MIN–MAX HYPERSURFACE IN MANIFOLD OF POSITIVE RICCI CURVATURE

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Abstract

In this paper, we study the shape of the min–max minimal hypersurface produced by Almgren–Pitts–Schoen–Simon [**AF62**, **AF65**, **P81**, **SS81**] in a Riemannian manifold (M^{n+1}, g) of positive Ricci curvature for all dimensions. The min–max hypersurface has a singular set of Hausdorff codimension 7. We characterize the Morse index, area and multiplicity of this singular min–max hypersurface. In particular, we show that the min–max hypersurface is either orientable and has Morse index one, or is a double cover of a non-orientable stable minimal hypersurface.

As an essential technical tool, we prove a stronger version of the discretization theorem. The discretization theorem, first developed by Marques–Neves in their proof of the Willmore conjecture [**MN12**], is a bridge to connect sweepouts appearing naturally in geometry to sweepouts used in the min–max theory. Our result removes a critical assumption of [**MN12**], called the no mass concentration condition, and hence confirms a conjecture by Marques–Neves in [**MN12**].

1. Introduction

Given an $(n + 1)$-dimensional closed Riemannian manifold M^{n+1}, minimal hypersurfaces are critical points of the area functional. When M has certain topology, a natural way to produce minimal hypersurface is to minimize area among its homology class. This idea leads to the famous existence and regularity theory for area minimizing hypersurfaces by De Giorgi, Federer, Fleming, Almgren and Simons, etc. (cf. [**FH**, **Gi**, **Si83**]). In general cases, when every hypersurface is homologically trivial, e.g., if the Ricci curvature of the ambient manifold is positive, the minimization method fails. This motivates F. Almgren [**AF62**, **AF65**], followed up by J. Pitts [**P81**], to develop a Morse theoretical method for the area functional in the space of hypersurfaces, namely the min–max theory. The heuristic idea of developing a Morse theory is to associate a nontrivial 1-cycle in the space of hypersurfaces

with a critical point of the area functional, i.e., a minimal hypersurface. In particular, denote $\mathcal{Z}_n(M)$ by the space of all closed hypersurfaces with a natural topology in geometric measure theory, called the flat topology. Now consider a one-parameter family $\Phi : [0,1] \to \mathcal{Z}_n(M)$. Let $[\Phi]$ be the set of all maps $\Psi : [0,1] \to \mathcal{Z}_n(M)$ which are homotopic to Φ in $\mathcal{Z}_n(M)$. The min–max value can be associated with $[\Phi]$ as

$$(1.1) \qquad \mathbf{L}([\Phi]) = \inf \big\{ \max_{x \in [0,1]} Area(\Psi(x)) : \Psi \in [\Phi] \big\}.$$

Almgren [**AF62**] showed that there is a nontrivial Φ with $\mathbf{L}([\Phi]) > 0$ in any closed manifold M; together with Pitts [**AF65**, **P81**], they showed that when $2 \leq n \leq 5$, there is a disjoint collection of closed, smooth, embedded, minimal hypersurfaces $\{\Sigma_i\}_{i=1}^l$ with integer multiplicity $k_i \in \mathbb{N}$ such that $\sum_{i=1}^l k_i Area(\Sigma_i) = \mathbf{L}([\Phi])$. Schoen and Simon [**SS81**] extended the regularity results to $n \geq 6$. Note that for $n \geq 7$, the min–max hypersurface Σ_i has a singular set of codimension 7. Later on, there are other variations of the Almgren–Pitts min–max theory, cf. [**Sm82**, **CD03**, **DT09**].

  However, besides the existence and regularity, much is unknown about these min–max hypersurfaces. For instance, a natural question is how large can the area and multiplicity be? Moreover, in this Morse theoretical approach, one key open problem, raised by Almgren [**AF65**] and emphasized by F. Marques [**M14**, §4.1] and A. Neves [**N14**, §8], is to bound the Morse index of the min–max minimal hypersurface by the number of parameters. It is conjectured that generically the Morse index is equal to the number of parameters, and the multiplicity is one. The importance of this problem lies in several aspects. First, finding minimal hypersurfaces with bounded (or prescribed) Morse index is a central motivation for Almgren [**AF65**] to develop the min–max theory. Also the bound of Morse index plays an important role in application to geometric problems. In his famous open problems section [**Y**, Problems 29 and 30], S. T. Yau stressed the importance of the estimates of Morse index in several conjectures. In the recent celebrated proof of the Willmore conjecture by Marques and Neves [**MN12**], a key part is to prove that the Morse index of certain min–max minimal surface in the standard three-sphere is bounded by 5. The major challenge of bounding the Morse index comes from the fact that the min–max hypersurface is constructed as a very weak limit (i.e., varifold limit), therefore, classical methods in nonlinear analysis (cf. [**St00**]) do not extend to this situation. Here one difficulty of understanding the weak limit is due to the existence of multiplicity (see [**I95**] for similar issue in studying the singularity of mean curvature flow).

  The current progress of understanding the min–max hypersurfaces mainly focused on the case of one-parameter families. Marques and

Neves [**MN11**] have confirmed the Morse index conjecture in three dimension when the Ricci curvature of the ambient manifold is positive, where they proved the existence of minimal Heegaard surface of Morse index 1 in certain 3-manifolds. This was extended to manifold M^{n+1} with positive Ricci curvature in dimensions $2 \leq n \leq 6$, when the min–max hypersurfaces are smooth, by the author [**Z12**]. In [**Z12**], we also gave a general characterization of the multiplicity, area and Morse index of the min–max hypersurface. In particular, the min–max hypesurface is either orientable of Morse index 1, or is a double cover of a nonorientable least area minimal hypersurface. Recently, the methods in [**MN11**, **Z12**] were used by Mazet and Rosenberg [**MR15**] to study the minimal hypersurfaces of least area in an arbitrary closed Riemannian manifold M^{n+1} with $2 \leq n \leq 6$. They gave several characterizations of the least area minimal hypersurfaces similar to [**Z12**]. The work in this paper will generalize the characterization of the min–max hypersurface to all dimensions, even allowing singularities. Several new ingredients are developed to deal with the presence of singularities.

Let (M^{n+1}, g) be an $(n+1)$-dimensional, connected, closed, orientable Riemannian manifold. We consider singular hypersurfaces which share the same regularity properties as the min–max hypersurfaces. To be precise, we set up some terminology. By a *singular hypersurface* with a singular set of Hausdorff co-dimension no less than k ($k \in \mathbb{N}, k < n$), we mean a closed subset $\overline{\Sigma}$ of M with finite n-dimensional Hausdorff measure $\mathcal{H}^n(\overline{\Sigma}) < \infty$, where the *regular part* of $\overline{\Sigma}$ is defined as:

$$reg(\Sigma) = \{x \in \overline{\Sigma} : \ \overline{\Sigma} \text{ is a smooth, embedded, hypersurface near } x\};$$

and the *singular part* of Σ is $sing(\Sigma) = \overline{\Sigma} \backslash reg(\Sigma)$ (see [**SS81**, **I96**]), with the $(n-k+\epsilon)$-dimensional Hausdorff measure $\mathcal{H}^{n-k+\epsilon}\big(sing(\Sigma)\big) = 0$ for all $\epsilon > 0$. Clearly the regular part $reg(\Sigma)$ is an open subset of $\overline{\Sigma}$. Later on, we will denote $\Sigma = reg(\Sigma)$ and also call Σ a singular hypersurface. Given such a singular hypersurface Σ, it represents an integral varifold, denoted by $[\Sigma]$ (cf. [**Si83**, §15]). We say Σ is *minimal* if $[\Sigma]$ is stationary (cf. [**Si83**, 16.4]). In fact, this is equivalent to the fact that the mean curvature of $reg(\Sigma)$ is zero and the density of $[\Sigma]$ is finite everywhere (cf. [**I96**, (3)(4)]). To simplify the presentation, in the following we simply assume that the tangent cones (cf. [**Si83**, §42]) of $[\Sigma]$ have multiplicity one everywhere (which is satisfied by min–max hypersurfaces by Lemma 6.3). We use $\mathrm{Ind}(\Sigma)$ to denote the Morse index of Σ (see §2.4). Denote

(1.2)
$$\mathcal{S} = \{\Sigma^n : \ \overline{\Sigma} \text{ is a connected, closed, minimal, hypersurface with a}$$
$$\text{singular set } sing(\Sigma) \text{ of Hausdorff co-dimension no less than 7}\}.$$

Let

$$(1.3) \qquad A_M = \inf_{\Sigma \in \mathcal{S}} \left\{ \begin{array}{ll} \mathcal{H}^n(\Sigma), & \text{if } \Sigma \text{ is orientable} \\ 2\mathcal{H}^n(\Sigma), & \text{if } \Sigma \text{ is non-orientable} \end{array} \right\}.$$

If the Ricci curvature of M is positive, then the min–max hypersurface has only one connected component (Theorem 2.10), and we denote it by Σ. Our main result is as follows:

Theorem 1.1. *Assume that the Ricci curvature of M is positive; then the min–max hypersurface Σ*

  (i) *either is orientable of multiplicity one, which has Morse index $Ind(\Sigma) = 1$, and $\mathcal{H}^n(\Sigma) = A_M$;*
 (ii) *or is non-orientable with multiplicity two, which is stable, i.e., $Ind(\Sigma) = 0$, and $2\mathcal{H}^n(\Sigma) = A_M$.*

REMARK 1.2. The fact that $\mathcal{H}^n(\Sigma) = A_M$ or $2\mathcal{H}^n(\Sigma) = A_M$ says that the min–max hypersurface has least area among all singular minimal hypersurfaces (if counting non-orientable minimal hypersurface with multiplicity two).

The main idea contains two parts. First, given a minimal hypersurface Σ, we will embed Σ into a one parameter family $\{\Sigma_t\}_{t \in [-1,1]}$ with $\Sigma_0 = \Sigma$, such that the area of Σ achieves a strict maximum, i.e., $Area(\Sigma_t) < Area(\Sigma)$ if $t \neq 0$. Second, we will show that all of such one parameter families obtained in this way (from a minimal hypersurface) belong to the same homotopy class. Then from the definition of the min–max value (1.1), the family $\{\Sigma_t\}$ corresponding to the min–max hypersurface Σ must be optimal, i.e., $\max_t Area(\Sigma_t) \leq \max_t Area(\Sigma'_t)$, where $\{\Sigma'_t\}$ is generated by any other minimal hypersurface Σ' in the first step. The characterization of Morse index, multiplicity and area of Σ will then follow from this optimality condition. Specifically, in the first part, we will choose the one parameter family as the level sets of the distance function to Σ. Note that the minimal hypersurface Σ has a singular set of Hausdorff codimension 7. To deal with the presence of singularities, we will use an idea explored by Gromov [**Gr**] in his study of isoperimetric inequalities. To show the homotopic equivalence of these one parameter families, we need to use an isomorphism constructed by Almgren in [**AF62**], under which the homotopy groups of the space of hypersurfaces in M are mapped isometrically to the homology groups of M.

One main difficulty is caused by the fact that two different topology are used on the space of hypersurfaces $\mathcal{Z}_n(M)$. The geometric method in the first part produces families of hypersurfaces which are continuous under the flat topology. However, the Almgren–Pitts min–max theory works under another topology, called the mass norm topology, which is much stronger than the flat topology. A bridge is desired to connect

the two topology. In fact, this is a very common problem in the study of min–max theory (cf. [**MN12**, **Z12**, **MN13**, **Mo14**]). Pitts already developed some tools in his book [**P81**]. Marques–Neves, in their proof of the Willmore conjecture [**MN12**], first gave a complete theory to connect families continuous under flat topology to families satisfying the requirement of the Almgren–Pitts setting (see also [**Z12**, **MN13**, **Mo14**]). Marques–Neves need a critical technical assumption for the starting family, called no mass concentration condition, which means that there is no point mass in the measure-theoretical closure of the family. However, in our situation the one parameter family does not necessarily satisfy the no mass concentration condition due to the presence of singular set. In fact, in the same paper [**MN12**, §13.2], Marques–Neves conjectured that this assumption might not be necessary. Here we verify this conjecture under a very general condition. As this improvement will be useful in other situation, we present it here (in a simplified form).

Theorem 1.3. *(See Theorem 5.1 for a detailed version) Given a continuous (under the flat topology) one parameter family of hypersurfaces* $\Phi : [0,1] \to \mathcal{Z}_n(M)$*, such that for each* $x \in [0,1]$*,* $\Phi(x)$ *is represented by the boundary of some set* $\Omega_x \subset M$ *of finite perimeter, and such that* $\max_x Area\big(\Phi(x)\big) < \infty$*, then there exists a* $(1, \mathbf{M})$*-homotopy sequence* $\{\phi_i\}$ *(one parameter family in the sense of Almgren–Pitts, cf. §4.1), satisfying*

$$\max_x Area\big(\Phi(x)\big) = \limsup_{i \to \infty} \max_x Area(\phi_i(x)).$$

REMARK 1.4. The key step is to develop a new discretization procedure to connect the given family to a new family which satisfy the no mass concentration condition (see Lemma 5.8 and the discussions there). Under the same condition that the hypersurfaces are represented by boundary of sets of finite perimeter, the above result is also true for multi-parameter families.

The paper is organized as follows. In Section 2, we give several preliminary results concerning the topology, second variation and Morse index for singular hypersurfaces in a manifold of positive Ricci curvature. In Section 3, we show that the level sets of distance function to a singular minimal hypersurface is a good one parameter family. In Section 4, we introduce the Almgren–Pitts theory. In Section 5, we prove Theorem 1.3. Finally, we prove Theorem 1.1 in Section 6.

2. Preliminary results

In this section, we give several preliminary results about minimal hypersurfaces with a singular set of Hausdorff dimension less to or equal than $n - 7$.

2.1. Notions of geometric measure theory. For notions in geometric measure theory, we refer to [**Si83**] and [**P81**, §2.1].

Fix a connected, closed, oriented Riemannian manifold (M^{n+1}, g) of dimension $n + 1$. Assume that (M^{n+1}, g) is embedded in some $\mathbb{R}^N$ for N large. We denote by

- $\mathbf{I}_k(M)$ the space of k-dimensional integral currents in $\mathbb{R}^N$ with support in M;
- $\mathcal{Z}_k(A, B)$ the space of integral currents $T \in \mathbf{I}_k(M)$, with $spt(T) \subset A^1$ and $spt(\partial T) \subset B^2$, where A, B are compact subset of M, and $B \subset A$;
- $\mathcal{Z}_k(M)$ the space of integral currents $T \in \mathbf{I}_k(M)$ with $\partial T = 0$;
- $\mathcal{V}_k(M)$ the closure, in the weak topology, of the space of k-dimensional rectifiable varifolds in $\mathbb{R}^N$ with support in M;
- $\mathcal{F}$ and $\mathbf{M}$, respectively, the flat norm [**Si83**, §31] and mass norm [**Si83**, 26.4] on $\mathbf{I}_k(M)$;
- $\mathcal{C}(M)$ the space of sets $\Omega \subset M$ with finite perimeter [**Si83**, §14][**Gi**, §1.6].

Given $T \in \mathbf{I}_k(M)$, $|T|$ and $\|T\|$ denote, respectively, the integral varifold and Radon measure in M associated with T. $\mathbf{I}_k(M)$ and $\mathcal{Z}_k(M)$ are in general assumed to have the flat norm topology. $\mathbf{I}_k(M, \mathbf{M})$ and $\mathcal{Z}_k(M, \mathbf{M})$ are the same space endowed with the mass norm topology. Given $T \in \mathcal{Z}_k(M)$, $\mathcal{B}_s^{\mathcal{F}}(T)$ and $\mathcal{B}_s^{\mathbf{M}}(T)$ denote, respectively, balls in $\mathcal{Z}_k(M)$ centered at T, of radius s, under the flat norm $\mathcal{F}$ and the mass norm $\mathbf{M}$. Given a closed, orientable hypersurface Σ in M with a singular set of Hausdorff dimension no larger than $(n-7)$, or a set $\Omega \in \mathcal{C}(M)$ with finite perimeter, we use $[[\Sigma]], [[\Omega]]$ to denote the corresponding integral currents with the natural orientation, and $[\Sigma], [\Omega]$ to denote the corresponding integer-multiplicity varifolds.

2.2. Nearest point projection to $\overline{\Sigma}$. Here we recall the fact that the nearest point projection of any point in M to $\overline{\Sigma}$ (away from the singular set of Σ) is a regular point of $\overline{\Sigma}$ when Σ is minimal. Similar result for isoperimetric hypersurfaces appeared in [**Gr**].

Lemma 2.1. *Let $\Sigma \in \mathcal{S}$ be a singular minimal hypersurface in M. Take a point $p \in M \backslash \overline{\Sigma}$, and a minimizing geodesic γ connecting p to $\overline{\Sigma}$ in M, i.e., $\gamma(0) = p$, $\gamma(1) = q \in \overline{\Sigma}$, and $length(\gamma) = dist(p, \overline{\Sigma})$. Then q is a regular point of Σ.*

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[1] $spt(T)$ denotes the support of T [**Si83**, 26.11].
[2] $\partial T \in \mathbf{I}_{n-1}(M)$ denotes the boundary of T [**Si83**, 26.3].

Proof. Take the geodesic sphere of M center at $\gamma(\frac{1}{2})$ with radius $\frac{1}{2}dist(p,\overline{\Sigma})$. The sphere is a smooth hypersurface near q, and $\overline{\Sigma}$ lies in one side of the sphere. So the tangent cone of $\overline{\Sigma}$ (viewed as a rectifiable varifold with multiplicity 1 by assumption) at q is contained in a half-space of $\mathbb{R}^{n+1}$ (separated by the tangent plane of the sphere). As $\overline{\Sigma}$ is stationary, by [**Si83**, 36.5, 36.6], the tangent cone of $\overline{\Sigma}$ at q is equal to the tangent plane of the sphere (with multiplicity 1), and hence $\overline{\Sigma}$ is smooth at q by the Allard Regularity Theorem (cf. [**Al72**][**Si83**, 24.2]).
q.e.d.

2.3. Connectedness. For stationary hypersurface with a small singular set, the connectedness of the closure is the same as the connectedness of the regular part. In fact, this follows from the strong maximum principle for stationary singular hypersurfaces.

Theorem 2.2. [**I96**, Theorem A]

1) *If V_1 and V_2 are stationary integer rectifiable n-varifolds in an open subset $\Omega \subset M^{n+1}$, satisfying*

$$\mathcal{H}^{n-2}(spt(V_1) \cap spt(V_2) \cap \Omega) = 0,$$

   then $spt(V_1) \cap spt(V_2) \cap \Omega = \emptyset$.
2) *Assume that Σ is a stationary hypersurface in Ω with a singular set of Hausdorff dimension less than $n - 2$. If $\overline{\Sigma} \cap \Omega$ is connected, then $reg(\Sigma) \cap \Omega$ is connected.*

REMARK 2.3. By part 2, the closure of a singular hypersurface in our setting is connected if and only if the regular part is.

Definition 2.4. A singular minimal hypersurface Σ (with $dim(sing(\Sigma)) \leq n - 7$) is *connected* if its regular part is connected.

2.4. Orientation, second variation and Morse index.

Definition 2.5. A singular hypersurface Σ is *orientable* (or *non-orientable*) if the regular part is orientable (or non-orientable).

A singular hypersurface Σ is said to be *two-sided* if the normal bundle $\nu(\Sigma)$ of the regular part Σ inside M is trivial.

Lemma 2.6. *Let M^{n+1} be an $(n+1)$-dimensional, connected, closed, orientable manifold, and $\Sigma \subset M$ a connected, singular hypersurface with $dim(sing(\Sigma)) \leq n-2$, and with compact closure $\overline{\Sigma}$. Then Σ is orientable if and only if Σ is two-sided.*

Proof. The tangent bundle of M, when restricted to Σ, has a splitting into the tangent bundle $T\Sigma$ and normal bundle $\nu(\Sigma)$ of Σ, i.e., $TM|_{\Sigma} = T\Sigma \oplus \nu(\Sigma)$. By [**H**, Lemma 4.1], $T\Sigma$ is orientable if and only if $\nu(\Sigma)$

is orientable. By [**H**, Theorem 4.3][3] , $\nu(\Sigma)$ is orientable if and only if $\nu(\Sigma)$ is trivial.   q.e.d.

When Σ is two-sided, there exists a unit normal vector field ν. The *Jacobi operator* is

$$(2.1) \qquad L_\Sigma\phi = \triangle_\Sigma\phi + \big(Ric(\nu,\nu) + |A|^2\big)\phi,$$

where $\phi \in C_c^1(\Sigma)$, $\triangle_\Sigma$ is the Laplacian operator of the induced metric on Σ, and A is the second fundamental form of Σ along ν. Given an open subset Ω of Σ with smooth boundary $\partial\Omega$, we say that $\lambda \in \mathbb{R}$ is a *Dirichlet eigenvalue* of L_Σ on Ω if there exists a non-zero function $\phi \in C_0^\infty(\Omega)$ vanishing on $\partial\Omega$, i.e., $\phi|_{\partial\Omega} \equiv 0$, such that $L_\Sigma\phi = -\lambda\phi$. The *(Dirichlet) Morse index* of Ω, denoted by $\text{Ind}_D(\Omega)$, is the number of negative Dirichlet eigenvalues of L_Σ on Ω counted with multiplicity.

When Σ is non-orientable, we need to pass to the orientable double cover $\tilde{\Sigma}$ of Σ. Then there exists a unit normal vector field $\tilde{\nu}$ along $\tilde{\Sigma}$, satisfying $\tilde{\nu} \circ \tau = -\tilde{\nu}$, where $\tau : \tilde{\Sigma} \to \tilde{\Sigma}$ is the orientation-reversing involution, such that $\Sigma = \tilde{\Sigma}/\{id, \tau\}$. The *Jacobi operator* $L_{\tilde{\Sigma}}$ is well-defined using $\tilde{\nu}$. Given an open subset $\Omega \subset \Sigma$, and its lift-up $\tilde{\Omega}$ to $\tilde{\Sigma}$, we can define the *Dirichlet eigenvalue* and *(Dirichlet) Morse index* by restricting the Jacobi operator $L_{\tilde{\Sigma}}$ to functions $\tilde{\phi} \in C_0^1(\tilde{\Omega})$ which are anti-symmetric under τ, i.e., $\tilde{\phi} \circ \tau = -\tilde{\phi}$. (In this case, $\tilde{\phi}\tilde{\nu}$ descends to a vector field on Σ.) We refer to [**Ro**] for more discussions on Morse index in the non-orientable case.

Definition 2.7. The *Morse index* of Σ is defined as,

$$\text{Ind}(\Sigma) = \sup\{\text{Ind}_D(\Omega) : \ \Omega \text{ is any open subset of } \Sigma$$
$$\text{with smooth boundary}\}.$$

Σ is called *stable* if $\text{Ind}\Sigma \geq 0$, or equivalently, Σ is stable in the classical sense on any compactly supported open subsets.

2.5. Positive Ricci curvature. We need two properties for singular minimal hypersurfaces in manifolds of positive Ricci curvature. The first one says that there is no stable, two-sided, singular hypersurface with a small singular set. This generalizes an easy classical result for smooth hypersurfaces [**CM11**, Chap 1.8]. When Σ is two-sided, the fact that Σ is stable is equivalent to the following stability inequality:

$$(2.2) \qquad \int_\Sigma \big(Ric_g(\nu,\nu) + |A_\Sigma|^2\big)\varphi^2 d\mathcal{H}^n \leq \int_\Sigma |\nabla\varphi|^2 d\mathcal{H}^n,$$

for any $\varphi \in C_c^\infty(\Sigma)$.

Lemma 2.8. [**S10**] *Assume that (M^{n+1}, g) has positive Ricci curvature, i.e., $Ric_g > 0$, and Σ is a singular minimal hypersurface, with $\mathcal{H}^{n-2}(sing(\Sigma)) = 0$. If Σ is two-sided, then Σ is not stable.*

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[3]It is not hard to see that Σ is paracompact, so [**H**, Theorem 4.3] is applicable.

Proof. Suppose that Σ is stable. Since $\mathcal{H}^{n-2}(sing(\Sigma)) = 0$, for any $\epsilon > 0$, we can take a countable covering $\cup_i B_{r_i}(p_i)$ of $sing(\Sigma)$ using geodesics balls $\{B_{r_i}(p_i)\}_{i\in\mathbb{N}}$ of M, such that

$$\sum_{i\in\mathbb{N}} r_i^{n-2} < \epsilon.$$

For each i, we can choose a smooth cutoff function f_i, such that $f_i = 1$ outside $B_{2r_i}(p_i)$, $f_i = 0$ inside $B_{r_i}(p_i)$, and $|\nabla f_i| \leq \frac{2}{r_i}$ inside the annulus $B_{2r_i}(p_i)\backslash B_{r_i}(p_i)$. Let f_ϵ be the minimum of all f_i's (which is Lipschitz), and plug it into the stability inequality (2.2),

$$\int_\Sigma \left(Ric(\nu,\nu)+|A_\Sigma|^2\right)f_\epsilon^2 d\mathcal{H}^n \leq \int_\Sigma |\nabla f_\epsilon|^2 d\mathcal{H}^n$$

$$\leq 4\sum_{i\in\mathbb{N}} \int_{\Sigma\cap B_{2r_i}(p_i)} \frac{1}{r_i^2}d\mathcal{H}^n \leq 4\sum_{i\in\mathbb{N}} \frac{1}{r_i^2}\cdot Cr_i^n \leq 4C\epsilon.$$

Here we used the monotonicity formula [**Si83**, 17.6] to get the volume bound $\mathcal{H}^n(\Sigma\cap B_{2r_i}(p_i)) \leq Cr_i^n$ in the third "$\leq$". Now let ϵ tend to zero, we get a contradiction to the fact that $Ric(\nu,\nu) > 0$.   q.e.d.

