}{8}\alpha}$.
- (ii). $\| \varphi_{init} : C^{3, \beta}(D_p(4\delta'_p) \setminus D_p(\delta'_p), (\delta'_p)^{-2} g) \| \leq C \tau_p |\log \tau_p|$.
- (iii). $\| (\delta'_p)^2 H' : C^{0, \beta}(D_p(3\delta'_p) \setminus D_p(2\delta'_p), (\delta'_p)^{-2} g) \| \leq \tau_p^{1 + \frac{15}{8}\alpha}$, where H' denotes the pushforward of H to \mathbb{S}_{eq}^2 by $\Pi_{\mathbb{S}_{eq}^2}$ and H the mean curvature of the initial surface M .

Proof. By the definitions we have for each $p \in L$

$$\varphi_{init} = \tau_p G_p - \tau_p \log \frac{\tau_p}{2} \cos \circ \mathbf{d}_p + \Psi [2\delta'_p, 3\delta'_p; \mathbf{d}_p] (\varphi_-, \varphi_+),$$

on $\Omega_p := D_p(4\delta'_p) \setminus D_p(\delta'_p)$, where

$$\begin{aligned} \varphi_- &:= \underline{G}_{p, \tau_p} - \tau_p G_p + \tau_p \log(\tau_p/2) \cos \circ \mathbf{d}_p, \\ \varphi_+ &:= \widehat{\varphi}_p + \tau_p \log(\tau_p/2) \cos \circ \mathbf{d}_p + \varphi_{nl} - \varphi. \end{aligned}$$

By scaling now the ambient metric to $\tilde{g}' := (\delta'_p)^{-2} g$ and expanding in linear and higher order terms we have

$$(\delta'_p)^2 H' = (\Delta_{\tilde{g}'} + 2(\delta'_p)^2) \varphi_{init} + \delta'_p \tilde{Q}(\delta'_p)^{-1} \varphi_{init}.$$

Note that on Ω_p we have

$$\begin{aligned} \varphi_{init} - \underline{G}_{p,\tau_p} &= \Psi [2\delta'_p, 3\delta'_p; \mathbf{d}_p] (0, \varphi_+ - \varphi_-), \\ \mathcal{L}'\varphi_{init} &= \mathcal{L}'\Psi [2\delta'_p, 3\delta'_p; \mathbf{d}_p] (\varphi_-, \varphi_+). \end{aligned}$$

Using these (for the second and the third inequality below) and also 2.20.vii we clearly have

$$\begin{aligned} \|\varphi_{init}\| &\leq C (\tau_p |\log \tau_p| + \|\varphi_-\| + \|\varphi_+\|), \\ \|\varphi_{init} - \underline{G}_{p,\tau_p}\| &\leq C (\|\varphi_-\| + \|\varphi_+\|), \\ \|(\Delta_{\tilde{g}'} + 2(\delta'_p)^2)\varphi_{init} : C^{0,\beta}(\Omega_p, (\delta'_p)^{-2}g)\| &\leq C (\|\varphi_-\| + \|\varphi_+\|), \\ \|\delta'_p \tilde{Q}_{(\delta'_p)^{-1}}\varphi_{init} : C^{0,\beta}(\Omega_p, (\delta'_p)^{-2}g)\| &\leq C (\delta'_p)^{-1} \|\varphi_{init}\|^2, \end{aligned}$$

where in this proof when we do not specify the norm we mean the $C^{3,\beta}(\Omega_p, (\delta'_p)^{-2}g)$ norm. We conclude that if $\|\varphi_{\pm}\| \leq \delta'_p$ (to control the quadratic terms), then we have

$$\|(\delta'_p)^2 H' : C^{0,\beta}(\Omega_p, (\delta'_p)^{-2}g)\| \leq C ((\delta'_p)^{-1} \tau_p^2 |\log \tau_p|^2 + \|\varphi_-\| + \|\varphi_+\|).$$

By 2.30 we have

$$\|\varphi_-\| \leq C \tau_p^{1+2\alpha} |\log \tau_p|.$$

By the definition of φ_+ we have

$$\|\varphi_+\| \leq \|\widehat{\varphi}_p + \tau_p \log(\tau_p/2) \cos \circ \mathbf{d}_p\| + \|\varphi_{nl} - \varphi : C^{3,\beta}_{sym}(\mathbb{S}_{eq}^2, g)\|.$$

By standard theory (with interior regularity for the gain of derivative) and separation of variables the matching condition in 3.4 implies that

$$\|\widehat{\varphi}_p + \tau_p \log(\tau_p/2) \cos \circ \mathbf{d}_p\| \leq C (\delta'_p/\delta_p)^2 \|\widehat{\varphi}_p : C^{2,\beta}(\partial D_p(\delta_p), (\delta_p)^{-2}g)\|.$$

Using 3.15.iv, 3.18.ii, and 2.31, we conclude that

$$\|\varphi_+\| \leq C (\delta'_p)^2 \tau_p^{1-\alpha/9} + \tau_{min}^{3/2} \leq \tau_p^{1+\frac{17}{9}\alpha}.$$

Combining the above we complete the proof.

q.e.d.

Lemma 3.23. *If 3.15 holds then M is embedded. Moreover, the following estimates hold.*

- (i). On $\mathbb{S}_{eq}^2 \setminus \bigsqcup_{p \in L} D_p(\delta'_p)$ we have $\frac{8}{9} \tau_{max}^{1+\alpha/5} \leq \varphi_{init}$.
- (ii). $\|\varphi_{init} : C^{3,\beta}_{sym}(\mathbb{S}_{eq}^2 \setminus \bigsqcup_{p \in L} D_p(\delta'_p), g)\| \leq \frac{9}{8} \tau_{min}^{8/9}$.
- (iii). $\forall p \in L$ we have

$$\begin{aligned} &\|\varphi_{init} - \tau_p \log(2\mathbf{d}_p/\tau_p) : \\ &: C^{3,\beta}(D_p(4\tau_p^\alpha) \setminus D_p(9\tau_p), \mathbf{d}_p, g, \tau_p^{\frac{15}{8}\alpha} + \tau_p^2 \mathbf{d}_p^{-2})\| \leq C \tau_p. \end{aligned}$$

Proof. We first prove the estimates (i-iii): (i) on $\mathbb{S}_{eq}^2 \setminus \bigsqcup_{p \in L} D_p(3\delta'_p)$ follows from 3.15.vi, 3.18.ii, and 3.19.i, and on $D_p(4\delta'_p) \setminus D_p(\delta'_p)$ for $p \in L$ from 3.22.i, 2.29, and 3.15.iii. (ii) on $\mathbb{S}_{eq}^2 \setminus \bigsqcup_{p \in L} D_p(3\delta'_p)$ follows from 3.15.v, 3.18.ii, and 3.19.i, and on $D_p(4\delta'_p) \setminus D_p(\delta'_p)$ for $p \in L$ from

3.22.ii, 2.29, and 3.15.iii. 3.19.ii and 3.22.i allow us to replace φ_{init} with \underline{G}_{p,τ_p} in (iii). (iii) follows then from 2.29. Finally, the embeddedness of M follows from (i) and by comparing the rest of M with standard catenoids using 2.36. q.e.d.

Our general methodology requires that we subdivide the initial surfaces into various regions [20, 4, 16, 14, 11, 12, 10]. Because of the modified approach we only need some of the regions. Because of the linearized doubling approach we also need to define the projections of some regions by $\Pi_{S_{eq}^2}$ (recall 2.6):

Definition 3.24. We define the following for $x \in [0, 4]$.

$$(3.25a) \quad S'_x := S_{eq}^2 \setminus \bigsqcup_{p \in L} D_p(2\delta_p/(1+x)),$$

$$(3.25b) \quad \tilde{S}'_x := S_{eq}^2 \setminus \bigsqcup_{p \in L} D_p(b\tau_p(1+x)),$$

$$(3.25c) \quad S_x[p] := M \cap \Pi_{S_{eq}^2}^{-1} \left(\overline{D_p(b\tau_p(1+x))} \right) \quad \forall p \in L,$$

$$(3.25d) \quad S_x[L] := \bigsqcup_{p \in L} S_x[p],$$

$$(3.25e) \quad \hat{S}_x[p] := M \cap \Pi_{S_{eq}^2}^{-1} \left(\overline{D_p(2\delta'_p/(1+x))} \right) \subset \mathbb{K}_{p,\tau_p} \quad \forall p \in L,$$

$$(3.25f) \quad \hat{S}_x[L] := \bigsqcup_{p \in L} \hat{S}_x[p],$$

where b is a large constant independent of the τ parameters which is to be chosen appropriately later. When $x = 0$ we may omit the subscript.

We define now precise Euclidean catenoids approximating the appropriately scaled catenoidal regions of the initial surface M , and also auxiliary notation for future reference.

Definition 3.26. We define a map (recall 2.34)

$$\Pi_{\mathbb{K},p} := \tilde{\Pi}_{\mathbb{K},p} \circ \text{Exp}_{p,\tau_p}^{-1} : \hat{S}[p] \rightarrow \tilde{\mathbb{K}}_{p,0}.$$

We also define $\tilde{\mathbb{K}}_L := \bigsqcup_{p \in L} \tilde{\mathbb{K}}_{p,0}$ and $\Pi_{\mathbb{K}} : \hat{S}[L] \rightarrow \tilde{\mathbb{K}}_L$ by taking the restriction of $\Pi_{\mathbb{K}}$ to each $\hat{S}[p]$ to be $\Pi_{\mathbb{K},p}$.

Clearly by 2.36 $\Pi_{\mathbb{K},p}$ is a diffeomorphism from $\hat{S}[p]$ to a domain of $\tilde{\mathbb{K}}_{p,0}$. $\Pi_{\mathbb{K}}$ is also a diffeomorphism from $\hat{S}[L]$ to a domain of $\tilde{\mathbb{K}}_L$. To incorporate now the symmetries into the discussion observe that we can clearly define uniquely an action of $\mathcal{G}_{S^3,m}$ on $\bigsqcup_{p \in L} T_p S_{eq}^2 \supset \tilde{\mathbb{K}}_L$, so that $\Pi_{\mathbb{K}}$ is equivariant under the actions of the $\mathcal{G}_{S^3,m}$. Because of the importance of the scaling we will need the following.

Definition 3.27. We define $\tau : \tilde{\mathbb{K}}_L \rightarrow \mathbb{R}$ by $\tau = \tau_p$ on $\tilde{\mathbb{K}}_{p,0}$.

We extend now the notation in 2.17 to apply to functions on domains of M or $\tilde{\mathbb{K}}_L$ as follows.

NOTATION 3.28. Suppose X is a function space consisting of functions defined on a domain $\Omega \subset M$ or $\Omega \subset \widetilde{\mathbb{K}}_L$. If Ω is invariant under the action of $\mathcal{G}_{\mathbb{S}^3, m}$ acting on M or $\widetilde{\mathbb{K}}_L$ (recall 2.14), then we use a subscript “sym” to denote the subspace $X_{sym} \subset X$ consisting of those functions in X which are invariant under the action of $\mathcal{G}_{\mathbb{S}^3, m}$. \square

4. The linearized equation and the nonlinear terms on the initial surfaces

The definition of $\mathcal{R}_{M,appr}$. In this section, we state and prove proposition 4.17 and lemma 4.24. In 4.17 we solve with estimates the linearized equation on an initial surface $M = M[L, \tau, w]$ defined as in 3.20, where $\varphi[L, \tau, w]$ is a $\mathcal{G}_{\mathbb{S}_{eq}^2, m}$ -symmetric MLD solution of configuration (L, τ, w) defined as in 3.4. In 4.24 we estimate the nonlinear terms on the initial surface M . To streamline the presentation we have the following.

CONVENTION 4.1. From now on we assume that b (recall 3.24) is as large as needed in absolute terms. We also fix some $\beta \in (0, 1)$ and $\gamma \in (1, 2)$ satisfying $1 - \frac{\gamma}{2} > 2\alpha$ and $(1 - \alpha)(\gamma - 1) > 2\alpha$, for example, $\gamma = \frac{3}{2}$. We will suppress the dependence of various constants on β and γ . \square

We construct now a linear map (recall 3.28)

$$(4.2) \quad \mathcal{R}_{M,appr} : C_{sym}^{0,\beta}(M) \rightarrow C_{sym}^{2,\beta}(M) \oplus \mathcal{K}_{sym}[L] \oplus C_{sym}^{0,\beta}(M),$$

where if $E \in C_{sym}^{0,\beta}(M)$ and $\mathcal{R}_{M,appr}E = (u_1, w_{E,1}, E_1)$, then u_1 is an approximate solution to the linearized equation modulo the “extended substitute kernel”, that is the equation

$$(4.3) \quad \mathcal{L}u = E + w_E \circ \Pi_{\mathbb{S}_{eq}^2} \quad \text{where} \quad w_E \in \mathcal{K}_{sym}[L], \quad \mathcal{L} := \Delta + |A|^2 + 2,$$

$w_{E,1}$ is the $\mathcal{K}_{sym}[L]$ term, and E_1 is the approximation error defined by

$$(4.4) \quad E_1 := \mathcal{L}u_1 - E - w_{E,1} \circ \Pi_{\mathbb{S}_{eq}^2}.$$

The approximate solution u_1 is constructed by combining semi-local approximate solutions. Before we proceed with the construction we define some cut-off functions we will need.

Definition 4.5. We define $\psi' \in C_{sym}^\infty(\mathbb{S}_{eq}^2)$ and $\widetilde{\psi}, \widehat{\psi} \in C_{sym}^\infty(M)$ by requesting the following.

- (i). $\widetilde{\psi} = (1 - \psi') \circ \Pi_{\mathbb{S}_{eq}^2}$ on M .
- (ii). $\widetilde{\psi}$ is supported on $S_1[L]$, $\widehat{\psi}$ is supported on $\widehat{S}[L]$, and ψ' on \widetilde{S}' (recall 3.24).

(iii). $\psi' = 1$ on \tilde{S}'_1 and for each $p \in L$ we have

$$\begin{aligned} \psi' &= \Psi [b\tau_p, 2b\tau_p; \mathbf{d}_p] (0, 1) \quad \text{on } D_p(2b\tau_p), \\ \hat{\psi} &= \Psi \left[2\delta'_p, \delta'_p; \mathbf{d}_p \circ \Pi_{\mathbb{S}^2_{eq}} \right] (0, 1) \quad \text{on } \hat{S}[p]. \end{aligned}$$

Given now $E \in C^{0,\beta}_{sym}(M)$, we define $E' \in C^{0,\beta}_{sym}(\mathbb{S}^2_{eq})$ by requiring that it is supported on \tilde{S}' , and that on M we have the decomposition

$$(4.6) \quad E = \tilde{\psi} E + E' \circ \Pi_{\mathbb{S}^2_{eq}}.$$

Because of 3.12 there are unique $u' \in C^{2,\beta}_{sym}(\mathbb{S}^2_{eq})$ and $w_{E,1} \in \mathcal{K}_{sym}[L]$ such that

$$(4.7) \quad \mathcal{L}' u' = E' + w_{E,1} \text{ on } \mathbb{S}^2_{eq} \quad \text{and} \quad \forall p \in L \quad u'(p) = 0, \quad d_p u' = 0.$$

We define now $\tilde{E} \in C^{0,\beta}_{sym}(\tilde{\mathbb{K}}_L)$, supported on $\Pi_{\mathbb{K}}(S_1[L])$, by

$$(4.8) \quad \tilde{E} \circ \Pi_{\mathbb{K}} = \tilde{\psi} E + \{ [\psi', \mathcal{L}'] u' + (1 - \psi') E' \} \circ \Pi_{\mathbb{S}^2_{eq}} \quad \text{on } S_1[L].$$

We introduce a decomposition

$$(4.9) \quad \tilde{E} = \tilde{E}_{low} + \tilde{E}_{high},$$

where $\tilde{E}_{low} \in C^{0,\beta}_{sym,low}(\tilde{\mathbb{K}}_L)$ and $\tilde{E}_{high} \in C^{0,\beta}_{sym,high}(\tilde{\mathbb{K}}_L)$ are supported on $\Pi_{\mathbb{K}}(S_1[L])$. Note that here we use subscripts “low” and “high” to denote subspaces of functions which satisfy the condition that their restrictions to a meridian of a $\tilde{\mathbb{K}}_{p,0}$ belong or are orthogonal respectively to the span of the constants and the first harmonics on the meridian. Let $\mathcal{L}_{\tilde{\mathbb{K}}}$ denote the linearized operator on $\tilde{\mathbb{K}}_L$, and let $\tilde{u}_{low} \in C^{2,\beta}_{sym,low}(\tilde{\mathbb{K}}_L)$ and $\tilde{u}_{high} \in C^{2,\beta}_{sym,high}(\tilde{\mathbb{K}}_L)$ be solutions of (recall 3.27)

$$(4.10) \quad \mathcal{L}_{\tilde{\mathbb{K}}} \tilde{u}_{low} = \tau^2 \tilde{E}_{low}, \quad \mathcal{L}_{\tilde{\mathbb{K}}} \tilde{u}_{high} = \tau^2 \tilde{E}_{high},$$

determined uniquely as follows. By separating variables the first equation amounts to uncoupled ODE equations which are solved uniquely by assuming vanishing initial data on the waist of the catenoids. For the second equation we can as usual change the metric conformally to $h = \frac{1}{2}|A|^2 g = \nu^* g_{\mathbb{S}^2_{eq}}$, and then we can solve uniquely because the inhomogeneous term is clearly orthogonal to the kernel. We conclude now the definition of $\mathcal{R}_{M,appr}$:

Definition 4.11. We define $\mathcal{R}_{M,appr}$ as in 4.2 by taking $\mathcal{R}_{M,appr} E = (u_1, w_{E,1}, E_1)$, where $w_{E,1}$ was defined in 4.7, E_1 in 4.4, and $u_1 := \hat{\psi} \tilde{u} \circ \Pi_{\mathbb{K}} + (\psi' u') \circ \Pi_{\mathbb{S}^2_{eq}} \in C^{2,\beta}_{sym}(M)$, where $\tilde{u} := \tilde{u}_{low} + \tilde{u}_{high} \in C^{2,\beta}_{sym}(\tilde{\mathbb{K}}_L)$.

