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BOGOMOLOV–TIAN–TODOROV THEOREMS FOR LANDAU–GINZBURG MODELS

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to the memory of Andrey Todorov

Abstract

In this paper we prove the smoothness of the moduli space of Landau–Ginzburg models. We formulate and prove a Bogomolov–Tian–Todorov theorem for the deformations of Landau–Ginzburg models, develop the necessary Hodge theory for varieties with potentials, and prove a double degeneration statement needed for the unobstructedness result. We discuss the various definitions of Hodge numbers for non-commutative Hodge structures of Landau–Ginzburg type and the role they play in mirror symmetry. We also interpret the resulting families of de Rham complexes attracted to a potential in terms of mirror symmetry for one parameter families of symplectic Fano manifolds and argue that modulo a natural triviality property the moduli spaces of Landau–Ginzburg models posses canonical special coordinates.

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1. Introduction

In this paper we study the local structure of the moduli space of complex Landau–Ginzburg models. Such a Landau–Ginzburg model is determined by a pair (Y, w), where Y is a complex quasi-projective variety, and $\mathsf{w} : Y \to \mathbb{A}^1$ is a holomorphic function on Y. Our main objective is to prove the unobstructedness of the deformations of (Y, w) in the case when Y has a trivial canonical class $K_Y \cong \mathcal{O}_Y$. At a first glance, such a statement is not likely to hold since the non-compactness of Y will often cause the moduli space of the pair (Y, w) to be infinite dimensional and to have a complicated and unwieldy local behavior in general.

Before we address this difficulty it is useful to look at the model example provided by the classical unobstructedness statement for the deformations of compact Calabi–Yau varieties. This statement was proven by different methods by Bogomolov [**Bog81**], Tian [**Tia87**], and Todorov [**Tod89**]. Recall from [**Bog81**], [**Tia87**] and [**Tod89**], that if X is a smooth compact Calabi–Yau manifold of dimension $\dim_{\mathbb{C}} X = d$, then the (formal) versal deformation space $\mathscr{M}_X$ of X is smooth and of dimension $h^{d-1,1}(X)$. Moreover, a choice of a splitting of the Hodge filtration on $H^d_{DR}(X, \mathbb{C})$ defines an analytic affine structure on $\mathscr{M}_X$. This theorem has many variants establishing the unobstructedness of deformations of log Calabi–Yau varieties or Deligne–Mumford stacks, or of weak Fano varieties or Deligne–Mumford stacks, see, e.g., [**Ran92**, **Kaw92**, **Man04**, **KKP08**, **IM13**, **Iac13**, **San13**]. The log version of the Bogomolov–Tian–Todorov theorem suggests that if we want to attain a good control of the deformations of a Landau–Ginzburg model (Y, w), we should look at a nice, e.g., log Calabi–Yau, compactification Z of Y and consider only deformations that fix the boundary divisor $D_{\mathsf{Z}} = \mathsf{Z} - Y$. To streamline this discussion it will be convenient to distinguish notationally the varying and the fixed parts in any given deformation problem. Our convention in that regard will be that when the deformations of some collection of geometric data are studied, the moving part of the data will be listed in parentheses, while the part of the data that is kept fixed will be listed in a subscript. Thus when we say that we are analyzing the deformations of $(\mathsf{Z}, \mathsf{f})_{D_{\mathsf{Z}}}$, we mean that we consider deformations of the pair (Z, f) together with compatible *trivial* deformations of the divisor D_{Z}.

In this framework prove the following unobstructedness result:

Theorem A. *Let* Z *be a smooth projective variety,* $\mathsf{f} : \mathsf{Z} \to \mathbb{P}^1$ *a flat morphism, and* $D_{\mathsf{Z}} \subset \mathsf{Z}$ *a reduced anti-canonical divisor with strict normal crossings. Assume, moreover, that* $\mathrm{crit}(\mathsf{f})$ *does not intersect the horizontal part of* D_{Z}*, and that the vertical part of* D_{Z} *coincides with the scheme theoretic fiber* $\mathsf{f}^{-1}(\infty)$ *of* f *over* $\infty \in \mathbb{P}^1$*. Then the versal deformation space* $\mathscr{M}_{(\mathsf{Z},\mathsf{f})_{D_{\mathsf{Z}}}}$ *of* $(\mathsf{Z}, \mathsf{f})_{D_{\mathsf{Z}}}$ *is smooth.*

This theorem can be viewed as an unobstructedness result for the Calabi–Yau Landau–Ginzburg model (Y, w) where $Y = \mathsf{Z} - D_{\mathsf{Z}}$, $\mathsf{w} = \mathsf{f}_{|Y} : Y \to \mathbb{A}^1$. Indeed, the theorem asserts that if (Y, w) admits a compactification (Z, f) with normal crossing boundary D_{Z}, then the deformations of (Y, w) that are "anchored at infinity", i.e., the deformations of the compactification that keep the boundary fixed, are unobstructed. To prove Theorem A we identify the L_∞-algebra that controls the deformation theory of $(\mathsf{Z}, \mathsf{f})_{D_{\mathsf{Z}}}$ and show in Theorem 2.10 that this L_∞-algebra is homotopy abelian. We argue that, as in the case of compact Calabi–Yau manifolds, the latter statement can be reduced to a Hodge theoretic property: the double degeneration property for the Hodge-to-De Rham spectral sequence associated with the complex of f-adapted logarithmic forms (see Definition 2.11). This double degeneration is then established in Theorem 2.18.

The setup and conclusion of Theorem A are natural from the point of view of mirror symmetry. To elaborate on this, note first that a Landau–Ginzburg pair (Y, w) as above will typically arise as the mirror of a symplectic manifold (X, ω_X) underlying a projective Fano variety. Now the homological mirror symmetry conjecture predicts that the Fukaya category $\mathbf{Fuk}(X, \omega_X)$ of (X, ω_X) will be equivalent to the category $\mathbf{MF}(Y, \mathsf{w})$ of matrix factorizations of the potential $\mathsf{w} : Y \to \mathbb{A}^1$. In particular, the deformation theories of the Fukaya category and of the category of matrix factorizations will be identified. The heuristics motivating Theorem A comes from the comparison of the corresponding moduli spaces. The versal deformation space of the Fukaya category is manifestly smooth since it is an open cone in the space of harmonic 2-forms on X. Thus mirror symmetry predicts that the versal deformation space of the category of matrix factorizations will also be smooth. Next recall [**Orl04**, **Orl05**, **Orl12**] that $\mathbf{MF}(Y, \mathsf{w})$ is the coproduct $\coprod_{\lambda \in \operatorname{crit} \mathsf{w}} \mathbf{D}^b_{\mathrm{sing}}(Y_\lambda)$ of the categories of singularities of the singular fibers of w. This interpretation indicates that flat deformations of the geometric data (Y, w) will not necessarily give rise to flat deformations of $\mathbf{MF}(Y, \mathsf{w})$. Indeed, when we deform (Y, w) geometrically, the singularities of fibers of w can coalesce and more importantly can run away to infinity. This will happen for instance if we deform a compactification of (Y, w) so that some interior singular fiber gets absorbed in the fiber at infinity. Because of this phenomenon we will have flat families of Landau–Ginzburg models which give us families of categories of matrix factorizations whose periodic cyclic homologies jump. This suggests that we should only consider geometric deformations of (Y, w) that are anchored at infinity. Indeed, if $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}})$ is a compactification of (Y, w), then the deformations of (Z, f) that fix the boundary divisor D_{Z} will give deformations of (Y, w) without jumps in the global vanishing cohomology. In this setting the corresponding categories of matrix factorizations will move in a flat family and we expect that the deformations of a compactification with a fixed boundary will provide enough parameters to cover the full versal deformation space $\mathscr{M}_{\mathsf{MF}(Y,\mathsf{w})}$ of the category $\mathsf{MF}(Y,\mathsf{w})$. In fact, in the process of proving the double degeneration property Theorem 2.18 we will check that, under the hypothesis of Theorem A, the natural map of versal deformation spaces $\mathscr{M}_{(\mathsf{Z},\mathsf{f})_{D_\mathsf{Z}}} \to \mathscr{M}_{\mathsf{MF}(Y,\mathsf{w})}$ is étale. More precisely, it is not hard to see that the composition of the isomorphism constructed in Lemma 2.21 with Efimov's comparison isomorphism [**Efi12**] can be identified with the differential of the map $\mathscr{M}_{(\mathsf{Z},\mathsf{f})_{D_\mathsf{Z}}} \to \mathscr{M}_{\mathsf{MF}(Y,\mathsf{w})}$ at the closed point. In particular, this differential is an isomorphism and so the map is étale. Altogether this heuristic reasoning explains why the unobstructedness of the deformation theory of $((\mathsf{Z},\mathsf{f}), D_\mathsf{Z})$ is indeed the expected behavior.

Since the compactified Landau–Ginzburg model $((\mathsf{Z},\mathsf{f}), D_\mathsf{Z})$ plays a central role in the above heuristics it is natural to expect that this compactification should also have a mirror interpretation. A closer look at the associated Hodge/de Rham data and the double degeneration property of $((\mathsf{Z},\mathsf{f}), D_\mathsf{Z})$ suggests that the mirror of $((\mathsf{Z},\mathsf{f}), D_\mathsf{Z})$ is an anticanonical pencil on the symplectic manifold (X, ω_X). In Section 3.2 we discuss this mirror picture in detail and compare the Hodge theoretic data appearing on the two sides of this extended mirror correspondence. We use this analysis to explain how the commutative pure Hodge structure of the compactified Landau–Ginzburg model arising from the double degeneration property can be reconstructed from the non-commutative Hodge structure of the original Landau–Ginzburg model (Y,w). Through the extended mirror symmetry picture we rewrite this reconstruction process for the Fano mirror and use the resulting structure to propose a definition of Hodge numbers for X which is formulated entirely in symplectic terms.

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2. Moduli of Landau–Ginzburg models

In this section we study variations of pure nc Hodge structures that arise from universal families of Landau–Ginzburg models. We focus on the components of the universal variation that encode geometric properties of the Landau–Ginzburg moduli space and investigate the Hodge theoretic input in the Landau–Ginzburg deformation theory.

The relevant class of Landau–Ginzburg models appears naturally in the context of mirror symmetry for Fano manifolds. Since this context is a primary source of examples for us, we recall it next.

2.1. Mirrors of Fano manifolds. Mirror symmetry is a duality that identifies seemingly different two dimensional supersymmetric quantum field theories. Geometrically such theories arise as linear or non-linear sigma models with Kähler targets, or as Landau–Ginzburg models with targets given by Kähler manifolds equipped with holomorphic superpotentials. The mirror map matches the target geometries that produce mirror symmetric models into mirror pairs. Typically a sigma model or a Landau–Ginzburg model with a given target geometry admits two topological twists – the A and B twists – each of which gives rise to a category of boundary field theories or D-branes (see, e.g., [**HKK$^+$03**, **ABC$^+$09**]). According to the homological mirror conjecture from [**Kon95**], the mirror correspondence can be generalized to an identification of the categories of boundary field theories. Specifically the homological mirror conjecture predicts that in a mirror pair, the category of A-branes for one side of the pair must be equivalent to the category of B-branes for the other side. Such an equivalence induces non-obvious isomorphisms between the various invariants that one can extract from the categories. In particular, we get a conjectural matching of the cohomology of the two categories; matching of the nc Hodge structures on the cohomology of the two categories; matching of the deformation spaces of the two categories; and matching of the natural variations of nc motives over these deformation spaces. We will exploit these conjectural identifications to deduce interesting predictions for the properties of the moduli spaces and the Hodge theory of the requisite geometric backgrounds and will eventually prove these predictions directly. To set things up we begin by recalling the basic geometric framework for the mirror correspondence.

We will indicate that two geometries $(X, \cdots)$ and $(Y, \cdots)$ are mirror equivalent by writing $(X, \cdots) \mid (Y, \cdots)$. Mirror pairs of geometries fall naturally into three classes: mirror pairs of Calabi–Yau, Fano, and general type. Here we will discuss in detail only the mirror pairs of Fano type.

By definition a ***mirror pair of Fano type*** is a pair

$$(X, \omega_X, s_X) \mid ((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y),$$

where:

- X is a projective Fano manifold;
- (Y, w) is a holomorphic Landau–Ginzburg model consisting of a quasi-projective Calabi–Yau manifold Y with $\dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} X = n$, and a surjective algebraic function $\mathsf{w} : Y \to \mathbb{A}^1$ with a compact critical locus $\mathrm{crit}(\mathsf{w}) \subset Y$;
- $\omega_X \in A^2_{\mathbb{C}}(X)$ and $\omega_Y \in A^2_{\mathbb{C}}(Y)$ are (complexified) Kähler forms on X and Y;
- $s_X \in H^0(X, K_X^{-1})$ is an anti-canonical section of X, and $\mathsf{vol}_Y \in H^0(Y, K_Y)$ is a trivialization of the canonical bundle of Y, i.e., a holomorphic volume form on Y.

The anti-canonical section $s_X \in H^0(X, K_X^{-1})$ defines a Calabi–Yau hypersurface $D_X = \mathrm{divisor}(s_X) \subset X$ and a nowhere vanishing section $s_{X|X-D_X} \in H^0(X - D_X, K_X^{-1})$. We will write $\mathsf{vol}_{X-D_X} = 1/s_X$ for the corresponding holomorphic volume form on $X - D_X$.

Remark 2.1. Mirror pairs of Fano type can be qualified/refined in different ways:

(i) Requiring that D_X is smooth is mirrored by the requirement that w is proper.

(ii) Requiring that D_X has strict normal crossings is mirrored by the requirement that the fibers of w are Zariski open subsets in projective $(n-1)$-dimensional Calabi–Yau manifolds.

It is helpful to examine the shape of the geometry of a mirror pair in examples. Many explicit and detailed descriptions of mirror pairs of Fano type are discussed in, e.g., [**Giv98**, **HV00**, **AKO08**, **AKO06**, **Abo09**]. Here we just briefly recall Givental's picture [**Giv98**] of mirrors of projective spaces.