REMARK 2.9. If we only require $Ric_g \geq 0$, the above proof will show that the stable hypersurface must be smooth and totally geodesic, and the restriction of Ric_g to Σ is zero.

The second property says that any two such singular minimal hypersurfaces in manifold with positive Ricci curvature must intersect, which generalizes the classical Frankel's theorem [**Fr66**] for smooth minimal hypersurfaces.

Theorem 2.10. *(Generalized Frankel Theorem) Assume that (M^{n+1}, g) has positive Ricci curvature. Given any two connected, singular, minimal hypersurfaces Σ and Σ' with singular sets of Hausdorff co-dimension no less than 2, then $\overline{\Sigma}$ and $\overline{\Sigma'}$ must intersect on a set of Hausdorff dimension no less than $n-2$. Therefore, $\Sigma\cap\Sigma' \neq \emptyset$.*

Proof. First if $\overline{\Sigma}\cap\overline{\Sigma'} = \emptyset$, then we can find two points $p \in \overline{\Sigma}$, $p' \in \overline{\Sigma'}$, such that $d(p,p') = dist(\overline{\Sigma},\overline{\Sigma'})$. By the argument as in Lemma 2.1, both p, p' are regular points of Σ, Σ'. Then as in [**Fr66**, §2], we can get a contradiction by looking at the second variational formula of the length functional along the minimizing geodesic connecting p to p' when (M, g) has positive Ricci curvature.

Then $\overline{\Sigma} \cap \overline{\Sigma'} \neq \emptyset$, so Theorem 2.2 implies that $\overline{\Sigma} \cap \overline{\Sigma'}$ must have Hausdorff dimension no less than $n-2$.   q.e.d.

2.6. Orientation and singular hypersurfaces. Now we list a few properties related to the orientation of singular hypersurfaces. Similar properties for smooth hypersurfaces were discussed in [**Z12**, §3].

Proposition 2.11. *Given a connected, minimal, singular hypersurface Σ^n with a singular set of Hausdorff dimension less than $n-2$, then*

1) *Σ is orientable if and only if $\overline{\Sigma}$ represents an integral n-cycle.*
2) *If $\overline{\Sigma}$ separates M, i.e., $M\backslash\overline{\Sigma}$ contains two connected components, then Σ is orientable.*
3) *When M has positive Ricci curvature, if Σ is orientable, then $\overline{\Sigma}$ separates M.*

Proof. Part 1. $\overline{\Sigma}$ is a rectifiable set, and when Σ is orientable, it can represent an integer-multiplicity rectifiable current $[\overline{\Sigma}]$ as follows:

$$[\overline{\Sigma}](\omega) = \int_\Sigma \langle \xi(x), \omega(x)\rangle d\mathcal{H}^n = \int_\Sigma \omega,$$

where $\xi(x)$ is the orientation form of Σ, and ω is any smooth n-form on M. Now we will show that $[\overline{\Sigma}]$ is a cycle, i.e., $\partial[\overline{\Sigma}] = 0$. Given any smooth $(n-1)$-form ω on M, take the sequence of cutoff functions f_ϵ, $\epsilon \to 0$, as in the proof of Lemma 2.8,

$$\partial[\overline{\Sigma}](\omega) = [\overline{\Sigma}](d\omega) = \int_\Sigma d\omega = \lim_{\epsilon\to 0}\int_\Sigma f_\epsilon d\omega$$

$$= \lim_{\epsilon\to 0}\int_\Sigma d(f_\epsilon \omega) - df_\epsilon \wedge \omega.$$

The first term is zero by the Stokes Theorem, and the second term can be estimated as:

$$\left|\int_\Sigma df_\epsilon \wedge \omega\right| \leq \int_\Sigma |df_\epsilon \wedge \omega| d\mathcal{H}^n \leq C\sum_{i\in\mathbb{N}}\int_{\Sigma\cap B_{r_i}(p_i)}\frac{1}{r_i}d\mathcal{H}^n \leq C\sum_{i\in\mathbb{N}}r_i^{n-1} \to 0.$$

Now assume that $\overline{\Sigma}$ represents an integral cycle, and we will show that Σ is orientable. In fact, assume that $[\overline{\Sigma}] = \langle \Sigma, \xi(x), \theta(x) = 1\rangle$ is an integral cycle, where $\xi(x)$ is locally an orientation form. Given any open subset $U \subset M\backslash sing(\Sigma)$, then $\partial\big([\overline{\Sigma}]\llcorner U\big) = 0$ in U by definition. By the same argument in [**Z12**, Proposition 6, Claim 4], $[\overline{\Sigma}]\llcorner U$ represents an integral n-cycle in $\Sigma \cap U$, hence by the Constancy Theorem [**Si83**, 26.27], $[\overline{\Sigma}]\llcorner U = [\Sigma \cap U]$. Let U exhaust the whole regular part Σ, then $[\overline{\Sigma}]\llcorner(M\backslash sing(\Sigma)) = [\Sigma]$; hence the orientation of $[\overline{\Sigma}]$ gives a global orientation of Σ.

Part 2. The case for smooth $\overline{\Sigma}$ is given in [**H**, §4 Theorem 4.5]. Now we modify the proof to our case. Take a connected component U of $M\backslash\overline{\Sigma}$, the (topological) boundary ∂U of U is then a closed subset of $\overline{\Sigma}$. By using local coordinate charts of $(M, \overline{\Sigma})$ around any smooth point of Σ, it is easy to see that $\partial U \cap \Sigma$ is an open subset of Σ. Hence as a subset of Σ, $\partial U \cap \Sigma$ is both open and closed, so $\partial U \cap \Sigma = \Sigma$ since Σ is connected, and then $\partial U = \overline{\Sigma}$. Using the same argument as in [**H**, Theorem 4.2], the orientation of U induces an orientation for the normal

bundle N of the regular part of ∂U, i.e., Σ. Note the splitting of the tangent bundle TM restricted on Σ: $TM|_\Sigma = T\Sigma \bigoplus N$; hence $T\Sigma$ is orientable by [**H**, Lemma 4.1].

Part 3. By Part 1, $\overline{\Sigma}$ represents an integral cycle $[\overline{\Sigma}]$, hence it represents an integral homology class $[[\overline{\Sigma}]]$ in $H_n(M, \mathbb{Z})$ [**FH**, 4.4.1]. If $\overline{\Sigma}$ does not separate M, i.e., $M\backslash\overline{\Sigma}$ is connected, we claim that $[[\overline{\Sigma}]]$ is non-trivial in $H_n(M, \mathbb{Z})$. In fact, if $[[\overline{\Sigma}]] = 0$, then there exists an integral $(n + 1)$-current $C \in \mathbf{I}_{n+1}(M, \mathbb{Z})$, such that $\partial C = [\overline{\Sigma}]$. Given any connected open subset $U \subset M\backslash\overline{\Sigma}$, then $\partial(C\llcorner U) = 0$ in U by definition. The Constancy Theorem [**Si83**, 26.27] implies that $C\llcorner U = m[U]$, for some $m \in \mathbb{Z}$, where $[U]$ denotes the integral $(n+1)$-current represented by U. As $M\backslash\overline{\Sigma}$ is connected ($\overline{\Sigma}$ does not separate M), we can take $U = M\backslash\overline{\Sigma}$, and hence $C\llcorner(M\backslash\overline{\Sigma}) = m[M\backslash\overline{\Sigma}]$. As $\overline{\Sigma}$ has zero $(n + 1)$-dimensional Hausdorff measure, then $C = m[M]$, hence $\partial C = m\partial[M] = 0$, which is a contradiction to the fact that $\partial C = [\overline{\Sigma}]$.

Now we can take the mass minimizer $T_0 \in [[\overline{\Sigma}]]$ inside the homology class [**FH**, 4.4.4][**Si83**, 34.3]. The codimension one regularity theory ([**Si83**, Theorem 37.7]) says that T_0 is represented by a minimal hypersurface $\overline{\Sigma}_0$ (possibly with multiplicity) with a singular set of Hausdorff dimension no larger than $n - 7$, i.e., $T_0 = m[\overline{\Sigma}_0]$, where $m \in \mathbb{Z}$, $m \neq 0$. Since $m[\overline{\Sigma}_0]$ represents a nontrivial integral homology class, Σ_0 is orientable by Part 1. Hence Σ_0 is two-sided by Lemma 2.6. By the nature of mass minimizing property of T, Σ_0 must be locally volume minimizing, and hence Σ_0 is stable, contradicting the positive Ricci curvature condition via Lemma 2.8. q.e.d.

3. Min–max family

In this section, by using the volume comparison result in [**HK**], we show that every singular minimal hypersurface in a manifold with positive Ricci curvature lies in a nice "mountain-pass" type family. In particular, the family sweeps out the whole manifold, and the area of the minimal hypersurface (when it is orientable), or the area of its double cover (when the hypersurface is non-orientable) achieves a strict maximum among the family. Actually, in manifold with positive Ricci curvature, the level sets of distance function towards the singular minimal hypersurface will play the role.

3.1. A volume comparison result in [HK]. Let (M^{n+1}, g) be a closed, oriented manifold. Given a singular minimal hypersurface $\Sigma \in \mathcal{S}$, denote $\nu(\Sigma)$ by the normal bundle of the regular part Σ in M. Let $exp_\nu : \nu(\Sigma) \to M$ be the normal exponential map. Given $\xi \in \nu(\Sigma)$, the *focal distance* in the direction of ξ means the first time $t > 0$ such that the derivative of the normal exponential map at $t\xi$, i.e., $dexp_\nu(t\xi)$, becomes degenerate. Denote Ω by the sets of all vectors ξ in $\nu(\Sigma)$, which is

no longer than the diameter of M or the focal distance in the direction of ξ.

Lemma 3.1. $exp_\nu : \Omega \to M \backslash sing(\Sigma)$ *is surjective.*

Proof. Any point $x \in M \backslash \overline{\Sigma}$ can be connected to $\overline{\Sigma}$ by a minimizing geodesic. Also by Lemma 2.1, the nearest point of x in $\overline{\Sigma}$ is a regular point of Σ; then the minimizing geodesic meets Σ orthogonally, and hence exp_ν is surjective to $M \backslash sing(\Sigma)$. Moreover, if ξ is the tangent vector of the minimizing geodesic (parametrized on $[0,1]$) connecting x to Σ, then the length of ξ is no more than the focal distance in the direction of ξ.   q.e.d.

Now we will introduce a Riemannian metric on $\nu(\Sigma)$ (see also [**HK**, §3]), such that $\nu(\Sigma)$ is locally isomorphic to the product of Σ with the fiber. Let $\pi : \nu(\Sigma) \to \Sigma$ be the projection map. Denote D by the Riemannian connection of M, and $D^\perp$ the normal connection of $\nu(\Sigma)$. The tangent bundle of $\nu(\Sigma)$ can be split as a sum of "vertical" and "horizontal" sub-bundles $T\nu(\Sigma) = V + H$ as follows. Given $\xi \in \nu(\Sigma)$, the vertical tangent space V_ξ contains tangent vectors of $\nu(\Sigma)$ which are tangent to the fibers and hence killed by π_*, so V_ξ is canonically isometric to the fiber space $\nu_{\pi(\xi)}(\Sigma)$. The horizontal tangent space H_ξ contains tangent vectors of $\nu(\Sigma)$ which are tangent to $D^\perp$-parallel curves—viewed as vector fields along their base curves (projected to Σ by π), so H_ξ is canonically isometric to $T_{\pi(\xi)}\Sigma$ under π_*. The metric on $\nu(\Sigma)$ can be defined as:

$$\|v\|^2 = \|\pi_* v\|^2 + \|v_{ver}\|^2, \quad v \in T_\xi \nu(\Sigma),$$

where v_{ver} denotes the vertical component of v. It is easily seen that under this metric, $\nu(\Sigma)$ is locally isometric to the product of Σ with the fibers.

We need the following estimate of the volume form along normal geodesics by [**HK**, §3]. Fix $p \in \Sigma$ and a normal vector $\xi \in \nu_p(\Sigma)$. Given an orthonormal basis $e_1, \cdots, e_n$ of $T_p\Sigma$, they can be lifted up to $T\nu(\Sigma)$ as horizontal vector fields $u_1(s), \cdots, u_n(s)$ along the normal vectors $s\xi$. By our construction above, $u_1(s), \cdots, u_n(s)$ form an orthonormal basis of $T_{s\xi}\nu(\Sigma)$, as $\pi_*(u_i(s)) = e_i$. The distortion of the n-dimensional volume element under the normal exponential map $exp_\nu : T\nu(\Sigma) \to M$ is given by $\|dexp_\nu u_1(s) \wedge \cdots \wedge dexp_\nu u_n(s)\|$. Assume that the Ricci curvature of (M, g) satisfies $Ric_g \geq n\Lambda$ for some $\Lambda > 0$. Consider an $(n+1)$-dimensional manifold $\tilde{M}$ of constant curvature Λ, and a totally geodesic hypersurface $\tilde{\Sigma}$. Fix an arbitrary point $\tilde{p} \in \tilde{\Sigma}$, with a unit normal $\nu(\tilde{p})$. Choose an orthonormal basis $\tilde{e}_1, \cdots, \tilde{e}_n$ of $T_{\tilde{p}}(\tilde{\Sigma})$, and a frame $\tilde{u}_1(s), \cdots, \tilde{u}_n(s)$ along $s\nu(\tilde{p})$ constructed as above. We have the following comparison estimates:

Lemma 3.2. [**HK**, §3.2.1, Case (d)]. *Let s_0 be no larger than the first focal distance of Σ in the direction of ξ, then for $0 \le s \le s_0$,*

$$\|dexp_\nu u_1(s) \wedge \cdots \wedge dexp_\nu u_n(s)\| \le \|dexp_\nu \tilde{u}_1(s) \wedge \cdots \wedge dexp_\nu \tilde{u}_n(s)\|.$$

It is easy to calculate that the n-dimensional volume distortion of the constant curvature manifold $\tilde{M}$ is given by $\|dexp_\nu \tilde{u}_1(s) \wedge \cdots \wedge dexp_\nu \tilde{u}_n(s)\| = \cos^n(\sqrt{\Lambda}s)\|dexp_\nu \tilde{u}_1(0) \wedge \cdots \wedge dexp_\nu \tilde{u}_n(0)\| = \cos^n(\sqrt{\Lambda}s)$.

Corollary 3.3. *Under the above setting,*

$$\|dexp_\nu u_1(s) \wedge \cdots \wedge dexp_\nu u_n(s)\| \le \cos^n(\sqrt{\Lambda}s).$$

3.2. Orientable case. Let $\Sigma \in \mathcal{S}$ be orientable, then Σ is two-sided. Denote ν by the unit normal vector field along Σ. When $Ric_g > 0$, $\overline{\Sigma}$ separates M by Proposition 2.11, i.e., $M\backslash\overline{\Sigma} = M_1 \cup M_2$. Now the signed distance function $d_\pm^\Sigma$ is well-defined by

$$(3.1) \qquad d_\pm^\Sigma(x) = \begin{cases} dist(x, \overline{\Sigma}), & \text{if } x \in M_1, \\ -dist(x, \overline{\Sigma}), & \text{if } x \in M_2, \\ 0, & \text{if } x \in \overline{\Sigma}. \end{cases}$$

Consider the levels sets of the signed distance function: $\Sigma_t = \{x \in M : d_\pm^\Sigma(x) = t\}$ for $-d(M) \le t \le d(M)$. Denote

$$(3.2) \qquad \mathcal{S}_+ = \{\Sigma^n \in \mathcal{S} : \Sigma^n \text{ is orientable}\}.$$

We collect several properties of the distance family as follows:

Proposition 3.4. *Assume that $Ric_g > 0$. For any $\Sigma \in \mathcal{S}_+$, the distance family $\{\Sigma_t\}_{t \in [-d(M), d(M)]}$ satisfy that:*

(a) $\Sigma_0 = \Sigma$;

(b) $\mathcal{H}^n(\Sigma_t) \le \mathcal{H}^n(\Sigma)$, *with equality only if $t = 0$;*

(c) *For any open set $U \subset M\backslash sing(\Sigma)$ with compact closure $\overline{U}$, $\{\Sigma_t \llcorner U\}_{t \in [-\epsilon, \epsilon]}$ forms a smooth foliation of a neighborhood of Σ in U, i.e.,*

$$\Sigma_t \llcorner U = \{exp_\nu(t\nu(x)) : x \in \Sigma \cap U\}, \quad t \in [-\epsilon, \epsilon].$$

Proof. (a) is trivial by construction.

To prove (b), consider the height-t section $S_t(\Sigma) = \{\xi \in \nu(\Sigma) : \xi = t\nu\}$ of $\nu(\Sigma)$ for $-d(M) \le t \le d(M)$.

Lemma 3.5. *Under the canonical metric of $\nu(\Sigma)$, $S_t(\Sigma)$ is isometric to Σ.*

Proof. First, it is easy to see that the projection map $\pi : \nu(\Sigma) \to \Sigma$ restricts to be a one to one map $\pi : S_t(\Sigma) \to \Sigma$. Also the tangent plane $T_\xi S_t(\Sigma)$ of $S_t(\Sigma)$ at $\xi = t\nu$ consists all horizontal vectors of $T_\xi \nu(\Sigma)$. Then $\pi_* : T_\xi S_t(\Sigma) \to T_{\pi(\xi)}\Sigma$ gives the isometry by the construction of the metric on $\nu(\Sigma)$. q.e.d.

Recall that $exp_\nu : \Omega \subset \nu(\Sigma) \to M \backslash sing(\Sigma)$ is surjective, so the pre-image $exp_\nu^{-1}(\Sigma_t)$ is totally contained in $S_t(\Sigma) \cap \Omega$, and hence by Corollary 3.3,

$$
\mathcal{H}^n(\Sigma_t) \leq \int_{S_t(\Sigma) \cap \Omega} \|(dexp_\nu)_* dvol_{S_t(\Sigma)}\|
$$

(3.3)
$$
= \int_{S_t(\Sigma) \cap \Omega} \|dexp_\nu u_1(s) \wedge \cdots \wedge dexp_\nu u_n(s)\|
$$

$$
\leq \int_\Sigma \cos^n(\sqrt{\Lambda}t) d\mathcal{H}^n \leq \cos^n(\sqrt{\Lambda}t)\mathcal{H}^n(\Sigma).
$$

To prove (c), we first realize that $\nu(\Sigma)$ is globally isometric to $\Sigma \times \mathbb{R}$ when Σ is orientable, so that $\nu(\Sigma)$ has a global smooth foliation structure. When restricted to the zero section, the normal exponential map $exp_\nu : \nu(\Sigma) \to M$ is the identity map, and has non-degenerate tangent map. As the closure $\overline{U}$ is a compact subset of $M \backslash sing(\Sigma)$, we can use the Inverse Function Theorem to infer that exp_ν is a diffeomorphism in a small neighborhood of $exp_\nu^{-1}(\Sigma \cap U)$. Hence (c) follows.   q.e.d.

3.3. Non-orientable case. Given $\Sigma \in \mathcal{S}$ non-orientable, $\overline{\Sigma}$ does not separate M by Proposition 2.11. Denote $d^\Sigma(x) = dist(x, \overline{\Sigma})$ by the distance function (without sign). Consider the level sets of d^Σ: $\Sigma_t = \{x \in M : d^\Sigma(x) = t\}$ for $0 \leq t \leq d(M)$. Denote

(3.4) $\mathcal{S}_- = \{\Sigma^n \in \mathcal{S} : \Sigma^n \text{ is non-orientable}\}.$

We have:

Proposition 3.6. *Assume that $Ric_g > 0$. For any $\Sigma \in \mathcal{S}_-$, the distance family $\{\Sigma_t\}_{0 \leq 0 \leq d(M)}$ satisfy that:*
  (a) $\Sigma_0 = \Sigma$;
  (b) $\mathcal{H}^n(\Sigma_t) < 2\mathcal{H}^n(\Sigma)$, for all $0 \leq t \leq d(M)$;
  (c) When $t \to 0$, $\mathcal{H}^n(\Sigma_t) \to 2\mathcal{H}^n(\Sigma)$, and Σ_t converge smoothly to a double cover of Σ in any open set $U \subset M \backslash sing(\Sigma)$ with compact closure $\overline{U}$.

Proof. (a) is by construction.

For (b), let the height-t section of $\nu(\Sigma)$ be $\tilde{S}_t(\Sigma) = \{\xi \in \nu(\Sigma) : |\xi| = t\}$ for $0 \leq t \leq d(M)$. Similar as the proof of Lemma 3.5, the projection map $\pi : \tilde{S}_t(\Sigma) \to \Sigma$ is locally isometric. Also as the fiber of $\nu(\Sigma)$ is one dimensional, π is a 2-to-1 map. Hence $\pi : \tilde{S}_t(\Sigma) \to \Sigma$ is an isometric double cover. The pre-image of the exponential map $exp_\nu^{-1}(\Sigma_t)$ is then contained in $\tilde{S}_t \cap \Omega$, with Ω as above. By the volume comparison estimates in (3.3),

$$
\mathcal{H}^n(\Sigma_t) \leq \int_{\tilde{S}_t(\Sigma) \cap \Omega} \|(dexp_\nu)_* dvol_{\tilde{S}_t(\Sigma)}\| \leq 2 \int_\Sigma \cos^n(\sqrt{\Lambda}t) d\mathcal{H}^n
$$

(3.5)
$$
\leq 2 \cos^n(\sqrt{\Lambda}t)\mathcal{H}^n(\Sigma).
$$

For (c), to prove that $\mathcal{H}^n(\Sigma_t) \to 2\mathcal{H}^n(\Sigma)$, as $t \to 0$, by (3.5), we only need to prove that $\lim_{t \to 0} \mathcal{H}^n(\Sigma_t) \geq 2\mathcal{H}^n(\Sigma)$, and this follows from the smooth convergence $\Sigma_t \to 2\Sigma$ on any open set $U \subset\subset M \backslash sing(\Sigma)$. By similar argument as Proposition 3.4(c), when restricted to a small neighborhood of $exp_\nu^{-1}(\Sigma \cap U)$, $exp_\nu : \nu(\Sigma) \to M$ is a diffeomorphism. Therefore, the convergence $\Sigma_t \to 2\Sigma$ on U follows from the fact that $\tilde{S}_t(\Sigma)$ converge smoothly to a double cover of the zero section, as $t \to 0$.

<div style="text-align: right">q.e.d.</div>

4. Almgren–Pitts min–max theory

In this section, we will introduce the min–max theory developed by Almgren and Pitts [**AF62**, **AF65**, **P81**]. We will mainly follow [**Z12**, §4] [**P81**, 4.1] and [**MN12**, §7 and §8]. We refer to §2.1 for the notions of Geometric Measure Theory. At the end of this section, we will recall the characterization of the orientation structure of the min–max hypersurfaces proved by the author in [**Z12**].

4.1. Homotopy sequences.

Definition 4.1. (Cell complex.)
1) For $m \in \mathbb{N}$, $I^m = [0,1]^m$, $I_0^m = \partial I^m = I^n \backslash (0,1)^m$;
2) For $j \in \mathbb{N}$, $I(1,j)$ is the cell complex of I, whose 1-cells are all intervals of form $[\frac{i}{3^j}, \frac{i+1}{3^j}]$, and 0-cells are all points $[\frac{i}{3^j}]$; $I(m,j) = I(1,j) \otimes \cdots \otimes I(1,j)$ (m times) is a cell complex on I^m;
3) For $p \in \mathbb{N}$, $p \leq m$, $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_m$ is a p-cell if for each i, α_i is a cell of $I(1,j)$, and $\sum_{i=1}^m dim(\alpha_i) = p$. 0-cell is called a vertex;
4) $I(m,j)_p$ denotes the set of all p-cells in $I(m,j)$, and $I_0(m,j)_p$ denotes the set of p-cells of $I(m,j)$ supported on I_0^m;
5) Given a p-cell $\alpha \in I(m,j)_p$, and $k \in \mathbb{N}$, $\alpha(k)$ denotes the p-dimensional sub-complex of $I(m,j+k)$ formed by all cells contained in α. For $q \in \mathbb{N}$, $q \leq p$, $\alpha(k)_q$ and $\alpha_0(k)_q$ denote, respectively, the set of all q-cells of $I(m,j+k)$ contained in α, or in the boundary of α;
6) $T(m,j) = I(m-1,j) \otimes \{[1]\}$, $B(m,j) = I(m-1,j) \otimes \{[0]\}$ and $S(m,j) = I_0(m-1,j) \otimes I(1,j)$ denote the top, bottom and side sub-complexes of $I(m,j)$, respectively;
7) The boundary homeomorphism $\partial : I(m,j) \to I(m,j)$ is given by

$$\partial(\alpha_1 \otimes \cdots \otimes \alpha_m) = \sum_{i=1}^m (-1)^{\sigma(i)} \alpha_1 \otimes \cdots \otimes \partial\alpha_i \otimes \cdots \otimes \alpha_m,$$

where $\sigma(i) = \sum_{l<i} dim(\alpha_l)$, $\partial[a,b] = [b] - [a]$ if $[a,b] \in I(1,j)_1$, and $\partial[a] = 0$ if $[a] \in I(1,j)_0$;
8) The distance function $\mathbf{d} : I(m,j)_0 \times I(m,j)_0 \to \mathbb{N}$ is defined as $\mathbf{d}(x,y) = 3^j \sum_{i=1}^m |x_i - y_i|$;

9) The map $\mathbf{n}(i,j) : I(m,i)_0 \to I(m,j)_0$ is defined as: $\mathbf{n}(i,j)(x) \in I(m,j)_0$ is the unique element of $I(m,j)_0$, such that $\mathbf{d}\big(x, \mathbf{n}(i,j)(x)\big) = \inf\big\{\mathbf{d}(x,y) : y \in I(m,j)_0\big\}$.