Norms and approximations. We introduce now some abbreviated notation for the norms we will be using.

Definition 4.12. For $k \in \mathbb{N}$, $\widehat{\beta} \in (0, 1)$, $\widehat{\gamma} \in \mathbb{R}$, and Ω a domain in \mathbb{S}_{eq}^2 , M , or $\widetilde{\mathbb{K}}_L$ (recall 3.26), we define

$$\|u\|_{k, \widehat{\beta}, \widehat{\gamma}; \Omega} := \|u : C^{k, \widehat{\beta}}(\Omega, \rho, g, \rho^{\widehat{\gamma}})\|,$$

where $\rho := \mathbf{d}_L$ and g is the standard metric on \mathbb{S}_{eq}^2 when $\Omega \subset \mathbb{S}_{eq}^2$, $\rho := \mathbf{d}_L \circ \Pi_{\mathbb{S}_{eq}^2}$ and g is the metric induced on M by the standard metric on $\mathbb{S}^3(1)$ when $\Omega \subset M$, and $\rho = \widetilde{r}$ (recall 2.35) and g is the metric induced by the Euclidean metric h_p on $T_p\mathbb{S}^3(1)$ as in 2.32 when $\Omega \subset \widetilde{\mathbb{K}}_L$.

Note that these definitions are equivalent to more popular definitions but we find these definitions more intuitive. We compare now norms on some nearby surfaces.

Lemma 4.13. (i). *If τ_{max} is small enough in terms of given $\epsilon > 0$, $\widetilde{\Omega}$ is a domain in $\Pi_{\mathbb{K}}(\widehat{S}[L])$, $\Omega := \Pi_{\mathbb{K}}^{-1}(\widetilde{\Omega}) \subset \widehat{S}[L] \subset M$, $k = 0, 2$, $\widehat{\gamma} \in \mathbb{R}$, and $f \in C^{k, \beta}(\widetilde{\Omega})$, then we have (recall 1.1 and 3.27):*

$$\|f \circ \Pi_{\mathbb{K}}\|_{k, \beta, \widehat{\gamma}; \Omega} \sim_{1+\epsilon} \|\tau^{-\widehat{\gamma}} f\|_{k, \beta, \widehat{\gamma}; \widetilde{\Omega}}.$$

(ii). *If b is large enough in terms of given $\epsilon > 0$, τ_{max} is small enough in terms of ϵ and b , Ω' is a domain in $\widetilde{S}' = \mathbb{S}_{eq}^2 \setminus \bigsqcup_{p \in L} D_p(b\tau_p)$ (recall 3.25b), $\Omega := \Pi_{\mathbb{S}_{eq}^2}^{-1}(\Omega') \cap M$, $k = 0, 2$, $\widehat{\gamma} \in \mathbb{R}$, and $f \in C^{k, \beta}(\Omega')$, then*

$$\|f \circ \Pi_{\mathbb{S}_{eq}^2}\|_{k, \beta, \widehat{\gamma}; \Omega} \sim_{1+\epsilon} \|f\|_{k, \beta, \widehat{\gamma}; \Omega'}.$$

Proof. Note that by assuming τ_{max} small enough we can ensure that $9C(k, \alpha)\tau_{max}^{2\alpha} \leq \epsilon$. (i) follows then from the definitions, 2.39, and 2.40. To prove (ii) let $q \in \widetilde{S}'$ and consider the metric $\widehat{g}_q := (\mathbf{d}_L(q))^{-2}g$ on $\mathbb{S}^3(1)$, where g is the standard metric on $\mathbb{S}^3(1)$. In this metric M is the union of the graphs of $\pm\varphi_{:q}$ where $\varphi_{:q} := (\mathbf{d}_L(q))^{-1}\varphi_{init}$. Let B'_q be the geodesic disc in $(\mathbb{S}_{eq}^2, \widehat{g}_q)$ of center q and radius $1/10$. Note that

$$\|\log(2r/\tau) : C^k((9\tau, 9\tau^\alpha), r, dr^2, \log(r/\tau))\| \leq C(k).$$

By 3.23 we have then that

$$(4.14) \quad \|\varphi_{:q} : C^{3, \beta}(B'_q, \widehat{g}_q)\| \leq C f_{weight}(q) \leq C b^{-1} \log b,$$

where $f_{weight}(q) = \frac{\log(\mathbf{d}_p(q)/\tau_p)}{\mathbf{d}_p(q)/\tau_p}$ if $q \in D_p(3\delta'_p)$ for some $p \in L$ (where we used that if $b > 10$ then $\tau_p^{1+\frac{15}{8}\alpha}\mathbf{d}_p^{-1}(q) + \tau_p^3\mathbf{d}_p^{-3} \leq \frac{\log(\mathbf{d}_p(q)/\tau_p)}{\mathbf{d}_p(q)/\tau_p}$)

and $f_{weight}(q) = 2\tau_{min}^{8/9}$ otherwise (where we used 3.18.ii and 3.15.v). By comparing the metrics and using the definitions we complete the proof. q.e.d.

We reformulate now the estimate for the mean curvature from 3.22 to an estimate stated in terms of the global norm we just defined.

Lemma 4.15. *The function $H - w \circ \Pi_{\mathbb{S}_{eq}^2}$ on the initial surface $M = M[L, \tau, w]$ is supported on $\Pi_{\mathbb{S}_{eq}^2}^{-1} \left(\bigsqcup_{p \in L} (D_p(3\delta'_p) \setminus D_p(2\delta'_p)) \right)$. Moreover, it satisfies the estimate*

$$\| H - w \circ \Pi_{\mathbb{S}_{eq}^2} \|_{0, \beta, \gamma-2; M} \leq \tau_{max}^{1+\alpha/3}.$$

Proof. The statement on the support follows from 2.26.ii, 3.18.i, and the definitions. Combining now 3.22.iii, 4.12, and 4.13.ii we complete the proof. q.e.d.

Lemma 4.16. (i). *If τ_{max} is small enough and $f \in C^{2, \beta}(\Pi_{\mathbb{K}}(\widehat{S}[L]))$, then we have*

$$\begin{aligned} \| \mathcal{L}(f \circ \Pi_{\mathbb{K}}) - \tau^{-2} (\mathcal{L}_{\widetilde{\mathbb{K}}} f) \circ \Pi_{\mathbb{K}} \|_{0, \beta, \widehat{\gamma}-2; \widehat{S}[L]} &\leq \\ &\leq C \tau_{max}^{2\alpha} \| \tau^{-\widehat{\gamma}} f \|_{2, \beta, \widehat{\gamma}; \Pi_{\mathbb{K}}(\widehat{S}[L])}. \end{aligned}$$

(ii). *If τ_{max} is small enough and $f \in C^{2, \beta}(\widetilde{S}')$, then for $\epsilon_1 \in [0, 1/2]$ we have*

$$\begin{aligned} \| \mathcal{L} \{ f \circ \Pi_{\mathbb{S}_{eq}^2} \} - \{ \mathcal{L}' f \} \circ \Pi_{\mathbb{S}_{eq}^2} \|_{0, \beta, \widehat{\gamma}-2; \Pi_{\mathbb{S}_{eq}^2}^{-1}(\widetilde{S}')} &\leq \\ &\leq C b^{\epsilon_1-1} \log b \tau_{max}^{\epsilon_1} \| f \|_{2, \beta, \widehat{\gamma}+\epsilon_1; \widetilde{S}'}. \end{aligned}$$

Proof. (i). In analogy with 3.26 we define the map $\widetilde{\Pi}_{\mathbb{K}} : \bigsqcup_{p \in L} \widetilde{\mathbb{K}}_{\tau_p, p} \rightarrow \widetilde{\mathbb{K}}_L$ by requesting that its restriction to $\widetilde{\mathbb{K}}_{\tau_p, p}$ for $p \in L$ is the restriction of $\widetilde{\Pi}_{\mathbb{K}, p}$ to $\widetilde{\mathbb{K}}_{\tau_p, p}$ (recall 2.36 and 2.35). We also define $\widetilde{\mathcal{L}}$ to be the linearized operator on $\bigsqcup_{p \in L} \widetilde{\mathbb{K}}_{\tau_p, p}$ with respect to the ambient metric which $\forall p \in L$ on $B_{p, \tau_p} \subset T_p \mathbb{S}^3(1)$ equals $\widetilde{g}_{p, \tau_p}$ defined as in 2.32. We have then

$$\begin{aligned} \tau^{2-\widehat{\gamma}} \{ \mathcal{L}(f \circ \Pi_{\mathbb{K}}) \} \circ \Pi_{\mathbb{K}}^{-1} - \tau^{-\widehat{\gamma}} (\mathcal{L}_{\widetilde{\mathbb{K}}} f) &= \\ &= \tau^{-\widehat{\gamma}} [\{ \widetilde{\mathcal{L}}(f \circ \widetilde{\Pi}_{\mathbb{K}}) \} \circ \widetilde{\Pi}_{\mathbb{K}}^{-1} - \mathcal{L}_{\widetilde{\mathbb{K}}} f]. \end{aligned}$$

Using then 4.13.i and that τ is locally constant proving (i) reduces to proving

$$\| \{ \widetilde{\mathcal{L}}(f \circ \widetilde{\Pi}_{\mathbb{K}}) \} \circ \widetilde{\Pi}_{\mathbb{K}}^{-1} - \mathcal{L}_{\widetilde{\mathbb{K}}} f \|_{0, \beta, \widehat{\gamma}-2; \Pi_{\mathbb{K}}(\widehat{S}[L])} \leq C \tau_{max}^{2\alpha} \| f \|_{2, \beta, \widehat{\gamma}; \Pi_{\mathbb{K}}(\widehat{S}[L])}.$$

We fix now a $p \in L$ and we apply 2.39 and the notation and the observations in its proof (with τ_p instead of τ) including 2.40: We have then that the $C^{0, \beta}$ norm on B'_q with respect to the metric induced by \widehat{h}_q of the corresponding difference of linearized operators applied on f is bounded by

$$C \tau_{max}^{2\alpha} \| f : C^{2, \beta}(B'_q, h_p) \|.$$

Using scaling and the definitions we conclude the proof of (i).

(ii). In this case we apply the notation and observations in the proof of 4.13.ii. By 4.14 and by using scaling for the left hand side, we conclude that for $q \in \tilde{S}'$, we have

$$\begin{aligned} (\mathbf{d}_L(q))^2 \|\mathcal{L}\{f \circ \Pi_{\mathbb{S}_{e_q}^2}\} - \{\mathcal{L}' f\} \circ \Pi_{\mathbb{S}_{e_q}^2}\| &: C^{0,\beta}(\Pi_{\mathbb{S}_{e_q}^2}^{-1}(B'_q), \hat{g}_p) \|\leq \\ &\leq C f_{weight}(q) \|f : C^{2,\beta}(B'_q, \hat{g}_p) \|. \end{aligned}$$

By the definitions it is enough then to check that $\forall q \in \tilde{S}'$ we have

$$f_{weight}(q) (\mathbf{d}_L(q))^{\epsilon_1} \leq C b^{\epsilon_1-1} \log b \tau_{max}^{\epsilon_1}.$$

This follows from the definition of f_{weight} (given in the proof of 4.13) and the observation that $x^{\epsilon_1-1} \log x$ is decreasing in x for $x \geq b$. This completes the proof. q.e.d.

The main Proposition.

Proposition 4.17. *Recall that we assume that 2.31, 3.15, 4.1, and 3.12 hold. Suppose further that*

$$(4.18) \quad \delta_{min}^{-4} \tau_{max}^\alpha \|\mathcal{E}_L^{-1}\| \leq 1.$$

A linear map $\mathcal{R}_M : C_{sym}^{0,\beta}(M) \rightarrow C_{sym}^{2,\beta}(M) \times \mathcal{K}_{sym}[L]$ can be defined then by

$$\mathcal{R}_M E := (u, w_E) := \sum_{n=1}^\infty (u_n, w_{E,n}) \in C_{sym}^{2,\beta}(M) \times \mathcal{K}_{sym}[L],$$

for $E \in C_{sym}^{0,\beta}(M)$, where the sequence $\{(u_n, w_{E,n}, E_n)\}_{n \in \mathbb{N}}$ is defined inductively for $n \in \mathbb{N}$ by

$$(u_n, w_{E,n}, E_n) := -\mathcal{R}_{M,appr} E_{n-1}, \quad E_0 := -E.$$

Moreover, the following hold.

- (i). $\mathcal{L}u = E + w_E \circ \Pi_{\mathbb{S}_{e_q}^2}$.
- (ii). $\|u\|_{2,\beta,\gamma;M} \leq C(b) \delta_{min}^{\gamma-2-\beta} \|\mathcal{E}_L^{-1}\| \|E\|_{0,\beta,\gamma-2;M}$.
- (iii). $\|w_E : C^{0,\beta}(\mathbb{S}_{e_q}^2, g)\| \leq C \delta_{min}^{\gamma-2-\beta} \|\mathcal{E}_L^{-1}\| \|E\|_{0,\beta,\gamma-2;M}$.
- (iv). \mathcal{R}_M depends continuously on the parameters of φ .

Proof. We subdivide the proof into five steps:

Step 1: Estimates on u' and $w_{E,1}$: We start by decomposing E' and u' (defined as in 4.6 and 4.7) into various parts which will be estimated separately. We clearly have by the definitions and the equivalence of the norms as in 4.13 that

$$\|E'\|_{0,\beta,\gamma-2;\mathbb{S}_{e_q}^2} \leq C \|E\|_{0,\beta,\gamma-2;M}.$$

We first solve uniquely for each $p \in L$ the equation $\mathcal{L}'u'_p = E'$ on $D_p(2\delta_p)$ by requiring that $u'_p(p) = 0$, $d_p u'_p = 0$, and that the restriction of u_p

on $\partial D_p(2\delta_p)$ is a linear combination of constants and first harmonics. Clearly then by standard theory and separation of variables we have

$$\|u'_p\|_{2,\beta,\gamma;D_p(2\delta_p)} \leq C \|E'\|_{0,\beta,\gamma-2;D_p(2\delta_p)}.$$

We define now $u'' \in C_{sym}^{2,\beta}(\mathbb{S}_{eq}^2)$ supported on $\bigsqcup_{p \in L} D_p(2\delta_p)$ by requesting that for each $p \in L$ we have

$$u'' = \Psi[2\delta_p, \delta_p; \mathbf{d}_p](0, u'_p) \quad \text{on } D_p(2\delta_p).$$

We clearly have then

$$\|u''\|_{2,\beta,\gamma;\mathbb{S}_{eq}^2} \leq C \|E\|_{0,\beta,\gamma-2;M}.$$

$E' - \mathcal{L}'u''$ vanishes on $\bigsqcup_{p \in L} D_p(\delta_p)$ and, therefore, it is supported on $\mathbb{S}_{eq}^2 \setminus \bigsqcup_{p \in L} D_p(\delta_p) = S'_1$ (recall 3.25a). Moreover, it satisfies

$$\|E' - \mathcal{L}'u''\|_{0,\beta,\gamma-2;\mathbb{S}_{eq}^2} \leq C \|E\|_{0,\beta,\gamma-2;M}.$$

Using the definition of the norms and the restricted support S'_1 we conclude that

$$\|E' - \mathcal{L}'u'' : C^{0,\beta}(\mathbb{S}_{eq}^2, g)\| \leq C \delta_{min}^{\gamma-2-\beta} \|E' - \mathcal{L}'u''\|_{0,\beta,\gamma-2;\mathbb{S}_{eq}^2}.$$

The last two estimates and standard linear theory imply that the unique by symmetry solution $u''' \in C_{sym}^{2,\beta}(\mathbb{S}_{eq}^2)$ to $\mathcal{L}'u''' = E' - \mathcal{L}'u''$ satisfies

$$\|u'''\| : C^{2,\beta}(\mathbb{S}_{eq}^2, g)\| \leq C \delta_{min}^{\gamma-2-\beta} \|E\|_{0,\beta,\gamma-2;M}.$$

By 3.12 there is a unique $v \in \widehat{\mathcal{K}}_{sym}[L]$ (recall 3.9) such that $u''' + v$ and $d(u''' + v)$ vanish at each $p \in L$. Moreover, by the last estimate and 3.13 v satisfies the estimate

$$\|v : C^{2,\beta}(\mathbb{S}_{eq}^2, g)\| + \|\mathcal{L}'v : C^{0,\beta}(\mathbb{S}_{eq}^2, g)\| \leq C \delta_{min}^{\gamma-2-\beta} \|\mathcal{E}_L^{-1}\| \|E\|_{0,\beta,\gamma-2;M}.$$

By the definition of u''' we conclude that $\mathcal{L}'(u'' + u''' + v) = E' + \mathcal{L}'v$. By the definitions of u'' and v we clearly have that $u'' + u''' + v$ satisfies also the vanishing conditions in 4.7 and hence

$$u' = u'' + u''' + v \quad \text{and} \quad w_{E,1} = \mathcal{L}'v.$$

Note now that $\mathcal{L}'u''' = E' - \mathcal{L}'u''$ vanishes on $\bigsqcup_{p \in L} D_p(\delta_p)$ and by 3.7 and 3.9 so does $\mathcal{L}'v \in \mathcal{K}_{sym}[L]$. We conclude that for each $p \in L$ we have $\mathcal{L}'(u''' + v) = 0$ on $D_p(\delta_p)$, and since we know already that $u''' + v$ and $d(u''' + v)$ vanish at p , we can use standard theory and separation of variables to estimate with decay $u''' + v$ on $D_p(\delta_p)$ in terms of the Dirichlet data on $\partial D_p(\delta_p)$. Combining with the earlier estimates for u''' and v we conclude that

$$\|u''' + v\|_{2,\beta,\gamma';\mathbb{S}_{eq}^2} \leq C \delta_{min}^{\gamma-\gamma'-2-\beta} \|\mathcal{E}_L^{-1}\| \|E\|_{0,\beta,\gamma-2;M},$$

where $\gamma' = \frac{\gamma+2}{2} \in (\gamma, 2)$. We need the stronger decay for estimating E_1 later. A similar estimate holds with γ instead of γ' . Note that by 3.13 $\|\mathcal{E}_L^{-1}\| \geq 1$. Combining with the earlier estimate for u'' we conclude that

$$\|u'\|_{2,\beta,\gamma;\mathbb{S}_{eq}^2} \leq C \delta_{min}^{-2-\beta} \|\mathcal{E}_L^{-1}\| \|E\|_{0,\beta,\gamma-2;M}.$$