Example 2.2. The first instance of a Fano type mirror pair was described by Givental [**Giv98**]. In the most basic setting $X = \mathbb{P}^n$ is a projective space with homogeneous coordinates $u_0, \ldots, u_n$, ω_X is the Fubini–Studi form, $D_X \subset \mathbb{P}^n$ is the union of the $(n+1)$ coordinate hyperplanes, and s_X is given by the product of the homogeneous coordinate functions. On the mirror side $Y = (\mathbb{C}^\times)^n$ is an n-dimensional affine torus with coordinates $z_1, \ldots, z_n$, the potential $\mathsf{w} : Y \to \mathbb{A}^1$ is given by

$$\mathsf{w}(z_1, \ldots, z_n) = \sum_{i=1}^n z_i + \frac{1}{z_1 \cdots z_n},$$

the symplectic form is

$$\omega_Y = \sum_{i=1}^n \frac{1}{|z_i|^2} dz_i \wedge d\bar{z}_i,$$

the point a is the point at infinity, i.e., $a = \infty$, and the holomorphic volume form is

$$\mathsf{vol}_Y = \bigwedge_{i=1}^{n} \frac{dz_i}{z_i}.$$

If we change the setting so that on the left hand side of the pair D_X is not the toric divisor of $\mathbb{P}^n$ but rather is a smooth Calabi–Yau hypersurface, the mirror Y is a partial compactification of the torus so that w becomes a proper map with $n-1$ dimensional Calabi–Yau fibers. Accordingly the symplectic form ω_Y and holomorphic volume form vol_Y have to be extended to the compactification.

The mirror correspondence gives a non-trivial matching [**HKK+03**] of the various ingredients of the mirror pair. The complexified Kähler structure ω_X is identified with a combination of the complex structure on Y, the potential w, and the volume form vol_Y. In the other direction ω_Y is identified with a combination of the complex structure on X and the section s_X [**HV00**, **HKK+03**].

A Fano type mirror pair gives rise to a pair of mirror non-compact Calabi–Yau manifolds:

$$\left(X - D_X, \omega_{X|X-D_X}, \mathsf{vol}_{X-D_X}\right) \mid (Y, \omega_Y, \mathsf{vol}_Y)\,.$$

Under a convergence assumption on the quantum product on X the category of A-branes for the background (X, ω_X, s_X) can be identified with the Fukaya category $\mathbf{Fuk}(X, \omega_X)$ of the symplectic manifold underlying the Fano variety X [**ABC+09**]. $\mathbf{Fuk}(X, \omega_X)$ is a $\mathbb{C}$-linear A_∞ category which is only $\mathbb{Z}/2$-graded [**FOOO09a**, **FOOO09b**]. The category of B-branes associated with (X, ω_X, s_X) is identified with a dg enhancement of the bounded derived category $D^b(X)$ of coherent sheaves on X. We will write $\mathbf{D}^b(X)$ for this $\mathbb{Z}$-graded $\mathbb{C}$-linear dg category. There are many choices for $\mathbf{D}^b(X)$, e.g., the homotopy category of complexes of injective $\mathcal{O}_X$-modules with coherent cohomology, or Block's category of graded C^∞ complex vector bundles on X with $(0, \bullet)$ superconnections [**Blo10**]. By a theorem of Lunts and Orlov [**LO10**] all dg enhancements of $D^b(X)$ are quasi-equivalent so one can work with any of those enhancements. By definition $\mathbf{D}^b(X)$ depends only on the complex structure on X and is independent of the complexified Kähler structure ω_X or the section s_X. Both D_X and s_X are of course essential for defining the associated Calabi–Yau pair and its categories of branes.

On the right hand side of the mirror pair the definition of the categories of branes is modified to incorporate the potential w. The category of A-branes associated with the background $((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y)$ is the Fukaya–Seidel category $\mathbf{FS}\,((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y)$ [**Sei08**] and the category of B-branes for $((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y)$ is defined as the category $\mathbf{MF}(Y, \mathsf{w})$ of matrix factorizations of the holomorphic function $\mathsf{w} : X \to \mathbb{A}^1$ [**Orl04**, **KKP08**, **Orl12**, **LP11**, **Pre11**, **EP15**]. By construction

$\mathsf{FS}\,((Y,\mathsf{w}),\omega_Y,\mathsf{vol}_Y)$ is a $\mathbb{C}$-linear $\mathbb{Z}$-graded A_∞ category. Again the $\mathbb{Z}/2$-folding of $\mathsf{FS}\,((Y,\mathsf{w}),\omega_Y,\mathsf{vol}_Y)$ depends only on the C^∞ manifold underlying Y, on the function w, and on the complexified symplectic structure ω_Y while the $\mathbb{Z}$-graded version $\mathsf{FS}\,((Y,\mathsf{w}),\omega_Y,\mathsf{vol}_Y)$ depends also on vol_Y viewed as a C^∞ form on Y. Similarly $\mathsf{MF}(Y,\mathsf{w})$ is a d($\mathbb{Z}/2$)graded $\mathbb{C}$-linear category which depends only on the complex structure of Y and on the holomorphic function w.

Homological mirror symmetry now predicts several conjectural equivalences of categories of branes for the Fano mirror pair and for the associated Calabi–Yau mirror pair. These equivalences are summarized in Table 1. In this table D^b_c denotes dg enhancements of the derived categories of coherent sheaves with compact support and $\mathsf{Fuk}^{\mathrm{wr}}$ denotes the wrapped version of the Fukaya category [**AS10**, **Abo12**]. Additionally, our convention is that whenever the notation for a Fukaya or a Fukaya–Seidel category includes a holomorphic volume form, the objects of this category are graded spin Lagrangians or Lagrangian thimbles, and so the category is $\mathbb{Z}$-graded. In particular, aside from $\mathsf{Fuk}(X,\omega_X)$ and $\mathsf{MF}(Y,\mathsf{w})$ all categories appearing in Table 1 are $\mathbb{Z}$-graded.

Table 1. Homological mirror symmetry for a mirror pair
$$(X,\omega_X,s_X)\mid((Y,\mathsf{w}),\omega_Y,\mathsf{vol}_Y)$$
of Fano type

A-*branes*	B-*branes*
$\mathsf{Fuk}(X,\omega_X)$	$\mathsf{MF}(Y,\mathsf{w})$
$\mathsf{Fuk}^{\mathrm{wr}}(X-D_X,\omega_X,\mathsf{vol}_{X-D_X})$	$\mathsf{D}^b(Y)$
$\mathsf{Fuk}(X-D_X,\omega_X,\mathsf{vol}_{X-D_X})$	$\mathsf{D}^b_c(Y)$

HMS

B-*branes*	A-*branes*
$\mathsf{D}^b(X)$	$\mathsf{FS}((Y,\mathsf{w}),\omega_Y,\mathsf{vol}_Y)$
$\mathsf{D}^b(X-D_X)$	$\mathsf{Fuk}^{\mathrm{wr}}(Y,\omega_Y,\mathsf{vol}_Y)$
$\mathsf{D}^b_c(X-D_X)$	$\mathsf{Fuk}(Y,\omega_Y,\mathsf{vol}_Y)$

HMS

Remark 2.3. **(i)** Homological mirror symmetry predicts that the equivalence

$$\mathsf{Fuk}(X, \omega_X) \cong \mathsf{MF}(Y, \mathsf{w}),$$

of $\mathbb{Z}/2$-graded A_∞ categories in Table 1 will respect the natural additional structures on these categories of branes. In particular, mirror symmetry will respect the natural decompositions of these categories. It is known from the work of Orlov [**Orl04**, **Orl05**, **Orl12**] that the category of matrix factorizations decomposes

$$\mathsf{MF}(Y, \mathsf{w}) = \coprod_{\substack{\lambda \in \mathbb{A}^1 \\ \text{a critical value} \\ \text{of } \mathsf{w}}} \mathbf{D}^b_{\text{sing}}(Y_\lambda),$$

into a sum of categories of singularities of the singular fibers of w.

Similarly (see [**KKP08**]) the Fukaya category of the Fano manifold X decomposes

$$\mathsf{Fuk}(X, \omega_X) = \coprod_{\substack{\lambda \in \mathbb{C} \\ \text{an eigenvalue} \\ \text{of } c_1(T_X)*_1(\bullet)}} \mathsf{Fuk}(X, \omega_X)_\lambda,$$

corresponding to the eigenvalues of quantum multiplication[1] with $c_1(T_X)$ on $H^\bullet(X, \mathbb{C})$.

(ii) The equivalences of the categories of branes listed in Table 1 induce respective mirror identifications of cohomology groups. For future reference we collect these identifications in Table 2. In this table $Y_{\text{sm}} \subset Y$ denotes a smooth fiber of $\mathsf{w} : Y \to \mathbb{A}^1$ taken "near infinity" as explained in [**KKP08**, Section 4.5.2(2)].

(iii) The B-to-A homological mirror correspondence in Table 1 can be extended to one more case. Let $Y_{-\infty}$ denote the fiber $\mathsf{w}^{-1}(z)$ over $z \in \mathbb{C}$ with $\operatorname{Re} z \ll 0$. We will also write $\omega_{-\infty}$ for the restriction $\omega_{Y|Y_{-\infty}}$ of the symplectic form, and $\mathsf{vol}_{-\infty}$ for the induced holomorphic volume form on the fiber. The parallel transport for the Erhesmann symplectic connection on $\mathsf{w} : Y \to \mathbb{A}^1$ identifies symplectically all fibers of w over points $z \in \mathbb{A}^1$ with $\operatorname{Re} z \ll 0$. So the dg category $\mathsf{Fuk}(Y_{-\infty}, \omega_{-\infty}, \mathsf{vol}_{-\infty})$ is well defined up to quasi-equivalence. Now, we can supplement Table 1 by the statement that the category of perfect complexes (= the

[1] By assumption we are working here with a convergent version of the quantum product $*_q$. The value of q corresponding to the particular complex structure on Y under the mirror map is normalized in the flat coordinates to be $q = 1$. This is why we use the $*_1$ quantum product in the decomposition above.

Table 2. Matching of cohomology for a mirror pair $(X, \omega_X, s_X) \mid ((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y)$ of Fano type

A-*brane charges*	B-*brane charges*
$H^\bullet(X, \mathbb{C})$	$H^\bullet(Y, Y_{\text{sm}}; \mathbb{C})$
$H^\bullet(X - D_X, \mathbb{C})$	$H^\bullet(Y, \mathbb{C})$
$H_c^\bullet(X - D_X, \mathbb{C})$	$H_c^\bullet(Y, \mathbb{C})$

HMS

B-*brane charges*	A-*brane charges*
$H^\bullet(X, \mathbb{C})$	$H^\bullet(Y, Y_{\text{sm}}; \mathbb{C})$
$H^\bullet(X - D_X, \mathbb{C})$	$H^\bullet(Y, \mathbb{C})$
$H_c^\bullet(X - D_X, \mathbb{C})$	$H_c^\bullet(Y, \mathbb{C})$

HMS

category of topological B branes) on the Calabi–Yau variety D_X is quasi-equivalent to the Fukaya category (= the category o f A-branes) on the fiber $Y_{-\infty}$:

B-*branes*	A-*branes*
$\mathbf{Perf}(D_X)$	$\mathbf{Fuk}(Y_{-\infty}, \omega_{-\infty}, \mathsf{vol}_{-\infty})$

HMS

Again this induces an identification of the associated brane charges, i.e., of the periodic cyclic homologies of the two categories. Since $HP_\bullet(\mathbf{Perf}(D_X)) \cong H^\bullet(D_X, \mathbb{C})$ and conjecturally

$$HP_\bullet(\mathbf{Fuk}(Y_{-\infty}, \omega_{-\infty}, \mathsf{vol}_{-\infty})) \cong H_c^\bullet(Y_{-\infty}, \mathbb{C}),$$

we get a mirror identification

B-*brane charges*	A-*brane charges*
$H^\bullet(D_X, \mathbb{C})$	$H^\bullet_c(Y_{\mathrm{sm}}; \mathbb{C})$

HMS

Note that it is not clear how to extend the A-to-B homological mirror correspondence in a similar manner. If on the A-side we consider the symplectic data $\left(D_X, \omega_{X|D_X}, \mathsf{vol}_{D_X}\right)$, there is no obvious complex fiber Y_c of $\mathsf{w} : Y \to \mathbb{A}^1$ for which we can hope to get a quasi-equivalence $\mathbf{Fuk}\left(D_X, \omega_{X|D_X}, \mathsf{vol}_{D_X}\right) \cong \mathbf{D}^b(Y_c)$. The problem is that the mirror of $\left(D_X, \omega_{X|D_X}, \mathsf{vol}_{D_X}\right)$ is normally understood in terms of the large volume degeneration, so for an A-to-B mirror statement we will need to understand the large complex structure degeneration of $\mathsf{w} : Y \to \mathbb{A}^1$.

2.2. Families of Landau–Ginzburg models. As we explained in Section 2.1 the mirror of a symplectic manifold underlying a Fano variety is a quasi-projective Landau–Ginzburg model $((Y, \mathsf{w}), \mathsf{vol}_Y)$ equipped with a holomorphic volume form. Such Landau–Ginzburg models admit a natural class of compactifications.

Definition 2.4. A **compactified Landau–Ginzburg model** is the datum $((\mathsf{Z}, \mathsf{f}), D_\mathsf{Z}, \mathsf{vol}_\mathsf{Z})$, where:

(a) Z is a smooth projective variety and $\mathsf{f} : \mathsf{Z} \to \mathbb{P}^1$ is a flat projective morphism.

(b) $D_\mathsf{Z} = \left(\cup_i D_i^\mathsf{h}\right) \cup \left(\cup_j D_j^\mathsf{v}\right) \subset \mathsf{Z}$ is a reduced normal crossings divisor, such that
  - $D^\mathsf{v} = \cup_j D_j^\mathsf{v}$ is the reduced pole divisor of f, i.e., $\left(\mathsf{f}^{-1}(\infty)\right)_{\mathrm{red}} = \cup_j D_j^\mathsf{v}$;
  - each component D_i^h of $D^\mathsf{h} = \cup_i D_i^\mathsf{h}$ is a smooth divisor which is horizontal for w, i.e., $\mathsf{w}_{|D_i^\mathsf{h}}$ is a flat projective morphism;
  - the critical locus $\mathrm{crit}(\mathsf{f}) \subset \mathsf{Z}$ does not intersect D^h.

(c) vol_Z is a meromorphic section of K_Z with no zeroes and with poles at most at D_Z, i.e., $\mathsf{vol}_\mathsf{Z} \in H^0\left(\mathsf{Z}, K_\mathsf{Z}(*D_\mathsf{Z})\right)$.

With every $((\mathsf{Z}, \mathsf{f}), D_\mathsf{Z}, \mathsf{vol}_\mathsf{Z})$ we associate its 'open part' $((Y, \mathsf{w}), \mathsf{vol}_Y)$ where $Y := \mathsf{Z} - D_\mathsf{Z}$, $\mathsf{w} : Y \to \mathbb{A}^1$ is defined to be the restriction $\mathsf{w} := \mathsf{f}_{|Y}$ of f to Y, and $\mathsf{vol}_Y := \mathsf{vol}_{\mathsf{Z}|Y}$ is the restriction of vol_Z to Y. The condition on vol_Z ensures that Y is a quasi-projective variety with a trivial canonical class and that vol_Y is a holomorphic volume form on Y. The condition that the critical locus of f does not intersect the horizontal part of D_Z, in particular, implies that $\mathrm{crit}(\mathsf{w})$ is proper and so $((Y, \mathsf{w}), \mathsf{vol}_Y)$ is exactly the type of Landau–Ginzburg model that we considered in the previous section.