As we are mainly interested in applying the Almgren–Pitts theory to the 1-parameter families, in the following of this section, our notions will be restricted to the case $m = 1$.

Consider a map to the space of integral cycles: $\phi : I(1,j)_0 \to \mathcal{Z}_n(M^{n+1})$. The *fineness of* ϕ is defined as:

$$(4.1) \qquad \mathbf{f}(\phi) = \sup\left\{\frac{\mathbf{M}\big(\phi(x) - \phi(y)\big)}{\mathbf{d}(x,y)} : x, y \in I(1,j)_0, x \neq y\right\}.$$

$\phi : I(1,j)_0 \to \big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$ denotes a map such that $\phi\big(I(1,j)_0\big) \subset \mathcal{Z}_n(M^{n+1})$ and $\phi|_{I_0(1,j)_0} = 0$, i.e., $\phi([0]) = \phi([1]) = 0$.

Definition 4.2. Given $\delta > 0$ and $\phi_i : I(1,k_i)_0 \to \big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$, $i = 1, 2$, we say ϕ_1 *is 1-homotopic to* ϕ_2 *in* $\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$ *with fineness* δ, if $\exists\, k_3 \in \mathbb{N}$, $k_3 \geq \max\{k_1, k_2\}$, and

$$\psi : I(1,k_3)_0 \times I(1,k_3)_0 \to \mathcal{Z}_n(M^{n+1}),$$

such that

- $\mathbf{f}(\psi) \leq \delta$;
- $\psi([i-1], x) = \phi_i\big(\mathbf{n}(k_3, k_i)(x)\big)$, $i = 1, 2$;
- $\psi\big(I(1,k_3)_0 \times I_0(1,k_3)_0\big) = 0$.

Definition 4.3. A $(1, \mathbf{M})$-*homotopy sequence of mappings into* $\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$ is a sequence of mappings $\{\phi_i\}_{i \in \mathbb{N}}$,

$$\phi_i : I(1,k_i)_0 \to \big(\mathcal{Z}_n(M^{n+1}), \{0\}\big),$$

such that ϕ_i is 1-homotopic to ϕ_{i+1} in $\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$ with fineness δ_i, and

- $\lim_{i\to\infty} \delta_i = 0$;
- $\sup_i \big\{\mathbf{M}(\phi_i(x)) : x \in I(1,k_i)_0\big\} < +\infty$.

Definition 4.4. Given two $(1, \mathbf{M})$-homotopy sequences of mappings $S_1 = \{\phi_i^1\}_{i \in \mathbb{N}}$ and $S_2 = \{\phi_i^2\}_{i \in \mathbb{N}}$ into $\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$, S_1 *is homotopic with* S_2 if $\exists\, \{\delta_i\}_{i \in \mathbb{N}}$, such that

- ϕ_i^1 is 1-homotopic to ϕ_i^2 in $\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$ with fineness δ_i;
- $\lim_{i\to\infty} \delta_i = 0$.

The relation "is homotopic with" is an equivalent relation on the space of $(1, \mathbf{M})$-homotopy sequences of mapping into $\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$ (see [**P81**, §4.1.2]). An equivalent class is a $(1, \mathbf{M})$ *homotopy class of mappings into* $\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$. Denote the set of all equivalent classes by $\pi_1^{\#}\big(\mathcal{Z}_n(M^{n+1}, \mathbf{M}), \{0\}\big)$. Similarly we can define the $(1, \mathcal{F})$-homotopy class (using another fineness associated with the $\mathcal{F}$-norm in

place of the $\mathbf{M}$-norm in (4.1)), and denote the set of all equivalent classes by $\pi_1^\#\big(\mathcal{Z}_n(M^{n+1}, \mathcal{F}), \{0\}\big)$.

4.2. Almgren's isomorphism. Almgren [**AF62**] showed that the homotopy groups of $\mathcal{Z}_n(M)$ (under $\mathbf{M}$ and $\mathcal{F}$ topology) are all isomorphic to the top homology group of M by constructing an isomorphism as follows.

By [**AF62**, Corollary 1.14], there exists a small number $\nu_M > 0$ (depending only on M), such that for any two n-cycles $T_1, T_2 \in \mathcal{Z}_n(M^{n+1})$, if $\mathcal{F}(T_2 - T_1) \leq \nu_M$, then there exists an $(n + 1)$-dimensional integral current $Q \in \mathbf{I}_{n+1}(M)$, with $\partial Q = T_2 - T_1$, and $\mathbf{M}(Q) = \mathcal{F}(T_2 - T_1)$. Q is called the *isoperimetric choice* for $T_2 - T_1$.

Given $\phi : I(1, k)_0 \to \mathcal{Z}_n(M^{n+1})$, with $\mathbf{f}(\phi) \leq \delta \leq \nu_M$, then for any 1-cell $\alpha \in I(1, k)_1$, with $\alpha = [t_\alpha^1, t_\alpha^2]$, $\mathcal{F}\big(\phi(t_\alpha^1) - \phi(t_\alpha^2)\big) \leq \mathbf{M}\big(\phi(t_\alpha^1) - \phi(t_\alpha^2)\big) \leq \mathbf{f}(\phi) \leq \nu_M$. So there exists an isoperimetric choice $Q_\alpha \in \mathbf{I}_{n+1}(M^{n+1})$, with

$$\mathbf{M}(Q_\alpha) = \mathcal{F}\big(\phi(t_\alpha^1) - \phi(t_\alpha^2)\big), \text{ and } \partial Q_\alpha = \phi(t_\alpha^2) - \phi(t_\alpha^1).$$

Now the sum of the isoperimetric choices for all 1-cells is an $(n + 1)$-dimensional integral current, i.e., $\sum_{\alpha \in I(1,k)_1} Q_\alpha \in \mathbf{I}_{n+1}(M^{n+1})$. We call the map:

$$(4.2) \qquad\qquad F_A : \phi \to \sum_{\alpha \in I(1,k)_1} Q_\alpha$$

Almgren's isomorphism (the name comes from Theorem 4.5).

Given a $(1, \mathbf{M})$-homotopy sequence of mappings $S = \{\phi_i\}_{i \in \mathbb{N}}$ into $\big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$, take i large enough, and $\phi_i : I(1, k_i)_0 \to \big(\mathcal{Z}_n(M^{n+1}), \{0\}\big)$, such that $\mathbf{f}(\phi_i) \leq \delta_i \leq \nu_M$. Then

$$F_A(\phi_i) = \sum_{\alpha \in I(1,k_i)_1} Q_\alpha$$

is an $(n + 1)$-dimensional integral cycle as $\phi_i([0]) = \phi_i([1]) = 0$, and represents an $(n + 1)$-dimensional integral homology class

$$\big[\sum_{\alpha \in I(1,k_i)_1} Q_\alpha \big] \in H_{n+1}(M^{n+1}).$$

Moreover, Almgren [**AF62**, §3.2] showed that this homology class depends only on the homotopy class of $\{\phi_i\}$. Hence it reduces to a map

$$F_A : \pi_1^\#\big(\mathcal{Z}_n(M^{n+1}, \mathbf{M}), \{0\}\big) \to H_{n+1}(M^{n+1}),$$

defined in [**AF62**, §3.2] as:

$$(4.3) \qquad\qquad F_A : [\{\phi_i\}_{i \in \mathbb{N}}] \to \big[\sum_{\alpha \in I(1,k_i)_1} Q_\alpha \big].$$

Almgren also proved that this mapping is an isomorphism.

Theorem 4.5. ([**AF62**, Theorem 13.4] and [**P81**, Theorem 4.6]) *The followings are all isomorphic under F_A:*

$$H_{n+1}(M^{n+1}), \ \pi_1^{\#}\big(\mathcal{Z}_n(M^{n+1}, \mathbf{M}), \{0\}\big), \ \pi_1^{\#}\big(\mathcal{Z}_n(M^{n+1}, \mathcal{F}), \{0\}\big).$$

We also call this map *Almgren's isomorphism.*

4.3. Existence of min–max hypersurface.

Definition 4.6. (Min–max definition) Given $\Pi \in \pi_1^{\#}\big(\mathcal{Z}_n(M^{n+1}, \mathbf{M}), \{0\}\big)$, define:

$$\mathbf{L} : \Pi \to \mathbb{R}^{+},$$

as a function given by:

$$\mathbf{L}(S) = \mathbf{L}(\{\phi_i\}_{i \in \mathbb{N}})$$
$$= \limsup_{i \to \infty} \max \big\{\mathbf{M}\big(\phi_i(x)\big) : \ x \text{ lies in the domain of } \phi_i\big\}.$$

The *width of* Π is defined as

(4.4) $$\mathbf{L}(\Pi) = \inf\{\mathbf{L}(S) : \ S \in \Pi\}.$$

$S \in \Pi$ is call a *critical sequence*, if $\mathbf{L}(S) = \mathbf{L}(\Pi)$. Let $K : \Pi \to \{\text{compact subsets of } \mathcal{V}_n(M^{n+1})\}$ be defined by

$$K(S) = \{V : \ V = \lim_{j \to \infty} |\phi_{i_j}(x_j)| : \ x_j \text{ lies in the domain of } \phi_{i_j}\}.$$

A *critical set* of S is $C(S) = K(S) \cap \{V : \ \mathbf{M}(V) = \mathbf{L}(S)\}$.

The celebrated min–max theorem of Almgren–Pitts (Theorem 4.3, 4.10, 7.12, Corollary 4.7 in [**P81**]) and Schoen–Simon (for $n \geq 6$ [**SS81**, Theorem 4]) is as follows:

Theorem 4.7. *Given a nontrivial $\Pi \in \pi_1^{\#}\big(\mathcal{Z}_n(M^{n+1}, \mathbf{M}), \{0\}\big)$, then $\mathbf{L}(\Pi) > 0$, and there exists a stationary integral varifold V, whose support is a disjoint collection of connected, closed, singular, minimal hypersurfaces $\{\Sigma_i\}_{i=1}^{l}$, with singular sets of Hausdorff dimension no larger than $n - 7$, (which may have multiplicity, say m_i), such that $V = \sum_{i=1}^{l} m_i[\Sigma_i]$, and*

$$\sum_{i=1}^{l} m_i \mathcal{H}^n(\Sigma_i) = \mathbf{L}(\Pi).$$

In particular, V lies in the critical set $C(S)$ of some critical sequence $S \in \Pi$.

4.4. Orientation and multiplicity.
As V lies in the critical set $C(S)$, V is a varifold limit of a sequence of integral cycles $\{\phi_{i_j}(x_j)\}_{j \in \mathbb{N}}$. It has been conjectured that V should inherit some orientation structures from $\{\phi_{i_j}(x_j)\}_{j \in \mathbb{N}}$. In fact, we verified this conjecture and gave a characterization of the orientation structure of V in low dimensions (where the support of V is the smooth) in [**Z12**, Proposition 6.1]. Some straight-

forward modifications of the proof will give similar characterization for singular min–max hypersurfaces (in all dimensions) as follows.

Proposition 4.8. *Let V be the stationary varifold in Theorem 4.7, with $V = \sum_{i=1}^{l} m_i[\Sigma_i]$. If Σ_i is non-orientable, then the multiplicity m_i must be an even number.*

REMARK 4.9. When a connected component Σ_i is orientable, it represents an integral cycle by Proposition 2.11. While a connected component Σ_i is non-orientable, an even multiple of it also represents an integral cycle—a zero cycle. This result will play a key role in the characterization of the multiplicity in Theorem 1.1. (This result was also used in [**Z12**, **MR15**] to characterize the multiplicity of min–max hypersurfaces).

5. Discretization and construction of sweepouts

The purpose of this section is to adapt the families of currents constructed by geometric method (in §3) to the Almgren–Pitts setting (in §4). Usually families constructed by geometric method are continuous under the flat norm topology, but the Almgren–Pitts theory applies only to discrete family continuous under the mass norm topology. Therefore, we need to discretize our families and to make them continuous under the mass norm topology. Similar issue was also an essential technical difficulty in the celebrated proof of the Willmore conjecture [**MN12**], and also in a previous paper by the author [**Z12**] which deals with the same problem in low dimensions. A key technical condition in these discretization type theorems in [**MN12**, **MN13**, **Z12**] is the no local mass concentration assumption. Roughly speaking, it means that the weak measure-theoretical closure of the family of currents does not contain any point mass. However, the families used here do not necessarily satisfy this technical assumption, so we will build up a stronger version of the discretization theorem without assuming the no mass concentration condition. Actually, this issue was originally considered by Pitts [**P81**, §3.5, §3.7] in another setting. Our strategy is motivated by Pitts's method, and is simpler than Pitts's discretization procedure. In this paper, we only deal with families of currents which are boundaries of sets of finite perimeter. This is already enough for the purpose of many geometric applications, as all the known interesting geometric families (cf. [**MN12**, **MN13**, **Z12**]) belong to this class. In fact, it is conjectured by Marque and Neves [**MN12**, §13.2] that the no mass concentration assumption is not necessary, and our result confirms this conjecture in the co-dimension one case. For the purpose of simplicity, we only present the discretization theorem for one-parameter families. The case for multi-parameter families is still true by similar arguments as in [**MN12**, Theorem 13.1] using our technical results Proposition 5.3

and Proposition 5.10 in place of [**MN12**, Proposition 13.3, 13.5], and will be addressed elsewhere.

Another key ingredient which utilizes the big machinery by Almgren–Pitts is an identification type result. We will show that all the discretized families corresponding to those families constructed in §3 belong to the same homotopy class in the sense of Almgren–Pitts. This type of result was proved in [**Z12**] under the no mass concentration assumption, and we will extend this identification type result to the case without no mass concentration assumption. We prove this by showing that the image of the discretized families under the Almgren's isomorphism represent the top homology class of M. Then these families must be homotopic to each other by Theorem 4.5.

The main result can be summarized as the following theorem. Recall that $\mathcal{C}(M)$ is consisted by all subsets of M of finite perimeter.

Theorem 5.1. *Given a continuous mapping*

$$\Phi : [0,1] \to \big(\mathcal{Z}_n(M^{n+1}, \mathcal{F}), \{0\}\big),$$

satisfying

(a) $\Phi(x) = \partial[[\Omega_x]]$, $\Omega_x \in \mathcal{C}(M)$, *for all* $x \in [0,1]$;
(b) $\sup_{x \in [0,1]} \mathbf{M}(\Phi(x)) < \infty$;

then there exists a $(1, \mathbf{M})$*-homotopy sequence*

$$\phi_i : I(1, k_i)_0 \to \big(\mathcal{Z}_n(M^{n+1}, \mathbf{M}), \{0\}\big),$$

and a sequence of homotopy maps

$$\psi_i : I(1, k_i)_0 \times I(1, k_i)_0 \to \mathcal{Z}_n(M^{n+1}, \mathbf{M}),$$

with $k_i < k_{i+1}$, *and* $\{\delta_i\}_{i \in \mathbb{N}}$ *with* $\delta_i > 0$, $\delta_i \to 0$, *and* $\{l_i\}_{i \in \mathbb{N}}$, $l_i \in \mathbb{N}$ *with* $l_i \to \infty$, *such that* $\psi_i([0], \cdot) = \phi_i$, $\psi_i([1], \cdot) = \phi_{i+1}|_{I(1,k_i)_0}$, *and*

(i)

$$\mathbf{M}\big(\phi_i(x)\big) \leq \sup\big\{\mathbf{M}\big(\Phi(y)\big) : x, y \in \alpha, \text{ for some 1-cell } \alpha \in I(1, l_i)\big\} + \delta_i,$$

and hence

(5.1) $$\mathbf{L}(\{\phi_i\}_{i \in \mathbb{N}}) \leq \sup_{x \in [0,1]} \mathbf{M}\big(\Phi(x)\big);$$

(ii) $\mathbf{f}(\psi_i) < \delta_i$;
(iii) $\sup\big\{\mathcal{F}\big(\phi_i(x) - \Phi(x)\big) : x \in I(1, k_i)_0\big\} < \delta_i$;
(iv) *If* $\Omega_0 = \emptyset$, $\Omega_1 = M$, *then*

$$F_A(\{\phi_i\}) = [[M]],$$

where F_A *is the Almgren's isomorphism, and* $[[M]]$ *is the fundamental class of* M.

REMARK 5.2. The proof of properties $(i)(ii)(iii)$ is based on the proof of [**MN12**, Theorem 13.1] and [**P81**, §3.5, 3.7]. The idea to deal with the existence of mass concentration is motivated by [**P81**, §3.5, 3.7]. We actually simplify the discretization procedure in [**P81**, §3.5] for currents which can be represented by boundary of sets of finite perimeter using some new observations (cf. Lemma 5.8). The proof of property (iv) is based on the ideas in [**Z12**, Theorem 5.8].

Upon first perusal of this section, the reader might skip the following technical proof and move to §6.

5.1. Technical preliminaries. The following two technical results are parallel to [**MN12**, Propositions 13.3, 13.5], while without assuming the no mass concentration condition.

The first result is parallel to [**MN12**, 13.3], and it says that given $T \in \mathcal{Z}_n(M^{n+1})$, and $l, m \in N$, there exists $k \in \mathbb{N}$, $k > l$, such that any ϕ which maps $I_0(m, l)_0$ into a small neighborhood of T (with respect to the flat topology) can be extended to a map $\tilde{\phi}$ which maps $I(m, k)_0$ into a slightly larger neighborhood of T (with respect to the flat topology), such that the fineness and maximal mass of $\tilde{\phi}$ are not much bigger than those of ϕ. Compared to [**MN12**, 13.3], we do not require the no mass concentration condition, but we need to assume that the image of ϕ are represented by boundary of sets of finite perimeter. Also, the extension $\tilde{\phi}$ will be mapped to a slightly large neighborhood. The idea to deal with the mass concentration traces back to [**P81**, 3.5]. We will first deform ϕ to certain local cones around the mass concentration points (cf. Lemma 5.8), and then apply similar extension process as [**MN12**, 13.3].

Fix an integer $n_0 \in \mathbb{N}$.

Proposition 5.3. *Given $\delta, L > 0$, $l, m \in \mathbb{N}$, $m \leq n_0 + 1$, and*

$$T \in \mathcal{Z}_n(M) \cap \{S : \mathbf{M}(S) \leq 2L\}, \text{with } T = \partial[[\Omega_T]],$$

$\Omega_T \in \mathcal{C}(M)$, then there exist $0 < \epsilon = \epsilon(l, m, T, \delta, L) < \delta$, and $k = k(l, m, T, \delta, L) \in \mathbb{N}$, $k > l$, and a function $\rho = \rho_{(l,m,T,\delta,L)} : \mathbb{R}_+^1 \to \mathbb{R}_+^1$, with $\rho(s) \to 0$, as $s \to 0$, such that: for any $0 < s < \epsilon$, and

$$(5.2) \quad \phi : I_0(m, l)_0 \to \mathcal{B}_s^{\mathcal{F}}(T) \cap \{S : \mathbf{M}(S) \leq 2L\}, \text{with } \phi(x) = \partial[[\Omega_x]],$$

$\Omega_x \in \mathcal{C}(M)$, $x \in I_0(m, l)_0$, there exists

$$\tilde{\phi} : I(m, k)_0 \to \mathcal{B}_{\rho(s)}^{\mathcal{F}}(T), \text{with } \tilde{\phi}(y) = \partial[[\Omega_y]],$$

$\Omega_y \in \mathcal{C}(M)$, $y \in I(m, k)_0$, and satisfying

(i) $\mathbf{f}(\tilde{\phi}) \leq \delta$ if $m = 1$, and $\mathbf{f}(\tilde{\phi}) \leq m(\mathbf{f}(\phi) + \delta)$ if $m > 1$;
(ii) $\tilde{\phi} = \phi \circ \mathbf{n}(k, l)$ on $I_0(m, k)_0$;
(iii)

$$\sup_{x \in I(m,k)_0} \mathbf{M}\big(\tilde{\phi}(x)\big) \leq \sup_{x \in I_0(m,l)_0} \mathbf{M}\big(\phi(x)\big) + \frac{\delta}{n_0 + 1};$$

(iv) If $m = 1$, $\delta < \nu_M$,[4] $\phi([0]) = \partial[[\Omega_0]]$, $\phi([1]) = \partial[[\Omega_1]]$, then

$$F_A(\tilde{\phi}) = [[\Omega_1 - \Omega_0]],$$

where F_A is the Almgren's isomorphism (4.2).

REMARK 5.4. (i) controls the fineness of the extension $\tilde{\phi}$; (ii) says that on the boundary vertices $I_0(m,k)_0$ of the cell complex $I(m,k)$, the extension $\tilde{\phi}$ directly inherits from ϕ; (iii) controls the increase of the mass; (iv) calculates the image of $\tilde{\phi}$ under the Almgren's isomorphism when $m = 1$.

Proof. We use the contradiction argument. If the statement is not true, by Section 7.1, there exists $k_0 \in \mathbb{N}$ large enough, $\rho_0 > 0$, and a sequence of $\epsilon_k < 1/k$, and

$$\phi_k : I_0(m,l)_0 \to \mathcal{B}^{\mathcal{F}}_{\epsilon_k}(T) \cap \{S : \mathbf{M}(S) \leq 2L\},$$

$\phi_k(x) = \partial[[\Omega^k_x]]$, $\Omega^k_x \in \mathcal{C}(M)$, such that there is no extension $\tilde{\phi}_k$ of ϕ_k from $I(m,k_0)$ to $\mathcal{B}^{\mathcal{F}}_{\rho_0}(T)$, i.e., $\tilde{\phi}_k : I(m,k_0)_0 \to \mathcal{B}^{\mathcal{F}}_{\rho_0}(T)$, satisfying all the above properties (i)(ii)(iii)(iv).

The next lemma is an analog to [**MN12**, Lemma 13.4] without assuming the no mass concentration condition, and uses some new ideas motivated from [**P81**, §3.5]. Proposition 5.3 will be proved using Lemma 5.5.

Lemma 5.5. *With ϕ_k, ϵ_k as above, there exist $N = N(l,m,T,\delta,L) \in \mathbb{N}$, $N > l$, and a subsequence $\{\phi_j\}$, and a sequence of positive numbers $\rho_j \to 0$, as $j \to \infty$, such that we can construct*

$$\psi_j : I(1,N)_0 \times I_0(m,l)_0 \to \mathcal{B}^{\mathcal{F}}_{\rho_j}(T),$$

satisfying

(0) $\psi_j(y,x) = \partial[[\Omega^j_{y,x}]]$, $\Omega^j_{y,x} \in \mathcal{C}(M)$, $(y,x) \in I(1,N)_0 \times I_0(m,l)_0$;
(i) $\mathbf{f}(\psi_j) \leq \delta$ if $m = 1$, and $\mathbf{f}(\psi_j) \leq \mathbf{f}(\phi_j) + \delta$ if $m > 1$;
(ii) $\psi_j([0], \cdot) = \phi_j$, $\psi_j([1], \cdot) = T$;
(iii)

$$\sup\{\mathbf{M}(\psi_j(y,x)), (y,x) \in I(1,N)_0 \times I_0(m,l)_0\}$$

$$\leq \sup_{x \in I_0(m,l)_0} \mathbf{M}(\phi_j(x)) + \frac{\delta}{n_0 + 1};$$

(iv) If $m = 1$, $\delta < \nu_M$, $\phi_j([0]) = \partial[[\Omega_{j,0}]]$, $\phi_j([1]) = \partial[[\Omega_{j,1}]]$, then

$$F_A(\psi_j|_{I(1,N)_0 \times \{[0]\}}) = [[\Omega_T - \Omega_{j,0}]], \quad F_A(\psi_j|_{I(1,N)_0 \times \{[1]\}}) = [[\Omega_T - \Omega_{j,1}]],[5]$$

where F_A is the Almgren's isomorphism (4.2).

---

[4] ν_M is defined in Section 4.2.
[5] Here we identify $I(1,N)_0 \times \{[0]\}$ and $(I(1,N))_0 \times \{[1]\}$ with $I(1,N)_0$.

Proof. As a subset in $\mathcal{V}_n(M)$ with uniformly bounded mass is weakly compact, we can find a subsequence $\{\phi_j\}$ of $\{\phi_k\}$, and a map

$$V : I_0(m,l)_0 \to \mathcal{V}_n(M),$$

such that $\lim_{j\to\infty} |\phi_j(x)| = V(x)$ as varifolds, $\|V(x)\|(M) \le 2L$, for all $x \in I_0(m,l)_0$. Also as $\epsilon_j \to 0$, $\lim_{j\to\infty} \phi_j(x) = T$ as currents.