Step 2: Estimates on \tilde{u} : By the definitions and 4.13 (with $\epsilon = 1$) we have that

$$\|\tau^{2-\gamma} \tilde{E}\|_{0,\beta,\gamma-2;\tilde{\mathbb{K}}_L} \leq C (\|E\|_{0,\beta,\gamma-2;M} + \|u'\|_{2,\beta,\gamma;\mathbb{S}_{eq}^2}).$$

By considering the standard conformal parametrization of the catenoid on a cylinder it is easy to conclude that

$$\|\tau^{-\gamma} \tilde{u}_{low}\|_{2,\beta,1;\tilde{\mathbb{K}}_L} \leq C(b) \|\tau^{2-\gamma} \tilde{E}\|_{0,\beta,\gamma-2;\tilde{\mathbb{K}}_L}.$$

Similarly, by standard linear theory and the obvious C^0 bound on \tilde{u}_{high} we conclude

$$\|\tau^{-\gamma} \tilde{u}_{high}\|_{2,\beta,0;\tilde{\mathbb{K}}_L} \leq C(b) \|\tau^{2-\gamma} \tilde{E}\|_{0,\beta,\gamma-2;\tilde{\mathbb{K}}_L}.$$

Combining the above we conclude that

$$\|\tau^{-\gamma} \tilde{u}\|_{2,\beta,\gamma;\tilde{\mathbb{K}}_L} \leq \|\tau^{-\gamma} \tilde{u}\|_{2,\beta,1;\tilde{\mathbb{K}}_L} \leq C(b) \delta_{min}^{-2-\beta} \|\mathcal{E}_L^{-1}\| \|E\|_{0,\beta,\gamma-2;M}.$$

Step 3: A decomposition of E_1 : Using 4.4 and 4.11, 4.6, 4.10, and 4.9, we obtain

$$(4.19) \quad E_1 = E_{1,I} + \hat{\psi} \mathcal{L}(\tilde{u} \circ \Pi_{\mathbb{K}}) + E_{1,III} + \{\mathcal{L}'(\psi' u')\} \circ \Pi_{\mathbb{S}_{eq}^2} - E - w_{E,1} \circ \Pi_{\mathbb{S}_{eq}^2},$$

where $E_{1,I}, E_{1,III} \in C_{sym}^{0,\beta}(M)$ are supported on $\widehat{S}[L] \setminus \widehat{S}_1[L]$ and \tilde{S}' respectively by 4.5.ii, and where they are defined by

$$(4.20) \quad \begin{aligned} E_{1,I} &:= [\mathcal{L}, \hat{\psi}](\tilde{u} \circ \Pi_{\mathbb{K}}), \\ E_{1,III} &:= \mathcal{L}\{(\psi' u') \circ \Pi_{\mathbb{S}_{eq}^2}\} - \{\mathcal{L}'(\psi' u')\} \circ \Pi_{\mathbb{S}_{eq}^2}. \end{aligned}$$

Using 4.6, 4.10, 4.9, and 4.8, we obtain

$$(4.21) \quad \begin{aligned} E_1 &= E_{1,I} + E_{1,II} + \tilde{\psi} E + \{[\psi', \mathcal{L}'] u' + (1 - \psi') E'\} \circ \Pi_{\mathbb{S}_{eq}^2} + \\ &+ E_{1,III} + \{[\mathcal{L}', \psi'] u' + \psi' \mathcal{L}' u'\} \circ \Pi_{\mathbb{S}_{eq}^2} - \tilde{\psi} E - E' \circ \Pi_{\mathbb{S}_{eq}^2} - w_{E,1} \circ \Pi_{\mathbb{S}_{eq}^2}, \end{aligned}$$

where $E_{1,II} \in C_{sym}^{0,\beta}(M)$ is supported by 4.5.ii on $\widehat{S}[L]$ where it satisfies

$$(4.22) \quad E_{1,II} = \hat{\psi} (\mathcal{L}(\tilde{u} \circ \Pi_{\mathbb{K}}) - \tau^{-2} (\mathcal{L}_{\tilde{\mathbb{K}}} \tilde{u}) \circ \Pi_{\mathbb{K}}) = \hat{\psi} \mathcal{L}(\tilde{u} \circ \Pi_{\mathbb{K}}) - \tilde{E} \circ \Pi_{\mathbb{K}}.$$

By 4.5.ii and 4.7 we have $\psi' \mathcal{L}' u' = \psi' E' + w_{E,1}$. Using this and canceling terms we conclude that

$$(4.23) \quad E_1 = E_{1,I} + E_{1,II} + E_{1,III}.$$

Step 4: Estimates on u_1 and E_1 : Using the definitions, 4.13 with $\epsilon = 1$, and the estimates for u' and \tilde{u} above we conclude that

$$\|u_1\|_{2,\beta,\gamma;M} \leq C(b) \delta_{min}^{-2-\beta} \|\mathcal{E}_L^{-1}\| \|E\|_{0,\beta,\gamma-2;M}.$$

By 4.13 we have

$$\|(\tau^{1-\gamma}\tilde{u}) \circ \Pi_{\mathbb{K}}\|_{2,\beta,1;\widehat{S}[L]\backslash\widehat{S}_1[L]} \sim_2 \|\tau^{-\gamma}\tilde{u}\|_{2,\beta,1;\Pi_{\mathbb{K}}(\widehat{S}[L]\backslash\widehat{S}_1[L])}.$$

Using definitions 4.12 and 3.24 we conclude that

$$\|\tilde{u} \circ \Pi_{\mathbb{K}}\|_{2,\beta,\gamma;\widehat{S}[L]\backslash\widehat{S}_1[L]} \leq C \tau_{max}^{(1-\alpha)(\gamma-1)} \|\tau^{-\gamma}\tilde{u}\|_{2,\beta,1;\widetilde{\mathbb{K}}_L},$$

and, therefore, we have by the definition of $E_{1,I}$ that

$$\|E_{1,I}\|_{0,\beta,\gamma-2;M} \leq C \tau_{max}^{(1-\alpha)(\gamma-1)} \|\tau^{-\gamma}\tilde{u}\|_{2,\beta,1;\widetilde{\mathbb{K}}_L}.$$

Applying now 4.16.i with $f = \tilde{u}$ and $\widehat{\gamma} = \gamma$ and using the definition of $\widehat{\psi}$ we conclude that

$$\|E_{1,II}\|_{0,\beta,\gamma-2;M} \leq C \tau_{max}^{2\alpha} \|\tau^{-\gamma}\tilde{u}\|_{2,\beta,\gamma;\widetilde{\mathbb{K}}_L}.$$

We decompose now $E_{1,III} = E''_{1,III} + E'''_{1,III}$ where $E''_{1,III}$ and $E'''_{1,III}$ are defined the same way as $E_{1,III}$ but with u' replaced by u'' and $u''' + v$ respectively. Applying 4.16.ii with $\epsilon_1 = 0$, $f = u''$, and $\widehat{\gamma} = \gamma$, we conclude that

$$\|E''_{1,III}\|_{0,\beta,\gamma-2;M} \leq C b^{-1} \log b \|u''\|_{2,\beta,\gamma;\mathbb{S}_{eq}^2}.$$

Applying 4.16.ii with $\epsilon_1 = \gamma' - \gamma$, $f = u''$, and $\widehat{\gamma} = \gamma$,

$$\|E'''_{1,III}\|_{0,\beta,\gamma-2;M} \leq C b^{\gamma'-\gamma-1} \log b \tau_{max}^{\gamma'-\gamma} \|u''' + v\|_{2,\beta,\gamma';\mathbb{S}_{eq}^2}.$$

Combining the above with the earlier estimates and using 4.18 and 4.1 we conclude that

$$\begin{aligned} &\|E_1\|_{0,\beta,\gamma-2;M} \leq \\ &\leq (C(b) \tau_{max}^{\alpha/2} + C b^{-1} \log b + C b^{-1/2} \log b \tau_{max}^{\gamma'-\gamma-\alpha}) \|E\|_{0,\beta,\gamma-2;M}. \end{aligned}$$

Step 5: The final iteration: By assuming b large enough and τ_{max} small enough in terms of b we conclude using $\gamma' - \gamma - \alpha > 0$ and induction that

$$\|E_n\|_{0,\beta,\gamma-2;M} \leq 2^{-n} \|E\|_{0,\beta,\gamma-2;M}.$$

The proof is then completed by using the earlier estimates. q.e.d.

The nonlinear terms. If $\phi \in C^1_{sym}(M)$ is appropriately small, we denote by M_ϕ the perturbation of M by ϕ , defined as the image of $I_\phi : M \rightarrow \mathbb{S}^3$, where $I : M \rightarrow \mathbb{S}^3(1)$ is the inclusion map of M and I_ϕ is defined by $I_\phi(x) := \exp_x(\phi(x)\nu(x))$ where $\nu : M \rightarrow T\mathbb{S}^3(1)$ is the unit normal to M . Clearly then (recall 3.28) M_ϕ is invariant under the action of $\mathcal{G}_{\mathbb{S}^3,m}$ on the sphere $\mathbb{S}^3(1)$. Using now rescaling we prove a

global estimate for the nonlinear terms of the mean curvature of M_ϕ as follows (see [20, Lemma 5.1] for a similar statement):

Lemma 4.24. *If M is as in 4.17 and $\phi \in C_{sym}^{2,\beta}(M)$ satisfies $\|\phi\|_{2,\beta,\gamma;M} \leq \tau_{max}^{1+\alpha/4}$, then M_ϕ is well defined as above, is embedded, and if H_ϕ is the mean curvature of M_ϕ pulled back to M by I_ϕ and H is the mean curvature of M , then we have*

$$\|H_\phi - H - \mathcal{L}\phi\|_{0,\beta,\gamma-2;M} \leq C \|\phi\|_{2,\beta,\gamma;M}^2.$$

Proof. Note that such a strong bound on ϕ is only needed for ensuring the embeddedness of M_ϕ . Following the notation in the proof of 4.13 and by 4.14 we have that for $q \in \tilde{S}'$ the graph B_q'' of $\varphi_{:q}$ over B_q' in $(\mathbb{S}_{eq}^2, \hat{g}_q)$ can be described by an immersion $X_{:q} : B_q' \rightarrow B_q'' = X_{:q}(B_q')$ such that there are coordinates on B_q' and a neighborhood in $\mathbb{S}^3(1)$ of B_q'' which are uniformly bounded and the immersion in these coordinates has uniformly bounded $C^{3,\beta}$ norms, the standard Euclidean metric on the domain is bounded by $CX_{:q}^*\hat{g}_q$, and the coefficients of \hat{g}_q in the target coordinates have uniformly bounded $C^{3,\beta}$ norms. By the definition of the norm and since $\|\phi\|_{2,\beta,\gamma;M} \leq \tau_{max}^{1+\alpha/4}$, we have that the restriction of ϕ on B_q'' satisfies

$$\|\mathbf{d}_L^{-1}(q)\phi : C^{2,\beta}(B_q'', \hat{g}_q)\| \leq C \mathbf{d}_L^{\gamma-1}(q) \|\phi\|_{2,\beta,\gamma;M}.$$

Since the right hand side is small in absolute terms we conclude that I_ϕ is well defined on B_q'' , its restriction to B_q'' is an embedding, and by using scaling for the left hand side that

$$\|\mathbf{d}_L(q)(H_\phi - H - \mathcal{L}\phi) : C^{0,\beta}(B_q'', \hat{g}_q)\| \leq C \mathbf{d}_L^{2\gamma-2}(q) \|\phi\|_{2,\beta,\gamma;M}^2.$$

Since $2\gamma - 3 - (\gamma - 2) = \gamma - 1 > 0$ we conclude that

$$\mathbf{d}_L^{2-\gamma}(q) \|H_\phi - H - \mathcal{L}\phi : C^{0,\beta}(B_q'', \hat{g}_q)\| \leq C \|\phi\|_{2,\beta,\gamma;M}^2.$$

Note now that B_q'' is very close to the geodesic disc of radius $1/10$ in \hat{g}_q and with a center a point of M which projects by $\Pi_{\mathbb{S}_{eq}^2}$ to q . It remains to establish similar estimates for such discs $B_x'' \subset M$ with centers at points $x \in S[L]$. Note that the components of $S[L]$ appropriately scaled are small perturbations of a fixed compact region of the standard catenoid by smooth dependence on each τ_p . This allows us to repeat the arguments above in this case and combining with the earlier estimates we conclude by the definitions the estimate in the statement of the lemma.

It remains to prove the global embeddedness of M_ϕ . Given the local embeddedness we already know, global embeddedness could only fail if there was a nontrivial intersection between M_ϕ and \mathbb{S}_{eq}^2 . Using the estimates in 3.23 we can exclude this possibility and the proof is complete.

q.e.d.

5. LD and MLD solutions in the two-circle case

Basic definitions. We concentrate now to the case where L_{par} consists of only two circles. Because of the invariance under $\mathcal{G}_{S_{eq}^2, m}$ there exists $x_1 \in (0, \pi/2)$ such that

$$(5.1) \quad L_{par} = L_{par}[x_1] := (\mathbb{P}_{\sin x_1} \cup \mathbb{P}_{-\sin x_1}) = \Theta(\{x = \pm x_1, z = 0\}).$$

We define

$$(5.2) \quad L := L[x_1, m] = L_{mer}[m] \cap L_{par}[x_1] = \mathcal{G}_{S_{eq}^2, m} p_1,$$

where $p_1 := \Theta(x_1, 0, 0)$. Clearly L consists of $2m$ points, m of them at latitude x_1 , and the other m at latitude $-x_1$. We define (recall 3.6)

$$(5.3) \quad \delta_p := \delta_1 := \frac{1}{9m} \cos x_{root} \quad (p \in L),$$

where we assume from now on

$$(5.4) \quad x_1 \in (x_{balanced}/2, (x_{root} + x_{balanced})/2),$$

where x_{root} was defined in 2.19 and $x_{balanced}$ will be defined in 5.12. 5.4 ensures that the condition in 3.6 is satisfied. Clearly $\mathcal{K}_{sym}[L]$ is two-dimensional and spanned by $W, W' \in \mathcal{K}_{sym}[L]$, both of which are supported on $D_L(2\delta_1) \setminus D_L(\delta_1)$, and are defined by requesting that on $D_{p_1}(2\delta_1)$ we have

$$(5.5) \quad \begin{aligned} W &= W[x_1, m] := \mathcal{L}' \Psi [2\delta_1, \delta_1; \mathbf{d}_{p_1}] (G_{p_1}, \log \delta_1 \cos \circ \mathbf{d}_{p_1}), \\ W' &= W'[x_1, m] := \mathcal{L}' \Psi [2\delta_1, \delta_1; \mathbf{d}_{p_1}] (0, u), \end{aligned}$$

where u is the first harmonic on S_{eq}^2 characterized by $u(p_1) = 0$ and $d_{p_1}u = d_{p_1}x$. Because of the symmetries we only consider constant $\tau : L \rightarrow \mathbb{R}$.

Definition 5.6. We define an LD solution

$$\Phi = \Phi[x_1, m] := \varphi[L[x_1, m], 1] \in C_{sym}^\infty(S_{eq}^2 \setminus L),$$

and $V = V[x_1, m], V' = V'[x_1, m] \in \widehat{\mathcal{K}}[L]$ by $\mathcal{L}'V = W$ and $\mathcal{L}'V' = W'$ (recall 3.10, 3.9, and 5.5).

Clearly $\widehat{\mathcal{K}}_{sym}[L]$ is spanned by V and V' .

The rotationally invariant part $\phi := \Phi_{avg}$. To help with the presentation we first introduce some notation.

NOTATION 5.7. If a function u is defined on a neighborhood of L_{par} and has one-sided partial derivatives at $x = x_1$, then we use the notation (so that if u is C^1 then $\partial_{1+}u + \partial_{1-}u = 0$)

$$\partial_{1+}u := \left. \frac{\partial u}{\partial x} \right|_{x=x_1+}, \quad \partial_{1-}u := - \left. \frac{\partial u}{\partial x} \right|_{x=x_1-}. \quad \square$$

Lemma 5.8. For x_1 as in 5.4 we have that $\phi := \Phi_{avg}[x_1, m]$ is given by (recall 2.18 and 2.21)

$$\phi = \begin{cases} \frac{\phi_1}{\phi_{even}(x_1)} \phi_{even} & \text{on } \{|x| \leq x_1\} \subset \mathbb{S}_{eq}^2, \\ \frac{\phi_1}{\phi_{odd}(x_1)} \phi_{odd} & \text{on } \{x_1 \leq x\} \subset \mathbb{S}_{eq}^2, \end{cases}$$

where $\phi_1 = \frac{m}{\cos x_1 (h_{1+} + h_{1-})}$,

$$h_{1+} := \frac{1}{\phi_{odd}(x_1)} \frac{\partial \phi_{odd}}{\partial x}(x_1) = \frac{1}{\phi_1} \partial_{1+} \phi > 0,$$

$$h_{1-} := -\frac{1}{\phi_{even}(x_1)} \frac{\partial \phi_{even}}{\partial x}(x_1) = \frac{1}{\phi_1} \partial_{1-} \phi > 0.$$

Moreover, we have $\phi \geq \phi_1 > 0$ on \mathbb{S}_{eq}^2 .