In addition we will often require that the datum $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}}, \mathsf{vol}_{\mathsf{Z}})$ satisfies the following tameness assumption which bounds the orders of poles of $\mathsf{vol}_{\mathsf{Z}}$ and f along D_{Z}:

$$
(\mathsf{T}) \qquad \begin{aligned}
\operatorname{ord}_{D_i^{\mathsf{h}}}(\mathsf{vol}_{\mathsf{Z}}) &= -1, \\
\operatorname{ord}_{D_i^{\mathsf{h}}}(\mathsf{f}) &= 0, \\
\operatorname{ord}_{D_j^{\mathsf{v}}}(\mathsf{vol}_{\mathsf{Z}}) &= -1, \\
\operatorname{ord}_{D_j^{\mathsf{v}}}(\mathsf{f}) &= -1,
\end{aligned} \qquad \text{for all } i \text{ and } j.
$$

Remark 2.5. (i) The assumption that $\mathsf{vol}_{\mathsf{Z}}$ has poles of order exactly 1 along all the components of D_{Z}, in particular, implies that the reduced divisor D_{Z} is an anti-canonical divisor on Z and so $(\mathsf{Z}, D_{\mathsf{Z}})$ is a log Calabi–Yau pair.

Note also that if we start with a Calabi–Yau quasi-projective Landau–Ginzburg model $((Y, \mathsf{w}), \mathsf{vol}_Y)$ and we choose a smooth normal-crossing compactification $\mathsf{f} : \mathsf{Z} \to \mathbb{P}^1$ of $\mathsf{w} : Y \to \mathbb{A}^1$, then the condition that the holomorphic volume form has a first order pole along the divisor at infinity is a tight constraint which is rather unnatural since it is not invariant under semi-stable reduction. Nevertheless, this condition is often satisfied in mirror symmetry examples and so it is not unreasonable to impose.

(ii) The assumption that w has first order poles at the vertical boundary divisor D^{v}, i.e., that the scheme theoretic fiber $\mathsf{f}^{-1}(\infty)$ is actually reduced, is more natural and can be justified by mirror symmetry considerations.

Indeed, if $\mathsf{f}^{-1}(\infty) = \sum_j m_j D_j^{\mathsf{v}}$, then by Landman's theorem [**Lan73**] we know that the least common multiple m of the m_j's is the order of the semi-simple part of the local monodromy transformation around infinity. Concretely, choose a small disk $\Delta \subset \mathbb{P}^1$ centered at $\infty \in \mathbb{P}^1$ and such that ∞ is the only critical value of f in Δ. Fix a base point $c_0 \in \partial\Delta$, and orient $\partial\Delta$ with the orientation on Δ. Consider the monodromy transformation $\mathsf{mon}_{c_0} : H^\bullet(\mathsf{Z}_{c_0}, \mathbb{C}) \to H^\bullet(\mathsf{Z}_{c_0}, \mathbb{C})$ corresponding to going around $\partial\Delta$ once in the positive direction. By Landman's theorem mon_{c_0} is a quasi-unipotent operator and the minimal power of mon_{c_0} which is unipotent is m. In other words $\left(\mathsf{mon}_{c_0}^m - \mathrm{id}\right)^{n-1} = 0$, where $n = \dim_{\mathbb{C}} Y$, and m is the minimal number with this property. Similarly, going once around $\partial\Delta$ gives a monodromy transformation $T : H^\bullet(Y, Y_{c_0}; \mathbb{C}) \to H^\bullet(Y, Y_{c_0}; \mathbb{C})$. From our assumption on equisingularity of D^{h} and from the compatibility of the long exact sequence of the pair with the action of monodromy we get that T will also be quasi-unipotent with eigenvalues which are m-th roots of unity with at least one eigenvalue being a primitive m-th root of unity. Next observe that the cycle class map that assigns a relative cohomology class in $H^\bullet(Y, Y_{c_0}; \mathbb{C})$ to each Lefschetz thimble will identify the periodic cyclic homology $HP_\bullet(\mathsf{FS}((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y))$ of the Fukaya–Seidel category with $H^\bullet(Y, Y_{c_0}; \mathbb{C})$. From this point of view the operator T is induced from the inverse of the monodromy auto-equivalence of $\mathsf{FS}((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y)$. When $((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y)$ is the mirror of a Fano datum (X, ω_X, s_X), the mirror equivalence (see Table 1) $\mathsf{FS}((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y) \cong \mathsf{D}^b(X)$ identifies the monodromy auto-equivalence with the Serre functor $\otimes K_X[n] : \mathsf{D}^b(X) \to \mathsf{D}^b(X)$. But on cohomology $H^\bullet(Y, Y_{c_0}; \mathbb{C}) \cong H^\bullet(X, \mathbb{C})$ the Serre functor induces multiplication with $\exp((-1)^n c_1(K_X))$. In other words T is a unipotent operator, and so we must have $m = 1$. For future reference note that this mirror symmetry description also predicts that under the identification $H^\bullet(X, \mathbb{C}) \cong H^\bullet(Y, Y_{c_0}; \mathbb{C})$ the nilpotent endomorphism

$$((-1)^n c_1(K_X)) \cup (\bullet) : H^\bullet(X, \mathbb{C}) \to H^\bullet(X, \mathbb{C}),$$

becomes identified with the logarithm of monodromy:

$$-\log T : H^\bullet(Y, Y_{c_0}; \mathbb{C}) \to H^\bullet(Y, Y_{c_0}; \mathbb{C}).$$

(iii) It is possible and useful to allow for $D^\vee$ to be a smooth divisor, i.e., for ∞ not to be a critical value of f. Such Landau–Ginzburg models arise naturally as mirrors of quasi-Fano varieties and can be studied in the same manner.

Our goal is to understand the moduli spaces of compactified complex Landau–Ginzburg models satisfying the tameness assumption. If such a model $((\mathsf{Z}, \mathsf{f}), D_\mathsf{Z}, \mathsf{vol}_\mathsf{Z})$ compactifies the mirror of a Fano datum (X, ω_X, s_X), then its moduli space will be identified with the symplectic moduli of (X, ω_X) and in fact will look like a conical open subset in $H^2(X, \mathbb{C}) \oplus H^0(X, \mathbb{C})$. In particular, when $((\mathsf{Z}, \mathsf{f}), D_\mathsf{Z}, \mathsf{vol}_\mathsf{Z})$ arises from a mirror situation, we expect its moduli space to be smooth. This motivates the following purely algebraic–geometric statement:

Theorem 2.6. *Let* $((\mathsf{Z}, \mathsf{f}), D_\mathsf{Z}, \mathsf{vol}_\mathsf{Z})$ *be a compactified Landau–Ginzburg model satisfying the tameness assumption* (T) *and the assumption* $H^1(\mathsf{Z}, \mathbb{Q}) = 0$*. Then the deformation theory of* $((\mathsf{Z}, \mathsf{f})_{D_\mathsf{Z}}, \mathsf{vol}_\mathsf{Z})$ *is unobstructed.*

Remark 2.7. The requirement that $H^1(\mathsf{Z}, \mathbb{Q}) = 0$ is a technical requirement that simplifies the Teichmüller theory of Z. It is very likely unnecessary but we will not pursue this here.

Theorem 2.6 extends the classical unobstructedness results of Bogomolov [**Bog79**, **Bog81**], Tian [**Tia87**], and Todorov [**Tod89**] to the setup of log Calabi–Yau varieties with potentials. It also gives the following more direct generalization of Bogomolov–Tian–Todorov unobstructedness:

Corollary 2.8. *Let* Z *be a smooth projective variety of dimension* n *satisfying* $H^1(\mathsf{Z},\mathbb{Q})=0$ *and such that the anti-canonical linear system on* Z *gives a flat projective morphism* $\mathsf{f}:\mathsf{Z}\to\mathbb{P}^1$*. Then the deformation theory of* Z *is unobstructed.*

Proof. By assumption the variety Z determines the morphism $\mathsf{f}:\mathsf{Z}\to\mathbb{P}^1$. Let $D_{\mathsf{Z}}\subset\mathsf{Z}$ be a smooth anti-canonical divisor in Z. And let $\mathsf{vol}_{\mathsf{Z}}$ be a trivialization of $K_{\mathsf{Z}}(D_{\mathsf{Z}})$. Then the datum $((\mathsf{Z},\mathsf{f}),D_{\mathsf{Z}},\mathsf{vol}_{\mathsf{Z}})$ is a tame compactified Landau–Ginzburg model satisfying the hypotheses of Theorem 2.6. Locally in the analytic topology the versal deformation space of the datum $((\mathsf{Z},\mathsf{f}),D_{\mathsf{Z}},\mathsf{vol}_{\mathsf{Z}})$ is the product of the versal deformation space of Z and the moduli of pairs $(D_{\mathsf{Z}},\mathsf{vol}_{\mathsf{Z}})$ for a fixed Z. But the moduli of such pairs is isomorphic to $(\mathbb{P}^1-\mathrm{crit}(\mathsf{f}))\times\mathbb{C}^{\times}$ and is, therefore, smooth. Combined with the unobstructedness of Theorem 2.6 this implies that the deformations of Z are unobstructed. q.e.d.

Remark 2.9. If Z satisfies the conditions of Corollary 2.8, then the family $\mathsf{f}:\mathsf{Z}\to\mathbb{P}^1$ is classified by a holomorphic map from $\mathbb{P}^1$ to the compactified moduli space of $(n-1)$-dimensional projective Calabi–Yau varieties. By the classical Bogomolov–Tian–Todorov theorem [**Tia87**, **Tod89**] we know that the moduli space M of $(n-1)$-dimensional projective Calabi–Yau varieties is smooth. By Corollary 2.8 we know that a certain component M_1 of the moduli space of rational curves in $\overline{M}$ is also smooth. Considerations of mirrors of hybrid Landau–Ginzburg models suggest that this process can be iterated: a component M_2 of the moduli of rational curves in a compactification $\overline{M}_1$ will be smooth, and so on. It will be very interesting to analyze this problem from purely algebraic–geometric point of view and to construct iteratively the sequence of L_∞ algebras controlling the corresponding deformation problems.

Before we proceed with the proof of Theorem 2.6 we will need to establish some general facts about the deformation theory of varieties with potentials.

2.3. Deformations of compactified Landau–Ginzburg models. Let $((\mathsf{Z},\mathsf{f}),D_{\mathsf{Z}},\mathsf{vol}_{\mathsf{Z}})$ be a compactified Landau–Ginzburg model satisfying $H^1(\mathsf{Z},\mathbb{Q})=0$ and the tameness condition (T). Since $H^1(\mathsf{Z},\mathbb{Q})=0$ implies $\mathrm{Pic}^0(\mathsf{Z})=0$ and since the Neron–Severi class $[D_{\mathsf{Z}}]\in H^2(\mathsf{Z},\mathbb{Z})$ is preserved under small deformations of Z, it follows that the condition $D_{\mathsf{Z}}\in|K_{\mathsf{Z}}^{-1}|$ is also preserved under small deformations of the pair $(\mathsf{Z},D_{\mathsf{Z}})$. By (T) the meromorphic volume form $\mathsf{vol}_{\mathsf{Z}}$ is a trivialization of the line bundle $K_{\mathsf{Z}}(D_{\mathsf{Z}})$ and so the versal deformation space of $((\mathsf{Z},\mathsf{f})_{D_\mathsf{Z}}, \mathsf{vol}_\mathsf{Z})$ is a principal $\mathbb{C}^\times$-bundle over the versal deformation space of $(\mathsf{Z},\mathsf{f})_{D_\mathsf{Z}}$. Therefore, it suffices to prove the unobstructedness of the deformation theory of $(\mathsf{Z},\mathsf{f})_{D_\mathsf{Z}}$.

As usual the deformation theory of $(\mathsf{Z},\mathsf{f})_{D_\mathsf{Z}}$ is controlled by an L_∞ algebra [**KS05**, **Hin01**, **KS09**, **Lur11**]. By standard Kodaira–Spenser theory the deformations of the map $\mathsf{f} : \mathsf{Z} \to \mathbb{P}^1$ are computed [**Ill71**, **Ill72**, **Hor74**, **Ser06**] by the sheaf of dg Lie algebras

$$\begin{array}{ccc} [\, T_\mathsf{Z} & \xrightarrow{d\mathsf{f}} & \mathsf{f}^* T_{\mathbb{P}^1} \,]. \\ 0 & & 1 \end{array}$$

Here $\mathsf{f}^* T_{\mathbb{P}^1}$ denotes the $\mathcal{O}$-module pullback, and this is a complex of locally free coherent sheaves with an $\mathcal{O}_\mathsf{Z}$-linear differential and $\mathbb{C}$-bilinear (graded) Lie bracket.

Similarly the deformations of $\mathsf{f} : \mathsf{Z} \to \mathbb{P}^1$ which preserve the boundary divisor D_Z are computed by the sheaf of dg Lie algebras

$$\begin{array}{cccc} \mathfrak{g}^\bullet := [\, & T_{\mathsf{Z},D_\mathsf{Z}} & \xrightarrow{d\mathsf{f}} & \mathsf{f}^* T_{\mathbb{P}^1,\infty} \,], \\ & \| & & \| \\ & \mathfrak{g}^0 & & \mathfrak{g}^1 \end{array}$$

where for any smooth variety M and any closed reduced subscheme $S \subset M$ we write $T_{M,S}$ for the coherent sheaf of vector fields on M that are tangent to S at the points of S. Since $D_\mathsf{Z} \subset \mathsf{Z}$ and $\{\infty\} \subset \mathbb{P}^1$ are reduced normal crossings divisors, it follows that $T_{\mathsf{Z},D_\mathsf{Z}} \subset T_\mathsf{Z}$ and $T_{\mathbb{P}^1,\infty} \subset T_{\mathbb{P}^1}$ are locally free subsheaves.

Recall (see, e.g., [**GM90**, **KS05**, **Man04**]) that the unobstructedness of the deformation theory defined by an L_∞ algebra follows from the stronger property that this L_∞ algebra is homotopy abelian. Therefore, Theorem A and Theorem 2.6 will follow immediately from the following:

Theorem 2.10. *Suppose* $((\mathsf{Z},\mathsf{f}), D_\mathsf{Z}, \mathsf{vol}_\mathsf{Z})$ *is a compactified Landau–Ginzburg model satisfying the tameness condition* (T)*. Then the* L_∞ *algebra*

$$R\Gamma(\mathsf{Z}, \mathfrak{g}^\bullet) = R\Gamma\left(\mathsf{Z}, \left[T_{\mathsf{Z},D_\mathsf{Z}} \xrightarrow{d\mathsf{f}} \mathsf{f}^* T_{\mathbb{P}^1,\infty} \right]\right)$$

is homotopy abelian.