Now we need to separate our discussion into two cases:

Case 1: $\|V(x)\|(p) \le \delta/5$, for all $p \in M$, $x \in I_0(m,l)_0$;

Case 2: The set $S_{con}{}^6 = \{q \in M : \|V(x)\|(q) > \delta/5$ for some $x \in I_0(m,l)_0\} \ne \emptyset$.

Lemma 5.6. *In **Case 1**, there exist $N_1 = N_1(l,m,T,\delta,L) \in \mathbb{N}$, and*

$$\psi_j : I(1,N_1)_0 \times I_0(m,l)_0 \to \mathcal{B}^{\mathcal{F}}_{\epsilon_j}(T),$$

satisfying properties $(0)(i)(ii)(iii)(iv)$ *in Lemma 5.5.*

REMARK 5.7. The proof is a straightforward adaption of [**P81**, 3.7] [**MN12**, Lemma 13.4][**Z12**, Theorem 5.8], so we omit some identical details. See Figure 1 for illustration of notions.

Proof. By the lower semi-continuity of weak convergence $\lim_{j\to\infty} \phi_j(x) \to T$,

$$\|T\|\big(B_r(p)\big) \le \|V(x)\|\big(B_r(p)\big), \quad \forall p \in M, r > 0.$$

As $\|V(x)\|(\{p\}) \le \delta/5$ for all $x \in I_0(m,l)_0$, $p \in M$, we can find a finite collection of pairwise disjoint open balls $\{B_{r_i}(p_i) : 1 \le i \le v\}$, $p_i \in M$, $r_i > 0$, $v \in \mathbb{N}$, such that for all $x \in I_0(m,l)_0$,

Fact 1.   1) $\|T\|\big(B_{r_i}(p_i)\big) \le \|V(x)\|\big(B_{r_i}(p_i)\big) < \delta/3$;

2) $\|T\|\big(M\setminus \cup_{i=1}^v B_{r_i}(p_i)\big) \le \|V(x)\|\big(M\setminus \cup_{i=1}^v B_{r_i}(p_i)\big) < \delta/3$;

3) $\|T\|\big(\partial B_{r_i}(p_i)\big) = \|V(x)\|\big(\partial B_{r_i}(p_i)\big) = 0$;

4) v depends only on l,m,T,δ,L by compactness of varifolds with bounded mass.

By [**AF62**, Corollary 1.14], for $j \gg 1$, $x \in I_0(m,l)_0$, there exists isoperimetric choices $Q_j(x) \in \mathbf{I}_{n+1}(M^{n+1})$, such that

(5.3) $\partial Q_j(x) = \phi_j(x) - T$, $\quad \mathbf{M}\big(Q_j(x)\big) = \mathcal{F}\big(\phi_j(x) - T\big) \le \epsilon_j < 1/j$.

For each $i = 1, \cdots, v$, let $d_i(x) = dist(p_i, x)$ be the distance function to p_i on (M,g). Using the Slicing Theorem [**Si83**, 28.5], for each $i = 1, \cdots, v$, we can find a sequence of positive numbers $\{r_i^j\}$, such that $r_i^j \searrow r_i$, such that for all $x \in I_0(m,l)_0$, the slices $\langle Q_j(x), d_i, r_i^j \rangle \in \mathbf{I}_n(M)$, and

(5.4) $\langle Q_j(x), d_i, r_i^j \rangle = \partial\big(Q_j(x) \llcorner B_{r_i^j}(p_i)\big) - \big(\phi_j(x) - T\big) \llcorner B_{r_i^j}(p_i).$
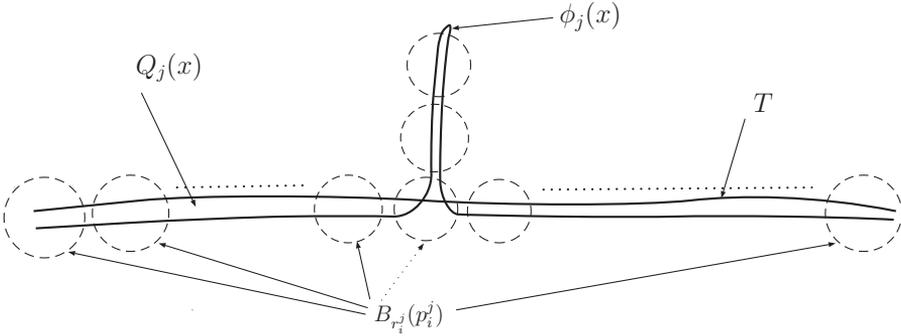

Figure 1. This figure illustrates the geometric objects using in Lemma 5.6.

Also as $\lim_{j\to\infty} \mathbf{M}\big(Q_j(x)\big) = 0$, by [**Si83**, 28.5(1)], we can choose $\{r_i^j\}$ so that for j large enough,

$$(5.5) \qquad \sum_{x\in I_0(m,l)_0} \sum_{i=1}^{v} \mathbf{M}\big(\langle Q_j(x), d_i, r_i^j\rangle\big) \leq \frac{\delta}{2(n_0+1)}.$$

Using Fact 1 and the lower semi-continuity of mass functional, for j large enough,

$$(5.6) \qquad \|\phi_j(x)\|\big(B_{r_i^j}(p_i)\big) < \delta/3, \qquad \|T\|\big(B_{r_i^j}(p_i)\big) < \delta/3;$$

$$(5.7) \quad \|\phi_j(x)\|\big(M\backslash\cup_{i=1}^{v} B_{r_i^j}(p_i)\big) < \delta/3, \quad \|T\|\big(M\backslash\cup_{i=1}^{v} B_{r_i^j}(p_i)\big) < \delta/3;$$

$$(5.8) \qquad \big(\|T\| - \|\phi_i^j(x)\|\big)\big(B_{r_i^j}(p_i)\big) \leq \frac{\delta}{2(n_0+1)v},$$

for all $i = 1, \cdots, v$, and $x \in I_0(m,l)_0$.

Let $v+1 = 3^{N_1}$, $N_1 \in \mathbb{N}$, then N_1 depends only on l, m, T, δ, L. Define $\psi_j : I(1, N_1)_0 \times I_0(m,l)_0 \to \mathcal{Z}_n(M^{n+1})$ by,

(5.9)

$$\psi_j([\frac{i}{3^{N_1}}], x) = \phi_j(x) - \sum_{a=1}^{i} \partial\big(Q_j(x)\llcorner B_{r_a^j}(p_a)\big), \text{ for } 0 \leq i \leq 3^{N_1} - 1,$$

$$\psi_j([1], x) = T.$$

Similar arguments as in the proof of [**MN12**, Lemma 13.4] using $(5.4)(5.5)(5.6)(5.7)(5.8)$ in place of [**MN12**, $(67)(68)(69)(70)(71)$] show that $\psi_j([\frac{i}{3^{N_1}}], x) \in \mathcal{B}_{\epsilon_j}^{\mathcal{F}}(T)$ for all $1 \leq i \leq 3^{N_1}$, $x \in I_0(m,l)_0$, and that $\{\psi_j\}$ satisfy properties (i)(ii)(iii) in Lemma 5.5.

Now let us check property (0) in Lemma 5.5. We assume that $T \neq 0$ (the case $T = 0$ is easier). Denote $\phi_j(x) = \partial[[\Omega_j(x)]]$, $\Omega_j(x) \in \mathcal{C}(M)$,

then by Lemma 7.3, for j large enough, the isoperimetric choices $Q_j(x)$ in (5.3) satisfy that:

$$Q_j(x) = [[\Omega_j(x) - \Omega_T]], \quad \text{for all } x \in I_0(m, l)_0.$$

Hence by (5.9), for $0 \le i \le 3^{N_1} - 1$,

$$\psi_j([\frac{i}{3^{N_1}}], x) = \partial[[\Omega_j(x)]] - \sum_{a=1}^{i} \partial\big([[\Omega_j(x) - \Omega_T]] \llcorner B_{r_a^j}(p_a)\big)$$

$$= \partial\Big\{ [[\Omega_j(x) \llcorner \big(M \setminus \cup_{a=1}^{i} B_{r_a^j}(p_a)\big)]]$$

$$+ [[\Omega_T \llcorner \big(\cup_{a=1}^{i} B_{r_a^j}(p_a)\big)]]\Big\}.$$

This proves Lemma 5.5(0) as $\Omega_j(x) \llcorner \big(M \setminus \cup_{a=1}^{i} B_{r_a^j}(p_a)\big) + \Omega_T \llcorner \big(\cup_{a=1}^{i} B_{r_a^j}(p_a)\big) \in \mathcal{C}(M)$.

Finally, let us check property (iv) in Lemma 5.5. Assume that $m = 1$, and $\epsilon_j < \nu_M$. Let us calculate $F_A(\psi_j|_{I(1,N_1)_0 \times \{[0]\}})$ and $F_A(\psi_j|_{I(1,N_1)_0 \times \{[1]\}})$. First we do $F_A(\psi_j|_{I(1,N_1)_0 \times \{[0]\}})$. By the definition of Almgren's isomorphism (4.2),

$$F_A(\psi_j|_{I(1,N_1)_0 \times \{[0]\}}) = \sum_{i=1}^{v+1} Q_{j,i}(0),$$

where $Q_{j,i}(0)$ is the isoperimetric choice of $\psi_j([\frac{i}{3^{N_1}}], [0]) - \psi_j([\frac{i-1}{3^{N_1}}], [0])$, $i = 1, \cdots, v$, and $Q_{j,v+1}(0)$ is the isoperimetric choice of $T - \psi_j([\frac{v}{3^{N_1}}], [0])$. By (5.9),

$$\psi_j([\frac{i}{3^{N_1}}], [0]) - \psi_j([\frac{i-1}{3^{N_1}}], [0]) = -\partial\big(Q_j(x) \llcorner B_{r_i^j}(p_i)\big),$$

and hence by Lemma 7.2, $Q_{j,i}(0) = -Q_j(x) \llcorner B_{r_i^j}(p_i) = [[\Omega_T - \Omega_j(0)]] \llcorner B_{r_i^j}(p_i)$. Similarly,

$$T - \psi_j([\frac{v}{3^{N_1}}], [0]) = -\partial\big(Q_j(x) \llcorner [M \setminus \cup_{i=1}^{v} B_{r_i^j}(p_i)]\big),$$

and hence by Lemma 7.2, $Q_{j,v+1}(0) = -Q_j(x) \llcorner [M \setminus \cup_{i=1}^{v} B_{r_i^j}(p_i)] = [[\Omega_T - \Omega_j(0)]] \llcorner [M \setminus \cup_{i=1}^{v} B_{r_i^j}(p_i)]$. Summing them together,

$$F_A(\psi_j|_{I(1,N_1)_0 \times \{[0]\}}) = \sum_{i=1}^{v} [[\Omega_T - \Omega_j(0)]] \llcorner B_{r_i^j}(p_i)$$

$$+ [[\Omega_T - \Omega_j(0)]] \llcorner [M \setminus \cup_{i=1}^{v} B_{r_i^j}(p_i)]$$

$$= [[\Omega_T - \Omega_j(0)]].$$

Similar arguments show that $F_A(\psi_j|_{I(1,N_1)_0 \times \{[1]\}}) = [[\Omega_T - \Omega_j(1)]]$, and hence property (iv) (in Lemma 5.5) is proved.   q.e.d.

Lemma 5.8. *In* **Case 2***, there exist* $N_2 = N_2(l, m, \delta, L) \in \mathbb{N}$, *and a subsequence (still denoted by)* $\{\phi_j\}$, *and a sequence of positive numbers* $\rho_j \to 0$, *as* $j \to \infty$, *and*

$$\psi_j : I(1, N_2)_0 \times I_0(m, l)_0 \to \mathcal{B}^{\mathcal{F}}_{\rho_j}(T),$$

satisfying:

(0) $\psi_j(y, x) = \partial[[\Omega^j_{y,x}]]$, $\Omega^j_{y,x} \in \mathcal{C}(M)$, $(y, x) \in I(1, N_2)_0 \times I_0(m, l)_0$;
(i) $\mathbf{f}(\psi_j) \leq \delta$ *if* $m = 1$, *and* $\mathbf{f}(\psi_j) \leq \mathbf{f}(\phi_j) + \delta$ *if* $m > 1$;
(ii) $\psi_j([0], \cdot) = \phi_j$,

$$\lim_{j \to \infty} |\psi_j([1], x)| = V(x) \llcorner G_n(M \backslash S_{con})^7 \text{ as varifolds for all } x \in I_0(m, l)_0;$$

(iii)

$$\sup\{\mathbf{M}(\psi_j(y, x)), (y, x) \in I(1, N_2)_0 \times I_0(m, l)_0\}$$

$$\leq \sup_{x \in I_0(m,l)_0} \mathbf{M}(\phi_j(x)) + \frac{\delta}{n_0 + 1};$$

(iv) *If* $m = 1$, $\delta < \nu_M$, $\phi_j([0]) = \partial[[\Omega_{j,0}]]$, $\phi_j([1]) = \partial[[\Omega_{j,1}]]$, $\psi_j([1] \otimes [0]) = \partial[[\Omega'_{j,0}]]$, $\psi_j([1] \otimes [1]) = \partial[[\Omega'_{j,1}]]$,[8] *then*

$$F_A(\psi_j|_{I(1,N_2)_0 \times \{[0]\}}) = [[\Omega'_{j,0} - \Omega_{j,0}]], \ F_A(\psi_j|_{I(1,N_2)_0 \times \{[1]\}}) = [\Omega'_{j,1} - \Omega_{j,1}]],^{9}$$

where F_A *is the Almgren's isomorphism (4.2).*

REMARK 5.9. This lemma is the key part towards Theorem 5.1. As the proof is very subtle, we sketch the main ideas here. Let us focus on a simpler case when S_{con} contains only one point q (Part I in the proof), and the general case (Part II) follows from straightforward induction. For j large, we will find points $p_j \to q$ and radii $r_j \to 0$, such that the mass of the slicing $\mathbf{M}[\partial(\phi_j(x) \llcorner B(p_j, r_j))] \to 0$ (Fact 2). To get rid of the mass concentration, we will connect $\phi_j(x)$ to local cones $0 \times \partial(\phi_j(x) \llcorner B(p_j, r_j))$ inside $B(p_j, r_j)$ in finitely many steps simultaneously for all $x \in I_0(m, l)_0$. To keep the fineness small during this procedure, we will find finitely many concentric annuli inside $B(p_j, r_j)$ (Fact 3), and do the deformation step by step on each annulus (Step 1 to 3). The number of annuli can be chosen to depend only on l, m, δ, L (Fact 3.4). All the properties (0)(i)(ii)(iii)(iv) are checked in Step 4 and 5.

As we are working on a manifold, so all the cone construction should be passed to the tangent plane using exponential map. We summarize the related formula for local exponential maps in §7.2.

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[7]$G_n(U)$, $U \subset M$ denotes the n-Grassmannian bundle over U [**Si83**, §38].
[8]We introduce new notions $\Omega'_{j,0}, \Omega'_{j,1}$ to simplify presentation, and according to (0), $\Omega'_{j,0} = \Omega^j_{[1],[0]}$ and $\Omega'_{j,1} = \Omega^j_{[1],[1]}$.
[9]Here we identify $I(1, N_2)_0 \times \{[0]\}$ and $(I(1, N_2))_0 \times \{[1]\}$ with $I(1, N_2)_0$.

Proof. For all basics facts about the local exponential map, we refer to §7.2.

$C(m,l)$ denotes the number of vertices in $I_0(m,l)_0$.

Denote $\alpha = \delta/5$, then the set S_{con} has at most $C(m,l)\frac{2L}{\alpha}$ points. Given $q \in S_{con}$, then $\|V(x)\|(q) > \alpha$, for some $x \in I_0(m,l)_0$. Choose a neighborhood $Z = Z_q$ of q satisfying the requirement of §7.2, with respect to some fixed $\epsilon \le n/2$. We can make sure that the sets $\{Z_q : q \in S_{con}\}$ are pairwise disjoint by possibly shrinking Z_q.

Part I: First assume that S_{con} has a single point, i.e., $S_{con} = \{q\}$, and write $Z = Z_q$. We will discuss the general cases using induction method later.

We need the following facts:

(A) By basic measure theory,
$$\lim_{r \to 0} \|V(x)\|\big(B(q,r)\backslash\{q\}\big) = 0, \ \forall x \in I_0(m,l)_0.$$

(B) Given a set of integral currents $\{T(x) \in \mathcal{Z}_n(M^{n+1}) : x \in I_0(m,l)_0\}$, by [**P81**, 3.6], the set
$$\{p \in Z : \ \|T(x)\|\big(\partial B(p,t)\big) = 0, \forall t > 0, B(p,t) \subset Z\}$$
has a full measure in Z;

(C) Fix $p \in Z$, and $s > 0$, with $B(p,2s) \subset Z$. Then by the slicing theorem [**Si83**, 28.5] and §7.2(d), $\partial(T(x)\llcorner B(p,t)) \in \mathcal{Z}_{n-1}(M)$ for (L^1 almost all) $t \in [s/2, 2s]$, and
$$2\|T(x)\|\big(A(p,s/2,2s)\big) \ge Lip(r_p)\|T(x)\|\big(A(p,s/2,2s)\big)$$
$$\ge \int_{s/2}^{2s} \mathbf{M}\big[\partial(T(x)\llcorner B(p,t))\big]dt.$$

Hence by the Pigeonhole Principle, there exists $r \in [s/2, 2s]$, such that for all $x \in I_0(m,l)_0$,
$$- \partial\big(T(x)\llcorner B(p,r)\big) = \langle T(x), r_p, r \rangle \in \mathcal{Z}_{n-1}(M^{n+1});^{[10]}$$
$$-$$
$$2C(m,l)\|T(x)\|\big(A(p,s/2,2s)\big) \ge \frac{3}{2}s\mathbf{M}\big[\partial(T(x)\llcorner B(p,r))\big]$$
$$\ge \frac{3}{4}r\mathbf{M}\big[\partial(T(x)\llcorner B(p,r))\big].$$

Now denote $T_j(x) = \phi_j(x)$, $x \in I_0(m,l)_0$,

Claim 1. *We can find (possibly up to a further subsequence of $\{\phi_j\}$),*
- *a sequence of points $p_j \in Z$, $p_j \to q$ as $j \to \infty$;*
- *sequences of numbers $s_j, r_j \in \mathbb{R}$, with $0 < s_j/2 < r_j < 2s_j$, $\lim_{j \to \infty} s_j = 0$;*

satisfying

---

[10] $\langle T, r_p, r \rangle$ denotes the slicing of T by the function r_p (see §7.2) at r [**Si83**, 28.4].

(i) $B(q, s_j/8) \subset B(p_j, s_j/4) \subset B(p_j, 2s_j) \subset B(q, 4s_j)$;

(ii) $\|T_j(x)\|\big(\partial B(p_j, t)\big) = 0$, for all $x \in I_0(m, l)_0$, $0 < t < 2s_j$;

(iii) $\lim_{j \to \infty} \max_{x \in I_0(m,l)_0} \|T_j(x)\|\big[A(p_j, s_j/2, 2s_j)\big] = 0$;

(iv) $\partial\big(T_j(x) \llcorner B(p_j, r_j)\big) = \langle T_j(x), r_{p_j}, r_j \rangle \in \mathcal{Z}_{n-1}(M)$;

(v) $r_j \mathbf{M}\big[\partial(T_j(x) \llcorner B(p_j, r_j))\big] \leq 8/3 C(m, l)\|T_j(x)\|\big(A(p_j, s_j/2, 2s_j)\big)$;

(vi) $\lim_{j \to \infty} |T_j(x)| \llcorner G_n\big(B^c(p_j, r_j)\big) = V(x) \llcorner G_n(M\backslash\{q\})$ as varifolds.[11]

Now let us check the claim. By fact (A), we can find $s_j > 0$, $s_j \to 0$, as $j \to \infty$, such that

$$\lim_{j \to \infty} \max_{x \in I_0(m,l)_0} \|V(x)\|\big(B(q, 4s_j)\backslash\{q\}\big) = 0.$$

As $|T_j(x)| = |\phi_j(x)|$ converge to $V(x)$ as varifolds, we can possibly take a subsequence of $\{\phi_j\}$, still denoted by $\{\phi_j\}$, such that

$$\lim_{j \to \infty} \max_{x \in I_0(m,l)_0} \|T_j(x)\|\big(A(q, s_j/8, 4s_j)\big) = 0, \quad \text{and}$$

$$\lim_{j \to \infty} |T_j(x)| \llcorner G_n(B^c(q, s_j/8)) = V(x) \llcorner G_n(M\backslash\{q\}),$$

as varifolds, for all $x \in I_0(m, l)_0$.

(In fact, for any j one can find $j' \geq j$, such that $\|T_{j'}(x)\|\big(A(q, s_j/8, 4s_j)\big) \leq 2\|V(x)\|\big(B(q, 4s_j)\backslash\{q\}\big)$ and $\|T_{j'}(x)\|B(q, s_j/8) \leq \|V(x)\|B(q, s_j/8) + \frac{1}{j}$, and $\{\phi_{j'}(x) = T_{j'}(x)\}$ satisfies the requirement.)

By fact (B), we can find a sequence $p_j \in Z$, $p_j \to q$, such that $B(q, s_j/8) \subset B(p_j, s_j/4) \subset B(p_j, 2s_j) \subset B(q, 4s_j)$, and $\|T_j(x)\|\big(\partial B(p_j, s)\big) = 0$, for all $x \in I_0(m, l)_0$ and $s > 0$ with $B(p_j, s) \subset Z$. Hence (i)(ii) are true. (iii) is true as $A(p_j, s_j/2, 2s_j) \subset A(q, s_j/8, 4s_j)$. Now for each j, by fact (C), we can find $r_j \in [s_j/2, 2s_j]$, such that (iv)(v) are true. (vi) is true as $B^c(p_j, r_j) \subset B^c(q, s_j/8)$ and $B(p_j, r_j)\backslash B(q, s_j/8) \subset A(q, s_j/8, 4s_j)$.

Then we have the following facts:

Fact 2. *Given $\delta_1 > 0$ (to be determined later), $\delta_1 < \delta$, by Claim 1(iii)(v), there exists J large enough, such that if $j \geq J$,*

$$(5.10) \qquad \frac{2r_j}{n} \mathbf{M}\big[\partial\big(T_j(x) \llcorner B(p_j, r_j)\big)\big] \leq \delta_1/5;$$

$$(5.11) \qquad vol\big(B(p_j, r_j)\big) \leq \delta_1/5;$$

$$(5.12) \qquad vol\big(\partial B(p_j, r)\big) \leq \delta_1/5, \text{ for all } r \leq r_j.$$

---

[11] $B^c(p, r)$ denotes the complement of $B(p, r)$ in M.

Now we are going to connect $T_j(x) \llcorner B(p_j, r_j)$ to the cones $E_\# \left[\delta_0 \bar{\times} E_\#^{-1} \partial \left(T_j(x) \llcorner B(p_j, r_j) \right) \right]$ using discrete sequences with controlled fineness simultaneously for all $x \in I_0(m, l)_0$.

We separate the whole procedure into several steps. For notions $E, \mu(\lambda), h(r)$, we refer to §7.2.

Step 0: Now fix $j \geq J$, and forget the subscript "j" now. So $T(x)$ and $B(p, r)$ satisfy (5.10)(5.11)(5.12). Recall that $T(x) = \partial[[\Omega(x)]]$, $\Omega(x) \in \mathcal{C}(M)$. For simplicity, we will identify $\Omega(x)$ with $[[\Omega(x)]]$ in the following of the proof. By the Pigeonhole Principle and the Slicing Theorem [**Si83**, 28.5], we have that

Fact 3. *we can find finitely many numbers* $r_i > 0^{12}$, $i = 1, \cdots, \nu$, *for some* $\nu \in \mathbb{N}$, *with* $r > r_1 > r_2 > \cdots > r_\nu > 0$, *such that for all* $x \in I_0(m, l)_0$, $1 \leq i \leq \nu - 1$,

1) $\|T(x)\| A(p, r_{i+1}, r_i) \leq \delta/5$, $\|T(x)\| B(p, r_\nu) \leq \delta/5$;
2) $\partial \left(T(x) \llcorner B(p, r_i) \right) \in \mathcal{Z}_{n-1}(M^{n+1})$;
3) $\langle \Omega(x), r_p, r_i \rangle = \partial \left(\Omega(x) \llcorner B(p, r_i) \right) - T(x) \llcorner B(p, r_i) \in \mathbf{I}_n(M^{n+1})$;
4) ν *can be any integer no less than* $C(m, l)(\delta/6)^{-1} \times \max_{x \in I_0(m, l)_0} \mathbf{M}(T(x) \llcorner Z)$, *and hence depends only on* m, l, δ, L.

<u>Step 1</u>: (See Figure 2) For each $x \in I_0(m, l)_0$, let

$$S_1(x) = E_\# \left\{ \delta_0 \bar{\times} \left[E_\#^{-1} \partial (T(x) \llcorner B(p, r)) - \mu(\frac{r_1}{r})_\# E_\#^{-1} \partial (T(x) \llcorner B(p, r)) \right] \right\};$$

then by (5.10) and §7.2(k), $spt(S_1(x)) \subset A(p, r_1, r)$, and

(5.13)
$$\mathbf{M}(S_1(x)) \leq 2rn^{-1}(1 - (\frac{r_1}{r})^n)\mathbf{M}\left(\partial(T(x) \llcorner B(p, r)) \right)$$
$$\leq 2rn^{-1}\mathbf{M}\left(\partial(T(x) \llcorner B(p, r)) \right) \leq \delta_1/5.$$

For each $x \in I_0(m, l)_0$, define

(5.14) $\quad R_1(x) = \begin{cases} S_1(x), & \text{in } A(p, r_1, r), \\ (E \circ h(r_1) \circ E^{-1})_\# T(x) \llcorner B(p, r), & \text{in } B(p, r_1), \\ T(x), & \text{outside } B(p, r). \end{cases}$

Claim 2. *For each* $x \in I_0(m, l)_0$, $R_1(x) = \partial \Omega_1(x)$ *for some* $\Omega_1(x) \in \mathcal{C}(M)$.