Proof. To simplify the notation for this proof we define domains of \mathbb{S}_{eq}^2 , $\Omega_N := \{x_1 \leq x\}$ and $\Omega_{eq} := \{|x| \leq x_1\}$, which are neighborhoods of the North pole and the equator respectively. Because of the symmetries it is clear that $\phi = A_+ \phi_{odd}$ on Ω_N and $\phi = A_- \phi_{even}$ on Ω_{eq} for some constants A_+ and A_- . Because of the continuity of ϕ at $P := \Omega_N \cap \Omega_{eq} = \mathbb{P}_{\sin x_1}$ by 3.10.ii we have

$$A_- \phi_{even}(x_1) = A_+ \phi_{odd}(x_1).$$

For $0 < \epsilon_1 \ll \epsilon_2$ we consider now the domain $\Omega_{\epsilon_1, \epsilon_2} := D_P(\epsilon_2) \setminus D_L(\epsilon_1)$. By integrating $\mathcal{L}'\Phi = 0$ on $\Omega_{\epsilon_1, \epsilon_2}$ and integrating by parts we obtain

$$\int_{\partial\Omega_{\epsilon_1, \epsilon_2}} \frac{\partial}{\partial \eta} \Phi + 2 \int_{\Omega_{\epsilon_1, \epsilon_2}} \Phi = 0.$$

By taking the limit as $\epsilon_1 \rightarrow 0$ first and then as $\epsilon_2 \rightarrow 0$ we obtain using the logarithmic behavior near L that

$$2\pi m = 2\pi \cos x_1 (A_+ \phi_{odd}(x_1) h_{1+} + A_- \phi_{even}(x_1) h_{1-}).$$

Solving the system of the two equations for A_{\pm} and using the monotonicity of ϕ_{even} and ϕ_{odd} (recall 2.19) we conclude the proof. q.e.d.

Motivated by the above lemma we have the following definition. Note for later applications that $\underline{\phi}$ depends linearly on $(\tilde{a}, \tilde{b}) \in \mathbb{R}^2$ and \underline{j} on $\tilde{b} \in \mathbb{R}$:

Definition 5.9. Given $\tilde{a}, \tilde{b} \in \mathbb{R}$ we define

$$\underline{\phi} = \underline{\phi}[\tilde{a}, \tilde{b}; x_1] \in C_x^\infty(\{x \in [0, \pi/2)\}) \cap C_{|x|}^0(\mathbb{S}_{eq}^2 \setminus \{p_N, p_S\}),$$

$$\underline{j} = \underline{j}[\tilde{b}; x_1] \in C_x^\infty(\{x \in [x_1, \pi/2)\}) \cap C_x^\infty(\{x \in [0, x_1]\})$$

$$\cap C_{|x|}^0(\mathbb{S}_{eq}^2 \setminus \{p_N, p_S\}),$$

by requesting they satisfy the initial data

$$\begin{aligned} \underline{\phi}(x_1) &= \tilde{a}, & \left. \frac{\partial \underline{\phi}}{\partial \underline{x}} \right|_{x=x_1} &= \frac{1}{m} \frac{\partial \underline{\phi}}{\partial x}(x_1) = \tilde{b}, \\ \underline{j}(x_1) &= 0, & \partial_{1+} \underline{j} = \partial_{1-} \underline{j} &= m\tilde{b}, \end{aligned}$$

and the ODEs $\mathcal{L}'\underline{\phi} = 0$ on $\{x \in [0, \pi/2)\}$

and $\mathcal{L}'\underline{j} = 0$ on $\{x \in [x_1, \pi/2)\} \subset \mathbb{S}_{eq}^2$ and on $\{x \in [0, x_1]\} \subset \mathbb{S}_{eq}^2$.

To simplify the presentation we define also the function $\widehat{h} : (0, \pi/2) \rightarrow \mathbb{R}$ by

$$(5.10) \quad \widehat{h}(x) := \frac{1}{2 \cos x} \frac{\frac{1}{\phi_{odd}(x)} \frac{\partial \phi_{odd}}{\partial x}(x) + \frac{1}{\phi_{even}(x)} \frac{\partial \phi_{even}}{\partial x}(x)}{\frac{1}{\phi_{odd}(x)} \frac{\partial \phi_{odd}}{\partial x}(x) - \frac{1}{\phi_{even}(x)} \frac{\partial \phi_{even}}{\partial x}(x)}.$$

Corollary 5.11. *On $\{x \in (-x_1, \pi/2)\}$ we have that*

$$\phi := \Phi_{avg}[x_1, m] = \underline{\phi}[\phi_1, \widehat{h}(x_1); x_1] + \underline{j}\left[\frac{1}{2 \cos x_1}; x_1\right].$$

Proof. By 5.8 on $\mathbb{P}_{\sin x_1}$ we have

$$\phi = \phi_1, \quad \partial_{1+} \phi = \phi_1 h_{1+}, \quad \partial_{1-} \phi = \phi_1 h_{1-}.$$

By 5.9 the corresponding initial data for the right hand side are

$$\phi_1, \quad m \left(\widehat{h}(x_1) + \frac{1}{2 \cos x_1} \right), \quad m \left(-\widehat{h}(x_1) + \frac{1}{2 \cos x_1} \right).$$

Using the definitions we calculate that

$$\widehat{h}(x_1) = \frac{1}{2m}(h_{1+} - h_{1-})\phi_1, \quad \frac{1}{2 \cos x_1} = \frac{1}{2m}(h_{1+} + h_{1-})\phi_1.$$

This implies that the initial data are the same for both sides and, therefore, the corollary follows by the uniqueness of ODE solutions. q.e.d.

The following is important for horizontal balancing considerations.

Lemma 5.12. $\frac{d\widehat{h}}{dx} < 0$ on $(0, x_{root})$. \widehat{h} has a unique root in $(0, x_{root})$ which we will denote by $x_{balanced}$.

Proof. By direct calculation using 2.18 and 2.19 we have

$$\begin{aligned} 2\widehat{h}(x) &= \cos x - \frac{\sin^2 x}{\cos x} - \sin 2x \log \frac{1 + \sin x}{\cos x}, \\ \lim_{x \rightarrow 0^+} \widehat{h}(x) &= 1/2, & \widehat{h}(x_{root}) &= -\frac{\cos x_{root}}{2} - \frac{\sin^2 x_{root}}{2 \cos x_{root}} < 0, \\ 2\frac{d\widehat{h}}{dx} &= -5 \sin x - \frac{\sin^3 x}{\cos^2 x} - 2 \log \frac{1 + \sin x}{\cos x} + 4 \sin^2 x \log \frac{1 + \sin x}{\cos x}. \end{aligned}$$

We clearly have $\log \frac{1+\sin x}{\cos x} > 0$ and by 2.18 and 2.19 $\sin x \log \frac{1+\sin x}{\cos x} < 1$ on $(0, x_{root})$. It follows that on $(0, x_{root})$ $2\frac{d\widehat{h}}{dx} < -\sin x < 0$ which allows us to complete the proof. q.e.d.

It will be very important that we often work with solutions which are “almost” symmetric with respect to reflection in the x coordinate across x_1 in a sense made precise later. In this spirit we define an “antisymmetrization” \mathcal{A}_{x_1} as follows.

Definition 5.13. We define $\Omega_1 = \Omega_1[x_1, m] := D_{L_{par}[x_1]}(3/m)$ and given $u \in C_{sym}^0(\Omega_1[x_1, m])$ we define a function $\mathcal{A}_{x_1}u \in C_{sym}^0(\Omega_1[x_1, m])$ by requesting that for $x' \in (-3/m, 3/m)$ and $y \in \mathbb{R}$ we have

$$\mathcal{A}_{x_1}u(x_1 + x', y) = u(x_1 + x', y) - u(x_1 - x', y).$$

Lemma 5.14. *The following estimates hold.*

- (i). $\|\underline{\phi}[1, 0; x_1] - 1 : C_{sym}^2(\Omega_1[x_1, m], \tilde{g})\| \leq C/m^2$.
- (ii). $\|\underline{j}[1; x_1] - m|x| - x_1| : C_{sym}^2(\Omega_1[x_1, m] \setminus L_{par}[x_1], \tilde{g})\| \leq C/m$.
- (iii). $\|\mathcal{A}_{x_1}\underline{\phi}[1, 0; x_1] : C_{sym}^2(\Omega_1[x_1, m], \tilde{g})\| \leq C/m^2$.
- (iv). $\|\mathcal{A}_{x_1}\underline{j}[1; x_1] : C_{sym}^2(\Omega_1[x_1, m] \setminus L_{par}[x_1], \tilde{g})\| \leq C/m$.
- (v). $\|\underline{\phi}[0, 1; x_1] - m(|x| - x_1) : C_{sym}^2(\Omega_1[x_1, m], \tilde{g})\| \leq C/m$.

Proof. Let $\underline{\phi}_1 := \underline{\phi}[1, 0; x_1]$. $\underline{\phi}_1$ satisfies the ODE $\mathcal{L}'\underline{\phi}_1 = 0$ which in the notation of 5.22 amounts to

$$\partial_{\tilde{x}}^2 \underline{\phi}_1 - m^{-1} \tan(x_1 + m^{-1}\tilde{x}) \partial_{\tilde{x}} \underline{\phi}_1 + 2m^{-2} \underline{\phi}_1 = 0.$$

Consider Ω , the subset of $\Omega_1[x_1, m]$ where $|\underline{\phi}_1 - 1| < 1/2$ and $|\partial_{\tilde{x}} \underline{\phi}_1| < 1/m$. Using the equation we have $|\partial_{\tilde{x}}^2 \underline{\phi}_1| \leq C/m^2$ on that subset, and, therefore, we obtain a contradiction unless it is the whole $\Omega_1[x_1, m]$. This proves (i). The proof of (ii) is similar (note that $m \partial_{\tilde{x}} |x| - x_1 = \pm 1$). (iii) follows then from (i) and (iv) from (ii). (v) is equivalent to (ii). q.e.d.

Estimates on $\Phi = \Phi[x_1, m]$. Lemma 5.8 provides explicit information on Φ_{avg} . We need to estimate Φ_{osc} also. To this end we introduce a new decomposition $\Phi = \widehat{G} + \Phi''$ as follows.

Definition 5.15. We define $\widehat{G} \in C_{sym}^\infty(\mathbb{S}_{eq}^2 \setminus L)$ by requesting that

$$\widehat{G} := \{ \Psi[2\delta_1, 3\delta_1; I_{\mathbb{R}^+}](G, A_1) \} \circ \mathbf{d}_L, \quad \text{where } A_1 := \log \delta_1.$$

Observe that by 2.20 for each $p \in L$ we have on $D_p(2\delta_1)$ that $\widehat{G} = G_p$. This, 3.10, 3.1.ii, and the definition of Φ in 5.6, imply that $\Phi - \widehat{G}$ can be extended smoothly across L . This allows us to have the following.

Definition 5.16. We define $\Phi'', E'' \in C_{sym}^\infty(\mathbb{S}_{eq}^2)$ by requesting that on $\mathbb{S}_{eq}^2 \setminus L$

$$\Phi = \widehat{G} + \Phi'', \quad E'' := -\mathcal{L}'\widehat{G}.$$

Note that by 5.15 and 2.20 E'' vanishes on $D_L(2\delta_1)$. By 2.21 we have

$$(5.17) \quad \begin{aligned} & \Phi''_{avg}, \Phi''_{osc}, E''_{avg}, E''_{osc} \in C_{sym}^\infty(\mathbb{S}_{eq}^2), \\ & \text{and on } \mathbb{S}_{eq}^2 \quad \Phi'' = \Phi''_{avg} + \Phi''_{osc}, \quad E'' = E''_{avg} + E''_{osc}. \end{aligned}$$

Since $\mathcal{L}'\Phi$ vanishes by 3.10 and 5.6, and \mathcal{L}' is rotationally covariant, we conclude from 5.16 and 5.17 that

$$(5.18) \quad \mathcal{L}'\Phi'' = E'', \quad \mathcal{L}'\Phi''_{avg} = E''_{avg}, \quad \mathcal{L}'\Phi''_{osc} = E''_{osc} \quad \text{on } \mathbb{S}_{eq}^2.$$

Moreover, since $\Phi_{avg} \in C^0(\mathbb{S}_{eq}^2)$ by 3.10 and 5.6, and $\Phi''_{avg} \in C^\infty(\mathbb{S}_{eq}^2)$ by 5.17, we conclude by 5.16 that

$$(5.19) \quad \begin{aligned} \widehat{G}_{avg} \in C^0(\mathbb{S}_{eq}^2), \quad \Phi''_{avg} = \Phi_{avg} - \widehat{G}_{avg} = \phi - \widehat{G}_{avg} \quad \text{on } \mathbb{S}_{eq}^2, \\ \Phi = \phi + \widehat{G}_{osc} + \Phi''_{osc} \quad \text{on } \mathbb{S}_{eq}^2 \setminus L. \end{aligned}$$

Using 5.17 we conclude that

$$(5.20) \quad \Phi'' = \phi - \widehat{G}_{avg} + \Phi''_{osc} \quad \text{on } \mathbb{S}_{eq}^2.$$

Note that in this expression although ϕ and \widehat{G}_{avg} are not smooth because of a derivative jump at $x = x_1$, we do have $\phi - \widehat{G}_{avg} = \Phi''_{avg} \in C^\infty_{sym}(\mathbb{S}_{eq}^2)$ because the derivative jumps cancel out. Note also that neither $\mathcal{L}'\Phi''_{avg}$ nor $\mathcal{L}'\Phi''_{osc}$ have to vanish on $D_L(2\delta_1)$ but their sum does.

ϕ is known explicitly by 5.8. We need to estimate \widehat{G}_{avg} and Φ''_{osc} . \widehat{G} is almost explicit and Φ''_{osc} can be estimated by estimating E''_{osc} and using 5.18. We first estimate E'' as follows. Note that $(\mathcal{A}_{x_1}E'')_{osc} = \mathcal{A}_{x_1}(E''_{osc})$ by the definitions.

Lemma 5.21. *The following hold (recall 5.4 and 2.15).*

- (i). $\|\widehat{G} - A_1 : C^k_{sym}(\mathbb{S}_{eq}^2 \setminus D_L(\delta_1), \tilde{g})\| \leq C(k)$ and $\widehat{G} - A_1$ vanishes on $\mathbb{S}_{eq}^2 \setminus D_L(3\delta_1)$.
- (ii). $\|m^{-2}E'' : C^k_{sym}(\mathbb{S}_{eq}^2, \tilde{g})\| \leq C(k)$, $\|m^{-2}E''_{avg} : C^k_{sym}(\mathbb{S}_{eq}^2, \tilde{g})\| \leq C(k)$, and E'' vanishes on $D_L(2\delta_1)$.
- (iii). $\|m^{-2}E''_{osc} : C^k_{sym}(\mathbb{S}_{eq}^2, \tilde{g})\| \leq C(k)$ and E''_{osc} is supported on $D_{Lpar}(3\delta_1) \subset \Omega_1[x_1, m]$.
- (iv). $\|m^{-2}\mathcal{A}_{x_1}E'' : C^k_{sym}(\Omega_1[x_1, m], \tilde{g})\| \leq C(k)/m$ and $\|m^{-2}\mathcal{A}_{x_1}E''_{avg} : C^k_{sym}(\Omega_1[x_1, m], \tilde{g})\| \leq C(k)/m$.
- (v). $\|m^{-2}\mathcal{A}_{x_1}E''_{osc} : C^k_{sym}(\Omega_1[x_1, m], \tilde{g})\| \leq C(k)/m$ and $\mathcal{A}_{x_1}E''_{osc}$ is supported on $D_{Lpar}(3\delta_1)$.

Proof. (i) follows from 2.20.vi and the definitions. By 2.16 and 5.16 we have

$$m^{-2}E'' = -\mathcal{L}'_{\tilde{g}}\widehat{G} = -\mathcal{L}'_{\tilde{g}}(\widehat{G} - A_1) - 2m^{-2}A_1.$$

As mentioned earlier E'' vanishes on $D_L(2\delta_1)$ and clearly $-1 < m^{-2}A_1 < 0$ by 5.3. (ii) follows then from (i). The second part of (iii) follows from (i) which implies that $\widehat{G} = A_1$ on $\mathbb{S}_{eq}^2 \setminus D_L(3\delta_1)$. The first part of (iii) follows then from (ii) and the second part.

Recall now the coordinates defined in 2.15 and define a new coordinate $\widehat{x} := \tilde{x} - mx_1$. The metric \tilde{g} then in coordinates (\widehat{x}, \tilde{y}) is equal to

the metric

$$(5.22) \quad \widehat{g}_t := d\widehat{x}^2 + \cos^2(x_1 + t\widehat{x}) d\widehat{y}^2,$$

with $t = 1/m$. \widehat{g}_t clearly depends smoothly on t for $|t|$ small, and for $t = 0$ is the Euclidean metric with $(\widehat{x}, \widehat{y})$ the standard coordinates. This implies that $\mathbf{d}_{(0,0)}$ as a function of $(t, \widehat{x}, \widehat{y})$ is smooth for small $|t|$ and bounded $(\widehat{x}, \widehat{y})$ independently of m . Since $\mathcal{A}_{x_1} \mathbf{d}_{(0,0)}$ clearly vanishes for $t = 0$ we conclude that $\|\mathcal{A}_{x_1} \mathbf{d}_L : C_{sym}^k(D_L(3\delta_1) \setminus D_L(2\delta_1), \widetilde{g})\| \leq C(k)/m$.