As in the compact Calabi–Yau case we will deduce Theorem 2.10 from a Hodge theoretic statement – the degeneration of a “Hodge-to-de Rham spectral sequence” associated with the divisor D_Z and the potential f. Our main tool here is a new complex of logarithmic forms adapted to f:

Definition 2.11. Let $((\mathsf{Z},\mathsf{f}), D_\mathsf{Z})$ be a compactified Landau–Ginzburg model satisfying the conditions Definition 2.4(a) and Definition 2.4(b). For any $a \geq 0$ we define the **sheaf** $\Omega^a_\mathsf{Z}(\log D_\mathsf{Z}, \mathsf{f})$ **of** f**-adapted logarithmic forms on** $(\mathsf{Z}, D_\mathsf{Z})$ as the subsheaf of logarithmic a-forms that stay logarithmic after multiplication by $d\mathsf{f}$. Thus

$$\Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}) := \left\{ \alpha \in \Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}) \,\middle|\, d\mathsf{f} \wedge \alpha \in \Omega^{a+1}_{\mathsf{Z}}(\log D_{\mathsf{Z}}) \right\} \subset \Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}),$$

where f is viewed as a meromorphic function on Z and $d\mathsf{f}$ is viewed as a meromorphic one form.

The sheaves $\Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})$ have several interesting properties. As a first remark we have the following:

Lemma 2.12. (a) *The sheaf* $\Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})$ *of* f*-adapted logarithmic forms is a coherent* $\mathcal{O}_{\mathsf{Z}}$*-module which is locally free of rank equal to* rank $\Omega^a_{\mathsf{Z}} = \binom{n}{a}$.

(b) *Suppose* $\varepsilon : \widehat{\mathsf{Z}} \to \mathsf{Z}$ *is a blow-up of* Z *with smooth center contained in* D^{v} *and cleanly intersecting each component of* D^{v}*. Let* $\widehat{D}_{\widehat{\mathsf{Z}}} = \varepsilon^* D_{\mathsf{Z}}$ *and* $\hat{\mathsf{f}} = \varepsilon^* \mathsf{f}$ *denote the pullbacks of the divisor and potential to* $\widehat{\mathsf{Z}}$*. Then* $R\varepsilon_* \Omega^a_{\widehat{\mathsf{Z}}}\left(\log \widehat{D}_{\widehat{\mathsf{Z}}}, \hat{\mathsf{f}}\right) = \Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})$.

Proof. Indeed, let $\jmath_Y : Y \hookrightarrow \mathsf{Z}$ denote the inclusion of Y in Z. By definition $\Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})$ is the preimage of the coherent $\mathcal{O}_{\mathsf{Z}}$-submodule $\Omega^{a+1}_{\mathsf{Z}}(\log D_{\mathsf{Z}}) \subset \jmath_{Y*}\Omega^{a+1}_Y$ under the $\mathcal{O}_{\mathsf{Z}}$-linear map

$$d\mathsf{f}\wedge : \Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}) \to \jmath_{Y*}\Omega^{a+1}_Y.$$

Thus $\Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})$ is a torsion-free coherent submodule in $\Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}})$ of maximal rank.

The fact that $\Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})$ is locally free can be checked locally analytically on Z.

On the open set $Y \subset \mathsf{Z}$ we have by definition $\Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})_{|Y} = \Omega^a_Y$ and so it is locally free. Furthermore, if $p \in D^{\mathsf{h}} - D^{\mathsf{v}}$, then $d\mathsf{f}$ is holomorphic in a neighborhood of p. This implies that near p we have that $\Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})$ is isomorphic to $\Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}})$ and so is locally free.

Suppose next $p \in D^{\mathsf{v}}$. We can find local analytic coordinates $z_1, \ldots, z_n$ centered at p so that in a neighborhood of p:

- the divisor D^{v} is given by $\prod_{i=1}^k z_i = 0$, the divisor D^{h} is given by $\prod_{1=k+1}^{k+l} z_i = 0$;
- the potential f is given by $\mathsf{f}(z_1, \ldots, z_n) = \dfrac{1}{z_1^{m_1} \cdots z_k^{m_k}}$ for some $m_i \geq 1$.

Now for any a we have $\Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}) = \oplus_{p=0}^a \wedge^p V \otimes \wedge^{a-p} R$, where $V \subset \Omega^1_{\mathsf{Z}}(\log D_{\mathsf{Z}})$ is the sub $\mathcal{O}$-module spanned by $\{d\log z_i\}_{i=1}^k$, while $R \subset \Omega^1_{\mathsf{Z}}(\log D_{\mathsf{Z}})$ is the sub $\mathcal{O}$-module spanned by $\{d\log z_i\}_{i=k+1}^{k+l}$ and $\{dz_i\}_{i=k+l+1}^n$.

Since $d\mathsf{f}$ only has poles at the components of D^{v}, the condition that a logarithmic form $\alpha = \sum_p \nu_p \otimes \rho_{a-p} \in \Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}) = \oplus_{p=0}^a \wedge^p V \otimes \wedge^{a-p} R$

is f-adapted will only impose constraints on the pieces $\nu_p \in \wedge^p V$. Thus it is enough to understand which local sections of $\wedge^p V$ are f-adapted.

Write $W \subset V$ for the sub $\mathcal{O}$-module spanned by $\{d\log z_i\}_{i=1}^{k-1}$. In particular,

$$V = W \oplus \mathcal{O} \cdot d\log z_k, \text{ and } \wedge^p V = \wedge^p W \oplus \left(\wedge^{p-1} W \wedge d\log z_k\right),$$

and so given any $\nu \in \wedge^p V$, we can write ν and $d\mathsf{f} \wedge \nu$ uniquely as

$$\nu = \eta + \beta \wedge d\log z_k, \text{ with } \eta \in \wedge^p W, \text{ and } \beta \in \wedge^{p-1} W,$$

$$d\mathsf{f} \wedge \nu = \varphi + \psi \wedge d\log z_k, \text{ with } \varphi \in (\wedge^{p+1} W)(*D^{\mathsf{v}}), \text{ and } \psi \in (\wedge^p W)(*D^{\mathsf{v}}).$$

We have $d\mathsf{f} = \mathsf{f}\cdot d\log \mathsf{f}$. The logarithmic 1-form $d\log \mathsf{f}$ also decomposes as $d\log \mathsf{f} = \omega - m_k d\log z_k$ where $\omega = -\sum_{i=1}^{k-1} m_i d\log z_i$ is its W-component. This gives

$$\varphi = \mathsf{f}\cdot \omega \wedge \eta,$$

$$\psi = \mathsf{f}\cdot \left(\omega \wedge \beta - (-1)^p m_k \eta\right).$$

In particular, we can solve for η in terms of ψ and β. The condition that ν is f-adapted is simply the condition that $\varphi \in \wedge^{p+1} W$ and $\psi \in \wedge^p W$. But for any $\psi \in \wedge^p W$ and any $\beta \in \wedge^{p-1} W$ the form

$$\eta = \frac{1}{(-1)^p m_k} \cdot \left(\omega \wedge \beta - \frac{1}{\mathsf{f}} \cdot \psi\right),$$

automatically satisfies $\eta \in \wedge^p W$ and

$$\mathsf{f}\cdot \omega \wedge \eta = -\frac{1}{(-1)^p m_k} \cdot \omega \wedge \psi \in \wedge^{p+1} W.$$

In other words ν is f-adapted if and only if we can find a form $\psi \in \wedge^p W$ and a form $\beta \in \wedge^{p-1} W$ so that

$$\nu = \frac{1}{(-1)^p m_k} \cdot \left[d\log \mathsf{f} \wedge \beta - \frac{1}{\mathsf{f}} \cdot \psi\right].$$

This shows that the subsheaf $\wedge^p V \cap \Omega^p_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})$ in $\wedge^p V$ consisting of f-adapted forms is given by

$$\wedge^p V \cap \Omega^p_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}) = \frac{1}{\mathsf{f}} \wedge^p W \oplus d\log \mathsf{f} \wedge \left(\wedge^{p-1} W\right).$$

In particular, $\wedge^p V \cap \Omega^p_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})$ is locally free and hence $\Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})$ is locally free. Explicitly

(2.3.1)
$$\Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}) = \bigoplus_{p=0}^{a} \left[\frac{1}{\mathsf{f}} \wedge^p W \bigoplus d\log \mathsf{f} \wedge \left(\wedge^{p-1} W\right)\right] \bigotimes \wedge^{a-p} R.$$

This completes the proof of part (a) of the lemma. Part (b) follows immediately from the formula (2.3.1), the description of the exceptional divisor of $\varepsilon : \widehat{\mathsf{Z}} \to \mathsf{Z}$ as a projectivized normal bundle, and the Euler sequence of this projective bundle. q.e.d.

Remark 2.13. The f-adapted logarithmic forms are equipped with two natural differentials of degree one:

- the de Rham differential $d : \Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}) \to \Omega^{a+1}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})$, and
- the differential $d\mathsf{f}\wedge : \Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}) \to \Omega^{a+1}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})$.

Note that by definition the differential $d\mathsf{f}\wedge$ is $\mathcal{O}_{\mathsf{Z}}$-linear, while the de Rham differential satisfies the Leibnitz rule as usual. Note also that for any complex numbers c_1 and c_2 the linear combination $c_1 d + c_2 d\mathsf{f}\wedge$ is also a differential and so we get a family of complexes of f-adapted logarithmic forms

$$(\Omega^{\bullet}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}), c_1 d + c_2 d\mathsf{f}\wedge), \tag{2.3.2}$$

parametrized by $(c_1, c_2) \in \mathbb{C}^2$.

The previous discussion connects directly to the L_∞-algebra $R\Gamma(\mathsf{Z}, \mathfrak{g}^{\bullet})$ since in the Calabi–Yau case we can use the holomorphic volume form to convert f-adapted logarithmic forms to poly-vector fields. Suppose $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}}, \mathsf{vol}_{\mathsf{Z}})$ is a compactified Landau–Ginzburg model. The contraction with the meromorphic volume form gives a map of $\mathcal{O}_{\mathsf{Z}}$-modules

$$\begin{aligned} \boldsymbol{\iota}_{\mathsf{vol}_{\mathsf{Z}}}(\bullet) : \qquad \wedge^a T_{\mathsf{Z}} &\longrightarrow \jmath_{Y*}\Omega^{n-a}_Y, \\ \xi &\longmapsto \boldsymbol{\iota}_{\mathsf{vol}_{\mathsf{Z}}}(\xi). \end{aligned} \tag{2.3.3}$$

The preimage of $\Omega^{n-a}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})$ under the map (2.3.3) will be a coherent subsheaf in $\wedge^a T_{\mathsf{Z}}$. Furthermore, when $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}}, \mathsf{vol}_{\mathsf{Z}})$ satisfies the tameness condition (T), the explicit description of the local frames of $\Omega^{n-a}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})$ above gives that

$$(\wedge^a T_{\mathsf{Z}})(-\log D_{\mathsf{Z}}, \mathsf{f}) := (\boldsymbol{\iota}_{\mathsf{vol}_{\mathsf{Z}}}(\bullet))^{-1}\left(\Omega^{n-a}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})\right)$$

is a locally free subsheaf of maximal rank in $\wedge^a T_{\mathsf{Z}}$, and that $\boldsymbol{\iota}_{\mathsf{vol}_{\mathsf{Z}}}$ induces an isomorphism between $(\wedge^a T_{\mathsf{Z}})(-\log D_{\mathsf{Z}}, \mathsf{f})$ and $\Omega^{n-a}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})$.

With this notation we now have the following:

Lemma 2.14. *Let $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}}, \mathsf{vol}_{\mathsf{Z}})$ be a compactified LandauGinzburg model satisfying the tameness assumption (T). Then the subsheaf*

$$(\wedge^{\bullet} T_{\mathsf{Z}})(-\log D_{\mathsf{Z}}, \mathsf{f}) \subset \wedge^{\bullet} T_{\mathsf{Z}}$$

is closed under the Nijenhuis–Schouten bracket on $\wedge^{\bullet} T_{\mathsf{Z}}$.

Proof. Recall that the Nijenhuis–Schouten bracket on polyvector fields is a degree (-1) bracket that extends the Lie bracket, acts as a graded derivation for the wedge product, and is given on decomposable polyvector fields by

$$
\begin{aligned}
[\mathsf{g},\xi_1\wedge\cdots\wedge\xi_a] &= \boldsymbol{\iota}_{d\mathsf{g}}\left(\xi_1\wedge\cdots\wedge\xi_a\right),\\
[\xi_1\wedge\cdots\wedge\xi_a,\eta_1\wedge\cdots\wedge\eta_b] &= \sum_{i,j}(-1)^{i+j}\left[\xi_i,\eta_j\right]\\
&\quad\wedge\left(\xi_1\wedge\cdots\wedge\widehat{\xi_i}\wedge\cdots\wedge\xi_a\right)\\
&\quad\wedge\left(\eta_1\wedge\cdots\wedge\widehat{\eta_j}\wedge\cdots\wedge\eta_b\right),
\end{aligned}
\tag{2.3.4}
$$

for all $\mathsf{g}\in\mathcal{O}_{\mathsf{Z}}$, and all $\xi_i,\eta_j\in T_{\mathsf{Z}}$.

The statement of the lemma is local on Z and is obvious away from the points of D^{v}. Indeed, away from D^{v} we have that $\Omega^{n-a}_{\mathsf{Z}}(\log D_{\mathsf{Z}},\mathsf{f})$ is isomorphic to $\Omega^{n-a}_{\mathsf{Z}}(\log D_{\mathsf{Z}})$. Since $\mathsf{vol}_{\mathsf{Z}}$ has first order poles along the components of D_{Z} this implies that on $Y-D^{\mathsf{v}}$ we have an isomorphism $(\wedge^a T_{\mathsf{Z}})(-\log D_{\mathsf{Z}},\mathsf{f})\cong\wedge^a T_{\mathsf{Z},D_{\mathsf{Z}}}$. Since the subsheaf $T_{\mathsf{Z},D_{\mathsf{Z}}}\subset T_{\mathsf{Z}}$ is preserved by the Lie bracket we get that away from D^{v} the subsheaf $(\wedge^a T_{\mathsf{Z}})(-\log D_{\mathsf{Z}},\mathsf{f})$ is preserved by the Nijenhuis–Schouten bracket.