Proof. For each $x \in I_0(m, l)_0$, by the definition of slices [**Si83**, 28.4], the slices $\langle \Omega(x), r_p, r_i \rangle$ is represented by the set $\Omega(x) \cap \partial B(p, r_i)$, which has finite perimeter as $\partial \langle \Omega(x), r_p, r_i \rangle = -\partial \left(T(x) \llcorner B(p, r_i) \right)$. Denote

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^{12}Note that r_i's are different from the r_j's in Claim 1, and we will forget the subscript "j" of r_j until Step 5.
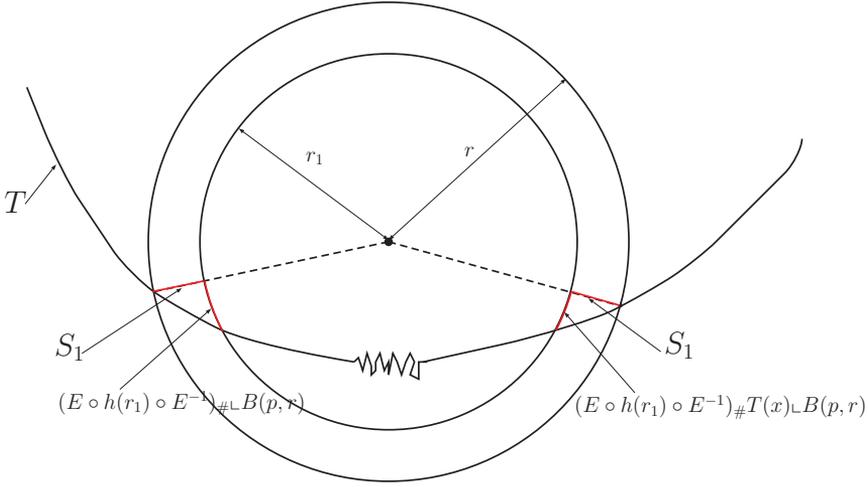
Figure 2.   This figure illustrates Step 1 in the discretization process with point mass. We omit the variable $x \in I(m,l)_0$.

$O_i(x) = \Omega(x) \cap \partial B(p, r_i) = \langle \Omega(x), r_p, r_i \rangle$, $O(x) = \Omega(x) \cap \partial B(p, r) = \langle \Omega(x), r_p, r \rangle$. Define a subset of M as[13]

$$(5.15) \quad \Omega_1(x) = \left\{ \begin{array}{ll} E\left\{0 \mathbb{x}\left[E^{-1}O(x) - \frac{r_1}{r}E^{-1}O(x)\right]\right\}, & \text{in } A(p, r_1, r), \\ \Omega(x), & \text{in } B_0(p, r_1) \text{ and outside } B(p, r). \end{array} \right.$$

Clearly $\Omega_1(x)$ is a set of finite perimeter, i.e., $\Omega(x) \in \mathcal{C}(M)$, as each part supported in $B(p, r_1)$, $B^c(p, r)$, $A(p, r_1, r)$ is. We will show that $R_1(x) = \partial \Omega_1(x)$. By [**Si83**, 28.5(2)],

$$\partial \Omega_1(x) = \partial \left[\Omega(x) \llcorner B^c(p, r)\right] + \partial E\left\{0 \mathbb{x}\left[E^{-1}O(x) - \frac{r_1}{r}E^{-1}O(x)\right]\right\}$$

$$+ \partial\left[\Omega(x) \llcorner B_0(p, r_1)\right]$$

$$= T(x) \llcorner B^c(p, r) - \langle \Omega(x), r_p, r \rangle + O(x)$$

$$- (E \circ \mu(\frac{r_1}{r}) \circ E^{-1})_\# O(x)$$

$$- E_\#\left\{\delta_0 \mathbb{x}\left[E_\#^{-1}\partial O(x) - \mu(\frac{r_1}{r})_\# E_\#^{-1}\partial O(x)\right]\right\}$$

$$+ T(x) \llcorner B(p, r_1) + \langle \Omega(x), r_p, r_1 \rangle$$

$$= T(x) \llcorner B^c(p, r) + T(x) \llcorner B(p, r_1)$$

$$- (E \circ \mu(\frac{r_1}{r}) \circ E^{-1})_\# O(x) + O_1(x)$$

$$+ E_\#\left\{\delta_0 \mathbb{x}\left[E_\#^{-1}\partial(T(x) \llcorner B(p, r))\right.\right.$$

$$\left.\left. - \mu(\frac{r_1}{r})_\# E_\#^{-1}\partial(T(x) \llcorner B(p, r))\right]\right\}.$$

_____

[13]$0 \mathbb{x} S$ denotes the cone in $\mathbb{R}^{n+1}$ over $S \subset \mathbb{R}^{n+1}$.

So together with Claim 1(ii),

$$R_1(x) - \partial\Omega_1(x) = (E \circ h(r_1) \circ E^{-1})_\# \big[T(x) \llcorner A(p, r_1, r)\big]$$
$$+ (E \circ \mu(\frac{r_1}{r}) \circ E^{-1})_\# O(x) - O_1(x)$$
$$= (E \circ h(r_1) \circ E^{-1})_\# \big(T(x) \llcorner A(p, r_1, r)$$
$$+ O(x) - O_1(x)\big)$$
$$= (E \circ h(r_1) \circ E^{-1})_\# \partial\big(\Omega(x) \llcorner A(p, r_1, r)\big)$$
$$= \partial(E \circ h(r_1) \circ E^{-1})_\# \big(\Omega(x) \llcorner A(p, r_1, r)\big)$$
$$= 0,$$

where we used the fact that $h(r_1) = \mu(\frac{r_1}{r})$ on $\partial B(p, r)$ in the second "=", and the fact that any integral $(n+1)$-current on an n-dimensional manifold $\partial B(p, r_1)$ is zero in the last "=". Hence we finish the proof of the claim.   q.e.d.

As $R_1(x) = \partial\Omega_1(x)$, using (5.12) it is easily seen that

$$(5.16) \qquad \mathbf{M}\big(R_1(x) \llcorner \partial B(p, r_1)\big) \leq vol\big(\partial B(p, r_1)\big) \leq \delta_1/5.$$

The set $\{R_1(x) : x \in I_0(m, l)_0\}$ satisfies the following properties. First using Claim 1(ii), Fact 3.1, (5.13)(5.16), we have the continuity estimate,

$$(5.17)$$
$$\mathbf{M}\big(R_1(x) - T(x)\big) \leq \mathbf{M}\big(T(x) \llcorner A(p, r_1, r)\big) + \mathbf{M}\big(R_1(x) \llcorner \partial B(p, r_1)\big)$$
$$+ \mathbf{M}\big(S_1(x)\big)$$
$$\leq \delta/5 + 2\delta_1/5.$$

Using Claim 1(ii), (5.13)(5.16), we have the mass estimate,

$$\mathbf{M}\big(R_1(x)\big) \leq \mathbf{M}\big(T(x) \llcorner B^c(p, r)\big) + \mathbf{M}\big(S_1(x)\big)$$
$$(5.18) \qquad\qquad + \mathbf{M}\big(R_1(x) \llcorner \partial B(p, r_1)\big) + \mathbf{M}\big(T(x) \llcorner B(p, r_1)\big)$$
$$\leq \mathbf{M}\big(T(x)\big) + 2\delta_1/5.$$

If $m > 1$, given $x, y \in I_0(m, l)_0$, such that $d(x, y) = 1$, then

$$R_1(x) - R_1(y) = \big(S_1(x) - S_1(y)\big) + \big(R_1(x) - R_1(y)\big) \llcorner \partial B(p, r_1)$$
$$+ \big(T(x) - T(y)\big) \llcorner B(p, r_1) \cup B^c(p, r);$$

hence using (5.13)(5.16), we have the fineness estimate,

$$(5.19)$$
$$\mathbf{M}\big(R_1(x) - R_1(y)\big) \leq \mathbf{M}\big(R_1(x) \llcorner \partial B(p, r_1)\big) + \mathbf{M}\big(R_1(y) \llcorner \partial B(p, r_1)\big)$$
$$+ \mathbf{M}\big(S_1(x)\big) + \mathbf{M}\big(S_1(y)\big) + \mathbf{M}\big(T(x) - T(y)\big)$$
$$\leq 4\delta_1/5 + \mathbf{f}(\phi),$$

where $\mathbf{f}(\phi)$ is the fineness (4.1) of ϕ.

Step 2: Now for $2 \leq i \leq \nu$, $x \in I_0(m, l)_0$, we can similarly define

$$S_i(x) = E_{\#}\Big\{ \delta_0 \mathbb{x} \big[E_{\#}^{-1} \partial(T(x) \llcorner B(p,r)) - \mu(\tfrac{r_i}{r})_{\#} E_{\#}^{-1} \partial(T(x) \llcorner B(p,r)) \big] \Big\};$$

then by (5.10) and §7.2(k), $spt(S_i(x)) \subset A(p, r_i, r)$, and

(5.20)
$$\mathbf{M}(S_i(x)) \leq 2rn^{-1}(1 - (\tfrac{r_i}{r})^n)\mathbf{M}(\partial(T(x) \llcorner B(p,r)))$$
$$\leq 2rn^{-1}\mathbf{M}(\partial(T(x) \llcorner B(p,r))) \leq \delta_1/5.$$

Similarly define

(5.21) $R_i(x) = \begin{cases} S_i(x), & \text{in } A(p, r_i, r), \\ (E \circ h(r_i) \circ E^{-1})_{\#} T(x) \llcorner B(p,r), & \text{in } B(p, r_i), \\ T(x), & \text{outside } B(p,r). \end{cases}$

The same argument as in Claim 2 with r_1 changed to r_i shows that $R_i(x) = \partial\Omega_i(x)$, $\Omega_i(x) \in \mathcal{C}(M)$ for all $2 \leq i \leq \nu$, $x \in I_0(m, l)_0$, with

(5.22) $\Omega_i(x) = \begin{cases} E\big\{ 0 \mathbb{x} \big[E^{-1}O(x) - \tfrac{r_i}{r} E^{-1}O(x) \big] \big\}, & \text{in } A(p, r_i, r), \\ \Omega(x), & \text{in } B_0(p, r_i) \text{ and outside } B(p,r). \end{cases}$

and hence by (5.12),

(5.23) $\mathbf{M}\big(R_i(x) \llcorner \partial B(p, r_i)\big) \leq vol\big(\partial B(p, r_i)\big) \leq \delta_1/5.$

Using (5.20)(5.23) in place of (5.13)(5.16) and similar estimates as in Step 1, the currents $\{R_i(x) : 2 \leq i \leq \nu, x \in I_0(m, l)_0\}$ satisfy the following properties:

(5.24)
$$\mathbf{M}\big(R_i(x) - R_{i-1}(x)\big) \leq \mathbf{M}\big(T(x) \llcorner A(p, r_i, r_{i-1})\big) + \mathbf{M}\big(R_i(x) \llcorner \partial B(p, r_i)\big)$$
$$+ \mathbf{M}\big(R_{i-1}(x) \llcorner \partial B(p, r_{i-1})\big) + \mathbf{M}\big(S_i(x) - S_{i-1}(x)\big)$$
$$\leq \delta/5 + 3\delta_1/5.$$

(5.25)
$$\mathbf{M}\big(R_i(x)\big) \leq \mathbf{M}\big(T(x) \llcorner B^c(p,r)\big) + \mathbf{M}\big(S_i(x)\big)$$
$$+ \mathbf{M}\big(R_i(x) \llcorner \partial B(p, r_i)\big) + \mathbf{M}\big(T(x) \llcorner B(p, r_i)\big)$$
$$\leq \mathbf{M}\big(T(x)\big) + 2\delta_1/5.$$

If $m > 1$, given $x, y \in I_0(m, l)_0$, such that $d(x, y) = 1$, then

(5.26)
$$\mathbf{M}\big(R_i(x) - R_i(y)\big) \leq \mathbf{M}\big(R_i(x) \llcorner \partial B(p, r_i)\big) + M\big(R_i(y) \llcorner \partial B(p, r_i)\big)$$
$$+ \mathbf{M}\big(S_i(x)\big) + \mathbf{M}\big(S_i(y)\big) + \mathbf{M}\big(T(x) - T(y)\big)$$
$$\leq 4\delta_1/5 + \mathbf{f}(\phi).$$

Step 3: Define the cones

$$S_{\nu+1}(x) = E_{\#}\Big\{ \delta_0 \mathbb{x} E_{\#}^{-1} \partial(T(x) \llcorner B(p,r)) \Big\};$$

then by (5.10) and §7.2(k), $spt(S_{\nu+1}(x)) \subset B(p, r)$, and

(5.27) $\mathbf{M}(S_{\nu+1}(x)) \leq 2rn^{-1}\mathbf{M}(\partial(T(x) \llcorner B(p, r))) \leq \delta_1/5.$

Define

(5.28) $R_{\nu+1}(x) = \begin{cases} S_{\nu+1}(x), & \text{in } B(p, r), \\ T(x), & \text{outside } B(p, r). \end{cases}$

Similar argument as in Claim 2 with r_1 changed to 0 shows that $R_{\nu+1}(x) = \partial\Omega_{\nu+1}(x)$, $\Omega_{\nu+1}(x) \in \mathcal{C}(M)$ for all $x \in I_0(m, l)_0$, with

(5.29) $\Omega_{\nu+1}(x) = \begin{cases} E\{0 \mathbb{x} [E^{-1}O(x)]\}, & \text{in } B(p, r), \\ \Omega(x), & \text{outside } B(p, r). \end{cases}$

Using Claim 1(ii), Fact 3.1, (5.23)(5.27), we have that

(5.30)
$$\mathbf{M}(R_{\nu+1}(x) - R_\nu(x)) \leq \mathbf{M}(T(x) \llcorner B(p, r_\nu)) + \mathbf{M}(R_\nu(x) \llcorner \partial B(p, r_\nu))$$
$$+ \mathbf{M}(S_{\nu+1}(x) - S_\nu(x))$$
$$\leq \delta/5 + 2\delta_1/5.$$

(5.31)
$$\mathbf{M}(R_{\nu+1}(x)) \leq \mathbf{M}(T(x) \llcorner B^c(p, r)) + \mathbf{M}(S_{\nu+1}(x))$$
$$\leq \mathbf{M}(T(x)) + \delta_1/5.$$

If $m > 1$, given $x, y \in I_0(m, l)_0$, such that $d(x, y) = 1$, then

(5.32)
$$\mathbf{M}(R_{\nu+1}(x) - R_{\nu+1}(y)) \leq \mathbf{M}(S_i(x)) + \mathbf{M}(S_i(y)) + \mathbf{M}(T(x) - T(y))$$
$$\leq 2\delta_1/5 + \mathbf{f}(\phi).$$

Step 4: Take $\nu + 1 = 3^{\tilde{N}}$ for $\tilde{N} \in \mathbb{N}$, then $\tilde{N}$ depends only on l, m, δ, L by Fact 3.4. We can define a map

$$\psi : I(1, \tilde{N})_0 \times I_0(m, l)_0 \to \mathcal{Z}_n(M^{n+1}),$$

by $\psi(0, x) = T(x) = \phi(x)$, $\psi([\frac{i}{3^{\tilde{N}}}], x) = R_i(x)$ for $1 \leq i \leq \nu + 1$. Now we check that ψ satisfy Lemma 5.8(0)(i)(iii)(iv). By combining (5.17)(5.18)(5.19)(5.24)(5.25)(5.26)(5.30)(5.31)(5.32) and our construction, we have

  (0) $\psi([\frac{i}{3^{\tilde{N}}}], x) = \partial[[\Omega_i(x)]], \Omega_i(x) \in \mathcal{C}(M);$
  (i) $\mathbf{f}(\psi) \leq \delta/5 + 3\delta_1/5$ if $m = 1$, and $\mathbf{f}(\psi) \leq \max\{\delta/5 + 3\delta_1/5, \mathbf{f}(\phi) + 4\delta_1/5\}$ if $m > 1$;
  (iii) $\max\{\mathbf{M}(\psi([\frac{i}{3^{\tilde{N}}}], x))\} \leq \max\{\mathbf{M}(\phi(x))\} + 2\delta_1/5.$

If $m = 1$, $\delta < \nu_M$, let us calculate $F_A(\psi|_{I(1,\tilde{N})_0 \times \{[0]\}})$ and $F_A(\psi|_{I(1,\tilde{N})_0 \times \{[1]\}})$. First focus on $F_A(\psi|_{I(1,\tilde{N})_0 \times \{[0]\}})$. We will use notions as above. By the definition of Almgren's isomorphism (4.2),

$$F_A(\psi|_{I(1,\tilde{N})_0 \times \{[0]\}}) = \sum_{i=1}^{\nu+1} Q_i(0),$$

where $Q_1(0)$ is the isoperimetric choice for $R_1(0) - T(0)$, and $Q_i(0)$ is the isoperimetric choice of $R_i(0) - R_{i-1}(0)$, $2 \leq i \leq \nu + 1$, with $R_i(0)$ given by (5.14)(5.21)(5.28). Recall that $T(0) = \partial\Omega(0)$, $R_i(0) = \partial\Omega_i(0)$, with $\Omega(0), \Omega_i(0) \in \mathcal{C}(M)$, and that $\Omega_i(0) - \Omega_{i-1}(0)$ are all supported in $B(p, r)$ by the construction (5.15)(5.22)(5.29), so $\mathbf{M}(\Omega_i(0) - \Omega_{i-1}(0)) \leq vol(B(p, r)) \leq 1/2 vol(M)$, as r is very small. By Lemma 7.2, $Q_1(0) = \Omega_1(0) - \Omega(0)$, $Q_i(0) = \Omega_i(0) - \Omega_{i-1}(0)$ for $2 \leq i \leq \nu + 1$, hence

$$F_A(\psi|_{I(1,\tilde{N})_0 \times \{[0]\}}) = \Omega_1(0) - \Omega(0) + \sum_{i=2}^{\nu+1}(\Omega_i(0) - \Omega_{i-1}(0))$$

$$= \Omega_{\nu+1}(0) - \Omega(0).$$

Similarly we can prove that $F_A(\psi|_{I(1,\tilde{N})_0 \times \{[1]\}}) = \Omega_{\nu+1}(1) - \Omega(1)$. By changing the notions, we showed that

(iv) If $m = 1$, $\delta < \nu_M$, $\phi([0]) = \partial[[\Omega_0]]$, $\phi([1]) = \partial[[\Omega_1]]$, $\psi([1] \otimes [0]) = \partial[[\Omega_0']]$, $\psi([1] \otimes [1]) = \partial[[\Omega_1']]$, then

$$F_A(\psi|_{I(1,\tilde{N})_0 \times \{[0]\}}) = [[\Omega_0' - \Omega_0]], \quad F_A(\psi|_{I(1,\tilde{N})_0 \times \{[1]\}}) = [[\Omega_1' - \Omega_1]].$$

Step 5: We now pick up the subscript "j". For each ϕ_j, $j \geq J$, we can construct $\psi_j : I(1, \tilde{N})_0 \times I_0(m, l)_0 \to \mathcal{Z}_n(M^{n+1})$ as above. Denote $\phi_j(x) = \partial[[\Omega_j(x)]]$, and $\psi_j(y, x) = R_{j,i}(x) = \partial[[\Omega_{j,i}(x)]]$ for $y = [\frac{i}{3^{\tilde{N}}}]$, with $\Omega_j(x), \Omega_{j,i}(x) \in \mathcal{C}(M)$. By the construction (5.15)(5.22)(5.29), $\Omega_{j,i}(x) - \Omega_j(x)$ are all supported in $B(p_j, r_j)$. Recall that $r_j \to 0$ by Claim 1, so

$$\mathcal{F}(\psi_j(y, x), \phi_j(x)) \leq \mathbf{M}(\Omega_{j,i}(x) - \Omega_j(x)) \leq vol(B(p_j, r_j)) \to 0,$$

uniformly for all $(y, x) \in I(1, \tilde{N})_0 \times I_0(m, l)_0$ as $j \to \infty$.

Define

$$\rho_j = \epsilon_j + \max\{\mathcal{F}(\psi_j(y, x), \phi_j(x)) : (y, x) \in I(1, \tilde{N})_0 \times I_0(m, l)_0\},$$

where ϵ_j is given in Lemma 5.5; then $\rho_j \to 0$, as $j \to \infty$, and $\mathcal{F}(\psi_j(y, x), T) \leq \mathcal{F}(\psi_j(y, x), \phi_j(x)) + \mathcal{F}(\phi_j(x), T) \leq \rho_j$, so

$$\psi_j : I(1, \tilde{N})_0 \times I_0(m, l)_0 \to \mathcal{B}_{\rho_j}^{\mathcal{F}}(T).$$

Finally, we claim that

(ii)

(5.33) $\lim_{j \to \infty} |\psi_j([1], x)| = V(x) \llcorner G_n(M \backslash \{q\})$, as varifolds.

In fact, by (5.28), $\psi_j([1], x) = \phi_j(x)$ outside $B(p_j, r_j)$, and inside $B(p_j, r_j)$, by (5.27) and Claim 1(iii)(v),

$$\mathbf{M}(\psi_j([1], x) \llcorner B(p_j, r_j)) \leq 2r_j n^{-1} \mathbf{M}(\partial(T_j(x) \llcorner B(p_j, r_j))) \to 0,$$
        as $j \to \infty$.

Therefore, (5.33) is a directly corollary of Claim 1(vi).

All the above properties show that $\{\psi_j\}$ satisfy Lemma 5.8 when $S_{con} = \{q\}$.

Part II: If S_{con} contains more than one point, we can construct ψ_j successively on the pairwise disjoint neighborhoods $\{Z_q : q \in S_{con}\}$ as above, as the construction is purely local. The only things to be taken care of are the increase of mass and fineness.

Write $S_{con} = \{q_a\}_{a=1}^{\kappa}$, $Z_a = Z_{q_a}$, $\kappa \in \mathbb{N}$. As mentioned above, $\kappa \leq C(m,l)\frac{2L}{\alpha}$ depends only on m, l, δ, L. We start by following the above process inside Z_1 to extend ϕ_j (possibly up to a subsequence) to $\psi_j^1 : I(1, \tilde{N})_0 \times I_0(m,l)_0 \to \mathcal{B}_{\rho_j^1}^{\mathcal{F}}(T)$, where $\{\rho_j^1\}$ is a sequence of positive numbers converging to zero. Denote $\phi_j^1(\cdot) = \psi_j^1([1], \cdot)$. Then $\{\psi_j^1\}$ satisfy (by Step 4 and Step 5 in Part 1): for all $x \in I_0(m,l)_0$

- $\psi_j^1([\frac{i}{3^{\tilde{N}}}]) = \partial[[\Omega_{j,i}^1(x)]]$, $\Omega_{j,i}^1(x) \in \mathcal{C}(M)$;
- $\mathbf{f}(\psi_j^1) \leq \delta/5 + 3\delta_1/5$ if $m = 1$, and $\mathbf{f}(\psi_j^1) \leq \max\{\delta/5 + 3\delta_1/5, \mathbf{f}(\phi_j) + 4\delta_1/5\}$ if $m > 1$;
- $\psi_j^1([0], x) = \phi_j(x)$, $\lim_{j \to \infty} |\psi_j^1([1], x)| = V(x) \llcorner G_n(M \backslash \{q_1\})$ as varifolds;
- $\max\{\mathbf{M}(\psi_j^1([\frac{i}{3^{\tilde{N}}}], x))\} \leq \max\{\mathbf{M}(\phi_j(x))\} + 2\delta_1/5$;
- If $m = 1$, and denote $\phi_j([0]) = \partial[[\Omega_{j,0}]]$, $\phi_j([1]) = \partial[[\Omega_{j,1}]]$, $\phi_j^1([0]) = \partial[[\Omega_{j,0}^1]]$, $\phi_j^1([1]) = \partial[[\Omega_{j,1}^1]]$; then

$$F_A(\psi_j^1|_{I(1,\tilde{N})_0 \times \{[0]\}}) = [[\Omega_{j,0}^1 - \Omega_{j,0}]], \quad F_A(\psi_j^1|_{I(1,\tilde{N})_0 \times \{[1]\}}) = [[\Omega_{j,1}^1 - \Omega_{j,1}]].$$

Also $\{\phi_j^1\}$ satisfy: for all $x \in I_0(m,l)_0$,

- $\phi_j^1(x) = \phi_j(x)$ outside Z_1, by (5.28);
- $\phi_j^1(x) = \partial[[\Omega_j^1(x)]]$, $\Omega_j^1(x) \in \mathcal{C}(M)$;
- $\lim_{j \to \infty} |\phi_j^1(x)| = V(x) \llcorner G_n(M \backslash \{q_1\})$, as varifolds;
- $\mathbf{M}(\phi_j^1(x)) \leq \mathbf{M}(\phi_j(x)) + \delta_1/5$, by (5.31);
- If $m > 1$, $\mathbf{f}(\phi_j^1) \leq \mathbf{f}(\phi_j) + 2\delta_1/5$, by (5.32).