Note now that $\mathcal{A}_{x_1} E''$ is supported on $D_L(3\delta_1) \setminus D_L(2\delta_1)$. By 5.15 and 5.16 E'' factors through \mathbf{d}_L . (iv) follows then from (ii) (with a loss of one derivative) by the estimate on $\mathcal{A}_{x_1} \mathbf{d}_L$ above. By averaging over the circles then and subtracting we obtain the estimate in (v). The statement on the support follows from the support of E''_{osc} in (iii). q.e.d.

Lemma 5.23. *Given $E \in C_{sym}^{0,\beta}(\mathbb{S}_{eq}^2)$ with $E_{avg} \equiv 0$ and E supported on $\Omega_1[x_1, m]$, there is a unique $v \in C_{sym}^{2,\beta}(\mathbb{S}_{eq}^2)$ characterized by (i) below and satisfying the following.*

- (i). $\mathcal{L}'v = E$, or equivalently $\mathcal{L}'_{\widetilde{g}} v = m^{-2}E$.
- (ii). $v_{avg} = 0$.
- (iii). $\|v : C_{sym}^{2,\beta}(\mathbb{S}_{eq}^2, \widetilde{g}, f_{\mathbb{S}_{eq}^2, x_1}^2)\| \leq C\|m^{-2}E : C_{sym}^{0,\beta}(\Omega_1[x_1, m], \widetilde{g})\|$ (recall 1.2), where we have $f_{\mathbb{S}_{eq}^2, x_1}^2 := e^{-c_1 m \min(|x| - x_1, c_2)}$ for some absolute constants $c_1, c_2 > 0$.
- (iv). $\|\mathcal{A}_{x_1} v : C_{sym}^{2,\beta}(\Omega_1[x_1, m], \widetilde{g})\| \leq C$
 $\left(m^{-1}\|m^{-2}E : C_{sym}^{0,\beta}(\Omega_1[x_1, m], \widetilde{g})\| \right.$
 $\left. + \|m^{-2}\mathcal{A}_{x_1} E : C_{sym}^{0,\beta}(\Omega_1[x_1, m], \widetilde{g})\| \right)$.

Proof. The existence and uniqueness of v is clear by the symmetries. (ii) follows also because $\mathcal{L}'v_{avg} = E_{avg} = 0$.

For (iii) observe first that $\mathbb{S}_{eq}^2 \setminus \{p_N, p_S\}$ equipped with the metric (recall 2.5)

$$\chi := m^2 \cos^{-2} x g = m^2 \cos^{-2} x dx^2 + m^2 dy^2,$$

can be isometrically identified with the cylinder $\mathbb{R} \times \mathbb{S}^1$ equipped with the metric $\chi = ds^2 + d\theta^2$, where (s, θ) (with θ defined modulo $2\pi m$) denotes the standard coordinates of the cylinder. Under this identification we can assume that s is an odd function of x and $\theta = my = \widetilde{y}$. By (ii) $u(p_N) = u(p_S) = 0$ and on the cylinder the equation (i) is equivalent to

$$(\Delta_\chi + 2m^{-2} \cos^2 x) v = m^{-2} \cos^2 x E.$$

Because of the symmetries we can work with θ modulo 2π instead of $2\pi m$. Let v' be the solution on the cylinder of

$$\Delta_\chi v' = m^{-2} \cos^2 x E,$$

subject to the condition $v' \rightarrow 0$ as $s \rightarrow \pm\infty$. By standard theory and separation of variables, and using also 5.4 to ensure the uniform equivalence of χ and \tilde{g} on the support of E , we have

$$\|v' : C_{sym}^{2,\beta}(\mathbb{R} \times \mathbb{S}^1, \chi, e^{-|s(x)-s(x_1)|/2})\| \leq C \|m^{-2}E : C_{sym}^{0,\beta}(\Omega_1[x_1, m], \tilde{g})\|.$$

$v - v'$ now satisfies the equation

$$(\Delta_\chi + 2m^{-2} \cos^2 x) (v - v') = -2m^{-2} \cos^2 x v'.$$

Note that v'_{avg} clearly vanishes. Using the smallness of the perturbation introduced by the coefficient $2m^{-2} \cos^2 x$, and the estimate on v' above, we conclude that

$$\begin{aligned} \|v - v' : C_{sym}^{2,\beta}(\mathbb{R} \times \mathbb{S}^1, \chi, e^{-|s(x)-s(x_1)|/2})\| &\leq \\ &\leq C m^{-2} \|m^{-2}E : C_{sym}^{0,\beta}(\Omega_1[x_1, m], \tilde{g})\|. \end{aligned}$$

We have then

$$\|v : C_{sym}^{2,\beta}(\mathbb{R} \times \mathbb{S}^1, \chi, e^{-|s(x)-s(x_1)|/2})\| \leq C \|m^{-2}E : C_{sym}^{0,\beta}(\Omega_1[x_1, m], \tilde{g})\|.$$

By choosing now c_1 and c_2 appropriately (iii) follows easily.

To prove (iv) recall first that we are working on the cylinder $\mathbb{R} \times \mathbb{S}^1$ equipped with the metric $\chi = ds^2 + d\theta^2$, where (s, θ) denotes the standard coordinates on the cylinder with θ defined modulo 2π . $\mathbb{S}_{eq}^2 \setminus \{p_N, p_S\}$ is then an m to 1 covering of the cylinder, and the coordinate x can be considered as a function on the cylinder as well with

$$\frac{ds}{dx} = \frac{m}{\cos x}.$$

We define following 1.7 $s_- : \mathbb{R} \rightarrow \mathbb{R}$ by

$$s_-(x) := \Psi[3/m, 6/m, |x - x_1|] (s(2x_1 - x), 2s(x_1) - s(x)).$$

For $u \in C^0(\mathbb{R} \times \mathbb{S}^1)$ we define $Au \in C^0(\Omega_1[x_1, m])$ by

$$Au(s, \theta) := u(s, \theta) - u(s_-(s), \theta).$$

Note that Au agrees on $\Omega_1 \cap \{x > 0\}$ with $\mathcal{A}_{x_1}u$ defined as in 5.13. We define now $v'_\pm, E'_\pm \in C^0(\mathbb{R} \times \mathbb{S}^1)$ by requesting

$$\Delta_\chi v'_\pm = m^{-2} \cos^2 x E'_\pm, \quad E = E^+ + E_-, \quad v = v_+ + v_-,$$

where $E = E_+$ on $\{s > 0\}$ and $v' \rightarrow 0$ as $s \rightarrow \pm\infty$. We have then that $v_+(s, \theta) \equiv v_-(-s, \theta)$ and

$$\Delta_\chi \mathcal{A}v'_+ = \mathcal{A}(m^{-2} \cos^2 x E'_+) + [\Delta_\chi, \mathcal{A}]v'_+.$$

Using the definitions it is clear that $\frac{ds_-}{ds} + 1$ and both terms on the right of the equation are supported on $\{|x - x_1| < 6/m\}$. Moreover, by an easy calculation

$$\left\| \frac{ds_-}{ds} + 1 : C^3 \right\| \leq C/m.$$

Estimating first v'_+ and then $\mathcal{A}v'_+$ (both with exponential decay $e^{-|s-s(x_1)|}$), we conclude that (iv) is valid with v replaced by v'_+ . Since $s(x_{balanced}) > mx_{balanced}$ we have $e^{-s(x_{balanced})} \ll m^{-2}$ and, therefore, we conclude that (iv) is valid with v replaced by v' . Combining with the earlier estimate for $v - v'$ we conclude the proof. q.e.d.

Lemma 5.24. Φ''_{osc} satisfies the following estimates.

- (i). $\|\Phi''_{osc} : C^{2,\beta}_{sym}(\mathbb{S}^2_{eq}, \tilde{g}, f_{\mathbb{S}^2_{eq},x_1})\| \leq C$, where $f_{\mathbb{S}^2_{eq},x_1}$ is as in 5.23.iii.
- (ii). $\|\mathcal{A}_{x_1}\Phi''_{osc} : C^{2,\beta}_{sym}(\Omega_1[x_1, m], \tilde{g})\| \leq C/m$ (recall 5.13).

Proof. Since $\mathcal{L}'\Phi''_{osc} = E''_{osc}$ by 5.18 we can use the estimates of 5.21 to conclude the proof by appealing to 5.23. q.e.d.

It helps with the presentation of our estimates to introduce one more decomposition which holds in the vicinity of L :

Definition 5.25. We define $\Phi' \in C^\infty_{sym}(\Omega_1[x_1, m])$ by requesting that (recall 5.9)

$$\Phi'' = \Phi' + \underline{\phi}[\phi_1 - A_1, \widehat{h}(x_1); x_1] \quad \text{on} \quad \Omega_1[x_1, m].$$

Using 5.9 we have then for $p \in L \cap \Omega_1[x_1, m]$

$$(5.26) \quad \Phi''(p) = \Phi'(p) + \phi_1 - A_1, \quad d_p\Phi'' = d_p\Phi' + \widehat{h}(x_1) d\tilde{x}.$$

Lemma 5.27. The following estimates hold.

- (i). $\|\Phi' : C^{2,\beta}_{sym}(\Omega_1[x_1, m], \tilde{g})\| \leq C$.
- (ii). $\|\mathcal{A}_{x_1}\Phi' : C^{2,\beta}_{sym}(\Omega_1[x_1, m], \tilde{g})\| \leq C/m$.

Proof. By combining 5.19, 5.25, and 5.11, we conclude

$$\Phi'_{avg} = \underline{\phi}[A_1, 0; x_1] + \underline{j}[\frac{1}{2\cos x_1}; x_1] - \widehat{G}_{avg}, \quad \Phi'_{osc} = \Phi''_{osc}.$$

Note that the discontinuities on the right hand side of the first equation cancel and the left hand side is smooth. Moreover, by 5.25 and 5.18 we have $\mathcal{L}'\Phi'_{avg} = E''_{avg}$ which in the notation of 5.22 amounts to the ODE

$$\partial_{\widehat{x}}^2 \Phi'_{avg} - m^{-1} \tan(x_1 + m^{-1}\widehat{x}) \partial_{\widehat{x}} \Phi'_{avg} + 2m^{-2} \Phi'_{avg} = m^{-2} E''_{avg}.$$

Using then 5.14 and that $\widehat{G}_{avg} = A_1$ on $\mathbb{S}^2_{eq} \setminus D_{Lpar}(3\delta_1)$ by 5.21.i, we conclude that at $\partial\Omega_1[x_1, m]$ we have $|\Phi'_{avg}| \leq C$ and $|\partial_{\widehat{x}} \Phi'_{avg}| \leq C$. Using these as initial data for the ODE and 5.21.ii we estimate Φ'_{avg} . By this estimate together with 5.24.i we conclude (i).

To prove (ii) it is enough to prove the estimate for $U := \mathcal{A}_{x_1}\Phi'_{avg}$ instead of $\mathcal{A}_{x_1}\Phi'$ because $\mathcal{A}_{x_1}\Phi'_{osc} = \mathcal{A}_{x_1}\Phi''_{osc}$ satisfies the estimate by 5.24.ii. To estimate U we calculate that it satisfies

$$\begin{aligned} &\partial_{\widehat{x}}^2 U - m^{-1} \tan(x_1 - m^{-1}\widehat{x}) \partial_{\widehat{x}} U + 2m^{-2} U + \\ &+ m^{-1} (\tan(x_1 - m^{-1}\widehat{x}) - \tan(x_1 + m^{-1}\widehat{x})) \partial_{\widehat{x}} \Phi'_{avg} = m^{-2} \mathcal{A}_{x_1} E''. \end{aligned}$$

Using 5.14 we obtain estimates for the initial data for U on $\partial\Omega_1[x_1, m]$. These estimates and 5.21.iv imply the required estimate. q.e.d.

Lemma 5.28. (i). $\|\Phi'' : C^{k,\beta}_{sym}(\mathbb{S}^2_{eq}, \tilde{g})\| \leq C(k)m$.
 (ii). $C'm \leq \frac{8}{9}\phi_1 \leq \Phi''$ on \mathbb{S}^2_{eq} .

Proof. By 5.25, 5.27.i, and 5.14, we have that $C'm \leq \frac{8}{9}\phi_1 \leq \Phi'' \leq Cm$ on $\Omega_1[x_1, m]$. By 5.20 and 5.21.i we have on $\mathbb{S}^2_{eq} \setminus D_L(3\delta_1)$ that $\Phi'' = \phi - A_1 + \Phi''_{osc}$. By 5.24.i and 5.8 we conclude that $C'm \leq \frac{8}{9}\phi_1 \leq \Phi'' \leq Cm$ on $\mathbb{S}^2_{eq} \setminus D_L(3\delta_1)$. Since $\Omega_1[x_1, m]$ and $\mathbb{S}^2_{eq} \setminus D_L(3\delta_1)$ cover \mathbb{S}^2_{eq} the proof of (ii) is complete and (i) follows by standard interior regularity theory and 5.21.ii. q.e.d.

Estimates on V, V', W, W' .

Lemma 5.29. $V, V' \in C^\infty_{sym}(\mathbb{S}^2_{eq})$ defined as in 5.6 satisfy the following.

- (i). On $\mathbb{S}^2_{eq} \setminus D_L(2\delta_1)$ we have $V = \Phi$ and $V' = 0$.
- (ii). With p_1 as in 5.2 and u as in 5.5 we have that on $D_{p_1}(2\delta_1)$ the following hold.

$$V = \Psi [2\delta_1, \delta_1; \mathbf{d}_{p_1}] (G_{p_1}, \log \delta_1 \cos \circ \mathbf{d}_{p_1}) + \Phi'',$$

$$V' = \Psi [2\delta_1, \delta_1; \mathbf{d}_{p_1}] (0, u).$$

- (iii). $V(p_1) = \phi_1 + \Phi'(p_1) \sim_C m$, $\frac{\partial V}{\partial \tilde{x}}(p_1) = \hat{h}(x_1) + \frac{\partial \Phi'}{\partial \tilde{x}}(p_1) \sim_C 1$
 and $\left| \frac{\partial V}{\partial \tilde{x}}(p_1) \right| \leq C$, for some absolute constant $C > 1$. We also

have $V'(p_1) = 0$, $\frac{\partial V'}{\partial \tilde{x}}(p_1) = m^{-1}$.

- (iv). $\|V : C^k_{sym}(\mathbb{S}^2_{eq}, \tilde{g})\| \leq C(k)m$ and $\|V' : C^k_{sym}(\mathbb{S}^2_{eq}, \tilde{g})\| \leq C(k)/m$.
- (v). 3.12 holds and $\|\mathcal{E}_L^{-1}\| \leq C m^{2+\beta}$ (recall 3.13).
- (vi). For $\mu, \mu' \in \mathbb{R}$ we have that $|\mu|m + |\mu'| \leq C \|\mu W + \mu' W' : C^{0,\beta}_{sym}(\mathbb{S}^2_{eq}, g)\|$.

Proof. Let V_{new} and V'_{new} be defined by the expressions for V and V' in (i) and (ii). To establish (i) and (ii) we need to prove that $V = V_{new}$ and $V' = V'_{new}$. Note first that the expressions for V_{new} and V'_{new} in (i) and (ii) match because by 5.15 and 5.16 $V_{new} = \Phi$ and $V'_{new} = 0$ on a neighborhood of $\partial D_{p_1}(2\delta_1)$. Since $\mathcal{L}'\Phi = 0$ by 5.6, we conclude that by 5.5 we have that $\mathcal{L}'V_{new} = W$ and $\mathcal{L}'V'_{new} = W'$, which characterize V and V' defined as in 5.6. This completes the proof of (i) and (ii). The equalities in (iii) follow from (ii) by using 5.26, the definition of u in 5.5, and the definition of A_1 in 5.15. Clearly by 5.8 and 5.4 we have $C'm < \phi_1 < Cm$. Because of 5.4 we have also that $|\hat{h}(x_1)| < C$. The proof of (iii) is completed then by using 5.27.

(iv) is implied by (i) and (ii) by using 5.28 and 5.21.i. By direct calculation using (iii) we have (recall 3.11)

$$\begin{aligned} \mathcal{E}_L^{-1}((1, 0)_{p \in L}) &= (\phi_1 + \Phi'(p_1))^{-1} \left\{ V - (\widehat{h}(x_1) + \frac{\partial \Phi'}{\partial \tilde{x}}(p_1))mV' \right\}, \\ \mathcal{E}_L^{-1}((0, d_p|\tilde{x}|)_{p \in L}) &= mV'. \end{aligned}$$

(v) follows then by using (iii) and (iv).

We have $|\mu|m + |\mu'| \leq C \|\mu V + \mu'V' : C_{sym}^{2,\beta}(\mathbb{S}_{eq}^2, g)\| \leq C \|\mu W + \mu'W' : C_{sym}^{0,\beta}(\mathbb{S}_{eq}^2, g)\|$, where the first inequality follows easily from (iii) and the second inequality from the symmetries and $\mathcal{L}(\mu V + \mu'V') = \mu W + \mu'W'$. (vi) follows then and the proof is complete. q.e.d.