Suppose next $p\in D^{\mathsf{v}}$. As before we choose local coordinates $z_1,\cdots,z_n$ centered at p so that near p we have:

- $D^{\mathsf{v}}: z_1\cdots z_k=0,\quad D^{\mathsf{h}}: z_{k+1}\cdots z_{k+l}=0;$
- $\mathsf{f}(z_1,\ldots,z_n)=\dfrac{1}{z_1z_2\cdots z_k};$
- $\mathsf{vol}_{\mathsf{Z}}=\dfrac{dz_1\wedge\cdots\wedge dz_n}{z_1\cdots z_{k+l}}.$

Using this formula for $\mathsf{vol}_{\mathsf{Z}}$ and the description (2.3.1) of the sheaf of f-adapted logarithmic forms, it is straightforward to compute $(\wedge^a T_{\mathsf{Z}})(-\log D_{\mathsf{Z}},\mathsf{f})$. Let

$$
M=\mathrm{Span}_{\mathcal{O}_{\mathsf{Z}}}\left(z_1\frac{\partial}{\partial z_1},\ldots,z_{k-1}\frac{\partial}{\partial z_{k-1}}\right)\subset T_{\mathsf{Z},D_{\mathsf{Z}}},
$$

$$
N=\mathrm{Span}_{\mathcal{O}_{\mathsf{Z}}}\left(z_{k+1}\frac{\partial}{\partial z_{k+1}},\ldots,z_{k+l}\frac{\partial}{\partial z_{k+l}},\frac{\partial}{\partial z_{k+l+1}},\ldots,\frac{\partial}{\partial z_n}\right)\subset T_{\mathsf{Z},D_{\mathsf{Z}}}.
$$

In terms of these sheaves we have

$$
\begin{aligned}
&(\wedge^a T_{\mathsf{Z}})(-\log D_{\mathsf{Z}},\mathsf{f})\\
&\quad=\bigoplus_{p=0}^{a}\left[\left(\frac{1}{\mathsf{f}}\wedge^{p-1}M\wedge z_k\frac{\partial}{\partial z_k}\right)\right.\\
&\qquad\left.\bigoplus\left(\boldsymbol{\iota}_{d\log\mathsf{f}}\left(\wedge^p M\wedge z_k\frac{\partial}{\partial z_k}\right)\right)\right]\bigotimes\wedge^{a-p}N.
\end{aligned}
\tag{2.3.5}
$$

From the formulas (2.3.4) it is now immediate that $(\wedge^\bullet T_{\mathsf{Z}})(-\log D_{\mathsf{Z}}, \mathsf{f})$ is preserved by the Nijenhuis–Schouten bracket. q.e.d.

Example 2.15. It is instructive to examine more carefully the simplest case of a one dimensional compactified Landau–Ginzburg model. Near a point p of $D^{\vee}$ we can choose a local coordinate z on Z so that $\mathsf{f}(z) = z^{-1}$, $\mathsf{vol}_{\mathsf{Z}} = dz/z$. Then locally near p we get

$$\Omega^\bullet_{\mathsf{Z}}(\log D_{\mathsf{Z}}) = \mathcal{O}_{\mathsf{Z}} \cdot 1 \oplus \mathcal{O}_{\mathsf{Z}} \cdot \frac{dz}{z};$$

$$\Omega^\bullet_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}) = \mathcal{O}_{\mathsf{Z}} \cdot z \oplus \mathcal{O}_{\mathsf{Z}} \cdot \frac{dz}{z};$$

$$\wedge^\bullet T_{\mathsf{Z}, D_{\mathsf{Z}}} = \mathcal{O}_{\mathsf{Z}} \cdot 1 \oplus \mathcal{O}_{\mathsf{Z}} \cdot z\frac{\partial}{\partial z};$$

$$(\wedge^\bullet T_{\mathsf{Z}})(-\log D_{\mathsf{Z}}, \mathsf{f}) = \mathcal{O}_{\mathsf{Z}} \cdot 1 \oplus \mathcal{O}_{\mathsf{Z}} \cdot z^2\frac{\partial}{\partial z}.$$

Using Lemma 2.14 we can organize the f-adapted polyvector fields into a sheaf of dg Lie algebras. For any $1 - n \leq b \leq 1$ set

$$\mathfrak{G}^b := \left(\wedge^{-b+1} T_{\mathsf{Z}}\right)(-\log D_{\mathsf{Z}}, \mathsf{f}).$$

The sheaves $\mathfrak{G}^b$ fit together with the Nijenhuis–Schouten bracket $[\bullet, \bullet]$ and the differential $[\mathsf{f}, \bullet] = \boldsymbol{\iota}_{d\mathsf{f}}$ into a sheaf of dg Lie algebras

$(\mathfrak{G}^\bullet, [\mathsf{f}, \bullet]) :=$

$$\begin{array}{ccccccccc} \big[\mathfrak{G}^{1-n} & \xrightarrow{[\mathsf{f},\bullet]} & \mathfrak{G}^{2-n} & \xrightarrow{[\mathsf{f},\bullet]} & \cdots & \xrightarrow{[\mathsf{f},\bullet]} & \mathfrak{G}^{0} & \xrightarrow{[\mathsf{f},\bullet]} & \mathfrak{G}^{1}\big] \\ (1-n) & & (2-n) & & \cdots & & 0 & & 1 \end{array}$$

This sheaf of dg Lie algebras is directly related to our unobstructedness problem. Indeed, note that any stupid truncation of $(\mathfrak{G}^\bullet, [\mathsf{f}, \bullet])$ will be a subsheaf of dg Lie algebras. In particular, we have a subsheaf of dg Lie algebras

$$\left[\mathfrak{G}^0 \xrightarrow{[\mathsf{f},\bullet]} \mathfrak{G}^1\right] = \sigma_{\geq 0}(\mathfrak{G}^\bullet, [\mathsf{f}, \bullet]) \hookrightarrow (\mathfrak{G}^\bullet, [\mathsf{f}, \bullet]).$$

On the other hand this subsheaf maps naturally to the sheaf of dg Lie algebras $\mathfrak{g}^\bullet = \left[\mathfrak{g}^0 \xrightarrow{d\mathsf{f}} \mathfrak{g}^1\right]$ that controls our deformation problem. So we get a diagram

$$\begin{tikzcd} & \sigma_{\geq 0}(\mathfrak{G}^\bullet, [\mathsf{f}, \bullet]) \arrow[dl] \arrow[dr] & \\ \mathfrak{g}^\bullet & & (\mathfrak{G}^\bullet, [\mathsf{f}, \bullet]) \end{tikzcd} \tag{2.3.6}$$

of sheaves of dg Lie algebras. In fact (2.3.6) is a roof diagram. More precisely we have the following:

Proposition 2.16. *The natural map of sheaves of dg Lie algebras*

$$\left[\mathfrak{G}^0 \xrightarrow{[\mathsf{f},\bullet]} \mathfrak{G}^1\right] \to \left[\mathfrak{g}^0 \xrightarrow{df} \mathfrak{g}^1\right] \tag{2.3.7}$$

is an L_∞ quasi-isomorphism.

Proof. The question is local on Z. From the definition of $\mathfrak{G}^\bullet$ it is clear that the map (2.3.7) is actually an isomorphism away from $D^\vee$. Thus it only remains to check the statement locally near a point $p \in D^\vee$.

Choose local coordinates $z_1, \ldots, z_n$ as in the proof of Lemma 2.14. In terms of these coordinates we can describe our dg Lie algebras explicitly. For the sheaves of f-adapted poly vector fields we have

$$\mathfrak{G}^0 = \left\{ \begin{array}{l|l} \sum_{i=1}^k \mathsf{g}_i \frac{z_i \partial}{\partial z_i} & \mathsf{g}_i, \mathsf{g}'_j, \mathsf{g}''_s \in \mathcal{O}_\mathsf{Z}, \quad \text{and} \\ + \sum_{j=k+1}^{k+l} \mathsf{g}'_j \frac{z_j \partial}{\partial z_j} & \displaystyle\sum_{i=1}^k \mathsf{g}_i \in z_1 \cdots z_k \mathcal{O}_\mathsf{Z} \\ + \sum_{s=k+l}^n \mathsf{g}''_s \frac{\partial}{\partial z_s} & \end{array} \right\},$$

$$\mathfrak{G}^1 = \mathcal{O}_\mathsf{Z}.$$

The differential $[\mathsf{f}, \bullet] = \boldsymbol{\iota}_{d\mathsf{f}} : \mathfrak{G}^0 \to \mathfrak{G}^1$ is given explicitly by the formula

$$\mathfrak{G}^0 \xrightarrow{\qquad [\mathsf{f},\bullet] \qquad} \mathfrak{G}^1, \tag{2.3.8}$$

$$(\mathsf{g}, \mathsf{g}', \mathsf{g}'') \longrightarrow \frac{\mathsf{g}_1 + \mathsf{g}_2 + \cdots + \mathsf{g}_k}{\mathsf{g}_1 \cdot \mathsf{g}_2 \cdot \cdots \cdot \mathsf{g}_k}.$$

We have an analogous local description of the deformation theory dg algebra:

$$\mathfrak{g}^0 = \left\{ \sum_{i=1}^k \mathsf{g}_i \frac{z_i \partial}{\partial z_i} + \sum_{j=k+1}^{k+l} \mathsf{g}'_j \frac{z_j \partial}{\partial z_j} + \sum_{s=k+l}^n \mathsf{g}''_s \frac{\partial}{\partial z_s} \;\middle|\; \mathsf{g}_i, \mathsf{g}'_j, \mathsf{g}''_s \in \mathcal{O}_\mathsf{Z} \right\},$$

$$\mathfrak{g}^1 = \frac{1}{z_1 \cdot \cdots \cdot z_k} \mathcal{O}_\mathsf{Z} = \mathsf{f}^* T_{\mathbb{P}^1, \infty} \cong \mathsf{f}^* \mathcal{O}_{\mathbb{P}^1}(1).$$

The differential $\mathfrak{g}^0 \to \mathfrak{g}^1$ is again given by the formula (2.3.8) and the map of dg Lie algebras $\left[\mathfrak{G}^0 \to \mathfrak{G}^1\right] \to \left[\mathfrak{g}^0 \to \mathfrak{g}^1\right]$ is given by the natural inclusions $\mathfrak{G}^0 \subset \mathfrak{g}^0$, $\mathfrak{G}^1 \subset \mathfrak{g}^1$. Thus we get a short exact sequence of complexes

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{G}^0 \arrow[r] \arrow[d, "{[\mathsf{f},\bullet]}"'] & \mathfrak{g}^0 \arrow[r] \arrow[d, "d\mathsf{f}"'] & \mathcal{O}_{\mathsf{Z}}/\left(z_1\cdots z_k\mathcal{O}_{\mathsf{Z}}\right) \arrow[r] \arrow[d, "\frac{1}{z_1\cdots z_k}"'] & 0 \\ 0 \arrow[r] & \mathfrak{G}^1 \arrow[r] & \mathfrak{g}^1 \arrow[r] & \left(\frac{1}{z_1\cdots z_k}\mathcal{O}_{\mathsf{Z}}\right)/\mathcal{O}_{\mathsf{Z}} \arrow[r] & 0 \end{tikzcd}$$

Since the last vertical map is clearly an isomorphism, this implies that $\left[\mathfrak{G}^0 \to \mathfrak{G}^1\right] \to \left[\mathfrak{g}^0 \to \mathfrak{g}^1\right]$ is a quasi-isomorphism. q.e.d.

Proposition 2.16 and the roof diagram 2.3.6 suggest that the unobstructedness statement in Theorem 2.10 is related to the unobstructedness of the L_∞ algebra $R\Gamma(\mathsf{Z}, (\mathfrak{G}^\bullet, [\mathsf{f}, \bullet]))$. In fact the standard formality yoga for L_∞ algebras allows us to deduce Theorem 2.10 from a stronger double degeneration statement for the cohomology of a two parameter family of L_∞ algebras. Specifically let $\mathsf{div}_{\mathsf{vol}_\mathsf{Z}} = \boldsymbol{\iota}_{\mathsf{vol}_\mathsf{Z}}^{-1} \circ d \circ \boldsymbol{\iota}_{\mathsf{vol}_\mathsf{Z}} : \mathfrak{G}^a \to \mathfrak{G}^{a+1}$ denote the divergence operator associated with vol_Z. Note that by definition the differentials $[\mathsf{f}, \bullet]$ and $\mathsf{div}_{\mathsf{vol}_\mathsf{Z}}$ anticommute and so for any pair of complex numbers (c_1, c_2) we will get a well defined complex $R\Gamma\left(\mathsf{Z}, (\mathfrak{G}^\bullet, c_1\, \mathsf{div}_{\mathsf{vol}_\mathsf{Z}} + c_2[\mathsf{f}, \bullet])\right)$. With this notation we now have the following:

Proposition 2.17. *Suppose that*

(2.3.9) *For all a the dimension* $\dim_{\mathbb{C}} \mathbb{H}^a\left(\mathsf{Z}, (\mathfrak{G}^\bullet, c_1\, \mathsf{div}_{\mathsf{vol}_\mathsf{Z}} + c_2[\mathsf{f}, \bullet])\right)$ *is independent of* $(c_1, c_2) \in \mathbb{C}^2$.

Then $R\Gamma(\mathsf{Z}, \mathfrak{g}^\bullet)$ *is homotopy abelian.*

Proof. By Proposition 2.16 we deduce that the L_∞ algebra $R\Gamma\left(\mathsf{Z}, \mathfrak{g}^\bullet\right)$ is homotopy abelian (i.e., unobstructed) if and only if the L_∞ algebra $R\Gamma\left(\mathsf{Z}, \left[\mathfrak{G}^0 \xrightarrow{[\mathsf{f},\bullet]} \mathfrak{G}^1\right]\right)$ is homotopy abelian. Now in view of [**KKP08**, Proposition 4.11(ii)] this reduces[2] the unobstructedness statement in Theorem 2.10 to showing that

(1) The L_∞ algebra $R\Gamma\left(\mathsf{Z}, (\mathfrak{G}^\bullet, [\mathsf{f}, \bullet])\right)$ is homotopy abelian;
(2) The induced map

$$R\Gamma\left(\mathsf{Z}, \sigma_{\geq 0}\left(\mathfrak{G}^\bullet, [\mathsf{f}, \bullet]\right)\right) \to R\Gamma\left(\mathsf{Z}, (\mathfrak{G}^\bullet, [\mathsf{f}, \bullet])\right)$$

is injective on cohomology.

[2]In [**KKP08**] Proposition 4.11 is formulated and proven for d($\mathbb{Z}/2$) graded algebras. However, the statement of the proposition and its proof transfer verbatim to the d($\mathbb{Z}$)graded case, and we use this d($\mathbb{Z}$)graded version here.