As $\phi_j^1(x) = \phi_j(x)$ outside Z_1, for all $x \in I_0(m,l)_0$, we can repeat the construction in Part I inductively on $Z_2, \cdots, Z_{\kappa}$, to get (possibly up to subsequences) $\{\psi_j^a\}$ and $\{\phi_j^a\}$, $2 \leq a \leq \kappa$, such that $\psi_j^a : I(1, \tilde{N})_0 \times I_0(m,l)_0 \to \mathcal{B}_{\rho_j^a}^{\mathcal{F}}(T)$, $\phi_j^a : I_0(m,l)_0 \to \mathcal{B}_{\rho_j^a}^{\mathcal{F}}(T)$, with $\{\rho_j^a\}$ a sequence of positive numbers converging to zero as $j \to \infty$ for each $2 \leq a \leq \kappa$, and $\phi_j^a(x) = \psi_j^a([1], x)$, and the following statements are true. For each $2 \leq a \leq \kappa$, $\{\psi_j^a\}$ satisfy that: for all $x \in I_0(m,l)_0$,

1) $\psi_j^a([\frac{i}{3^{\tilde{N}}}]) = \partial[[\Omega_{j,i}^a(x)]]$, $\Omega_{j,i}^a(x) \in \mathcal{C}(M)$;
2) $\mathbf{f}(\psi_j^a) \leq \delta/5 + 3\delta_1/5$ if $m = 1$, and if $m > 1$, $\mathbf{f}(\psi_j^a) \leq \max\{\delta/5 + 3\delta_1/5, \mathbf{f}(\phi_j^{a-1}) + 4\delta_1/5\}$, so by property 5 of ϕ_j^a (see below),

$$\mathbf{f}(\psi_j^a) \leq \max\{\delta/5 + 3\delta_1/5, \mathbf{f}(\phi_j) + 2(a+1)\delta_1/5\};$$

3) $\psi_j^a([0], x) = \phi_j^{a-1}(x)$, $\lim_{j\to\infty} |\psi_j^a([1], x)| = V(x) \llcorner G_n(M \backslash \{q_1, \cdots, q_a\})$ as varifolds;

4) $\max\{\mathbf{M}(\psi_j^a([\frac{i}{3^{\tilde{N}}}], x))\} \le \max\{\mathbf{M}(\phi_j^{a-1}(x))\} + 2\delta_1/5$, hence by property 4 of ϕ_j^a (see below),

$$\max\{\mathbf{M}(\psi_j^a([\frac{i}{3^{\tilde{N}}}], x))\} \le \max\{\mathbf{M}(\phi_j(x))\} + (a+1)\delta_1/5;$$

5) If $m = 1$, and denote $\phi_j^{a-1}([0]) = \partial[[\Omega_{j,0}^{a-1}]]$, $\phi_j^{a-1}([1]) = \partial[[\Omega_{j,1}^{a-1}]]$, $\phi_j^a([0]) = \partial[[\Omega_{j,0}^a]]$, $\phi_j^a([1]) = \partial[[\Omega_{j,1}^a]]$; then

$$F_A(\psi_j^a|_{I(1,\tilde{N})_0 \times \{[0]\}}) = [[\Omega_{j,0}^a - \Omega_{j,0}^{a-1}]],$$
$$F_A(\psi_j^a|_{I(1,\tilde{N})_0 \times \{[1]\}}) = [[\Omega_{j,1}^a - \Omega_{j,1}^{a-1}]].$$

$\{\phi_j^a\}$ satisfy: for all $x \in I_0(m, l)_0$,

1) $\phi_j^a(x) = \phi_j(x)$ outside $Z_1 \cup \cdots \cup Z_a$, by (5.28);
2) $\phi_j^a(x) = \partial[[\Omega_j^a(x)]]$, $\Omega_j^a(x) \in \mathcal{C}(M)$;
3) $\lim_{j\to\infty} |\phi_j^a(x)| = V(x) \llcorner G_n(M \backslash \{q_1, \cdots, q_a\})$, as varifolds;
4) $\mathbf{M}(\phi_j^a(x)) \le \mathbf{M}(\phi_j^{a-1}(x)) + \delta_1/5$ by (5.31), so

$$\mathbf{M}(\phi_j^a(x)) \le \mathbf{M}(\phi_j(x)) + a\delta_1/5;$$

5) If $m > 1$, $\mathbf{f}(\phi_j^a) \le \mathbf{f}(\phi_j^{a-1}) + 2\delta_1/5$ by (5.32), so

$$\mathbf{f}(\phi_j^a) \le \mathbf{f}(\phi_j) + 2a\delta_1/5.$$

Finally, let $\kappa(\nu + 1) = 3^{N_2}$, for some $N_2 \in \mathbb{N}$, with ν given in Fact 3; then N_2 depends only on m, l, δ, L. Recall that $\nu + 1 = 3^{\tilde{N}}$ (see Step 4 in Part I); then we can define $\psi_j : I(1, N_2)_0 \times I_0(m, l)_0 \to \mathcal{B}_{\rho_j^\kappa}^{\mathcal{F}}(T)$ as:

(5.34)
$$\psi_j([\frac{i}{3^{N_2}}], x) = \psi_j^a([\frac{i - (a-1)(\nu + 1)}{3^{\tilde{N}}}], x), \quad \text{if } (a-1)(\nu+1) \le i \le a(\nu+1).$$

Choose $\delta_1 < \delta$, such that

$$2(\kappa + 1)\delta_1/5 \le \delta, (\kappa + 1)\delta_1/5 \le \frac{\delta}{n_0 + 1},$$

and let $\rho_j = \rho_j^\kappa$; then ψ_j satisfy (0)(i)(ii)(iii) in Lemma 5.8. To check Lemma 5.8(iv), if $m = 1$, by the definition of Almgren's isomorphism (4.2),

$$F_A(\psi_j|_{I(1,N_2)_0 \times \{[0]\}}) = \sum_{a=1}^\kappa F_A(\psi_j^a|_{I(1,\tilde{N})_0 \times \{[0]\}})$$
$$= \sum_{a=1}^\kappa [[\Omega_{j,0}^a - \Omega_{j,0}^{a-1}]] = [[\Omega_{j,0}^\kappa - \Omega_{j,0}]].$$

Similarly, $F_A(\psi_j|_{I(1,N_2)_0 \times \{[1]\}}) = [[\Omega_{j,1}^\kappa - \Omega_{j,1}]]$. So Lemma 5.8(iv) is true by noticing that $\psi_j([1],[0]) = \partial[[\Omega_{j,0}^\kappa]]$ and $\psi_j([1],[1]) = \partial[[\Omega_{j,1}^\kappa]]$. The proof of Lemma 5.8 is now finished.   q.e.d.

Now let us go back to the proof of Lemma 5.5. If $S_{con} = \emptyset$, then ψ_j can be constructed by Lemma 5.6 with $\rho_j = \epsilon_j$, $N = N_1$. If $S_{con} \neq \emptyset$, let $\delta' = \delta/2$, and construct (possibly up to a subsequence) $\psi_j^2 : I(1,N_2)_0 \times I_0(m,l)_0 \to \mathcal{B}_{\rho_j}^{\mathcal{F}}(T)$ by Lemma 5.8 for the set of numbers l, m, δ', L. Then denote $\phi_j'(\cdot) = \psi_j^2([1],\cdot) : I_0(m,l)_0 \to \mathcal{B}_{\rho_j}^{\mathcal{F}}(T)$. By Lemma 5.8(ii), $\{\phi_j'\}$ satisfy the requirement of Lemma 5.6 for the set of numbers $l, m, \delta', L + \frac{\delta'}{n_0+1}$. Now we can apply Lemma 5.6 to $\{\phi_j'\}$, and construct (possibly up to a subsequence) $\psi_j^1 : I(1,N_1)_0 \times I_0(m,l)_0 \to \mathcal{B}_{\rho_j}^{\mathcal{F}}(T)$.

Assume $3^N = 3^{N_1} + 3^{N_2}$, $N \in \mathbb{N}$; then N depends only on (l,m,T,δ,L), as N_2 depends only on $(l,m,\delta/2,L)$ and N_1 depends only on $(l,m, T,\delta/2,L + \frac{\delta}{2(n_0+1)})$. Define $\psi_j : I(1,N)_0 \times I_0(m,l)_0 \to \mathcal{B}_{\rho_j}^{\mathcal{F}}(T)$ by

$$\psi_j([\frac{i}{3^N}],x) = \psi_j^2([\frac{i}{3^{N_2}}],x), \quad \text{if } 0 \le i \le 3^{N_2};$$

$$\psi_j([\frac{i}{3^N}],x) = \psi_j^1([\frac{i-3^{N_2}}{3^{N_1}}],x), \quad \text{if } 3^{N_2} \le i \le 3^N.$$

Then $\{\psi_j\}$ satisfy Lemma 5.5(0)(ii)(iv) by combining Lemma 5.6(0)(ii) (iv) with Lemma 5.8 (0)(ii)(iv). For Lemma 5.5(i), if $m = 1$,

$$\mathbf{f}(\psi_j) \le \max\{\mathbf{f}(\psi_j^1), \mathbf{f}(\psi_j^2)\} \le \delta/2;$$

if $m > 1$, then by Lemma 5.6(i) and Lemma 5.8(i),

$$\mathbf{f}(\psi_j) \le \max\{\mathbf{f}(\psi_j^1), \mathbf{f}(\psi_j^2)\} \le \mathbf{f}(\psi_j^2) + \delta/2 \le \mathbf{f}(\phi_j) + \delta.$$

For Lemma 5.5(iii), by Lemma 5.6(iii) and Lemma 5.8(iii),

$$\max\{\mathbf{M}(\psi_j(\cdot,\cdot))\} \le \max\{\max\{\mathbf{M}(\psi_j^1(\cdot,\cdot))\}, \max\{\mathbf{M}(\psi_j^2(\cdot,\cdot))\}\}$$

$$\le \max\{\mathbf{M}(\psi_j^2(\cdot,\cdot))\} + \frac{\delta}{2(n_0+1)}$$

$$\le \max\{\mathbf{M}(\phi_j(\cdot))\} + \frac{\delta}{n_0+1}.$$

So we finished checking that $\{\psi_j\}$ satisfy Lemma 5.5(0)(i)(ii)(iii)(iv).
                            q.e.d.

Now let us go back to the proof of Proposition 5.3. This part is similar to the final part of [**MN12**, 13.3]. We will use notions in Lemma 5.5. We are going to construct the extensions $\tilde{\phi}_j$ of ϕ_j from $I(m,k_0)_0$ to $\mathcal{B}_{\rho_j}^{\mathcal{F}}(T)$ for every j large enough, therefore, get a contradiction.

First let us discuss the case when $m > 1$. Let

$$\hat{\tilde{\phi}}_j : I_0(m,N)_0 \times I(1,N)_0 \to \mathcal{B}_{\rho_j}^{\mathcal{F}}(T),$$

be defined by $\hat{\phi}_j(x, y) = \psi_j(y, \mathbf{n}(N, l)(x))$, where ψ_j are constructed in Lemma 5.5. Recall that $S(m+1, N)_0 = I_0(m, N)_0 \times I(1, N)_0$. We can extend $\hat{\phi}_j$ to

$$S(m+1, N)_0 \cup T(m+1, N)_0,$$

by assigning it to T on $T(m+1, N)_0$.

Now recall the map $\mathbf{r}(N) : I(m, N+q)_0 \to S(m+1, N)_0 \cup T(m+1, N)_0$ defined in [**MN12**, Appendix C], which satisfies: q depends on m but not on N; if $x, y \in I(m, N+q)_0$, $\mathbf{d}(x, y) = 1$, then $\mathbf{d}\big(\mathbf{r}(N)(x), \mathbf{r}(N)(y)\big) \leq m$; if $x \in I_0(m, N+q)_0$, then $\mathbf{r}(N)(x) \in [0] \times I_0(m, N)_0$ and $\mathbf{r}(N)(x) = \mathbf{n}(N+q, N)(x)$.

With out loss of generality, we can assume $k_0 > N + q$; then the extension $\tilde{\phi}_j : I(m, k_0)_0 \to \mathcal{B}^{\mathcal{F}}_{\rho_j}(T)$ is defined by

$$\tilde{\phi}_j = \hat{\phi}_j \circ \mathbf{r}_m(N) \circ \mathbf{n}(k_0, N+q),$$

for which Proposition 5.3(i)(ii)(iii) are easily seen true by Lemma 5.5(i)(ii)(iii).

Finally when $m = 1$, define $\hat{\phi}_j : I(1, N+1)_0 \to \mathcal{B}^{\mathcal{F}}_{\rho_j}(T)$ by:

$$\hat{\phi}_j([\frac{i}{3^{N+1}}]) = \psi_j([\frac{i}{3^N}], [0]), \quad \text{if } 0 \leq i \leq 3^N;$$

$$\hat{\phi}_j([\frac{i}{3^{N+1}}]) = T, \quad \text{if } 3^N + 1 \leq i \leq 2 \cdot 3^N;$$

$$\hat{\phi}_j([\frac{i}{3^{N+1}}]) = \psi_j([\frac{3^{N+1} - i}{3^N}], [1]), \quad \text{if } 2 \cdot 3^N + 1 \leq i \leq 3^{N+1},$$

for which Proposition 5.3(i)(ii)(iii) are automatically true by Lemma 5.5(i)(ii)(iii). To check Proposition 5.3(iv), by the definition of Almgren's isomorphism (4.2) and Lemma 5.5(iv),

$$F_A(\hat{\phi}_j) = F_A\big(\psi_j|_{I(1,N)_0 \times \{[0]\}}\big) - F_A\big(\psi_j|_{I(1,N)_0 \times \{[1]\}}\big)$$
$$= [[\Omega_T - \Omega_{j,0}]] - [[\Omega_T - \Omega_{j,1}]] = [[\Omega_{j,1} - \Omega_{j,0}]].$$

For $k_0 > N + 1$, the extension $\tilde{\phi}_j : I(1, k_0)_0 \to \mathcal{B}^{\mathcal{F}}_{\rho_j}(T)$ is given by $\hat{\phi}_j \circ \mathbf{n}(k_0, N+1)$.   q.e.d.

The next result removes the dependence of ϵ and k on the parameters l, m in Proposition 5.3, which is analogous to [**MN12**, 13.5]. The idea is to apply Proposition 5.3 inductively along the p-skeletons of $I(m, l)$, $1 \leq p \leq m$. In the induction process, compared to [**MN12**, 13.5] where they need to pay attention to the increase of the parameter "$\mathbf{m}(\phi, r)$",[14] we need to take care of the increase of the size of the neighborhoods around T.

Fix $n_0 \in \mathbb{N}$. $b(n_0)$ is a constant depending only on n_0.

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[14]This parameter measures the local mass density. See [**MN12**, 4.2].

Proposition 5.10. *Given $\delta, L > 0$, and*

$$T \in \mathcal{Z}_n(M) \cap \{S : \mathbf{M}(S) \le 2L - \delta\}, \textit{with } T = \partial[[\Omega_T]],$$

$\Omega_T \in \mathcal{C}(M)$, *then there exist* $0 < \epsilon = \epsilon(T, \delta, L) < \delta$, *and* $k = k(T, \delta, L) \in \mathbb{N}$, *and a function* $\rho = \rho_{(T, \delta, L)} : \mathbb{R}^1_+ \to \mathbb{R}^1_+$, *with* $\rho(s) \to 0$, *as* $s \to 0$, *such that: given* $l, m \in \mathbb{N}$, $m \le n_0 + 1$, $0 < s < \epsilon$, *and*
(5.35)
$$\phi : I_0(m, l)_0 \to \mathcal{B}^{\mathcal{F}}_s(T) \cap \{S : \mathbf{M}(S) \le 2L - \delta\}, \ \textit{with } \phi(x) = \partial[[\Omega_x]],$$

$\Omega_x \in \mathcal{C}(M)$, $x \in I_0(m, l)_0$, *there exists*

$$\tilde{\phi} : I(m, l + k)_0 \to \mathcal{B}^{\mathcal{F}}_{\rho(s)}(T), \ \textit{with } \tilde{\phi}(y) = \partial[[\Omega_y]],$$

$\Omega_y \in \mathcal{C}(M)$, $y \in I(m, l + k)_0$, *and satisfying*

(i) $\mathbf{f}(\tilde{\phi}) \le \delta$ *if* $m = 1$, *and* $\mathbf{f}(\tilde{\phi}) \le b(n_0)(\mathbf{f}(\phi) + \delta)$ *if* $m > 1$;
(ii) $\tilde{\phi} = \phi \circ \mathbf{n}(l + k, l)$ *on* $I_0(m, l + k)_0$;
(iii)

$$\sup_{x \in I(m, l+k)_0} \mathbf{M}(\tilde{\phi}(x)) \le \sup_{x \in I_0(m, l)_0} \mathbf{M}(\phi(x)) + \delta;$$

(iv) *If* $m = 1$, $\delta < \nu_M$, $\phi([0]) = \partial[[\Omega_0]]$, $\phi([1]) = \partial[[\Omega_1]]$, *then*

$$F_A(\tilde{\phi}) = [[\Omega_1 - \Omega_0]],$$

where F_A *is the Almgren's isomorphism (4.2).*

Proof. The case $m = 1$ follows directly from Proposition 5.3. In fact, take $\epsilon = \epsilon(0, 1, T, \delta, L)$, $k = k(0, 1, T, \delta, L)$ and $\rho(s) = \rho_{(0,1,T,\delta,L)}(s)$ by Proposition 5.3, and denote the extension by $\tilde{\phi}_1 : I(1, k)_0 \to \mathcal{B}^{\mathcal{F}}_{\rho(s)}(T)$. Then $\tilde{\phi} : I(1, l + k)_0 \to \mathcal{B}^{\mathcal{F}}_{\rho(s)}(T)$ is given by $\tilde{\phi} = \tilde{\phi}_1 \circ \mathbf{n}(l + k, k)$. The fact that $\tilde{\phi}$ satisfies properties (i)(ii)(iii)(iv) follows from the fact that $\tilde{\phi}_1$ satisfies Proposition 5.3(i)(ii)(iii)(iv).

Now let us assume that $m > 1$. Using notations in Proposition 5.3, we can inductively define integers,

$$k_0 = 0, k_1 = k(0, 1, T, \delta, L), \cdots, k_i = k(k_{i-1}, i, T, \delta, L), \cdots,$$
$$k_m = k(k_{m-1}, m, T, \delta, L);$$

and positive numbers,

$$\epsilon_1 = \epsilon(0, 1, T, \delta, L), \cdots, \epsilon_i = \epsilon(k_{i-1}, i, T, \delta, L), \cdots,$$
$$\epsilon_m = \epsilon(k_{m-1}, m, T, \delta, L);$$

and functions from $\mathbb{R}^1_+$ to $\mathbb{R}^1_+$,

$$\rho_1 = \rho_{(0,1,T,\delta,L)}, \cdots, \rho_i = \rho_{(k_{i-1}, i, T, \delta, L)} \circ \rho_{i-1}, \cdots,$$
$$\rho_m = \rho_{(k_{m-1}, m, T, \delta, L)} \circ \rho_{m-1}.$$

As $\lim_{s\to 0}\rho_{(k_{i-1},i,T,\delta,L)}(s) = 0$, for $1 \le i \le m$, we know that $\lim_{s\to 0}\rho_i(s) = 0$, for all $1 \le i \le m$. Hence we can choose $\epsilon > 0$, such that $\epsilon \le \min\{\epsilon_1,\cdots,\epsilon_m\}$, and

$$\tilde{\epsilon}_i := \max_{0\le s\le\epsilon}\rho_i(s) \le \epsilon_{i+1}, \quad \text{for all } 1 \le i \le m-1.$$

Let $k = k_m$, and $\rho = \rho_m$; then ϵ, k, ρ depend only on T, δ, L. In the following, we will show that ϵ, k, ρ satisfy the requirement.

Fix a map $\phi : I_0(m,l)_0 \to \mathcal{B}_s^{\mathcal{F}}(T) \cap \{S : \mathbf{M}(S) \le 2L - \delta\}$, with $\phi(x) = \partial[[\Omega_x]]$, $\Omega_x \in \mathcal{C}(M)$, for all $x \in I_0(m,l)_0$. Assume that $s \le \epsilon$. Given $p \le m$, let V_p be the set of vertices of $I(m,l+k_p)$ that belong to the p-skeleton of $I(m,l)$, i.e., $V_p = \cup_{\alpha\in I(m,l)_p}\alpha(k_p)_0$. Clearly $V_m = I(m,l+k)_0$. Say a map $\phi_p : V_p \to \mathcal{B}_{\rho_p(s)}^{\mathcal{F}}(T) \cap \{S : \mathbf{M}(S) \le 2L\}$ is a p-extension of ϕ, if:

1) $\phi_p(y) = \partial[[\Omega_y]]$, $\Omega_y \in \mathcal{C}(M)$, for all $y \in V_p$;
2) $\phi_p = \phi \circ \mathbf{n}(l+k_p,l)$ on $V_p \cap I_0(m,l+k_p)_0$;
3) If $p = 1$, then $\mathbf{f}(\phi_p) \le \mathbf{f}(\phi) + \delta$; if $p > 1$, there exists a $(p-1)$-extension ϕ_{p-1} of ϕ, such that

$$\mathbf{f}(\phi_p) \le p\big(\mathbf{f}(\phi_{p-1}) + \delta\big);$$

4) $\sup_{y\in V_p}\mathbf{M}\big(\phi_p(y)\big) \le \sup_{x\in I_0(m,l)_0}\mathbf{M}\big(\phi(x)\big) + \frac{p\delta}{n_0+1}$.

We start with the construction of 1-extension ϕ_1 of ϕ. First construct a trivial extension of ϕ to $I(m,l)_0$, i.e., $\phi_0 : I(m,l)_0 \to \mathcal{B}_s^{\mathcal{F}}(T) \cap \{S : \mathbf{M}(S) \le 2L - \delta\}$ by

$$\phi_0(x) = \phi(x), \quad x \in I_0(m,l)_0;$$
$$\phi_0(x) = T, \qquad x \notin I_0(m,l)_0.$$

Then we can construct $\tilde{\phi}_0 : V_1 \to \mathcal{B}_{\rho_1(s)}^{\mathcal{F}}(T)$ as follows: given $\alpha \in I(m,l)_1$, $\tilde{\phi}_0|_{\alpha(k_1)_0}$ is gotten by extending $\phi_0|_{\alpha_0}$ on α_0 to $\alpha(k_1)_0$ using Proposition 5.3 for $l = 0, m = 1, T, \delta, L$ as $s \le \epsilon \le \epsilon_1$. Finally, we can define $\phi_1 : V_1 \to \mathcal{B}_{\rho_1(s)}^{\mathcal{F}}(T)$ by

$$\phi_1 = \phi_0 \circ \mathbf{n}(l+k_1,l), \quad \text{on } \alpha(k_1)_0, \text{ if } \alpha \text{ is a 1-cell of } I_0(m,l);$$

$$\phi_1 = \tilde{\phi}_0, \qquad \text{on } \alpha(k_1)_0, \text{ if } \alpha \text{ is not a 1-cell of } I_0(m,l).$$

It is easy to check that ϕ_1 is a 1-extension of ϕ.