The family of MLD solutions. We determine now the family of MLD solutions we need. The parameters of the family are $\zeta = (\zeta, \zeta') \in \mathbb{R}^2$ and their range is specified by

$$(5.30) \quad |\zeta|, |\zeta'| \leq \underline{c}_1,$$

where $\underline{c}_1 > 1$ is a constant independent of m and τ which will be specified later. We want to construct MLD solutions $\varphi[L, \tau, w]$ where $L = L[x_1[\zeta, m], m]$, $\tau = \tau[\zeta, m]$, and $w = w[\zeta, m]$, that is the parameters of φ are functions of ζ and m . Clearly then $\varphi = \tau\Phi + \mu V + \mu'V'$ where $w = \mu W + \mu'W'$ (see also 5.34 below). Note that by 3.1.ii, 5.15, 5.16, and 5.34, we have that

$$(5.31) \quad \forall p \in L \quad \widehat{\varphi}_p = \tau\Phi'' + \mu V + \mu'V' \quad \text{on} \quad D_p(2\delta_1).$$

The matching condition in 3.4 amounts to a system of two equations with x_1 and τ as the unknowns where we assume μ and μ' given in terms of ζ and m . We will write later this system explicitly by using 5.31, 5.26, and 5.29.iii. We consider now the following simplified approximate version which is obtained by treating Φ' as an error term to be ignored and making appropriate simple choices for μ and μ' (recall also $dx = \frac{1}{m}d\tilde{x}$).

$$\tau(\phi_1 - A_1) + \tau \log(\tau/2) = \tau\zeta, \quad \tau m \widehat{h}(x_1)dx = \tau\zeta'dx.$$

By straightforward calculation using 5.8 this is equivalent to $\tau = 2e^\zeta e^{A_1 - \phi_1}$ and $m\widehat{h}(x_1) = \zeta'$. To ensure a simplified expression we use a modified (by replacing A_1 with A'_1) version of these conditions to define x_1 and τ in 5.33 below.

Lemma 5.32. *For m large enough depending on \underline{c}_1 , and $\zeta = (\zeta, \zeta')$ as in 5.30, there are unique $x_1 = x_1[\zeta, m] \in (0, x_{root})$ and $\tau = \tau[\zeta, m] > 0$ satisfying*

$$(5.33) \quad \tau = 2e^\zeta e^{A'_1 - \phi_1} = \frac{1}{m} e^\zeta e^{-\frac{m}{\cos x_1 (h_{1+} + h_{1-})}}, \quad \widehat{h}(x_1) = \frac{\zeta'}{m},$$

where $A'_1 := -\log 2m$. Moreover, there is a unique

$$w = w[\zeta, m] := \mu[\zeta, m] W[x_1, m] + \mu'[\zeta, m] W'[x_1, m],$$

such that (recall 3.10)

$$(5.34) \quad \begin{aligned} \varphi &= \varphi[\zeta, m] := \tau[\zeta, m] \Phi[x_1, m] + \mu[\zeta, m] V[x_1, m] + \mu'[\zeta, m] V'[x_1, m] \\ &= \varphi[L[x_1[\zeta, m], m], \tau[\zeta, m], w[\zeta, m]] \end{aligned}$$

is an MLD solution as in 3.4. Furthermore, the following hold.

- (i). $x_1 = x_1[\zeta, m]$, $\tau = \tau[\zeta, m]$, $\mu = \mu[\zeta, m]$, and $\mu' = \mu'[\zeta, m]$ depend continuously on ζ .
- (ii). $|x_1[\zeta, m] - x_{balanced}| \leq C|\zeta'|/m \leq C\underline{c}_1/m$.
- (iii). In the notation of 1.1 we have

$$m \sim_2 |\log \tau| \cos x_{balanced} (h_{1+} + h_{1-})|_{x=x_{balanced}}, \quad \underline{\tau} \sim_{C(\underline{c}_1)} \tau[\zeta, m],$$
 where $\underline{\tau} := \tau[(0, 0), m]$ and $C(\underline{c}_1) > 1$ depends only on \underline{c}_1 .
- (iv). $|\zeta + \mu \phi_1/\tau| \leq C$ and $|\zeta' + \mu'/\tau| \leq C$.
- (v). $\|\varphi[[\zeta, m]] : C_{sym}^{3,\beta}(\mathbb{S}_{eq}^2 \setminus D_L(\delta'_1), g)\| \leq \tau^{1-4\alpha} \leq \tau^{8/9}$.
- (vi). $cm\tau \leq \frac{7}{8}\phi_1\tau \leq \varphi$ on $\mathbb{S}_{eq}^2 \setminus D_L(\delta'_1)$ for some absolute constant $c > 0$.
- (vii). φ satisfies the conditions in 3.15, 3.12, and 4.18.

Proof. The existence and uniqueness of x_1 and τ , their smoothness, and also (ii), follow from 5.33 and 5.12. (iii) follows from (ii), 5.33, and 5.30. Using 5.31, 5.26, and 5.29.iii, we conclude that the matching conditions in 3.4 amount to

$$\begin{aligned} \Phi'(p) + \phi_1 - A_1 + \frac{\mu}{\tau}(\phi_1 + \Phi'(p)) + \log \frac{\tau}{2} &= 0, \\ \left(1 + \frac{\mu}{\tau}\right) \left(\widehat{h}(x_1) + \frac{\partial \Phi'}{\partial \widetilde{x}}(p)\right) + \frac{\mu'}{m\tau} &= 0. \end{aligned}$$

By further calculation and 5.33 these conditions are equivalent to

$$\begin{aligned} \mu &= -\tau \frac{A'_1 - A_1 + \zeta + \Phi'(p)}{\phi_1 + \Phi'(p)}, \\ \mu' &= \tau \left(\frac{A'_1 - A_1 + \zeta + \Phi'(p)}{\phi_1 + \Phi'(p)} - 1 \right) \left(\zeta' + m \frac{\partial \Phi'}{\partial \widetilde{x}}(p) \right). \end{aligned}$$

w is uniquely determined by these conditions and (i) follows. Using 5.27 and the definition of ϕ_1 in 5.8 we conclude that

$$|\zeta + \mu \phi_1/\tau| \leq C + C\underline{c}_1/m, \quad |\zeta' + \mu'/\tau| \leq C + C\underline{c}_1/m.$$

(iv) then follows. By 5.16 we have that

$$\varphi = \tau \widehat{G} + \tau \Phi'' + \mu V + \mu' V'.$$

By (iv) and 5.29.iv we have

$$(5.35) \quad \|\mu V + \mu' V' : C_{sym}^{3,\beta}(\mathbb{S}_{eq}^2, \widetilde{g})\| \leq C\underline{c}_1\tau.$$

Using 2.20.vii we obtain

$$\|\widehat{G} : C_{sym}^{3,\beta}(\mathbb{S}_{eq}^2 \setminus D_L(\delta'_1), g)\| \leq C(\delta'_1)^{-3-\beta} |\log \delta'_1|.$$

Combining the above with 5.28.i for Φ'' we conclude that

$$\|\varphi : C_{sym}^{3,\beta}(\mathbb{S}_{eq}^2 \setminus D_L(\delta'_1), g)\| \leq C(\underline{c}_1 m^{3+\beta} + m^{4+\beta} + (\delta'_1)^{-3-\beta} |\log \delta'_1|) \tau.$$

Using (iii) and $\delta'_1 = \tau^\alpha$ we conclude (v) by assuming m is large enough (equivalently τ is small enough). By 2.20.vii we conclude that $|\widehat{G}| \leq C \alpha m$ on $\mathbb{S}_{eq}^2 \setminus D_L(\delta'_1)$. By 2.31 we can assume α small enough so that (vi) follows by using 5.28.ii and 5.35.

Finally, we prove (vii): 3.15.i follows from 5.3, 5.33, and by choosing m large enough. 3.15.ii–iii are obvious. 3.15.iv follows from 5.31 by using 5.28.i and 5.35. 3.15.v–vi follow from (v) and (vi). 3.12 and 4.18 follow from 5.29.v. q.e.d.

6. LD and MLD solutions in the equator-poles case

Basic definitions. We proceed now to study the LD and MLD solutions we need in the case that the catenoidal bridges are located on the equatorial circle and the two poles. The construction of these solutions parallels that of the LD and MLD solutions in the two-circle case as presented in the previous section. The main differences are that now we have two different catenoidal bridges modulo the symmetries. On the other hand, we have no horizontal forces and no horizontal sliding because the symmetries fix the location of the catenoidal bridges completely. The configuration now consists of $m + 2$ points, m of which lie on the equator and the other two are the poles:

(6.1)

$$L_{eq-pol} = L_{eq-pol}[m] := L_0[m] \cup L_2, \quad \text{where}$$

$$L_0 = L_0[m] := L_{mer}[m] \cap \mathbb{P}_0 = \mathcal{G}_{\mathbb{S}_{eq,m}^2} p_0, \quad L_2 := \{p_N, p_S\} = \mathcal{G}_{\mathbb{S}_{eq,m}^2} p_2,$$

where $p_0 := \Theta(0, 0, 0) = (1, 0, 0, 0)$ and $p_2 := p_N$ (recall 2.3). We define

(6.2)

$$\delta_p := \delta_0 := 1/9m \quad (p \in L_0), \quad \delta_p := \delta_2 := 1/100 \quad (p \in \{p_N, p_S\}).$$

Clearly $\mathcal{K}_{sym}[L_{eq-pol}]$ is two-dimensional and spanned by $W_j := W_j[m] \in \mathcal{K}_{sym}[L]$ for $j = 0, 2$, where W_j is defined by requesting (recall 3.7 and 3.9) that it is supported on $D_{L_j}(2\delta_j) \setminus D_{L_j}(\delta_j)$ and satisfies on $D_{p_j}(2\delta_j)$

(6.3)

$$W_j := \mathcal{L}' \Psi [2\delta_j, \delta_j; \mathbf{d}_{p_j}] (G_{p_j}, \log \delta_j \cos \circ \mathbf{d}_{p_j}).$$

Because of the symmetries each $\tau : L_{eq-pol} \rightarrow \mathbb{R}$ we consider takes only two values: $\tau_0 := \tau_{p_0}$ taken on L_0 and $\tau_2 := \tau_{p_N}$ taken on the poles. In analogy with 5.6 we have:

Definition 6.4. For $j = 0, 2$ we define LD solutions $\Phi_j = \Phi_j[m] := \varphi[L_j[m], 1, 0] \in C_{sym}^\infty(\mathbb{S}_{eq}^2 \setminus L_j)$. We also define $V_j = V_j[m] \in \widehat{\mathcal{K}}_{sym}[L_{eq-pol}]$ by $\mathcal{L}'V_j = W_j$ (recall 3.10, 3.9 and 5.5).

Clearly $\widehat{\mathcal{K}}_{sym}[L_{eq-pol}]$ is spanned by V_0 and V_2 .

The rotationally invariant parts.

Lemma 6.5. *We have that $\phi_{eq} := (\Phi_0[m])_{avg} = \frac{m}{2} \sin |x|$ on \mathbb{S}_{eq}^2 .*

Proof. The proof is similar to the one for 5.8 but simpler: Because of the smoothness on each hemisphere and the rotational symmetry it is clear that $\phi_{eq} = A \sin |x|$ for some $A \in \mathbb{R}$. For $0 < \epsilon_1 \ll \epsilon_2$ let $\Omega_{\epsilon_1, \epsilon_2} := D_{\mathbb{P}_0}(\epsilon_2) \setminus D_{L_0}(\epsilon_1)$. By integrating $\mathcal{L}'\Phi_0 = 0$ on $\Omega_{\epsilon_1, \epsilon_2}$, integrating by parts, and taking the limit as $\epsilon_1 \rightarrow 0$ first and then as $\epsilon_2 \rightarrow 0$, we obtain using the logarithmic behavior near L_0 that $2\pi m = 4\pi A$, which implies the lemma. q.e.d.

Note that if we extended the notation of 5.9 in the obvious way we would have $\phi_{eq} = \underline{j}[1/2; 0]$.

Lemma 6.6. *We have that (recall 2.18)*

$$\Phi_2[m] = \phi_{even} = G_{p_N} + (1 - \log 2)\phi_{odd} \in C_{|x|}^\infty(\mathbb{S}_{eq}^2 \setminus \{p_N, p_S\}).$$

Proof. The second equality is just 2.20.iii and it implies clearly the first equality by the definitions and 3.10. q.e.d.

Estimates on $\Phi_0 = \Phi_0[m]$. Since Φ_2 is rotational invariant and well understood by 6.6 and $(\Phi_0[m])_{avg} = \phi_{eq}$ is well understood by 6.5, the main remaining task is estimating $(\Phi_0[m])_{osc}$. Our approach for this is similar to the one for estimating Φ_{osc} in the previous section, except that the situation is simplified by the extra symmetry. In analogy with 5.15 and 5.16 we have now the following.

Definition 6.7. Let $\widehat{G}_0 \in C_{sym}^\infty(\mathbb{S}_{eq}^2 \setminus L_0)$, $\Phi''_0, E''_0, \Phi''_{0,avg}, \Phi''_{0,osc}, E''_{0,avg}, E''_{0,osc} \in C_{sym}^\infty(\mathbb{S}_{eq}^2)$, $\widehat{G}_2 \in C_{|x|}^\infty(\mathbb{S}_{eq}^2 \setminus \{p_N, p_S\})$, and $\Phi''_2, E''_2 \in C_{|x|}^\infty(\mathbb{S}_{eq}^2)$ be defined by requesting that for $j = 0, 2$

$$(6.8) \quad \widehat{G}_j := \{ \Psi[2\delta_0, 3\delta_0; I_{\mathbb{R}^+}](G, A_j) \} \circ \mathbf{d}_{L_j}, \quad \text{where } A_j := \log \delta_j,$$

$$(6.9) \quad \begin{aligned} \Phi_j &= \widehat{G}_j + \Phi''_j, & E''_j &:= -\mathcal{L}'\widehat{G}_j, & \text{on } \mathbb{S}_{eq}^2 \setminus L_j, \\ \Phi''_{0,avg} &:= (\Phi''_0)_{avg}, & \Phi''_{0,osc} &:= (\Phi''_0)_{osc}, & \text{on } \mathbb{S}_{eq}^2, \\ E''_{0,avg} &:= (E''_0)_{avg}, & E''_{0,osc} &:= (E''_0)_{osc}, & \text{on } \mathbb{S}_{eq}^2. \end{aligned}$$

Note that by 6.8 E''_0 vanishes on $D_{L_0}(2\delta_0)$. Moreover, E''_0 is constant on $\mathbb{S}_{eq}^2 \setminus D_{\mathbb{P}_0}(3\delta_0)$ and, therefore, $E''_{0,osc}$ is supported on $D_{\mathbb{P}_0}(3\delta_0)$. Since $\mathcal{L}'\Phi_0$ vanishes by 3.10 and 6.4, and \mathcal{L}' is rotationally covariant, we conclude from 6.9 that

$$(6.10) \quad \mathcal{L}'\Phi''_0 = E''_0, \quad \mathcal{L}'\Phi''_{0,avg} = E''_{0,avg}, \quad \mathcal{L}'\Phi''_{0,osc} = E''_{0,osc} \quad \text{on } \mathbb{S}_{eq}^2.$$

Moreover, since $\Phi_{0,avg} \in C_{sym}^0(\mathbb{S}_{eq}^2)$ by 3.10 and 6.4, and $\Phi_{0,avg}'' \in C_{sym}^\infty(\mathbb{S}_{eq}^2)$ by 6.9, we conclude by 6.9 that $\widehat{G}_{0,avg} \in C^0(\mathbb{S}_{eq}^2)$,

$$(6.11) \quad \begin{aligned} \Phi_{0,avg}'' &= \Phi_{0,avg} - \widehat{G}_{0,avg} = \phi_{eq} - \widehat{G}_{0,avg} \quad \text{on } \mathbb{S}_{eq}^2, \\ \Phi_0 &= \phi_{eq} + \widehat{G}_{0,osc} + \Phi_{0,osc}'' \quad \text{on } \mathbb{S}_{eq}^2 \setminus L_0. \end{aligned}$$

Using 6.9 we conclude that

$$(6.12) \quad \Phi_0'' = \phi_{eq} - \widehat{G}_{0,avg} + \Phi_{0,osc}'' \quad \text{on } \mathbb{S}_{eq}^2.$$

Note that in this expression although ϕ_{eq} and $\widehat{G}_{0,avg}$ are not smooth because of a derivative jump at the equator \mathbb{P}_0 , we do have $\phi_{eq} - \widehat{G}_{0,avg} = \Phi_{0,avg}'' \in C_{sym}^\infty(\mathbb{S}_{eq}^2)$ because the derivative jumps cancel out. Note also that neither $\mathcal{L}'\Phi_{0,avg}''$ nor $\mathcal{L}'\Phi_{0,osc}''$ have to vanish on $D_p(2\delta_1)$ but their sum does.

Lemma 6.13. *The following hold where $\Omega_{eq} := D_{\mathbb{P}_0}(3/m)$.*

- (i). $\|\widehat{G}_0 - A_0 : C_{sym}^k(\Omega_{eq} \setminus D_{L_0}(\delta_0), \tilde{g})\| \leq C(k)$ and $\widehat{G}_0 - A_0$ vanishes on $\mathbb{S}_{eq}^2 \setminus D_{L_0}(3\delta_0)$.
- (ii). $\|m^{-2}E_0'' : C_{sym}^k(\Omega_{eq}, \tilde{g})\| \leq C(k)$, $\|m^{-2}E_{0,avg}'' : C_{sym}^k(\mathbb{S}_{eq}^2, \tilde{g})\| \leq C(k)$, and E_0'' vanishes on $D_{L_0}(2\delta_0)$.
- (iii). $\|m^{-2}E_{0,osc}'' : C_{sym}^k(\mathbb{S}_{eq}^2, \tilde{g})\| \leq C(k)$ and $E_{0,osc}''$ is supported on $D_{\mathbb{P}_0}(3\delta_0) \subset \Omega_{eq}$.

Proof. (i) follows from 2.20.vi and the definitions. By 2.16 and 6.9 we have

$$m^{-2}E_0'' = -\mathcal{L}'_{\tilde{g}}\widehat{G}_0 = -\mathcal{L}'_{\tilde{g}}(\widehat{G}_0 - A_0) - 2m^{-2}A_0.$$

As mentioned earlier E_0'' vanishes on $D_{L_0}(2\delta_0)$ and clearly $-1 < m^{-2}A_0 < 0$ by 6.2. (ii) follows then from (i). The second part of (iii) follows from (i) which implies that $\widehat{G}_0 = A_0$ on $\mathbb{S}_{eq}^2 \setminus D_L(3\delta_0)$. The first part of (iii) follows then from (ii) and the second part. q.e.d.