First note that the stupid filtration $\sigma_{\geq\bullet}(\mathfrak{G}^\bullet, [\mathsf{f}, \bullet))$ gives rise to a spectral sequence which abuts to the spaces $\mathbb{H}^a(\mathsf{Z}, (\mathfrak{G}^\bullet, [\mathsf{f}, \bullet]))$. By assumption $\dim_{\mathbb{C}} \mathbb{H}^a(\mathsf{Z}, (\mathfrak{G}^\bullet, c[\mathsf{f}, \bullet]))$ is independent of $c \in \mathbb{C}$ and thus this spectral sequence will degenerate at E_1. This implies that

$$\mathbb{H}^a(\mathsf{Z}, \sigma_{\geq k}(\mathfrak{G}^\bullet, [\mathsf{f}, \bullet])) \to \mathbb{H}^a(\mathsf{Z}, (\mathfrak{G}^\bullet, [\mathsf{f}, \bullet]))$$

is injective for all k and, in particular, property (2) holds.

To prove property (1) we consider the flat family of L_∞ algebras over $\mathbb{C}[[\hbar]]$ given by $\mathfrak{k} := R\Gamma(\mathsf{Z}, (\mathfrak{G}^\bullet[[\hbar]], \hbar \cdot \mathsf{div}_{\mathsf{vol}_\mathsf{Z}} + [\mathsf{f}, \bullet]))$. According to [**KKP08**, Proposition 4.11(i)] it suffices to check that $\mathfrak{k}$ satisfies:

(A) $\mathfrak{k} \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}((\hbar))$ is homotopy abelian over $\mathbb{C}((\hbar))$, and
(B) $H^\bullet(\mathfrak{k}, d_{\mathfrak{k}})$ is a flat $\mathbb{C}[[\hbar]]$-module.

Condition (B) follows immediately from the flatness assumption (2.3.9). To check condition (A) we will use an observation from [**BK98**]: the map $\mathfrak{k} \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}((\hbar)) \to \mathfrak{k} \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}((\hbar))$, given by $\gamma \mapsto \exp(\gamma/\hbar)$, is a quasi-isomorphism between $\mathfrak{k} \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}((\hbar))$ and an abelian dg algebra over $\mathbb{C}((\hbar))$. The proposition is proven. q.e.d.

Proposition 2.17 finishes the proofs of Theorem 2.10 and Theorem 2.6 modulo the flatness assumption (2.3.9). Converting back to f-adapted logarithmic forms via $\iota_{\mathsf{vol}_\mathsf{Z}}$ the assumption (2.3.9) is equivalent to the statement that the dimension of the hypercohomology

$$\mathbb{H}^\bullet(\mathsf{Z}, (\Omega^\bullet_\mathsf{Z}(\log D_\mathsf{Z}, \mathsf{f}), c_1 d + c_2 d\mathsf{f}\wedge))$$

is independent of $(c_1, c_2) \in \mathbb{C}^2$. We investigate this Hodge theoretic statement in the next section.

2.4. The double degeneration property. In this section we complete the proof of the unobstructedness Theorem 2.6 by establishing the double degeneration property for the complex (2.3.2) of f-adapted logarithmic forms associated with a compactified tame Landau–Ginzburg model which is not necessarily of log Calabi–Yau type. An alternative proof and a generalization of this statement can be found in the recent work of Esnault–Sabbah–Yu [**ESY13**]. For the convenience of the reader we give our original argument here.

Theorem 2.18. *Let $((\mathsf{Z}, \mathsf{f}), D_\mathsf{Z})$ be geometric datum where*

(a) Z *is a smooth projective variety, and* $\mathsf{f} : \mathsf{Z} \to \mathbb{P}^1$ *is a flat projective morphism.*
(b) $D_\mathsf{Z} = (\cup_i D_i^\mathsf{h}) \cup (\cup_j D_i^\mathsf{v})$ *is a reduced normal crossing divisor, such that*
   - $D^\mathsf{v} = \cup_j D_j^\mathsf{v}$ *is the pole divisor of* f, *i.e.,* $D^\mathsf{v} = \mathsf{f}^{-1}(\infty)$ *is the scheme-theoretic fiber of* f *at* $\infty \in \mathbb{P}^1$.
   - $\mathrm{crit}(\mathsf{f}) \cap D_\mathsf{Z}^\mathsf{h} = \varnothing$.

Then the following flatness property holds:

(2.4.1) *For all* $a \geq 0$ $\dim_{\mathbb{C}} \mathbb{H}^a \left(\mathsf{Z}, (\Omega^{\bullet}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}), c_1 d + c_2 d\mathsf{f}\wedge)\right)$ *is independent of* $(c_1, c_2) \in \mathbb{C}^2$.

Proof. We will obtain the proof by checking the constancy of dimension of cohomology along various lines in $\mathbb{C}^2$. First we have the following:

Lemma 2.19. *For every* $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}})$ *satisfying the hypothesis if the theorem and every* $a \geq 0$ *we have*

(2.4.2)
$$\begin{aligned}\dim_{\mathbb{C}} \mathbb{H}^a(\mathsf{Z}, (\Omega^{\bullet}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}), d)) &= \dim_{\mathbb{C}} \mathbb{H}^a(\mathsf{Z}, (\Omega^{\bullet}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}), 0)) \\ &= \sum_{i+j=a} \dim_{\mathbb{C}} H^i(\mathsf{Z}, \Omega^j_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})).\end{aligned}$$

In particular, the spectral sequence corresponding to the stupid filtration on $(\Omega^{\bullet}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}), d)$ *degenerates at* E_1.

Proof. We will use the method of Deligne–Illusie [**DI87**, **EV92**, **Ill02**]. Here we only sketch the necessary modifications that make the method applicable to f-adapted logarithmic forms. More details can be found in the Esnault–Sabbah–Yu writeup in [**ESY13**, Appendix D].

By the standard spreading-out argument of [**DI87**, **EV92**, **Ill02**] it suffices to check the E_1 degeneration of the spectral sequence

$$H^i(\mathsf{Z}, \Omega^j_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}}, \mathsf{f})) \Rightarrow \mathbb{H}^{i+j}(\mathsf{Z}, (\Omega^{\bullet}_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}}, \mathsf{f}), d)), \tag{2.4.3}$$

in the case when the geometric datum $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}})$ satisfying the hypotheses of the lemma is defined over a perfect field $\Bbbk$ of characteristic $p > \dim X$ and admits a smooth lift to characteristic 0 (or at least to the second Witt vectors $W_2(\Bbbk)$ of $\Bbbk$).

Write $((\mathsf{Z}', \mathsf{f}'), D'_{\mathsf{Z}'})$ for the Frobenius twist of the datum $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}})$. In other words $((\mathsf{Z}', \mathsf{f}'), D'_{\mathsf{Z}'})$ is the base change of $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}})$ by the absolute Frobenius map $\varphi : \operatorname{Spec} \Bbbk \to \operatorname{Spec} \Bbbk$. Let $\Phi : ((\mathsf{Z}', \mathsf{f}'), D'_{\mathsf{Z}'}) \to ((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}})$ be the base change map and let

$$\mathsf{Fr} : ((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}}) \to ((\mathsf{Z}', \mathsf{f}'), D'_{\mathsf{Z}'})$$

denote the induced relative Frobenius morphism over $\Bbbk$.

The base change property for algebraic differential forms combined with the fact that Fr is a homeomorphism, and with the local description (2.3.1) of f-adapted forms implies that we have canonical isomorphisms $\Phi^* H^i\left(\mathsf{Z}, \Omega^j_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}}, \mathsf{f})\right) \cong H^i\left(\mathsf{Z}', \Omega^j_{\mathsf{Z}'/\Bbbk}(\log D_{\mathsf{Z}'}, \mathsf{f}')\right)$ and $H^a\left(\mathsf{Z}', \mathsf{Fr}_*\left(\Omega^{\bullet}_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}}, \mathsf{f}), d\right)\right) = H^a\left(\mathsf{Z}, \left(\Omega^{\bullet}_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}}, \mathsf{f}), d\right)\right)$. This gives equality of dimensions of these matching cohomology groups and so the E_1 degeneration of (2.4.3) will follow immediately (see, e.g., [**Ill02**, Section 4.8]) if we can show that the complex $\mathsf{Fr}_* \left(\Omega^\bullet_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}}, \mathsf{f}), d\right)$ is formal as an object in the derived category of quasi-coherent $\mathscr{O}_{\mathsf{Z}'}$-modules.

To that end, recall [**Car57**, **Kat70**] that the (inverse) Cartier map defined by $\boldsymbol{\gamma}(\Phi^* z) = z^p$ and $\boldsymbol{\gamma}(d\Phi^* z) = \left[z^{p-1}dz\right]$ on a local function z on Z, extends uniquely by multiplicativity and gives rise to an isomorphism

$$\boldsymbol{\gamma} : \bigoplus_{a\geq 0} \Omega^a_{\mathsf{Z}'/\Bbbk}(\log D'_{\mathsf{Z}'}) \xrightarrow{\cong} \bigoplus_{a\geq 0} \mathscr{H}^a \left(\mathsf{Fr}_* \left(\Omega^\bullet_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}}), d\right)\right),$$

of sheaves of super commutative algebras over $\mathscr{O}_{\mathsf{Z}'}$.

Using the explicit local description (2.3.1) of the f-adapted logarithmic forms one checks immediately that $\boldsymbol{\gamma}$ also restricts to an isomorphism

(2.4.4)
$$\boldsymbol{\gamma} : \bigoplus_{a\geq 0} \Omega^a_{\mathsf{Z}'/\Bbbk}(\log D'_{\mathsf{Z}'}, \mathsf{f}) \xrightarrow{\cong} \bigoplus_{a\geq 0} \mathscr{H}^a \left(\mathsf{Fr}_*\left(\Omega^\bullet_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}}, \mathsf{f}), d\right)\right),$$

of sheaves of super commutative algebras over $\mathscr{O}_{\mathsf{Z}'}$.

In view of the isomorphism (2.3.1) the formality of $\mathsf{Fr}_* \left(\Omega^\bullet_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}}, \mathsf{f}), d\right)$ as an object in $D(\mathsf{Z}')$ is equivalent to the existence of a morphism in $D(\mathsf{Z}')$:

$$\mathsf{for} : \quad \bigoplus_{a\geq 0} \Omega^a_{\mathsf{Z}'/\Bbbk}(\log D'_{\mathsf{Z}'}, \mathsf{f}')[-a] \longrightarrow \mathsf{Fr}_* \left(\Omega^\bullet_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}}, \mathsf{f}), d\right),$$

which induces $\boldsymbol{\gamma}$ on cohomology sheaves. Following [**DI87**] the construction of for can be carried out in three stages. Fix a lift $((\mathfrak{Z}, \mathfrak{f}), D_{\mathfrak{Z}})$ over $W_2(\Bbbk)$. We abuse notation and again write $\varphi : \operatorname{Spec} W_2(\Bbbk) \to \operatorname{Spec} W_2(\Bbbk)$ for the absolute Frobenius. Similarly we will write $\left((\mathfrak{Z}', \mathfrak{f}'), D'_{\mathfrak{Z}'}\right)$ for the pullback of $((\mathfrak{Z}, \mathfrak{f}), D_{\mathfrak{Z}})$ via φ and will write $\Phi : \mathfrak{Z}' \to \mathfrak{Z}$ for the base change map.

As a first step suppose that the relative Frobenius $\mathsf{Fr} : \mathsf{Z} \to \mathsf{Z}'$ admits a global lifting to a morphism $\mathfrak{Fr} : \mathfrak{Z} \to \mathfrak{Z}'$ of $W_2(\Bbbk)$-schemes which, furthermore, satisfies $\mathfrak{Fr}^*(\mathfrak{f}') = \mathfrak{f}^p$ and $\mathfrak{Fr}^* \mathscr{O}_{\mathfrak{Z}'} \left(D'_{\mathfrak{Z}'}\right) = \mathscr{O}_{\mathfrak{Z}} \left(p \cdot D_{\mathfrak{Z}}\right)$. With such a lifting we associate a formality morphism $\mathsf{for}_{\mathfrak{Fr}}$ as follows:

- For $a = 0$ we set $\mathsf{for}^0_{\mathfrak{Fr}} = \mathsf{Fr}^* : \mathscr{O}_{\mathsf{Z}'} \to \mathsf{Fr}_* \mathscr{O}_{\mathsf{Z}}$;
- For $a = 1$ we set $\mathsf{for}^1_{\mathfrak{Fr}} = ((1/p) \cdot \mathfrak{Fr}^* \mod p)$ viewed as a map $\mathsf{for}^1_{\mathfrak{Fr}} : \Omega^1_{\mathsf{Z}'/\Bbbk} \left(\log D'_{\mathsf{Z}'}, \mathsf{f}'\right) \to \mathsf{Fr}_* \Omega^1_{\mathsf{Z}/\Bbbk} \left(\log D_{\mathsf{Z}}\right)$;
- For $a > 1$ we define $\mathsf{for}^a_{\mathfrak{Fr}}$ to be the composition of $\wedge^a \mathsf{for}^1_{\mathfrak{Fr}}$ with the product map $\wedge^a \mathsf{Fr}_* \Omega^1_{\mathsf{Z}/\Bbbk} \left(\log D_{\mathsf{Z}}\right) \to \Omega^a_{\mathsf{Z}/\Bbbk} \left(\log D_{\mathsf{Z}}\right)$.

The key observation now is that for all a the map $\mathsf{for}^a_{\mathfrak{Fr}}$ sends $\Omega^a_{\mathsf{Z}'/\Bbbk}\left(\log D'_{\mathsf{Z}'}, \mathsf{f}'\right)$ to $\mathsf{Fr}_* \, \Omega^a_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}}, \mathsf{f})$. Once this is checked, the fact that $\mathsf{for}_{\mathfrak{Fr}}$ is a quasi-isomorphism inducing $\boldsymbol{\gamma}$ on all cohomology sheaves follows tautologically. To show that

$$\mathsf{for}^a_{\mathfrak{Fr}}\left(\Omega^a_{\mathsf{Z}'/\Bbbk}\left(\log D'_{\mathsf{Z}'}, \mathsf{f}'\right)\right) \subset \mathsf{Fr}_* \, \Omega^a_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}}, \mathsf{f}),$$

we argue locally on Z. By (2.3.1) we know that $\Omega^a_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}}, \mathsf{f})$ is a locally free sheaf which near $D^{\mathsf{v}}_{\mathsf{Z}}$ is equal to the sum of $d\log\mathsf{f}\wedge\Omega^{a-1}_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}})$ and $(1/\mathsf{f})\Omega^a_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}})$ inside $\Omega^a_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}})$.