To get p-extension inductively, we need the following lemma:

Lemma 5.11. *Given a p-extension ϕ_p of ϕ, $p \le m-1$, there exists a $(p+1)$-extension ϕ_{p+1} of ϕ.*

Proof. By assumption ϕ_p maps V_p into $\mathcal{B}_{\rho_p(s)}^{\mathcal{F}}(T) \cap \{S : \mathbf{M}(S) \le 2L\}$, so the image of ϕ_p also lie in $\mathcal{B}_{\tilde{\epsilon}_{p+1}}^{\mathcal{F}}(T) \cap \{S : \mathbf{M}(S) \le 2L\}$ as $\rho_p(s) \le \tilde{\epsilon}_p \le \epsilon_{p+1}$. Using the fact that $\phi_p(x) = \partial[[\Omega_x]]$, $\Omega_x \in \mathcal{C}(M)$ for all $x \in V_p$, we can apply Proposition 5.3 for each $(p+1)$-cell $\alpha \in I(m,l)_{p+1}$ to extend

$\phi_p|_{\alpha_0(k_p)_0}$ to $\tilde{\phi}_{p,\alpha} : \alpha(k_{p+1})_0 \to \mathcal{B}^{\mathcal{F}}_{\rho_{p+1}(s)}(T)$ for $l = k_p, m = p+1, T, \delta, L$. Given any two adjacent $(p+1)$-cells $\alpha, \bar{\alpha} \in I(m,l)_{p+1}$, by Proposition 5.3(ii), $\tilde{\phi}_{p,\alpha} = \tilde{\phi}_{p,\bar{\alpha}} = \phi_p \circ \mathbf{n}(k_{p+1}, k_p)$ on $\alpha(k_{p+1})_0 \cap \bar{\alpha}(k_{p+1})_0$, so we can construct a map

$$\tilde{\phi}_p : V_{p+1} \to \mathcal{B}^{\mathcal{F}}_{\rho_{p+1}(s)}(T),$$

by letting $\tilde{\phi}_p = \tilde{\phi}_{p,\alpha}$ on each $\alpha(k_{p+1})$, $\alpha \in I(m,l)_{p+1}$. By Proposition 5.3(i)(iii) and the inductive hypothesis 4,

$$\mathbf{f}(\tilde{\phi}_p) \le (p+1)\big(\mathbf{f}(\phi_p) + \delta\big);$$

$$\sup_{x \in V_{p+1}} \mathbf{M}\big(\tilde{\phi}_p(x)\big) \le \sup_{x \in V_p} \mathbf{M}\big(\phi_p(x)\big) + \frac{\delta}{n_0 + 1}$$

$$\le \sup_{x \in I_0(m,l)_0} \mathbf{M}\big(\phi(x)\big) + \frac{(p+1)\delta}{n_0 + 1}.$$

Finally, we define $\phi_{p+1} : V_{p+1} \to \mathcal{B}^{\mathcal{F}}_{\rho_{p+1}(s)}(T)$ by

$$\phi_{p+1} = \phi_p \circ \mathbf{n}(l + k_{p+1}, l + k_p), \quad \text{on } \alpha(k_{p+1})_0,$$
$$\text{if } \alpha \text{ is a } (p+1)\text{-cell of } I_0(m,l);$$

$$\phi_{p+1} = \tilde{\phi}_p, \qquad \text{on } \alpha(k_{p+1})_0, \text{ if } \alpha \text{ is not a } (p+1)\text{-cell of } I_0(m,l).$$

Now we check that ϕ_{p+1} satisfies all the requirements for a $(p+1)$-extension of ϕ. First, by construction $\phi_{p+1}(x) = \partial[[\Omega_x]]$, $\Omega_x \in \mathcal{C}(M)$, for all $x \in V_{p+1}$; second, given a $(p+1)$-cell α in $I_0(m,l)$, by inductive hypothesis 2, $\phi_{p+1} = \phi_p \circ \mathbf{n}(l + k_{p+1}, l + k_p) = \phi \circ \mathbf{n}(l + k_p, l) \circ \mathbf{n}(l + k_{p+1}, l + k_p) = \phi \circ \mathbf{n}(l + k_{p+1}, l)$ on $\alpha(k_{p+1})_0$; lastly, as ϕ_{p+1} is gotten by replacing $\tilde{\phi}_p$ by $\phi_p \circ \mathbf{n}(l + k_{p+1}, l + k_p)$ on $V_{p+1} \cap I_0(m, l + k_{p+1})_0$, hence $\mathbf{f}(\phi_{p+1}) \le \mathbf{f}(\tilde{\phi}_p) \le (p+1)\big(\mathbf{f}(\phi_p) + \delta\big)$, and $\sup_{x \in V_{p+1}} \mathbf{M}(\phi_{p+1}(x)) \le \sup_{x \in V_{p+1}} \mathbf{M}(\tilde{\phi}_p(x)) \le \sup_{x \in I_0(m,l)_0} \mathbf{M}(\phi(x)) + \frac{(p+1)\delta}{n_0+1}$.   q.e.d.

We can then inductively construct an m-extension $\phi_m : I(m, l + k_m)_0 \to \mathcal{B}^{\mathcal{F}}_{\rho_m(s)}(T)$. Let $\tilde{\phi} = \phi_m$; then it is easy to see that $\tilde{\phi}, \epsilon, k = k_m, \rho = \rho_m$ satisfy all the requirements of Proposition 5.10.   q.e.d.

5.2. Proof of Theorem 5.1. The idea is briefly as follows. Denote

$$L(\Phi) = \max_{x \in [0,1]} \mathbf{M}\big(\Phi(x)\big).$$

Given a $\delta > 0$, we can cover the set $\mathcal{Z}_n(M^{n+1}) \cap \{S : \mathbf{M}(S) \le 2L(\Phi)\} \cap \{S : S = \partial[[\Omega]] : \Omega \in \mathcal{C}(M)\}$ by finitely many balls $\{\mathcal{B}^{\mathcal{F}}_{\epsilon_i}(T_i)\}_{i=1}^{N}$, such that Proposition 5.10 can be applied on each ball for $n_0 = 1, T_i, \delta, L = L(\Phi)$.[15] Take j large enough, such that for each 1-cell $\alpha \in I(1,j)_1$, the image $\Phi(\alpha)$ lie in some $\mathcal{B}^{\mathcal{F}}_{\epsilon_i}(T_i)$; then we can apply Proposition 5.10 to each $\Phi|_{\alpha_0}$, and construct a discrete map ϕ_δ which has fineness controlled

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[15]Note that $n_0 = 1$ is the dimension of parameter space.

by δ, and total mass bounded by $L(\Phi) + \delta$. Finally, taking a sequence $\delta_i \to 0$, $i \to \infty$, we can construct the desired $(1, \mathbf{M})$-homotopy sequence $\{\phi_i\}_{i \in \mathbb{N}}$ by letting $\phi_i = \phi_{\delta_i}$. Detailed argument is given as below.

Proof of Theorem 5.1. In this part, we will repeatedly use notations and conclusions in Proposition 5.10 for $n_0 = 1$.[16]

Step I: Fix $\delta > 0$, such that $L = L(\Phi) < 2L - 2\delta$. By the weak compactness of the set $\mathcal{Z}_n(M^{n+1}) \cap \{S : \mathbf{M}(S) \leq 2L\} \cap \{S : S = \partial[[\Omega]] : \Omega \in \mathcal{C}(M)\}$ (see [**Si83**, §37.2][**Gi**, §1.20]), we can find a finite covering by balls $\{\mathcal{B}^{\mathcal{F}}_{\epsilon_i}(T_i) : i = 1, \cdots, N\}$, such that $T_i = \partial[[\Omega_i]]$, $\Omega_i \in \mathcal{C}(M)$, $\mathbf{M}(T_i) \leq 2L$, and

(5.36) $$3\epsilon_i + \sup_{0 \leq s \leq 3\epsilon_i} \rho_i(s) < \epsilon(T_i, \delta, L),$$

where $\epsilon(T_i, \delta, L)$, $k_i = k(T_i, \delta, L)$ and $\rho_i(s) = \rho_{(T_i, \delta, L)}(s)$ are given by Proposition 5.10. Assume that $\epsilon_1 \leq \epsilon_2 \leq \cdots \leq \epsilon_N \leq \delta$, and denote $k = \max\{k_i : 1 \leq i \leq N\}$.

By the continuity of Φ under the flat topology, we can take $j \in \mathbb{N}$ large enough, such that for any $\alpha \in I(1, j)_1$,

(5.37) $$\sup_{x,y \in \alpha} \mathcal{F}(\Phi(x) - \Phi(y)) < \epsilon_1 < \delta.$$

Define $c : I(1, j)_0 \to \{1, \cdots, N\}$ by $c(x) = \sup\{i : \Phi(x) \in \mathcal{B}^{\mathcal{F}}_{\epsilon_i}(T_i)\}$. Then define

$$c : I(1, j)_1 \to \{1, \cdots, N\},$$

by $c(\alpha) = \sup\{c(x) : x \in \alpha_0\}$.

Claim 3. $\Phi(\alpha) \subset \mathcal{B}^{\mathcal{F}}_{2\epsilon_{c(\alpha)}}(T_{c(\alpha)})$.

Proof. By definition, there exists $x \in \alpha_0$, such that $c(\alpha) = c(x)$, then $\Phi(x) \in \mathcal{B}^{\mathcal{F}}_{\epsilon_{c(\alpha)}}(T_{c(\alpha)})$. By (5.37), for any $y \in \alpha$, $\Phi(y) \in \mathcal{B}^{\mathcal{F}}_{\epsilon_1}(\Phi(x)) \subset \mathcal{B}^{\mathcal{F}}_{2\epsilon_{c(\alpha)}}(T_{c(\alpha)})$, as $\epsilon_1 \leq \epsilon_{c(\alpha)}$. q.e.d.

Let $\phi_0 : I(1, j)_0 \to \mathcal{Z}_n(M)$ be the restriction of Φ to $I(1, j)_0$, then $\phi_0(\alpha_0) \subset \mathcal{B}^{\mathcal{F}}_{2\epsilon_{c(\alpha)}}(T_{c(\alpha)})$ for all $\alpha \in I(1, j)_1$. By (5.36) and Theorem 5.1(a), we can apply Proposition 5.10 to each $\phi_0|_{\alpha_0}$, $\alpha \in I(1, j)_1$, and get

$$\tilde{\phi}_{0,\alpha} : \alpha(k_{c(\alpha)})_0 \to \mathcal{B}^{\mathcal{F}}_{\rho_{c(\alpha)}(2\epsilon_{c(\alpha)})}(T_{c(\alpha)}).$$

Define $\phi_\delta : I(1, j + k)_0 \to \mathcal{Z}_n(M)$ by

$$\phi_\delta = \tilde{\phi}_{0,\alpha} \circ \mathbf{n}(j + k, j + k_{c(\alpha)}), \quad \text{on } \alpha(k)_0.$$

Now we collect a few properties of ϕ_δ.

1) $\phi_\delta = \Phi$ on $I(1, j)_0$;
2) $\phi_\delta(x) = \partial[[\Omega_x]]$, $\Omega_x \in \mathcal{C}(M)$, for all $x \in I(1, j + k)_0$;

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[16] Again, $n_0 = 1$ is the dimension of parameter space.

3) $\mathbf{f}(\phi_\delta) \leq \delta$;

4) For any $\alpha \in I(1, j)_1$,
$$\sup_{x \in \alpha(k)_0} \mathbf{M}(\phi_\delta(x)) \leq \sup_{x \in \alpha_0} \mathbf{M}(\Phi(x)) + \delta < 2L - \delta;$$

5) $\sup\{\mathcal{F}(\phi_\delta(x) - \Phi(x)) : x \in I(1, j + k)_0\} \leq \delta$;

6) If $\delta < \nu_M$, then $F_A(\phi_\delta) = [[\Omega_1 - \Omega_0]]$, where $\Phi(0) = \partial[[\Omega_0]]$, $\Phi(1) = \partial[[\Omega_1]]$.

1 is by construction. 2,3,4,6 directly come from Proposition 5.10. 5 comes from (5.36), and the fact that $\phi_\delta(\alpha(k)_0) \subset \mathcal{B}^{\mathcal{F}}_{\rho_{c(\alpha)}(2\epsilon_{c(\alpha)})}(T_{c(\alpha)})$, $\Phi(\alpha) \subset \mathcal{B}^{\mathcal{F}}_{2\epsilon_{c(\alpha)}}(T_{c(\alpha)})$.

Step II: We say $\bar{\phi} : I(1, \bar{k})_0 \to \mathcal{Z}_n(M)$ is a $(\delta, \bar{k})$-extension of Φ, $\bar{k} \geq j + k$, if

1) $\bar{\phi} = \Phi$ on $I(1, j)_0$;

2) $\bar{\phi}(x) = \partial[[\Omega_x]]$, $\Omega_x \in \mathcal{C}(M)$, for all $x \in I(1, \bar{k})_0$;

3) $\mathbf{f}(\bar{\phi}) \leq \delta$;

4) For any $\alpha \in I(1, j)_1$,
$$\sup_{x \in \alpha(\bar{k}-j)_0} \mathbf{M}(\bar{\phi}(x)) \leq \sup_{x \in \alpha} \mathbf{M}(\Phi(x)) + \delta < 2L - \delta;$$

5) $\sup\{\mathcal{F}(\phi_\delta(x) - \Phi(x)) : x \in I(1, \bar{k})_0\} \leq \epsilon_1$.

The following lemma says that a $(\delta, \bar{k})$-extension $\bar{\phi}$ is 1-homotopic to ϕ_δ with fineness δ:

Lemma 5.12. *Given a $(\delta, \bar{k})$-extension $\bar{\phi}$ of Φ, with $\bar{k} \geq j + k$, then there exists*
$$\psi : I(1, \hat{k})_0 \times I(1, \hat{k})_0 \to \mathcal{Z}_n(M),$$
with $\hat{k} = \bar{k} + k$, such that

(a) $\psi(y, x) = \partial[[\Omega_{y,x}]]$, $\Omega_{y,x} \in \mathcal{C}(M)$, *for any $(y, x) \in I(1, \hat{k})_0 \times I(1, \hat{k})_0$;*

(b) $\psi([0], \cdot) = \phi_\delta \circ \mathbf{n}(\hat{k}, j + k)$, *and $\psi([1], \cdot) = \bar{\phi} \circ \mathbf{n}(\hat{k}, \bar{k})$;*

(c) $\mathbf{f}(\psi) \leq c_0\delta$, *for a fixed constant c_0;*

(d) $\mathbf{M}(\psi(y, x)) \leq \sup\{\mathbf{M}(\Phi(x')), x, x'$ *lie in some common 1-cell $\alpha \in I(1, j)_1\} + 2\delta$, for any $(y, x) \in I(1, \hat{k})_0 \times I(1, \hat{k})_0$.*

Proof. Given $\alpha \in I(1, j)_1$, using property 5 for $(\delta, \bar{k})$-extension and the fact that $\Phi(\alpha) \subset \mathcal{B}^{\mathcal{F}}_{2\epsilon_{c(\alpha)}}(T_{c(\alpha)})$, we have $\bar{\phi}(\alpha \cap I(1, \bar{k})_0) \subset \mathcal{B}^{\mathcal{F}}_{3\epsilon_{c(\alpha)}}(T_{c(\alpha)})$.

We will first construct ψ on $[0, \frac{1}{3^j}](\hat{k} - j)_0 \times I(1, \hat{k})_0$,[17] such that ψ satisfies:
$$\psi([0], \cdot) = \phi_\delta \circ \mathbf{n}(\hat{k}, j + k); \quad \psi([\frac{1}{3^j}], \cdot) = \bar{\phi} \circ \mathbf{n}(\hat{k}, \bar{k}),$$

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[17]Notice that $[0, \frac{1}{3^j}](\hat{k} - j)_0 = [0, \frac{1}{3^j}] \cap I(1, \hat{k})_0$.

and Lemma 5.12(a)(c)(d), where in (d) $(y, x) \in [0, \frac{1}{3^j}](\hat{k} - j)_0 \times I(1, \hat{k})_0$. Then we can extend ψ to $\left([\frac{1}{3^j}, 1] \cap I(1, \hat{k})_0\right) \times I(1, \hat{k})_0$ trivially by letting $\psi(y, x) = \bar{\phi} \circ \mathbf{n}(\hat{k}, \bar{k})(x)$ for $(y, x) \in \left([\frac{1}{3^j}, 1] \cap I(1, \hat{k})_0\right) \times I(1, \hat{k})_0$.

Let W_1 be the set of vertices of $[0, \frac{1}{3^j}](\bar{k} - j)_0 \times I(1, \bar{k})_0$ which belong to the 1-skeleton of $[0, \frac{1}{3^j}] \times I(1, j)$ (think $[0, \frac{1}{3^j}] \cong I(1, 0)$), and define $\psi_0 : W_1 \to \mathcal{Z}_n(M)$ by:

$$\psi_0([0], \cdot) = \phi_\delta \circ \mathbf{n}(\bar{k}, j + k); \quad \psi_0([\frac{1}{3^j}], \cdot) = \bar{\phi};$$

$$\psi_0(\cdot, x) \equiv \Phi(x), \quad \text{for all } x \in I(1, j)_0.$$

Then ψ_0 satisfies:

1) $\mathbf{f}(\psi_0) \leq \max\{\mathbf{f}(\phi_\delta), \mathbf{f}(\bar{\phi})\} \leq \delta$, as $\phi_\delta|_{I(1,j)_0} = \bar{\phi}|_{I(1,j)_0} = \Phi$;

2) Given any 2-cell β in $[0, \frac{1}{3^j}] \times I(1, j)$, with $\beta = [0, \frac{1}{3^j}] \otimes \alpha$, for some $\alpha \in I(1, j)_1$, then ψ_0 maps $\beta_0(\bar{k} - j)_0$[18] into $\mathcal{B}^{\mathcal{F}}_{\rho_{c(\alpha)}(2\epsilon_{c(\alpha)}) + 3\epsilon_{c(\alpha)}}(T_{c(\alpha)})$, so $\psi_0\left(\beta_0(\bar{k} - j)_0\right) \subset \mathcal{B}^{\mathcal{F}}_{\epsilon(T_{c(\alpha)}, \delta, L)}(T_{c(\alpha)})$ by (5.36);
   $--$This is because $\Phi(\alpha) \subset \mathcal{B}^{\mathcal{F}}_{2\epsilon_{c(\alpha)}}(T_{c(\alpha)})$, $\phi_\delta\left(\alpha(k)_0\right) \subset \mathcal{B}^{\mathcal{F}}_{\rho_{c(\alpha)}(2\epsilon_{c(\alpha)})}(T_{c(\alpha)})$, and $\bar{\phi}\left(\alpha(\bar{k} - j)_0\right) \subset \mathcal{B}^{\mathcal{F}}_{3\epsilon_{c(\alpha)}}(T_{c(\alpha)})$.[19]

3) $\psi_0(y, x) = \partial[[\Omega_{y,x}]]$, $\Omega_{y,x} \in \mathcal{C}(M)$, for all $(y, x) \in W_1$;
   $--$This comes from property 2 of ϕ_δ and $\bar{\phi}$.

4) $\sup_{(y,x) \in W_1} \mathbf{M}\left(\psi_0(y, x)\right) \leq \max\{\sup_{I(1,j+k)_0} \mathbf{M}(\phi_\delta), \sup_{I(1,\bar{k})_0} \mathbf{M}(\bar{\phi})\} \leq 2L - \delta$;
   $--$The last "$\leq$" comes from property 4 of ϕ_δ and $\bar{\phi}$.

Therefore, we can apply Proposition 5.10 for each 2-cell $\beta = [0, \frac{1}{3^j}] \otimes \alpha$ in $[0, \frac{1}{3^j}] \times I(1, j)$ to extend $\psi_0|_{\beta_0(\bar{k}-j)_0}$ to

$$\tilde{\psi}_{0,\beta} : \beta(\bar{k} - j + k_{c(\alpha)})_0 \to \mathcal{Z}_n(M),$$

which satisfies:

(a) $\mathbf{f}(\tilde{\psi}_{0,\beta}) \leq b(1)\left(\mathbf{f}(\psi_0) + \delta\right) \leq 2b(1)\delta$;

(b) $\tilde{\psi}_{0,\beta}([0], \cdot) = \psi_0([0], \cdot) \circ \mathbf{n}(\bar{k} - j + k_{c(\alpha)}, \bar{k} - j) = \phi_\delta \circ \mathbf{n}(\bar{k} - j + k_{c(\alpha)}, k)$ on $\alpha(\bar{k} - j + k_{c(\alpha)})_0$, and $\tilde{\psi}_{0,\beta}([\frac{1}{3^j}], \cdot) = \psi_0([\frac{1}{3^j}], \cdot) \circ \mathbf{n}(\bar{k} - j + k_{c(\alpha)}, \bar{k} - j) = \bar{\phi} \circ \mathbf{n}(\bar{k} - j + k_{c(\alpha)}, \bar{k} - j)$ on $\alpha(\bar{k} - j + k_{c(\alpha)})_0$;

(c)

$$\sup_{\beta(\bar{k}-j+k_{c(\alpha)})_0} \mathbf{M}(\tilde{\psi}_{0,\beta}) \leq \sup_{\beta_0(\bar{k}-j)_0} \mathbf{M}(\psi_0) + \delta$$

$$\leq \max\{\sup_{I(1,j+k)_0} \mathbf{M}(\phi_\delta), \sup_{I(1,\bar{k})_0} \mathbf{M}(\bar{\phi})\} + \delta$$

$$\leq \sup_{x \in \alpha} \mathbf{M}\left(\Phi(x)\right) + 2\delta.$$

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[18] Here $\beta_0(\bar{k} - j)_0 = \beta_0 \cap I(1, \bar{k})_0 \times I(1, \bar{k})_0$.

[19] Here $\alpha(\bar{k} - j)_0 = \alpha \cap I(1, \bar{k})_0$.

Also given any two adjacent 2-cells $\beta = [0, \frac{1}{3^j}] \otimes \alpha$ and $\bar{\beta} = [0, \frac{1}{3^j}] \otimes \bar{\alpha}$ in $[0, \frac{1}{3^j}] \times I(1, j)$, by Proposition 5.10(ii), we know that $\tilde{\psi}_{0,\beta} \circ \mathbf{n}(\hat{k} - j, \bar{k} - j + k_{c(\alpha)}) = \tilde{\psi}_{0,\bar{\beta}} \circ \mathbf{n}(\hat{k} - j, \bar{k} - j + k_{c(\bar{\alpha})}) = \psi_0 \circ \mathbf{n}(\hat{k} - j, \bar{k} - j)$ on $\beta(\hat{k} - j)_0 \cap \bar{\beta}(\hat{k} - j)_0$, so we can put all $\{\tilde{\psi}_{0,\beta}\}$ together and construct the desired map

$$\psi : [0, \frac{1}{3^j}](\hat{k} - j)_0 \times I(1, \hat{k})_0 \to \mathcal{Z}_n(M),$$

by letting $\psi = \tilde{\psi}_{0,\beta} \circ \mathbf{n}(\hat{k} - j, \bar{k} - j + k_{c(\alpha)})$ on $\beta(\hat{k} - j)_0$ for each 2-cell $\beta = [0, \frac{1}{3^j}] \otimes \alpha$. It is straightforward to check that ψ satisfies the requirement.   q.e.d.

Now let us go back to finish the proof of Theorem 5.1. Take a sequence of positive numbers $\{\delta_i\}$, $\delta_i \to 0$, as $i \to \infty$; then by Step I, we can construct a sequence of mappings $\{\phi_i\}$, with $\phi_i = \phi_{\delta_i/c_0} : I(1, j_i+k_i)_0 \to \mathcal{Z}_n(M)$.[20] After extracting a subsequence, we can assume that ϕ_{i+1} is a $(\delta_i, j_{i+1} + k_{i+1})$-extension of Φ. Then we can apply Lemma 5.12 to ϕ_i and ϕ_{i+1}, so as to construct ψ_i satisfying Theorem 5.1(ii). The fact that ϕ_i satisfy Theorem 5.1(i)(iii)(iv) come from properties 4,5,6 of ϕ_δ in Part I.   q.e.d.

6. Proof of the main theorem

The main idea for proving Theorem 1.1 is to apply the Almgren–Pitts min–max theory to the good families constructed in §3, so that we can obtain an optimal minimal hypersurface satisfying the requirement. The idea is similar to the proof of [**Z12**, Theorem 1.1], while we need a more delicate comparison argument when checking the min–max hypersurface has index one (cf. Claim 4).

Given $\Sigma \in \mathcal{S}$ (1.2), we can define a mapping into $(\mathcal{Z}_n(M^{n+1}), \{0\})$

(6.1) $\Phi^\Sigma : [0, 1] \to (\mathcal{Z}_n(M^{n+1}), \{0\}),$

as follows:

(i) When $\Sigma \in \mathcal{S}_+$ (3.2), let $\Phi^\Sigma(x) = \partial[[\Omega_x]]$, where $\Omega_x = \{p \in M : d_\pm^\Sigma(p) \le (2x - 1)d(M)\}$. Here $d_\pm^\Sigma$ is the signed distance function (3.1), and $d(M)$ is the diameter of M.

(ii) When $\Sigma \in \mathcal{S}_-$ (3.4), let $\Phi^\Sigma(x) = \partial[[\Omega_x]]$, where $\Omega_x = \{p \in M : d^\Sigma(p) \le xd(M)\}$. Here d^Σ is the distance function to $\overline{\Sigma}$.

By Proposition 3.4 and Proposition 3.6, Φ^Σ satisfies:

Proposition 6.1. $\Phi^\Sigma : [0, 1] \to (\mathcal{Z}_n(M^{n+1}), \{0\})$ *is continuous under the flat topology, and*

(a) $\Phi^\Sigma(x) = \partial[[\Omega_x]]$, $\Omega_x \in \mathcal{C}(M)$ *for all* $x \in [0, 1]$, *and* $\Omega_0 = \emptyset$, $\Omega_1 = M$;

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[20] c_0 is given in Lemma 5.12(c).

(b) $\sup_{x \in [0,1]} \mathbf{M}\big(\Phi^\Sigma(x)\big) = \mathcal{H}^n(\Sigma)$, if $\Sigma \in \mathcal{S}_+$;

(c) $\sup_{x \in [0,1]} \mathbf{M}\big(\Phi^\Sigma(x)\big) = 2\mathcal{H}^n(\Sigma)$, if $\Sigma \in \mathcal{S}_-$.

REMARK 6.2. Notice that Φ^Σ satisfies the requirement to apply Theorem 5.1.

We need one more elementary fact about min–max hypersurface, which is well-known to experts. A proof is included for completeness.

Lemma 6.3. *Let* $\{\Sigma_i\}_{i=1}^l$ *be the singular minimal hypersurfaces given in Theorem 4.7, then the associated integral varifolds* $[\Sigma_i]$ *have tangent cones with multiplicity one everywhere. Therefore,* $\Sigma_i \in \mathcal{S}$.

Proof. Given any point $p \in \overline{\Sigma}_i$, then $\overline{\Sigma}_i$ is stable (cf. [**P81**, 2.3][**I96**, (5)(6)]) in any small annuli neighborhood of p by [**P81**, 3.3]. A standard cutoff argument implies that $\overline{\Sigma}_i$ is stable near p, and [**I96**, Theorem B] implies that every tangent cone of $[\Sigma_i]$ has multiplicity one.   q.e.d.