Lemma 6.14. $\Phi_{0,osc}''$ satisfies $\|\Phi_{0,osc}'' : C_{sym}^{2,\beta}(\mathbb{S}_{eq}^2, \tilde{g}, f_{\mathbb{S}_{eq}^2,0})\| \leq C$ where we have $f_{\mathbb{S}_{eq}^2,0} := e^{-c_1 m \min(|x|, c_2)}$ for some absolute constants $c_1, c_2 > 0$.

Proof. The proof is similar to the proof of 5.24, where we apply an appropriately modified version of 5.23 on 6.13. Since the modifications are clear we omit the details. q.e.d.

Definition 6.15. We define $\Phi'_0 \in C_{sym}^\infty(\Omega_{eq})$ by requesting that $\Phi_0'' = \Phi'_0 - A_0\phi_{even}$ on Ω_{eq} .

6.15 corresponds to 5.25 with $-A_0$ and ϕ_{even} corresponding to $\phi_1 - A_1$ and $\underline{\phi}[1, 0; 0]$. Using now 6.9 we obtain

$$(6.16) \quad \Phi_0 = \widehat{G}_0 + \Phi'_0 - A_0\phi_{even} \quad \text{on } \Omega_{eq} \setminus L_0.$$

Lemma 6.17. Φ'_0 satisfies the estimate $\|\Phi'_0 : C_{sym}^{2,\beta}(\Omega_{eq}, \tilde{g})\| \leq C$.

Proof. By using the definitions we have

$$\Phi'_{0,avg} = \phi_{eq} + A_0\phi_{even} - \widehat{G}_{0,avg}, \quad \Phi'_{0,osc} = \Phi''_{0,osc}, \quad \mathcal{L}'\Phi'_{0,avg} = E''_{0,avg}.$$

Note that the discontinuities on the right hand side of the first equation cancel and the left hand side is smooth. Using then that $\widehat{G}_{0,avg} = A_0$ on $\mathbb{S}_{eq}^2 \setminus D_{\mathbb{P}_0}(3\delta_0)$ by 6.13.i and 2.18 we conclude that at $\partial\Omega_{eq}$ we have $|\Phi'_{0,avg}| \leq C$ and $|\partial_{\tilde{x}}\Phi'_{0,avg}| \leq C$. Using these as initial data for the ODE and 5.21.ii we estimate $\Phi'_{0,avg}$. By this estimate together with 6.14 we conclude the proof. q.e.d.

Lemma 6.18. (i). $\|\Phi_0 - \frac{m}{2} \sin|x| : C_{sym}^{k,\beta}(\mathbb{S}_{eq}^2 \setminus D_{\mathbb{P}_0}(\delta_0), \tilde{g})\| \leq C(k)$.

(ii). $\|\Phi_0 : C_{sym}^{k,\beta}(\Omega_{eq} \setminus D_{L_0}(\delta_0), \tilde{g})\| \leq C(k)$ (recall $\Omega_{eq} = D_{\mathbb{P}_0}(3/m) \supset D_{\mathbb{P}_0}(3\delta_0)$).

(iii). $\|\Phi''_0 + A_0 : C_{sym}^{k,\beta}(\Omega_{eq}, \tilde{g})\| \leq C(k)$.

Proof. By 6.11, 6.13.i, and 6.5 we have $\Phi_0 - \frac{m}{2} \sin|x| = \Phi''_{0,osc} + (\widehat{G}_0 - A_0)_{osc}$ on $\mathbb{S}_{eq}^2 \setminus D_{\mathbb{P}_0}(\delta_0)$. By 6.14 6.13.i we conclude (i) for $k = 2$.

By 6.16 we have $\Phi_0 = \widehat{G}_0 - A_0 + \Phi'_0 - A_0(\phi_{even} - 1)$ on $\Omega_{eq} \setminus L_0$ and by 6.15 $\Phi''_0 + A_0 = \Phi'_0 - A_0(\phi_{even} - 1)$ on Ω_{eq} . (ii) and (iii) follow then for $k = 2$ from 6.13.i, 6.17, and 2.18. Using interior regularity and 6.13.iii we complete the proof. q.e.d.

Corollary 6.19. (i). $\|\Phi_0 : C_{sym}^{k,\beta}(\mathbb{S}_{eq}^2 \setminus D_{L_0}(\delta_0), \tilde{g})\| \leq C(k) m$.

(ii). $\|\Phi''_0 : C_{sym}^{k,\beta}(\mathbb{S}_{eq}^2, \tilde{g})\| \leq C(k) m$.

(iii). $|\Phi_0 - \frac{m}{2} \sin|x|| \leq C$ on $\mathbb{S}_{eq}^2 \setminus D_{L_0}(\delta_0)$.

Proof. (i) follows from 6.18.i.ii. (ii) follows from (i) and 6.18.iii by using also that on $\mathbb{S}_{eq}^2 \setminus \Omega_{eq}$ we have $\Phi''_0 = \Phi_0 - A_0$ and $|A_0| < m$. (iii) follows from 6.18.i.ii. q.e.d.

Lemma 6.20. (i). $\|\Phi_2 : C_{sym}^{k,\beta}(\mathbb{S}_{eq}^2 \setminus D_{L_2}(\delta_2), g)\| \leq C(k)$.

(ii). $\|\Phi''_2 : C_{sym}^{k,\beta}(\mathbb{S}_{eq}^2, \tilde{g})\| \leq C(k)$ and, moreover, $\Phi''_2 = (1 - \log 2) \sin|x|$ on $D_{L_2}(\delta_2)$.

Proof. This follows easily from the definitions and 6.6. q.e.d.

Estimates on V_0, V_2, W_0, W_2 .

Lemma 6.21. $V_0, V_2 \in C_{sym}^\infty(\mathbb{S}_{eq}^2)$ satisfy the following.

(i). We have $V_0 = \Phi_0$ on $\mathbb{S}_{eq}^2 \setminus D_{L_0}(2\delta_0)$ and $V_2 = \Phi_2 = \phi_{even}$ on $\mathbb{S}_{eq}^2 \setminus D_{\{p_N, p_S\}}(2\delta_2)$.

(ii). We have on $D_{p_0}(2\delta_0)$ and on $D_{p_N}(2\delta_0)$ respectively that

$$V_0 = \Psi[2\delta_0, \delta_0; \mathbf{d}_{p_0}](G_{p_0}, \log \delta_0 \cos \circ \mathbf{d}_{p_0}) + \Phi'_0 - A_0\phi_{even},$$

$$V_2 = \Psi[2\delta_2, \delta_2; \mathbf{d}_{p_N}](G_{p_N}, \log \delta_2 \cos \circ \mathbf{d}_{p_N}) + (1 - \log 2)\phi_{odd}.$$

- (iii). $V_0(p_0) = \Phi'_0(p_0)$, $V_0(p_N) = m/2$, $V_2(p_0) = 1$, and $V_2(p_N) = 1 + \log(\delta_2/2)$. Moreover, $|V_0(p_0)| \leq C$ and $|V_2(p_N)| \leq C$.
- (iv). $\|V_0 : C^k_{sym}(D_{L_0}(\delta_0), \tilde{g})\| \leq C(k)$, $\|V_0 : C^k_{sym}(\mathbb{S}^2_{eq}, \tilde{g})\| \leq C(k)m$, and $\|V_2 : C^k_{sym}(\mathbb{S}^2_{eq}, \tilde{g})\| \leq C(k)$.
- (v). 3.12 holds and $\|\mathcal{E}^{-1}_{L_{eq-pol}}\| \leq C m^{2+\beta}$ (recall 3.13).
- (vi). For $\mu_0, \mu_2 \in \mathbb{R}$ we have that

$$|\mu_0|m + |\mu_2| \leq C \|\mu_0 W_0 + \mu_2 W_2 : C^{0,\beta}_{sym}(\mathbb{S}^2_{eq}, g)\|.$$

Proof. The proof is similar in structure to the one for 5.29. Let $V_{0,new}$ and $V_{2,new}$ be defined by the expressions for V_0 and V_2 in (i) and (ii). The expressions for $V_{0,new}$ and $V_{2,new}$ in (i) and (ii) match because by 6.7, 6.16, and 6.6 we have $V_{0,new} = \Phi_0 = \widehat{G}_0 + \Phi'_0 - A_0\phi_{even}$ on a neighborhood of $\partial D_{p_0}(2\delta_0)$ and $V_{2,new} = \Phi_2 = \phi_{even} = G_{p_N} + (1 - \log 2)\phi_{odd}$ on a neighborhood of $\partial D_{p_N}(2\delta_2)$. Since $\mathcal{L}'\Phi_0 = \mathcal{L}'\Phi_2 = 0$ by 6.4, we conclude that by 6.3 we have that $\mathcal{L}'V_{0,new} = W_0$ and $\mathcal{L}'V_{2,new} = W_2$, which characterize by 3.10 V_0 and V_2 . This completes the proof of (i) and (ii).

The equalities in (iii) now follow from (i), (ii), 6.5, and 2.18, where we used also $A_0 = \log \delta_0$ from 6.8. The estimates in (iii) follow from 6.17 and 6.2. (iv) is implied by (i), (ii), 6.19, 2.18, 6.20, and $A_0 = \log \delta_0$. Note now that if we use $\{V_0, V_2\}$ as the basis for $\widehat{\mathcal{K}}_{sym}[L_{eq-pol}]$ and the standard basis for $\mathcal{V}_{sym}[L_{eq-pol}]$, then the entries for the matrix of $\mathcal{E}_{L_{eq-pol}}$, defined as in 3.11, are given in (iv), and, therefore, using (iv) we can easily check that (v) holds. (vi) follows by the same argument we used for 6.21.vi. q.e.d.

The family of MLD solutions. We discuss now the family of MLD solutions which is converted to a family of initial surfaces by 3.20. The parameters of the family are $\zeta = (\zeta_0, \zeta_2) \in \mathbb{R}^2$ and their range is specified by

$$(6.22) \quad |\zeta_0|, |\zeta_2| \leq \underline{c}_2,$$

where $\underline{c}_2 > 1$ is a constant independent of m and τ which will be specified later. Given (ζ_0, ζ_2) as in 5.30 we define τ_0, τ_2 by

$$(6.23) \quad \begin{aligned} \tau_0 &= \tau_0[\zeta, m] := m^{-3/4} e^{\zeta_0} e^{-\sqrt{m/2}}, \\ \tau_2 &= \tau_2[\zeta, m] := \tau_0 \left(\zeta_2 - \frac{1}{4} \log m + \sqrt{\frac{m}{2}} \right). \end{aligned}$$

This definition is motivated by a straightforward calculation where various error terms are ignored. We skip this calculation because it is not needed for the proof and is similar to a precise calculation we present in the proof of 6.24.

Lemma 6.24. *For m large enough depending on \underline{c}_2 and $\zeta = (\zeta_0, \zeta_2)$ as in 6.22, we define $\tau : L_{eq-pol} \rightarrow \mathbb{R}^+$ to take the values τ_0 on L_0 and*

τ_2 on $\{p_N, p_S\}$, with τ_0, τ_2 defined as in 6.23, and also $w = w[\zeta, m] := \mu_0[\zeta, m] W_0[m] + \mu_2[\zeta, m] W_2[m]$ defined uniquely by the requirement that

$$(6.25) \quad \varphi = \varphi[[\zeta, m]] := \varphi[L_{eq-pol}[m], \tau[\zeta, m], w[\zeta, m]] = \tau_0 \Phi_0 + \tau_2 \Phi_2 + \mu_0 V_0 + \mu_2 V_2$$

is an MLD solution as in 3.4. Moreover, the following hold.

- (i). $\tau = \tau[\zeta, m]$ and $\mu = \mu[\zeta, m] := (\mu_0, \mu_2)$ depend continuously on ζ .
- (ii). $\left| \zeta_0 + \frac{\mu_0(m/2)^{1/2} + \mu_2}{2\tau_0} \right| \leq C$ and $\left| \zeta_2 + \frac{-\mu_0(m/2)^{1/2} + \mu_2}{2\tau_0} \right| \leq C$.
- (iii). $\|\varphi[[\zeta, m]] : C_{sym}^{3,\beta}(\mathbb{S}_{eq}^2 \setminus D', g)\| \leq \tau_0^{1-4\alpha} \leq \tau_0^{8/9}$ where $D' := \bigsqcup_{j=0,2} D_{L_j}(\delta'_j)$ where $\delta'_j := \tau_j^\alpha$.
- (iv). $c\tau_2 \leq \varphi$ on $\mathbb{S}_{eq}^2 \setminus D'$ for some absolute constant $c > 0$.
- (v). φ satisfies the conditions in 3.15, 3.12, and 4.18.

Proof. Note that by 3.1.ii and 6.7 for φ as in 6.25 we have that

$$(6.26) \quad \begin{aligned} \widehat{\varphi}_{p_0} &= \tau_0 \Phi_0'' + \tau_2 \Phi_2 + \mu_0 V_0 + \mu_2 V_2, & \text{on } D_{p_0}(2\delta_0), \\ \widehat{\varphi}_{p_N} &= \tau_0 \Phi_0 + \tau_2 \Phi_2'' + \mu_0 V_0 + \mu_2 V_2, & \text{on } D_{p_N}(2\delta_2), \end{aligned}$$

where motivated by 6.6 we define $\Phi_2'' := (1 - \log 2)\phi_{odd}$. Using 6.15, 6.6 we calculate

$$(6.27) \quad \begin{aligned} \Phi_0''(p_0) &= \Phi_0'(p_0) - A_0, & \Phi_2(p_0) &= 1, \\ \Phi_0(p_N) &= m/2, & \Phi_2''(p_N) &= 1 - \log 2. \end{aligned}$$

By straightforward calculation using 6.23 we obtain

$$(6.28) \quad \log \tau_0 = \zeta_0 - \frac{3}{4} \log m - \sqrt{\frac{m}{2}}, \quad \log \tau_2 = \zeta_0 - \frac{1}{4} \log m - \sqrt{\frac{m}{2}} + O(1),$$

where in this proof we use $O(1)$ to denote terms which are uniformly bounded (independently of \underline{c}_2) as $m \rightarrow \infty$. Using the above, 6.21.iii, and 6.17, we calculate the matching condition in 3.4 amounts to

$$\begin{aligned} \tau_0 (\zeta_0 + \zeta_2 + O(1)) + O(1) \mu_0 + \mu_2 &= 0, \\ \tau_0 (\zeta_0 - \zeta_2 + O(1)) \sqrt{\frac{m}{2}} + \frac{m}{2} \mu_0 + O(1) \mu_2 &= 0. \end{aligned}$$

Solving this linear system for μ_0, μ_2 we obtain its unique solution given by

$$\mu_0 = -\tau_0 (\zeta_0 - \zeta_2 + O(1)) (m/2)^{-1/2}, \quad \mu_2 = -\tau_0 (\zeta_0 + \zeta_2 + O(1)),$$

where we assumed that m is large enough in terms of \underline{c}_2 .

The above clearly imply (i) and (ii). They also imply that

$$(6.29) \quad |\mu_0| \sqrt{m} + |\mu_2| \leq C \underline{c}_2 \tau_0,$$

which together with 6.21.iv implies that

$$(6.30) \quad \|\mu_0 V_0 + \mu_2 V_2 : C_{sym}^{3,\beta}(\mathbb{S}_{eq}^2, \tilde{g})\| \leq C \underline{c}_2 \sqrt{m} \tau_0.$$

Using 2.20.vii we obtain that for $j = 0, 2$

$$\|\widehat{G}_j : C_{sym}^{3,\beta}(\mathbb{S}_{eq}^2 \setminus D', g)\| \leq C (\delta'_j)^{-3-\beta} |\log \delta'_j|.$$

Note that we have on $\mathbb{S}_{eq}^2 \setminus D'$

$$\varphi = \tau_0 \widehat{G}_0 + \tau_2 \widehat{G}_2 + \tau_0 \Phi_0'' + \tau_2 \Phi_2'' + \mu_0 V_0 + \mu_2 V_2.$$

Combining the above with 6.20.ii and 6.19.ii we conclude (iii) by assuming m large enough.

To prove (iv) observe that on $D_{L_0}(\delta_0) \setminus D_{L_0}(\delta'_0)$ we have $\varphi = \tau_0 \widehat{G}_0 + \tau_0 \Phi_0'' + (\tau_2 + \mu_2) \Phi_2 + \mu_0 V_0$, on $D_{L_2}(\delta_2) \setminus D_{L_2}(\delta'_2)$ we have $\varphi = (\tau_0 + \mu_0) \Phi_0 + \tau_2 \widehat{G}_2 + \tau_2 \Phi_2'' + \mu_2 V_2$, and on $\mathbb{S}_{eq}^2 \setminus \bigsqcup_{j=0,2} D_{L_j}(\delta_j)$ we have $\varphi = (\tau_0 + \mu_0) \Phi_0 + (\tau_2 + \mu_2) \Phi_2$. Using 6.29 and 6.23 we obtain bounds for the coefficients. (iv) on $\mathbb{S}_{eq}^2 \setminus \bigsqcup_{j=0,2} D_{L_j}(\delta_j)$ follows then by using 2.19. By 2.20.vii and 6.28 we have $|\widehat{G}_j| \leq C \alpha |\log \tau_j| \leq C \alpha \sqrt{m}$ on $D_{L_j}(\delta_j) \setminus D_{L_j}(\delta'_j)$ for $j = 0, 2$. Using 6.19.iii, 6.20, and 6.21.iv to estimate the remaining terms, we conclude the proof of (iv) by assuming α small enough as in 2.31.

Finally, we prove (v): 3.15.i follows from 6.2, 6.23, and by choosing m large enough. 3.15.ii–iii are obvious from the definitions and choosing m large enough. 3.15.iv follows from 6.26 by using 6.19.i,ii, 6.20, and 6.30. 3.15.v–vi follow from (iii) and (iv). 3.12 and 4.18 follow from 6.21.v. q.e.d.