Choosing an appropriate Zariski local etale map to an affine space we obtain local coordinates $\boldsymbol{z} = (z_1, \dots z_n)$ as in the proof of Lemma 2.12. In particular, we have that the divisor D_{Z} is given by the union of $z_i = 0$ for $i = 1, \dots, k+l$, and $\mathsf{f} = 1/(z_1 \cdot \ldots \cdot z_k)$. In these coordinates the lifted Frobenius $\mathfrak{Fr}$ has the form $\mathfrak{Fr}^*(\Phi^* z_i) = z_i^p + p \cdot z_i^p v_i(\boldsymbol{z})$ for $i = 1, \dots, k+l$ and $\mathfrak{Fr}^*(\Phi^* z_i) = z_i^p + p \cdot v_i(\boldsymbol{z})$ for $i = k+l+1, \dots, n$. Furthermore, we have $v_1(\boldsymbol{z}) + \cdots + v_k(\boldsymbol{z}) = 0$.

From these formulas we now see that for the forms in $\Omega^a_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}}, \mathsf{f})$ of type $d\log\mathsf{f}\wedge\Omega^{a-1}_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}})$ the pullback via $(1/p^a)\,\mathfrak{Fr}^*$ modulo p does not depend on the choice of $v_1, \dots, v_k$, and is, therefore, again a log form multiplied by $d\log\mathsf{f}$. For the forms of second type, i.e., forms in $(1/\mathsf{f})\Omega^a_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}}, \mathsf{f}) \subset \Omega^a_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}}, \mathsf{f})$ we note that these forms belong to the $\mathcal{O}$-module generated by products over all i of either z_i or dz_i. Hence the $(1/p^a)\,\mathfrak{Fr}^*$ pullback of such form modulo p will belong to the $\mathcal{O}$ module generated by products of either z_i^p or $z_i^{p-1}dz_i$ and is, therefore, again of second type.

In the second step one notes that locally in the Zariski topology we can always choose etale maps to an affine space and then use local coordinates as above to construct a lift of the relative Frobenius over $W_2(\Bbbk)$. Thus we have to analyze the relation between the formality isomorphisms associated to different local liftings of the relative Frobenius. Following [**DI87**] we want to show that for any two liftings $\mathfrak{Fr}_1 : \mathfrak{Z}_1 \to \mathfrak{Z}'$ and $\mathfrak{Fr}_2 : \mathfrak{Z}_2 \to \mathfrak{Z}'$ of Fr, we can find a canonical map of sheaves

$$h(\mathfrak{Fr}_1, \mathfrak{Fr}_2) : \Omega^1_{\mathsf{Z}'/\Bbbk}\left(\log D'_{\mathsf{Z}'}, \mathsf{f}'\right) \to \mathsf{Fr}_* \, \mathcal{O}_{\mathsf{Z}},$$

so that $\mathsf{for}^1_{\mathfrak{Fr}_1} - \mathsf{for}^1_{\mathfrak{Fr}_2} = dh(\mathfrak{Fr}_1, \mathfrak{Fr}_2)$. Furthermore, for a third lifting $\mathfrak{Fr}_3 : \mathfrak{Z}_3 \to \mathfrak{Z}'$ these maps should satisfy the cocycle condition $h(\mathfrak{Fr}_1, \mathfrak{Fr}_2) + h(\mathfrak{Fr}_2, \mathfrak{Fr}_3) = h(\mathfrak{Fr}_1, \mathfrak{Fr}_3)$.

To construct the maps h we repeat verbatim the reasoning in [**DI87**, **EV92**]. To show that the corresponding $dh(\mathfrak{Fr}_1, \mathfrak{Fr}_2)$ belongs again to $\mathsf{Fr}_* \, \Omega^1_{\mathsf{Z}'/\Bbbk}\left(\log D'_{\mathsf{Z}'}, \mathsf{f}'\right)$ one notes that by construction $h(\mathfrak{Fr}_1, \mathfrak{Fr}_2)$ is given by substitutions with vector fields of the form $\sum_{i=1}^{k+l} u_i(\boldsymbol{z}) \cdot z_i^p \cdot (\partial/\partial z_i) +$

$\sum_{i=k+l+1}^{n} u_i(\boldsymbol{z}) \cdot (\partial/\partial z_i)$ satisfying $\sum_{i=1}^{k} u_i = 0$. Such substitution vanish for forms divisible by $d\log \mathsf{f}$. For forms $\alpha = (1/\mathsf{f}) \cdot \beta$ with β being log form, the substitution of such vector field in β is again a log form, hence its pullback is a log form. Also note that the pullback of $1/\mathfrak{f}'$ is $1/\mathfrak{f}^p$ which is divisible by $1/\mathfrak{f}$. Thus we again get a form of the second type. The same argument should work for a triple of lifts. The key point here is that the forms $\Omega^a_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}}, \mathsf{f})$ are closed under contractions with vector fields preserving f.

From this point on the argument proceeds exactly as in [**DI87**]. First we cover Z by Zariski open sets U_i on which we can choose Frobenius lifts $\mathfrak{Fr}_i : \mathfrak{U}_i \to \mathfrak{U}'_i$ as above. Then on overlaps we use the maps $h(\mathfrak{Fr}_i, \mathfrak{Fr}_j)$ on overlaps to glue the formality morphisms $\mathsf{for}^1_{\mathfrak{Fr}_i}$ into a morphism in the derived category

$$\mathsf{for}^1_{\mathfrak{Z}} : \Omega^1_{\mathsf{Z}'/\Bbbk}(\log D'_{\mathsf{Z}'}, \mathsf{f}')[-1] \to \mathsf{Fr}_* \, \Omega^\bullet_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}}, \mathsf{f}),$$

which induces $\boldsymbol{\gamma}$ on $\mathcal{H}^1$.

In the last step we use the condition $n = \dim \mathsf{Z} < p$ and multiplicative structure on the de Rham complex to define a map

$$\mathsf{for}^a_{\mathfrak{Z}} : \Omega^a_{\mathsf{Z}'/\Bbbk}(\log D'_{\mathsf{Z}'}, \mathsf{f}')[-1] \to \mathsf{Fr}_* \, \Omega^\bullet_{\mathsf{Z}/\Bbbk}(\log D_{\mathsf{Z}}, \mathsf{f}),$$

by composing $\left(\mathsf{for}^1_{\mathfrak{Z}}\right)^{\otimes a}$ with the anti-symmetrization map $\Omega^a_{\mathsf{Z}'/\Bbbk}(\log D'_{\mathsf{Z}'}) \to \left(\Omega^1_{\mathsf{Z}'/\Bbbk}(\log D'_{\mathsf{Z}'})\right)^{\otimes a}$ given by

$$\alpha_1 \otimes \cdots \otimes \alpha_a \mapsto \frac{1}{a!} \sum_{\sigma \in S_a} \operatorname{sgn}(\sigma) \alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(a)}.$$

This completes the proof of the lemma. q.e.d.

Remark 2.20. Morihiko Saito recently found [**Sai13**] a different analytic proof of this lemma. Saito's argument uses Hodge theory with degenerating coefficients and takes place entirely in characteristic zero.

Lemma 2.19 implies that the dimension of the hypercohohomology of the complex $(\Omega^\bullet_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}), c_1 d + c_2 d\mathsf{f}\wedge\ )$ is constant on the line $\{c_2 = 0\} \subset \mathbb{C}^2$. Next we will show that this hypercohomology is also constant along the line $\{c_1 = c_2\} \subset \mathbb{C}^2$. First we have the following topological statement (see also [**ESY13**, Appendix C] where a more general statement allowing multiplicities is proven):

Lemma 2.21. *Consider* $\mathsf{w} : Y \to \mathbb{C}$. *Write* $Y_{-\infty}$ *for the fiber* $\mathsf{w}^{-1}(z)$ *over* $z \in \mathbb{C}$ *with* $\operatorname{Re} z \ll 0$. *Then for every* $a \geq 0$ *we have*

$$\dim_{\mathbb{C}} \mathbb{H}^a \left(\mathsf{Z}, (\Omega^\bullet_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}), d)\right) = \dim_{\mathbb{C}} H^a(Y, Y_{-\infty}; \mathbb{C}).$$

Proof. Before we address the statement of the lemma, it is instructive to look at the analogous statement in the classical Hodge theory of smooth open varieties. Let $\jmath_Y : Y \hookrightarrow \mathsf{Z}$ be the natural inclusion viewed as a continuous map in the analytic topology. The pushforward $R\jmath_{Y*}\mathbb{C}_Y$ is a constructible complex of sheaves of $\mathbb{C}$-vector spaces on Z. Recall that the log de Rham complex $(\Omega^\bullet_{\mathsf{Z}}(\log D_{\mathsf{Z}}), d)$ is naturally quasi-isomorphic to this constructible complex. Indeed, a direct local calculation [**Gri69**, **Del71**] shows that the natural map of complexes $(\Omega^\bullet_{\mathsf{Z}}(\log D_{\mathsf{Z}}), d) \to \jmath_{Y*}\mathcal{A}^\bullet_Y$ is a quasi-isomorphism. Composing this map with the augmentation quasi-isomorphism $\jmath_{Y*}\mathcal{A}^\bullet_Y \to R\jmath_{Y*}\mathbb{C}_Y$ gives an identification of $(\Omega^\bullet_{\mathsf{Z}}(\log D_{\mathsf{Z}}), d)$ and $R\jmath_{Y*}\mathbb{C}_Y$ in $D^b(\mathbb{C}_{\mathsf{Z}})$. Since $R\jmath_{Y*}\mathbb{C}_Y$ computes the Betti cohomology of the open variety Y this yields the classical statement that $\mathbb{H}^a(\mathsf{Z}, (\Omega^\bullet_{\mathsf{Z}}(\log D_{\mathsf{Z}}), d)) \cong H^a(Y, \mathbb{C})$.

The idea is to modify this reasoning to take into account relative cohomology and f-adapted forms. To that end consider the real oriented blow-up $\varepsilon : \widehat{\mathsf{Z}} \to \mathsf{Z}$ of Z along the reduced normal crossing divisor D_{Z}, and the real oriented blow-up $\pi : \widehat{\mathbb{P}}^1 \to \mathbb{P}^1$ of $\mathbb{P}^1$ at $\infty \in \mathbb{P}^1$. The morphism $\mathsf{f} : \mathsf{Z} \to \mathbb{P}^1$ lifts naturally to a real semi-algebraic map $\widehat{\mathsf{f}} : \widehat{\mathsf{Z}} \to \widehat{\mathbb{P}}^1$. The spaces $\widehat{\mathbb{P}}^1$ and $\widehat{\mathsf{Z}}$ are manifolds with boundary, and $\partial\widehat{\mathbb{P}}^1 = \pi^{-1}(\infty) \cong S^1$ and $\partial\widehat{\mathsf{Z}} = \varepsilon^{-1}(D_{\mathsf{Z}}) \supset \varepsilon^{-1}(D^{\mathsf{v}}_{\mathsf{Z}}) = \widehat{\mathsf{f}}^{-1}\left(\partial\widehat{\mathbb{P}}^1\right)$.

The boundary circle $\partial\widehat{\mathbb{P}}^1 = \pi^{-1}(\infty) \cong S^1$ is the circle of radial directions at $\infty \in \mathbb{P}^1$. If as before z denotes the affine coordinate on $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$, then this circle is parametrized by $\arg(1/z)$. Choose a point $\theta_0 \in \partial\widehat{\mathbb{P}}^1$ for which $\mathrm{Re}(z) \geq 0$ and let $\widehat{\mathsf{Z}}_{\theta_0} = \widehat{\mathsf{f}}^{-1}(\theta_0)$.

Consider the complex $\left(\mathcal{A}^\bullet_{\widehat{\mathsf{Z}},\widehat{\mathsf{Z}}_{\theta_0}}(\log D_{\mathsf{Z}}), d\right)$ of C^∞ logarithmic forms on $\widehat{\mathsf{Z}}$ that vanish along $\widehat{\mathsf{Z}}_{\theta_0}$. Let $\left(\mathcal{A}^\bullet_{\widehat{\mathsf{Z}}_{\theta_0}}(\log D_{\mathsf{Z}}), d\right)$ denote the cone (= quotient complex) of the natural map from $\left(\mathcal{A}^\bullet_{\widehat{\mathsf{Z}},\widehat{\mathsf{Z}}_{\theta_0}}(\log D_{\mathsf{Z}}), d\right)$ to $\left(\mathcal{A}^\bullet_{\widehat{\mathsf{Z}}}(\log D_{\mathsf{Z}}), d\right)$. Now from the explicit local description (see Lemma 2.12) of $(\Omega^\bullet_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}), d)$ one checks immediately that near the boundary $\partial\widehat{\mathsf{Z}}$ the quotient complex $(\Omega^\bullet_{\mathsf{Z}}(\log D)/\Omega^\bullet_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}), d)$ maps to $\varepsilon_*\left(\mathcal{A}^\bullet_{\widehat{\mathsf{Z}}_{\theta_0}}(\log D_{\mathsf{Z}}), d\right)$ and that the map is a quasi-isomorphism. This implies that the natural map from $\Omega^\bullet_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})\, \varepsilon_*\left(\mathcal{A}^\bullet_{\widehat{\mathsf{Z}},\widehat{\mathsf{Z}}_{\theta_0}}(\log D_{\mathsf{Z}}), d\right)$ is a quasi-isomorphism and thus gives the equality of dimensions claimed in the lemma.

A detailed writeup of this argument and an explicit check of the fact that the map of quotient complexes is a quasi-isomorphism can be found in [**ESY13**, Appendix C, Step 2]. Instead of repeating this calculation here we will give an alternative proof of the lemma which is be of independent interest.

To simplify the discussion let us first assume that $D_{\mathsf{Z}}^{\mathsf{h}}$ is empty. Consider the de Rham cohomology of the pair $\left(Y, \mathsf{w}^{-1}(\rho)\right)$ where ρ is real and $\rho \ll 0$. Using de Rham's theorem and the Gauss–Manin parallel transport along the ray $\rho \in \mathbb{R}_{<0}$ we can identify the cohomology $H^a\left(Y, Y_{-\infty}; \mathbb{C}\right)$ with the limit of $H^a_{DR}\left(Y, \mathsf{w}^{-1}(\rho); \mathbb{C}\right)$ as $\rho \to -\infty$.

Therefore, the statement of the lemma reduces to understanding the limit $\lim_{\rho\to-\infty} H^a_{DR}\left(Y, \mathsf{w}^{-1}(\rho); \mathbb{C}\right)$ in terms of the complex of f-adapted logarithmic forms. The relative de Rham cohomology $H^a_{DR}\left(Y, \mathsf{w}^{-1}(\rho); \mathbb{C}\right)$ is computed by the complex $\left(\Omega^\bullet_{\mathsf{Z}}\left(\log D_{\mathsf{Z}}, \operatorname{rel} \mathsf{f}^{-1}(\rho)\right), d\right)$ of holomorphic forms on Z that have logarithmic poles along $D_{\mathsf{Z}} = \mathsf{f}^{-1}(\infty)$ and vanish along the divisor $\mathsf{f}^{-1}(\rho) = \mathsf{w}^{-1}(\rho)$. We now have the following:

Claim 2.22. (a) *As $\rho \to -\infty$ the complex*

$$\left(\Omega^\bullet_{\mathsf{Z}}\left(\log D_{\mathsf{Z}}, \operatorname{rel} \mathsf{f}^{-1}(\rho)\right), d\right)$$

has a well defined limit, namely the complex $\left(\Omega^\bullet_{\mathsf{Z}}\left(\log D_{\mathsf{Z}}, \mathsf{f}\right), d\right)$ of f-adapted logarithmic forms on Z.