Proof of Theorem 1.1. Given $\Sigma \in \mathcal{S}$ and Φ^Σ (6.1), we can apply Theorem 5.1 to Φ^Σ and get a $(1, \mathbf{M})$-homotopy sequence $S^\Sigma = \{\phi_i^\Sigma\}_{i \in \mathbb{N}}$ into $\big(\mathcal{Z}_n(M^{n+1}, \mathbf{M}), \{0\}\big)$. By (5.1) and Proposition 6.1,

$$(6.2) \qquad \mathbf{L}(\{\phi_i^\Sigma\}_{i \in \mathbb{N}}) \leq \begin{cases} \mathcal{H}^n(\Sigma), & \text{if } \Sigma \in \mathcal{S}_+; \\ 2\mathcal{H}^n(\Sigma), & \text{if } \Sigma \in \mathcal{S}_-. \end{cases}$$

Also by Theorem 5.1(iv), $S^\Sigma \in F_A^{-1}\big([[M]]\big) \in \pi_1^\#\big(\mathcal{Z}_n(M^{n+1}, \mathbf{M}), \{0\}\big)$. Denote $F_A^{-1}\big([[M]]\big)$ by Π_M. By Theorem 4.7, $\mathbf{L}(\Pi_M) > 0$. Using (6.2), we have that

$$\mathbf{L}(\Pi_M) \leq A_M,$$

where A_M is defined in (1.3).

The Min–max Theorem 4.7 applied to Π_M gives a stationary varifold $V = \sum_{i=1}^l m_i[\Sigma_i]$, with $m_i \in \mathbb{N}$ and $\{\Sigma_i\}$ a disjoint collection of minimal hypersurfaces in $\mathcal{S}$, such that $\mathbf{L}(\Pi_M) = \|V\|(M) = \sum_{i=1}^l m_i \mathcal{H}^n(\Sigma_i)$. Notice that there is only one connected component, denoted by Σ_A, by Theorem 2.10 as M has positive Ricci curvature, i.e., $V = m[\Sigma_A]$ for some $m \in \mathbb{N}$, $m \neq 0$. Therefore,

$$(6.3) \qquad m\mathcal{H}^n(\Sigma_A) = \mathbf{L}(\Pi_M) \leq A_M \leq \begin{cases} \mathcal{H}^n(\Sigma_A), & \text{if } \Sigma_A \in \mathcal{S}_+; \\ 2\mathcal{H}^n(\Sigma_A), & \text{if } \Sigma_A \in \mathcal{S}_-, \end{cases}$$

where the last "$\leq$" follows from the definition (1.3) of A_M. Thus we have the following two cases:

Case 1: If $\Sigma_A \in \mathcal{S}_+$, orientable, then $m \leq 1$, so $m = 1$, and $\mathcal{H}^n(\Sigma_A) = A_M$;

Case 2: If $\Sigma_A \in \mathcal{S}_-$, non-orientable, then $m \leq 2$, so $m = 1$ or $m = 2$.

In Case 1 when $\Sigma_A \in \mathcal{S}_+$, to prove Theorem 1.1(i), we only need to show

Claim 4. *In this case,* Σ_A *has Morse index one.*

Assume that the claim is false, i.e., the index of Σ_A is no less than 2. By Definition 2.7, there exists an open set $\Omega \subset \Sigma_A$ with smooth boundary, such that $\mathrm{Ind}(\Omega) \geq 2$. Then we can find two nonzero L^2-orthonormal eigenfunctions $\{v_1, v_2\} \subset C_0^\infty(\Omega)$ of the Jacobi operator L_{Σ_A} with negative eigenvalues. A linear combination will give a $v_3 \in C_0^\infty(\Omega)$, such that

$$(6.4) \qquad \int_\Omega v_3 \cdot L_{\Sigma_A} 1 d\mu = \int_\Omega 1 \cdot L_{\Sigma_A} v_3 = 0, \quad v_3 \neq 0.$$

We can assume that $\Omega = U \cap \Sigma_A$ for some open set $U \subset M \backslash sing(\Sigma_A)$. Let $\tilde{X} = v_3 \nu$ with ν the unit normal of Σ_A, and extend it to a tubular neighborhood of Σ_A, such that $\tilde{X}$ has compact support in $\bar{U}$. Let $\{\tilde{F}_s\}_{s \in [-\epsilon, \epsilon]}$ be the flow of $\tilde{X}$, and denote $\Sigma_{t,s} = \tilde{F}_s(\Sigma_t)$, where $\{\Sigma_t\}$ is the family associated to Σ_A as in Proposition 3.4. Notice that $\Sigma_{t,s} = \Sigma_t$ outside $\bar{U}$, and $\{\Sigma_{t,s} \llcorner U\}_{(s,t) \in [-\epsilon,\epsilon] \times [-\epsilon,\epsilon]}$ is a smooth family for small ϵ by Proposition 3.4(c). Denote $\tilde{f}(t,s) = \mathcal{H}^n(\Sigma_{t,s} \cap \bar{U})$. Then $\nabla \tilde{f}(0,0) = 0$ (by minimality of Σ_A), $\frac{\partial^2}{\partial t \partial s} \tilde{f}(0,0) = -\int_\Omega v_3 L_{\Sigma_A} 1 d\mu = 0$ (by (6.4)), $\frac{\partial^2}{\partial t^2} \tilde{f}(0,0) = -\int_\Omega 1 \cdot L_{\Sigma_A} 1 d\mu < 0$ (by $Ric_g > 0$), and $\frac{\partial^2}{\partial s^2} \tilde{f}(0,0) = -\int_\Omega v_3 L_{\Sigma_A} v_3 d\mu < 0$ (as v_3 is a linear combination of eigenfunctions of L_{Σ_0} with negative eigenvalues).

Now consider $\mathcal{H}^n(\Sigma_{t,s}) = \mathcal{H}^n(\Sigma_{t,s} \cap \bar{U}) + \mathcal{H}^n(\Sigma_{t,s} \backslash \bar{U}) = \tilde{f}(t,s) + \mathcal{H}^n(\Sigma_t \backslash \bar{U})$. For $(t,s) \in [-\epsilon, \epsilon] \times [-\epsilon, \epsilon]$, $s \neq 0$, with ϵ small enough, by Taylor expansion,

$$\mathcal{H}^n(\Sigma_{t,s}) = \tilde{f}(t,0) + \frac{\partial}{\partial s}\tilde{f}(t,0)s + \frac{\partial^2}{\partial s^2}\tilde{f}(t,0)s^2 + o(s^2) + \mathcal{H}^n(\Sigma_t \backslash \bar{U})$$

$$= \tilde{f}(t,0) + \{\frac{\partial}{\partial s}\tilde{f}(0,0) + \frac{\partial^2}{\partial t \partial s}\tilde{f}(0,0)t + o(t)\}s + \frac{\partial^2}{\partial s^2}\tilde{f}(t,0)s^2$$
$$+ o(s^2) + \mathcal{H}^n(\Sigma_t \backslash \bar{U})$$

$$= \tilde{f}(t,0) + \frac{\partial^2}{\partial s^2}\tilde{f}(t,0)s^2 + o(ts + s^2) + \mathcal{H}^n(\Sigma_t \backslash \bar{U})$$

$$< \tilde{f}(t,0) + \mathcal{H}^n(\Sigma_t \backslash \bar{U})$$

$$= \mathcal{H}^n(\Sigma_t) \leq \mathcal{H}^n(\Sigma_A),$$

where the fourth "$<$" follows from the fact that $\frac{\partial^2}{\partial s^2}\tilde{f}(t,0) < 0$ for t small enough (as $\frac{\partial^2}{\partial s^2}\tilde{f}(0,0) < 0$). For $|t| \geq \epsilon$, as $\mathcal{H}^n(\Sigma_t) < \mathcal{H}^n(\Sigma_A)$, we can find $\delta > 0$, $\delta \leq \epsilon$ small enough, such that $\mathcal{H}^n(\Sigma_{t,\delta}) < \mathcal{H}^n(\Sigma_A)$. In summary,

$$\max\{\mathcal{H}^n(\Sigma_{t,\delta}) : -d(M) \leq t \leq d(M)\} < \mathcal{H}^n(\Sigma_A).$$

As $\{\Sigma_{t,\delta}\}$ are deformed from $\{\Sigma_t\}$ by the ambient isotopy $\tilde{F}_\delta : M \to M$, we can associate it with a mapping $\Phi_\delta : [0,1] \to (\mathcal{Z}_n(M^{n+1}, \mathcal{F}), \{0\})$ as in (6.1)(i), such that

- $\max_{x\in[0,1]} \mathbf{M}\big(\Phi_\delta(x)\big) = \max_t \mathcal{H}^n(\Sigma_{t,\delta}) < \mathcal{H}^n(\Sigma_A) = \mathbf{L}(\Pi_M)$;
- $\Phi_\delta(x) = \partial[[\tilde{\Omega}_x]]$, $\tilde{\Omega}_x = \tilde{F}_\delta(\Omega_x) \in \mathcal{C}(M)$, for all $x \in [0,1]$.

Applying Theorem 5.1 to Φ_δ gives a $(1, \mathbf{M})$-homotopy sequence $S_\delta = \{\phi_i^\delta\}_{i\in\mathbb{N}}$, such that $S_\delta \in \Pi_M$, and

$$\mathbf{L}(S_\delta) \leq \max_{x\in[0,1]} \mathbf{M}\big(\Phi_\delta(x)\big) < \mathbf{L}(\Pi_M),$$

which is a contradiction to the definition of $\mathbf{L}(\Pi_M)$ (4.4). So we finish the prove of Claim 4 and hence Theorem 1.1(i).

In Case 2 when $\Sigma_A \in \mathcal{S}_-$. By Proposition 4.8, m must be an even number. Hence $m = 2$, and $2\mathcal{H}^n(\Sigma_A) = A_M$. To prove Theorem 1.1(ii), we only need to show

Claim 5. *In this case, Σ_A is stable, i.e., $Ind(\Sigma_A) = 0$.*

The proof is similar to Claim 4. If the claim is false, then there exists an open set $\Omega \subset \Sigma_A$ with smooth boundary, such that $\mathrm{Ind}_D(\Omega) \geq 1$. Denote $\tilde{\Sigma}_A$ by the orientable double cover of Σ_A, and $\tilde{\Omega}$ the lift-up of Ω; then there exists an anti-symmetric eigenfunction $\tilde{\phi} \in C_0^\infty(\tilde{\Omega})$ of the Jacobi operator $L_{\tilde{\Sigma}_A}$ of $\tilde{\Sigma}_A$ with negative eigenvalue (cf. §2.4). The anti-symmetric condition directly implies that:

$$(6.5) \qquad \int_{\tilde{\Omega}} \tilde{\phi} \cdot L_{\tilde{\Sigma}_A} 1 d\mu = \int_{\tilde{\Omega}} 1 \cdot L_{\tilde{\Sigma}_A} \tilde{\phi} d\mu = 0.$$

Let $\tilde{\nu}$ be the unit normal of $\tilde{\Sigma}_A$, and $\pi : \tilde{\Sigma}_A \to \Sigma_A$ the covering map. The anti-symmetric condition of $\tilde{\phi}$ implies that $\tilde{\phi}\tilde{\nu}$ is symmetric on $\tilde{\Sigma}_A$ (cf. §2.4). Hence denote $\tilde{X} = \pi_*(\tilde{\phi}\tilde{\nu})$ by the push-forward of $\tilde{\phi}\tilde{\nu}$ to Σ_A under π. Similarly as above, extend $\tilde{X}$ to a neighborhood of Σ_A, and denote $\{\tilde{F}_s\}_{s\in[-\epsilon,\epsilon]}$ by the flow associated to $\tilde{X}$. Let $\{\Sigma_t\}$ be the family associated to Σ_A by Proposition 3.6, where we assume that Σ_0 is a double cover of Σ_A; then $\{\Sigma_t\}_{t\in[0,\epsilon]}$ is a smooth family away from $sing(\Sigma_A)$ for small ϵ by Proposition 3.6(c). Let $\Sigma_{t,s} = \tilde{F}_s(\Sigma_t)$; then $\Sigma_{t,s}$ are deformations of Σ_t away from $sing(\Sigma_A)$ by ambient isotopies. By similar argument as in Claim 4 using (6.5) instead of (6.4), we can find $\delta > 0$ small enough, such that

$$\max\{\mathcal{H}^n(\Sigma_{t,\delta}) : 0 \leq t \leq d(M)\} < 2\mathcal{H}^n(\Sigma_A).$$

Then we can get a contradiction by discretizing the family $\{\Sigma_{t,\delta}\}$ in the same way. Now we finish the proof.   q.e.d.

7. Appendix

7.1. Reverse statement of Proposition 5.3. Now we list the detailed argument to get the reverse statement of Proposition 5.3 used in the proof. In fact, Proposition 5.3 has another equivalent formulation as follows:

Proposition 7.1. *Given* δ, L, l, m, T *as in Proposition 5.3, there exists* $k = k(l, m, T, \delta, L) \in \mathbb{N}$, *such that for any* $\rho > 0$, *there exists an* $\epsilon = \epsilon(\rho, l, m, T, \delta, L) > 0$, *such that for any* $0 < s < \epsilon$, *and* ϕ *as in (5.2), there exists*

$$\tilde{\phi} : I(m, k)_0 \to \mathcal{B}_\rho^{\mathcal{F}}(T), \ \text{with} \ \tilde{\phi}(y) = \partial[[\Omega_y]],$$

$\Omega_y \in \mathcal{C}(M)$, $y \in I(m, l)_0$, *and satisfying* $(i)(ii)(iii)(iv)$ *in Proposition 5.3.*

Now we show that this formulation implies Proposition 5.3. In fact, under the assumption in the above proposition, we can fix an $\rho_0 = 1 > 0$, and take $\epsilon = \epsilon(\rho_0, l, m, T, \delta, L)$. Given $0 < s < \epsilon$, and ϕ as in (5.2), we can define

$$\rho_{\phi, s} = \inf\{\rho : \exists \ \rho > 0, \text{ and } \tilde{\phi} : I(m, k)_0 \to \mathcal{B}_\rho^{\mathcal{F}}(T),$$

(7.1) $$\text{with } \tilde{\phi}(y) = \partial[[\Omega_y]], \Omega_y \in \mathcal{C}(M), \text{satisfying}$$

$$(i)(ii)(iii)(iv) \text{ in Proposition 5.3}\}.$$

$\rho_{\phi, s}$ is well-defined since ρ_0 belongs to the above set, and $0 \leq \rho_{\phi, s} \leq \rho_0$. Now define the function $\rho : [0, \epsilon) \to \mathbb{R}_+^1$,

$$\rho(s) = 2 \sup\{\rho_{\phi, s} : \ \phi \text{ is any map as in (5.2)}\}.$$

$\rho(s)$ is well-defined, as $\rho(s) \leq 2\rho_0$. Also from the definition, the function ρ depends only on l, m, T, δ, L.

Claim 6. $\rho(s) \to 0$, *as* $s \to 0$.

Proof. For any $\sigma > 0$ small enough, by Proposition 7.1 we can find $\epsilon_\sigma = \epsilon(\sigma, l, m, T, \delta, L) > 0$, so that if $0 < s < \epsilon_\sigma$, then every ϕ as in (5.2) can be extended to $\tilde{\phi} : I(m, k)_0 \to \mathcal{B}_\sigma^{\mathcal{F}}(T)$ satisfying the requirement as in (7.1); hence $\rho_{\phi, s} \leq \sigma$, and $\rho(s) \leq 2\sigma$ by definition.   q.e.d.

By taking $k, \epsilon, \rho(s)$ as above, Proposition 7.1 implies Proposition 5.3. The reverse is trivial.

To get the reverse statement of Proposition 5.3, we can use the reverse statement of Proposition 7.1.

7.2. Some basic facts of exponential map. Here we collect a few basic facts about exponential maps summarized in [**P81**, §3.4] that we need to use for the discretization procedure in Lemma 5.8. We will use the following notions:

- $r_p(\cdot)$ denotes the distance function of M^{n+1} to $p \in M$, and $B(p, r)$ denotes the closed ball centered at p of radius r in M;
- Given $\lambda \geq 0$, $\mu(\lambda) : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ denotes the scaling map by: $\mu(\lambda) : x \to \lambda x$;
- Given a map $f : (W, g_1) \to (Z, g_2)$, $Lip(f)$ denotes the Lipschitz constant with respect the metrics g_1, g_2.

Given $p \in M$, let $exp_p : T_pM \cong \mathbb{R}^{n+1} \to M$ be the exponential map. First, let us list several basic facts in [**P81**, §3.4(4)]. Given $q \in M$, and $0 < \epsilon < 1$, there exists a neighborhood $Z \subset M$ of q, such that, if $p \in Z$, $W = exp_p^{-1}(Z) \subset T_pM \cong \mathbb{R}^{n+1}$, and $E = exp_p|_W$, then the following properties hold:

(a) E is a C^2 diffeomorphism onto Z;
(b) Z is strictly geodesic convex;
(c) $(LipE)^n(LipE^{-1})^n \le 2$;
(d) $Lip(r_p|_Z) \le 2$;
(e) If $x \in Z$ and $0 \le \lambda \le 1$, then $E \circ \mu(\lambda) \circ E^{-1}(x) \in Z$;
(f) if $x \in Z$, $0 \le \lambda \le 1$, and $v \in \Lambda_n T_xM$ (n-th wedge product of T_xM [**Si83**, §25]), then

$$\|D\big(E \circ \mu(\lambda) \circ E^{-1}\big)_* v\| \le \lambda^n\big(1 + \epsilon(1 - \lambda)\big)\|v\|.$$

Also $\lambda^n\big(1 + \epsilon(1 - \lambda)\big) \le 1$ for all $0 \le \lambda \le 1$, $\epsilon < n/2$.

Now we list a few facts about scaling of currents in Euclidean spaces as in [**P81**, §3.4(5)(6)(7)]. Given $r > 0$, $0 \le \lambda \le 1$, denote $B(0, r)$ by the closed ball of radius r in $\mathbb{R}^{n+1}$, and $T \in \mathcal{Z}_{n-1}(\partial B(0, r))$, then we can define the cone of T over the annulus $A(0, \lambda r, r) = B(0, r)\backslash B(0, \lambda r)$ as [**Si83**, 26.26]

$$S = \delta_0 \mathbb{x}(T - \mu(\lambda)_{\#}T) \in \mathcal{Z}_n(\mathbb{R}^{n+1}),$$

then

(g) $\partial S = T - \mu(\lambda)_{\#}T$;
(h) $\mathbf{M}(S) = rn^{-1}(1 - \lambda^n)\mathbf{M}(T)$;
(i) $spt(S) \subset A(0, \lambda r, r)$, where $spt(S)$ is the support of S [**Si83**, 26.11].

Given $\lambda \ge 0$, and $T \in \mathbf{I}_n(\mathbb{R}^{n+1})$, then it is easily seen that

$$\mathbf{M}(\mu(\lambda)_{\#}T) = \lambda^n\mathbf{M}(T).$$

Using notions as above,

(j) Given $r > 0$, $0 \le \lambda \le 1$, $B(p, r) \subset Z$, and $T \in \mathcal{Z}_n\big(B(p, r), \partial B(p, r)\big)$, then by (f),

(7.2) $\mathbf{M}\big((E \circ \mu(\lambda) \circ E^{-1})_{\#}T\big) \le \lambda^n(1 + \epsilon(1 - \lambda))\mathbf{M}(T) \le \mathbf{M}(T);$

(k) Denote $S_\lambda = E_{\#}\big(\delta_0 \mathbb{x}[E_{\#}^{-1}(\partial T) - (\mu(\lambda) \circ E^{-1})_{\#}(\partial T)]\big)$, then by (g)(h)(i),

$$\partial S_\lambda = \partial T - \partial\big[(E \circ \mu(\lambda) \circ E^{-1})_{\#}T\big],$$

$$spt(S_\lambda) \subset A(p, \lambda r, r) = \overline{B(p, r)\backslash B(p, \lambda r)},$$

(7.3)
$$\mathbf{M}(S_\lambda) \le (LipE)^n(LipE)^{-n}rn^{-1}(1 - \lambda^n)\mathbf{M}(\partial T)$$
$$\le 2rn^{-1}(1 - \lambda^n)\mathbf{M}(\partial T).$$

Finally, let us recall the contraction map in [**P81**, §3.4(8)]. For $r > 0$, define

$$h(r) : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1},$$

by $h(r)(x) = x$ if $|x| \leq r$, and $h(r)(x) = r|x|^{-1}x$ if $|x| > r$. If $V \in \mathcal{V}_n(\mathbb{R}^{n+1})$, then

  (l) $spt(h(r)_{\#}V) \subset B(0, r)$;
  (m) $(h(r)_{\#}V) \llcorner G_n(B_0(0, r)) = V \llcorner G_n(B_0(0, r))$;[21]
  (n) $\mathbf{M}(h(r)_{\#}V) \leq \mathbf{M}(V)$.

7.3. Isoperimetric choice. We refer the notions to §4.2.

Lemma 7.2. *Given $T_1, T_2 = \mathcal{Z}_n(M^{n+1})$, with $\mathcal{F}(T_1, T_2) \leq \nu_M$, assume that $T_1 = \partial[[\Omega_1]]$, $T_2 = \partial[[\Omega_2]]$, $\Omega_1, \Omega_2 \in \mathcal{C}(M)$, and $\mathbf{M}([[\Omega_2]] - [[\Omega_1]]) < vol(M)/2$, then the isoperimetric choice of $T_2 - T_1$ is $[[\Omega_2 - \Omega_1]]$.*

Proof. Let $Q \in \mathbf{I}_{n+1}(M)$ be the isoperimetric choice of $T_2 - T_1$, then $\mathbf{M}(Q) = \mathcal{F}(T_1, T_2) \leq \mathbf{M}([[\Omega_2 - \Omega_1]])$, and $\partial Q = T_2 - T_1$. As $T_2 - T_1 = \partial[[\Omega_2 - \Omega_1]]$, $\partial(Q - [[\Omega_2 - \Omega_1]]) = 0$ in $\mathbf{I}_{n+1}(M^{n+1})$. The Constancy Theorem [**Si83**, 26.27] implies that $Q - [[\Omega_2 - \Omega_1]] = n[[M]]$ for some $n \in \mathbb{Z}$. But $\mathbf{M}(Q - [[\Omega_2 - \Omega_1]]) \leq \mathbf{M}(Q) + \mathbf{M}([[\Omega_2 - \Omega_1]]) \leq 2\mathbf{M}([[\Omega_2 - \Omega_1]]) < vol(M)$, hence $n = 0$, and $Q = [[\Omega_2 - \Omega_1]]$.   q.e.d.

We will also need a more subtle technical lemma concerning the isoperimetric choice.

Lemma 7.3. *Given T_1, T_2 as above, with $T_1 \neq 0$, there exists $\delta > 0$ (depending on T_1), such that if $\mathcal{F}(T_1, T_2) \leq \delta$, then the isoperimetric choice of $T_2 - T_1$ is $[[\Omega_2 - \Omega_1]]$.*

Proof. We use the same notions as in the proof of the above Lemma. $T_1 \neq 0$ implies that $\Omega_1 \neq \emptyset$ and $\Omega_1 \neq M$. Take

$$\delta = \frac{1}{2}\min\{\mathcal{H}^{n+1}(\Omega_1), \mathcal{H}^{n+1}(M\backslash\Omega_1)\}.$$

Then $0 < \delta < vol(M)/2$. As we always assume that Ω_1, Ω_2 have the same orientation as M, hence $\mathbf{M}([[\Omega_2]] - [[\Omega_2]]) = \mathcal{H}^{n+1}(\Omega_1 \triangle \Omega_2)$, where $\Omega_1 \triangle \Omega_2$ is the symmetric difference, i.e., $\Omega_1 \triangle \Omega_2 = (\Omega_1\backslash\Omega_2) \cup (\Omega_2\backslash\Omega_1)$. Let Q be the isoperimetric choice of $T_2 - T_1$, by the above proof $Q - [[\Omega_2 - \Omega_1]] = n[[M]]$. If $n = 0$, the proof is done. If $n \neq 0$, then $|n|vol(M) = \mathbf{M}(Q - [[\Omega_2 - \Omega_1]]) \leq \mathbf{M}(Q) + \mathbf{M}([[\Omega_2 - \Omega_1]]) \leq \mathcal{F}(T_1, T_2) + \mathcal{H}^{n+1}(\Omega_1 \triangle \Omega_2) \leq \delta + vol(M) < 2vol(M)$, hence $n = \pm 1$. If $n = 1$, then $Q = [[M]] + [[\Omega_2 - \Omega_1]] = [[M - \Omega_1]] + [[\Omega_2]]$; hence $\mathbf{M}(Q) \geq \mathcal{H}^{n+1}(M\backslash\Omega_1) > \delta$ (as $M - \Omega_1$ has the same orientation as Ω_2), a contradiction. If $n = -1$, then $-Q = [[M]] - [[\Omega_2 - \Omega_1]] = [[M - \Omega_2]] + [[\Omega_1]]$; hence $\mathbf{M}(Q) \geq \mathcal{H}^{n+1}(\Omega_1) > \delta$ (as $M - \Omega_2$ has the same orientation as Ω_1), a contradiction.   q.e.d.

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[21] $B_0(0, r)$ denotes the open ball of radius r in $\mathbb{R}^{n+1}$.

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