REMARK 6.31. Note that if we only had bridges on the equatorial circle then 6.25 would have to be replaced by “ $\varphi = \tau_0 \Phi_0 + \mu_0 V_0$ ”. Clearly then it would be impossible to satisfy the vertical matching condition and construct an MLD solution in this way. \square

7. Main results

Theorem 7.1 (The two parallel circles case). *There is an absolute constant $\underline{c}_1 > 0$ such that if m is large enough depending on \underline{c}_1 , then there is $\widehat{\zeta} = (\widehat{\zeta}, \widehat{\zeta}') \in \mathbb{R}^2$ satisfying 5.30 such that (in the notation of 5.32) $\widehat{x}_1 := x_1[\widehat{\zeta}, m]$, $\widehat{\tau} := \tau[\widehat{\zeta}, m]$, $\widehat{w} := w[\widehat{\zeta}, m]$, and $\varphi[[\widehat{\zeta}, m]]$, satisfy 5.32.ii–viii, and, moreover, there is $\widehat{\phi} \in C^\infty(\widehat{M})$, where $\widehat{M} := M[L[\widehat{x}_1, m], \widehat{\tau}, \widehat{w}]$ in the notation of 3.20, such that in the notation of 4.12*

$$\|\widehat{\phi}\|_{2,\beta,\gamma;\widehat{M}} \leq \widehat{\tau}^{1+\alpha/4},$$

and, furthermore, $\widehat{M}_{\widehat{\phi}}$ (in the notation of 4.24) is a genus $2m - 1$ embedded minimal surface in $\mathbb{S}^3(1)$, which is invariant under the action of $\mathcal{G}_{\mathbb{S}^3,m}$ and has area $\text{Area}(\widehat{M}_{\widehat{\phi}}) \rightarrow 8\pi$ as $m \rightarrow \infty$.

Proof. Step 1: Construction of the diffeomorphisms F_ζ : We fix an $m \in \mathbb{N}$ which we assume as large in terms of \underline{c}_1 as needed. We will use the notation $\mathbf{0} := (0, 0) \in \mathbb{R}^2$, $\underline{\tau} := \tau[\mathbf{0}, m]$, and for $\zeta \in \mathbb{R}^2$ satisfying 5.30 $M[[\zeta]] := M[L[x_1[\zeta, m], m], \tau[\zeta, m], w[\zeta, m]]$ (recall 5.32 and 3.20) and $L[[\zeta]] := L[x_1[\zeta, m], m]$. We define for $\zeta \in \mathbb{R}^2$ satisfying 5.30 a smooth diffeomorphism $F_\zeta : M[[\mathbf{0}]] \rightarrow M[[\zeta]]$, covariant under the action of $\mathcal{G}_{\mathbb{S}^3, m}$, as follows. We start by constructing smooth diffeomorphisms $F'_{x_1} : \mathbb{S}_{eq}^2 \rightarrow \mathbb{S}_{eq}^2$ which depend smoothly on x_1 , are covariant under the action of $\mathcal{G}_{\mathbb{S}_{eq}^2, m}$, and satisfy the following.

(a). $F'_{x_1}(L[[\mathbf{0}]]) = L[x_1, m]$ and, moreover, if $p \in L[[\mathbf{0}]]$, then $F'_{x_1}(p)$ is the nearest point in $L[x_1, m]$ to p (which amounts to being on the same side of the equator and of the same longitude).

(b). $\forall p \in L[[\mathbf{0}]]$ we have on $D_p(4\delta_1)$ that $F'_{x_1} = R[x_1, p]$, where $R[x_1, p] \in \text{SO}(3)$ is characterized by $R[x_1, p](p) = F'_{x_1}(p)$ (as defined in (a) above), and $d_p R[x_1, p](\nabla_p x) = \nabla_{F'_{x_1}(p)} x$.

(c). If $q = \Theta(x, y, 0) \in D_{L_{par}[x_1[\mathbf{0}, m]]}(8\delta_1) \setminus D_{L[[\mathbf{0}]]}(5\delta_1)$ with $x \in (0, \pi)$ (recall 2.2), then $F'_{x_1}(q) = \Theta(x + x_1 - x_1[\mathbf{0}, m], y, 0)$.

(d). On $\mathbb{S}_{eq}^2 \setminus D_{L[[\mathbf{0}]]}(5\delta_1)$ F'_{x_1} is rotationally covariant in the sense that it maps a point $\Theta(x, y, 0)$ to $\Theta(f_{x_1}(x), y, 0)$ for a suitably chosen function f_{x_1} . Note this is consistent with (c) where f_{x_1} is implicitly specified on a smaller region.

(e). On $D_{L[[\mathbf{0}]]}(5\delta_1) \setminus D_{L[[\mathbf{0}]]}(4\delta_1)$ we interpolate between the definitions in (b) and (c) by using cut-off functions.

By choosing f_{x_1} carefully we can ensure that F'_{x_1} depends smoothly on x_1 and is close to the identity in all necessary norms. We proceed now to use $F'_{x_1[\zeta, m]}$ to define F_ζ by requesting the following.

(f). $\forall p \in L[[\mathbf{0}]]$ we define F_ζ to map $\Lambda_{\mathbf{0}} := \widehat{S}_1[p] \subset M[[\mathbf{0}]]$ onto $\Lambda_\zeta := \widehat{S}_1[q] \subset M[[\zeta]]$, where $q := F'_{x_1[\zeta, m]}(p)$, and to satisfy on $\Lambda_{\mathbf{0}}$ (recall 3.26 and 3.25e)

$$\widehat{F}_\zeta \circ Y_{\mathbf{0}} \circ \Pi_{\mathbb{K}, p} = Y_\zeta \circ \Pi_{\mathbb{K}, q} \circ F_\zeta,$$

where Y_ζ (and similarly for $Y_{\mathbf{0}}$) is the conformal isometric from $\Pi_{\mathbb{K}, q}(\Lambda_\zeta)$ equipped with the induced metric from the Euclidean metric $\tau^{-2}[\zeta, m] g|_p$, to the cylinder $[-\ell_\zeta, \ell_\zeta] \times \mathbb{S}^1(1)$ equipped with the standard flat metric, and

$$(7.2) \quad \widehat{F}_\zeta : [-\ell_{\mathbf{0}}, \ell_{\mathbf{0}}] \times \mathbb{S}^1(1) \rightarrow [-\ell_\zeta, \ell_\zeta] \times \mathbb{S}^1(1)$$

is of the form in standard coordinates

$$\widehat{F}_\zeta(t, \theta) = (\ell_\zeta t / \ell_{\mathbf{0}}, \theta),$$

where the ambiguity due to possibly modifying the θ coordinate by adding a constant is removed by the requirement that F_ζ is covariant with respect to the action of $\mathcal{G}_{\mathbb{S}^3, m}$.

(g). We define now the restriction of F_ζ on $M[[\mathbf{0}]] \setminus \widehat{S}[L[[\mathbf{0}]]] = M[[\mathbf{0}]] \cap \Pi_{\mathbb{S}_{eq}^2}^{-1}(\mathbb{S}_{eq}^2 \setminus D_{L[[\mathbf{0}]]}(\delta'_1))$ to be a map onto $M[[\zeta]] \setminus \widehat{S}[L[[\zeta]]] = M[[\zeta]] \cap \Pi_{\mathbb{S}_{eq}^2}^{-1}(\mathbb{S}_{eq}^2 \setminus D_{L[[\zeta]]}(\delta'_1))$ which preserves the sign of the z coordinate and satisfies

$$\Pi_{\mathbb{S}_{eq}^2} \circ F_\zeta = F'_{x_1[\zeta, m]} \circ \Pi_{\mathbb{S}_{eq}^2}.$$

(h). On the region $\widehat{S}[L[[\mathbf{0}]]] \setminus \widehat{S}_1[L[[\mathbf{0}]]] \subset M[[\mathbf{0}]]$ we apply the same definition as in (g) but with $F'_{x_1[\zeta, m]}$ appropriately modified by using cut-off functions and $\mathbf{d}_{L[[\zeta]]}$ so that the final definition provides an interpolation between (f) and (g).

Step 2: Equivalence of norms under F_ζ : Using 5.32.iii and 2.22 it is easy to check that

$$\ell_\zeta \sim_{1+C(\underline{c}_1)/m} \ell_{\mathbf{0}}.$$

Using this and arguing as in the proof of 4.13 we conclude that for $u \in C^{2,\beta}(M[[\zeta]])$ and $E \in C^{0,\beta}(M[[\zeta]])$ we have

$$\begin{aligned} \|u \circ F_\zeta\|_{2,\beta,\gamma;M[[\mathbf{0}]]} &\sim_2 \|u\|_{2,\beta,\gamma;M[[\zeta]]}, \\ \|E \circ F_\zeta\|_{0,\beta,\gamma-2;M[[\mathbf{0}]]} &\sim_2 \|E\|_{0,\beta,\gamma-2;M[[\zeta]]}. \end{aligned}$$

Step 3: The map \mathcal{J} : We define now a map $\mathcal{J} : B \rightarrow C_{sym}^{2,\beta}(M[[\mathbf{0}]]) \times \mathbb{R}^2$, where

$$\begin{aligned} B := \{v \in C_{sym}^{2,\beta}(M[[\mathbf{0}]]): \|v\|_{2,\beta,\gamma;M[[\mathbf{0}]]} \leq \underline{\tau}^{1+\alpha}\} &\times [-\underline{c}_1, \underline{c}_1]^2 \subset \\ &\subset C_{sym}^{2,\beta}(M[[\mathbf{0}]]) \times \mathbb{R}^2, \end{aligned}$$

as follows: We assume $(v, \zeta) \in B$ given. By 5.32.vii we can apply 4.17 to obtain $(u, w_H) := -\mathcal{R}_{M[[\zeta]]}(H - w \circ \Pi_{\mathbb{S}_{eq}^2})$. We define then $\phi \in C^{2,\beta}(M[[\zeta]])$ by $\phi := v \circ F_\zeta^{-1} + u$. We have then

(j). $\mathcal{L}u + H = (w + w_H) \circ \Pi_{\mathbb{S}_{eq}^2}$.

(k). By the definition of B, 5.32.iii, 4.17, 4.15, and 5.29.v we obtain

$$\|w_H : C^{0,\beta}(\mathbb{S}_{eq}^2, g)\| + \|\phi\|_{2,\beta,\gamma;M[[\zeta]]} \leq \tau^{1+\alpha/4}.$$

Applying 4.24 and 4.17 we obtain $(u_Q, w_Q) := -\mathcal{R}_{M[[\zeta]]}(H_\phi - H - \mathcal{L}\phi)$ which satisfies the following:

(l). $\mathcal{L}u_Q + H_\phi = H + \mathcal{L}\phi + w_Q \circ \Pi_{\mathbb{S}_{eq}^2}$.

(m). $\|w_Q : C^{0,\beta}(\mathbb{S}_{eq}^2, g)\| + \|u_Q\|_{2,\beta,\gamma;M[[\zeta]]} \leq \tau^{2+\alpha/4}$.

(n). $\mathcal{L}(u_Q - v \circ F_\zeta^{-1}) + H_\phi = (w + w_H + w_Q) \circ \Pi_{\mathbb{S}_{eq}^2}$, which follows by combining the definition of ϕ with (j) and (l).

This motivates us to define

$$\mathcal{J}(v, \zeta) = \left(u_Q \circ F_\zeta, \zeta + \frac{1}{\tau[\zeta, m]} (\mu_{sum} \phi_1, \mu'_{sum}) \right),$$

where $\mu_{sum}W + \mu'_{sum}W' = w + w_H + w_Q$.

Step 4: The fixed point argument: By using (k), (m), 5.29.vi and 5.32.iv, and by choosing \underline{c}_1 large enough in terms of an absolute constant, it is straightforward to check that $\mathcal{J}(B) \subset B$. B is clearly a compact convex subset of $C_{sym}^{2,\beta'}(M[[\mathbf{0}]]) \times \mathbb{R}^2$ for $\beta' \in (0, \beta)$, and it is easy to check that \mathcal{J} is a continuous map in the induced topology. By Schauder's fixed point theorem [3, Theorem 11.1] then, there is a fixed point $(\widehat{v}, \widehat{\zeta})$ of \mathcal{J} , which, therefore, satisfies $\widehat{v} = \widehat{u}_Q \circ F_{\widehat{\zeta}}$ and $\widehat{w} + \widehat{w}_H + \widehat{w}_Q = 0$, where we use “ $\widehat{}$ ” to denote the various quantities for $\zeta = \widehat{\zeta}$ and $v = \widehat{v}$. By (n) then we conclude the minimality of $\widehat{M}_{\widehat{\phi}}$. The smoothness follows from standard regularity theory and the embeddedness from 4.24 and (k). The genus follows because we are connecting two spheres with $2m$ bridges. Finally, the limit of the area as $m \rightarrow \infty$ follows from the available estimates for $\varphi_{init}[L[\widehat{x}_1, m], \widehat{\tau}, \widehat{w}]$ and the bound on the norm of $\widehat{\phi}$. q.e.d.

Theorem 7.3 (The equator and poles case). *There is an absolute constant $\underline{c}_2 > 0$ such that if m is large enough depending on \underline{c}_2 , then there is $\widehat{\zeta} = (\widehat{\zeta}_0, \widehat{\zeta}_2) \in \mathbb{R}^2$ satisfying 6.22 such that (in the notation of 6.23 and 6.24) $\widehat{\tau}_j := \tau_j[\widehat{\zeta}, m]$ for $j = 0, 2$, $\widehat{\tau} := \tau[\widehat{\zeta}, m]$, $\widehat{w} := w[\widehat{\zeta}, m]$, and $\varphi[[\widehat{\zeta}, m]]$, satisfy 5.32.ii–v, and, moreover, there is $\widehat{\phi} \in C^\infty(\widehat{M})$, where $\widehat{M} := M[L_{eq-pol}[m], \widehat{\tau}, \widehat{w}]$ in the notation of 3.20, such that in the notation of 4.12*

$$\|\widehat{\phi}\|_{2,\beta,\gamma;\widehat{M}} \leq \widehat{\tau}^{1+\alpha/4},$$

and, furthermore, $\widehat{M}_{\widehat{\phi}}$ (in the notation of 4.24) is a genus $m+1$ embedded minimal surface in $\mathbb{S}^3(1)$, which is invariant under the action of $\mathcal{G}_{\mathbb{S}^3,m}$ and has area $\text{Area}(\widehat{M}_{\widehat{\phi}}) \rightarrow 8\pi$ as $m \rightarrow \infty$.

Proof. The proof has the same structure as the proof of 7.1 and so we only provide a brief outline emphasizing the differences:

Step 1: Construction of the diffeomorphisms F_{ζ} : We fix an $m \in \mathbb{N}$ which we assume as large in terms of \underline{c}_2 as needed. We will use the notation $\mathbf{0} := (0, 0) \in \mathbb{R}^2$, $\underline{\tau} := \tau[\mathbf{0}, m]$ taking the value $\underline{\tau}_j := \tau_j[\mathbf{0}, m]$ on L_j for $j = 0, 2$, and for $\zeta \in \mathbb{R}^2$ satisfying 6.22 we write $M[[\zeta]] := M[L_{eq-pol}[m], \tau[\zeta, m], w[\zeta, m]]$ (recall 6.24 and 3.20). It is easy to modify the definition of F_{ζ} in the proof of 7.1 to define for $\zeta \in \mathbb{R}^2$ satisfying 6.22 a smooth diffeomorphism $F_{\zeta} : M[[\mathbf{0}]] \rightarrow M[[\zeta]]$ covariant under the action of $\mathcal{G}_{\mathbb{S}^3,m}$. Note that actually the definition is simpler because L_{eq-pol} does not depend on ζ and, therefore, we can skip the initial steps concerning the diffeomorphisms F'_{x_1} . The substantial step is to define maps analogous to the \widehat{F}_{ζ} 's which were defined in 7.2. In analogy we denote (half) the lengths of the corresponding cylinders by $\ell_{\mathbf{0}}[p]$ and $\ell_{\zeta}[p]$, where $p \in L_{eq-pol}$ is mentioned because the lengths depend on whether p is a pole or on the equator.

Step 2: Equivalence of norms under F_ζ : Using 6.28 and 2.22 it is easy to check that

$$\ell_\zeta[p] \sim_{1+Cm^{-1/2} \log m} \ell_0[p].$$

Using this and arguing as in the proof of 4.13 we conclude that for $u \in C^{2,\beta}(M[[\zeta]])$ and $E \in C^{0,\beta}(M[[\zeta]])$ we have

$$\begin{aligned} \|u \circ F_\zeta\|_{2,\beta,\gamma;M[[0]]} &\sim_2 \|u\|_{2,\beta,\gamma;M[[\zeta]]}, \\ \|E \circ F_\zeta\|_{0,\beta,\gamma-2;M[[0]]} &\sim_2 \|E\|_{0,\beta,\gamma-2;M[[\zeta]]}. \end{aligned}$$

Step 3: The map \mathcal{J} : By applying 6.24.v, 6.28, and 6.21.v we can repeat all definitions and estimates in step 3 of the proof of 7.1, except for using \underline{c}_2 instead of \underline{c}_1 and modifying the definition of \mathcal{J} as follows:

$$\mathcal{J}(v, \zeta) = \left(u_Q \circ F_\zeta, \zeta + \left(\frac{\tilde{\mu}_0(m/2)^{1/2} + \tilde{\mu}_2}{2\tau_0[\zeta, m]}, \frac{-\tilde{\mu}_0(m/2)^{1/2} + \tilde{\mu}_2}{2\tau_0[\zeta, m]} \right) \right),$$

where $\tilde{\mu}_0 W_0 + \tilde{\mu}_2 W_2 = w + w_H + w_Q$.

Step 4: The fixed point argument: Using 6.21.vi and 6.24.ii we can argue in the same way as in the proof of 7.1 to complete the proof. q.e.d.

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