(b) *The Gauss–Manin parallel transport along the ray $\rho \in \mathbb{R}_{<0}$ is well defined at the limit $\rho \to -\infty$ and identifies $H^a_{DR}\left(Y, \mathsf{w}^{-1}(\rho); \mathbb{C}\right)$ with $\mathbb{H}^a\left(\mathsf{Z}, \left(\Omega^\bullet_{\mathsf{Z}}\left(\log D_{\mathsf{Z}}, \mathsf{f}\right), d\right)\right)$.*

Proof. The statement is local on Z and is obvious at points of the open set $Y = \mathsf{Z} - D_{\mathsf{Z}}$. Suppose next $p \in D_{\mathsf{Z}}^{\mathsf{v}} = D_{\mathsf{Z}} \subset \mathsf{Z}$. As in the proof of Lemma 2.12 we can choose a local coordinate system $z_1, \ldots, z_n$ centered at p so that D_{Z} is given by the equation $\prod_{i=1}^k z_i = 0$ and $\mathsf{f} = \prod_{i=1}^k z_i^{-1}$. Now as in the proof of Lemma 2.12 we write $W \subset \Omega^1_{\mathsf{Z}}\left(\log D_{\mathsf{Z}}\right)$ for the sub $\mathcal{O}_{\mathsf{Z}}$-module spanned by $d\log z_1, \ldots, d\log z_{k-1}$, and $R \subset \Omega^1_{\mathsf{Z}}\left(\log D_{\mathsf{Z}}\right)$ for sub $\mathcal{O}_{\mathsf{Z}}$-module spanned by $dz_{k+1}, \ldots, dz_n$.

In these terms we have

$$\Omega^a_{\mathsf{Z}}\left(\log D_{\mathsf{Z}}\right) = \bigoplus_{p=0}^{a} \left[\wedge^p W \oplus d\log z_k \wedge \left(\wedge^{p-1} W\right)\right] \bigotimes \wedge^{a-p} R.$$

Write $\boldsymbol{\epsilon} = 1/\rho$, and let $Y_{\boldsymbol{\epsilon}} = \mathsf{f}^{-1}(\boldsymbol{\epsilon})$. Then for $\boldsymbol{\epsilon}$ close to 0 we can use $z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n$ as coordinates along the divisor $Y_{\boldsymbol{\epsilon}}$. In particular, the sheaf of holomorphic forms $\Omega^1_{Y_{\boldsymbol{\epsilon}}}$ is the $\mathcal{O}_{Y_{\boldsymbol{\epsilon}}}$-span of the forms $dz_1, \ldots, dz_{k-1}, dz_{k+1}, \ldots, dz_n$, and so

$$\Omega^a_{Y_{\boldsymbol{\epsilon}}} = \bigoplus_{p=0}^{a} \wedge^p W_{|Y_{\boldsymbol{\epsilon}}} \bigotimes \wedge^{a-p} R_{|Y_{\boldsymbol{\epsilon}}}.$$

From here we get

$$\begin{aligned}\Omega^a_{\mathsf{Z}}\left(\log D_{\mathsf{Z}}, \operatorname{rel} Y_{\boldsymbol{\epsilon}}\right) &= \ker\left[\Omega^a_{\mathsf{Z}}\left(\log D_{\mathsf{Z}}\right) \to \imath_{Y_{\boldsymbol{\epsilon}}*}\Omega^a_{Y_{\boldsymbol{\epsilon}}}\right] \\ &= \bigoplus_{p=0}^{a} \left[\left(z_1 \cdots z_k - \boldsymbol{\epsilon}\right) \wedge^p W \oplus d\log z_k \wedge \wedge^{p-1} W\right] \bigotimes \wedge^{a-p} R\end{aligned}$$

$$= \bigoplus_{p=0}^{a} \left[(z_1 \cdots z_k - \boldsymbol{\epsilon}) \wedge^p W + d \log \mathsf{f} \wedge \left(\wedge^{p-1} W \right) \right] \bigotimes \wedge^{a-p} R,$$

and so when $\boldsymbol{\epsilon} \to 0$ this sheaf specializes to the sheaf of f-adapted logarithmic forms (see (2.3.1))

$$\Omega^a_{\mathsf{Z}} \left(\log D_{\mathsf{Z}}, \mathsf{f} \right) = \bigoplus_{p=0}^{a} \left[z_1 \cdots z_k \cdot \wedge^p W \oplus d \log \mathsf{f} \wedge \left(\wedge^{p-1} W \right) \right] \bigotimes \wedge^{a-p} R.$$

This shows that as $\boldsymbol{\epsilon} \to 0$ the complex $\left(\Omega^\bullet_{\mathsf{Z}} \left(\log D_{\mathsf{Z}}, \mathrm{rel}\; Y_{\boldsymbol{\epsilon}} \right), d \right)$ will converge to the complex $\left(\Omega^\bullet_{\mathsf{Z}} \left(\log D_{\mathsf{Z}}, \mathsf{f} \right), d \right)$. In other words we have a family of complexes on Z parametrized by a small complex number $\boldsymbol{\epsilon}$, where $\left(\Omega^\bullet_{\mathsf{Z}} \left(\log D_{\mathsf{Z}}, \mathrm{rel}\; Y_{\boldsymbol{\epsilon}} \right), d \right)$ is the complex corresponding to $\boldsymbol{\epsilon} \neq 0$, while at $\boldsymbol{\epsilon} = 0$ we have the complex $\left(\Omega^\bullet_{\mathsf{Z}} \left(\log D_{\mathsf{Z}}, \mathsf{f} \right), d \right)$.

More invariantly, let $\boldsymbol{\Delta} \subset \mathbb{P}^1$ be a small disk centered at ∞ with coordinate $\boldsymbol{\epsilon}$. Let $\boldsymbol{\mathcal{Z}} := \mathsf{Z} \times \boldsymbol{\Delta}$, and let $p : \boldsymbol{\mathcal{Z}} \to \boldsymbol{\Delta}$ be the natural projection. The proper family $p : \boldsymbol{\mathcal{Z}} \to \boldsymbol{\Delta}$ is equipped with two relative divisors

$$\begin{aligned} D_{\boldsymbol{\mathcal{Z}}} &:= D_{\mathsf{Z}} \times \boldsymbol{\Delta}, \\ \boldsymbol{\Gamma} &:= (p \times \mathsf{f})^{-1} \left(\mathsf{graph} \left(\boldsymbol{\Delta} \hookrightarrow \mathbb{P}^1 \right) \right). \end{aligned}$$

By construction $\boldsymbol{\Gamma}$ is smooth, $D_{\boldsymbol{\mathcal{Z}}}$ has strict normal crossings, both $\boldsymbol{\Gamma}$ and $D_{\boldsymbol{\mathcal{Z}}}$ are flat over $\boldsymbol{\Delta}$, and the union of $\boldsymbol{\Gamma} \cup D_{\boldsymbol{\mathcal{Z}}}$ also has strict normal crossings. Write $D_{\boldsymbol{\Gamma}}$ for the normal crossing divisor in $\boldsymbol{\Gamma}$ given by $D_{\boldsymbol{\Gamma}} = D_{\boldsymbol{\mathcal{Z}}} \cap \Gamma$.

Consider now the sheaves of relative meromorphic forms, i.e., forms along the fibers of p, having logarithmic poles along $D_{\boldsymbol{\mathcal{Z}}}$ and vanishing along Γ:

$$\Omega^a_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}} \left(\log D_{\boldsymbol{\mathcal{Z}}}, \mathrm{rel}\; \boldsymbol{\Gamma} \right) := \ker \left[\Omega^a_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}} \left(\log D_{\boldsymbol{\mathcal{Z}}} \right) \to \imath_{\boldsymbol{\Gamma}*} \Omega^a_{\boldsymbol{\Gamma}/\boldsymbol{\Delta}} \left(\log D_{\boldsymbol{\Gamma}} \right) \right].$$

By definition these are locally free sheaves of (certain) relative logarithmic forms along the fibers of $p : \boldsymbol{\mathcal{Z}} \to \boldsymbol{\Delta}$, and the graded subsheaf $\Omega^\bullet_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}} \left(\log D_{\boldsymbol{\mathcal{Z}}}, \mathrm{rel}\; \boldsymbol{\Gamma} \right) \subset \Omega^\bullet_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}} \left(\log D_{\boldsymbol{\mathcal{Z}}} \right)$ is clearly preserved by the relative de Rham differential. The calculation in local coordinates above shows that the complex

$$\mathsf{E}^\bullet_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}} := \left(\Omega^\bullet_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}} \left(\log D_{\boldsymbol{\mathcal{Z}}}, \mathrm{rel}\; \boldsymbol{\Gamma} \right),\, d \right)$$

interpolates between relative logarithmic forms vanishing on $Y_{\boldsymbol{\epsilon}}$ and f-adapted relative logarithmic forms. In other words we have

$$\left(\mathsf{E}^\bullet_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}} \right)_{|\mathsf{Z} \times \{\boldsymbol{\epsilon} \neq 0\}} = \left(\Omega^\bullet_{\mathsf{Z}} \left(\log D_{\mathsf{Z}}, \mathrm{rel}\; Y_{\boldsymbol{\epsilon}} \right),\, d \right);$$

$$\left(\mathsf{E}^\bullet_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}} \right)_{|\mathsf{Z} \times \{0\}} = \left(\Omega^\bullet_{\mathsf{Z}} \left(\log D_{\mathsf{Z}}, \mathsf{f} \right),\, d \right).$$

This proves the first part of the claim.

The statement about the Gauss–Manin parallel transport follows form the homological description [**KO68**] of the Gauss–Manin connection. To spell this out one needs to describe the local system of relative cohomology via differential forms. To fix notation, we will use a superscript $(\bullet)^{\times}$ to indicate the removal of the fiber over $\boldsymbol{\epsilon} = 0$ in the various geometric and sheaf-theoretic objects we are dealing with. Thus we will write $\boldsymbol{\Delta}^{\times} = \boldsymbol{\Delta} - \{0\}$, $\boldsymbol{\mathcal{Z}}^{\times} = \boldsymbol{\mathcal{Z}} - p^{-1}(0)$, $D_{\boldsymbol{\mathcal{Z}}^{\times}} = \mathsf{Z} \times \boldsymbol{\Delta}^{\times}$, $\boldsymbol{\Gamma}^{\times} = \boldsymbol{\Gamma} \cap \boldsymbol{\mathcal{Z}}^{\times}$, and

$$\mathsf{E}^{\bullet}_{\boldsymbol{\mathcal{Z}}^{\times}/\boldsymbol{\Delta}^{\times}} = \left(\Omega^{\bullet}_{\boldsymbol{\mathcal{Z}}^{\times}/\boldsymbol{\Delta}^{\times}}\left(\log D_{\boldsymbol{\mathcal{Z}}^{\times}}, \mathrm{rel}\ \boldsymbol{\Gamma}^{\times}\right), d\right).$$

Also we set $\boldsymbol{\mathcal{Y}} = Y \times \boldsymbol{\Delta} = \boldsymbol{\mathcal{Z}} - D_{\boldsymbol{\mathcal{Z}}}$ and $\boldsymbol{\mathcal{Y}}^{\times} = Y \times \boldsymbol{\Delta}^{\times}$.

Let $\mathscr{E}^a_B$ denote the local system of $\mathbb{C}$-vector spaces on $\boldsymbol{\Delta}^{\times}$ whose fiber over $\boldsymbol{\epsilon} \in \boldsymbol{\Delta}^{\times}$ is the relative Betti cohomology $H^a(Y, Y_{\boldsymbol{\epsilon}}; \mathbb{C})$. The underlying coherent sheaf $\mathscr{E}^a_B \otimes_{\mathbb{C}} \mathcal{O}_{\boldsymbol{\Delta}^{\times}}$ can be identified with the sheaf $\mathcal{H}^a_{DR}\left(\boldsymbol{\mathcal{Y}}^{\times}/\boldsymbol{\Delta}^{\times}, \boldsymbol{\Gamma}^{\times}/\boldsymbol{\Delta}^{\times}; \mathbb{C}\right)$ of relative de Rham cohomology and is thus computed as the hyperderived image

$$\mathscr{E}^a_B \otimes_{\mathbb{C}} \mathcal{O}_{\boldsymbol{\Delta}^{\times}} \ \cong\ \mathcal{H}^a_{DR}\left(\boldsymbol{\mathcal{Y}}^{\times}/\boldsymbol{\Delta}^{\times}, \boldsymbol{\Gamma}^{\times}/\boldsymbol{\Delta}^{\times}; \mathbb{C}\right) \ =\ \mathbb{R}^a p_* \mathsf{E}^{\bullet}_{\boldsymbol{\mathcal{Z}}^{\times}/\boldsymbol{\Delta}^{\times}}.$$

Note that the hyper-derived image $\mathbb{R}^a p^{\times}_* \mathsf{E}^{\bullet}_{\boldsymbol{\mathcal{Z}}^{\times}/\boldsymbol{\Delta}^{\times}}$ is naturally an $\mathcal{O}_{\boldsymbol{\Delta}^{\times}}$-module since the de Rham differential on relative forms is linear over $p^{-1}\mathcal{O}_{\boldsymbol{\Delta}^{\times}}$. In these terms the Gauss–Manin connection is given by a $\mathbb{C}$-linear map of sheaves

$$\nabla^{\mathrm{GM}} : \mathbb{R}^a p_* \mathsf{E}^{\bullet}_{\boldsymbol{\mathcal{Z}}^{\times}/\boldsymbol{\Delta}^{\times}} \longrightarrow \mathbb{R}^a p_* \mathsf{E}^{\bullet}_{\boldsymbol{\mathcal{Z}}^{\times}/\boldsymbol{\Delta}^{\times}} \otimes_{\mathcal{O}_{\boldsymbol{\Delta}^{\times}}} \Omega^1_{\boldsymbol{\Delta}^{\times}},$$

satisfying the Leibnitz rule. The analysis of [**KO68**] applies verbatim to this setting and identifies ∇^{GM} with the connecting homomorphism in the long exact sequence of hyperderived direct images associated with the short exact sequence of complexes

$$\begin{tikzcd} 0 \arrow[d] \\ \mathsf{E}^{\bullet}_{\boldsymbol{\mathcal{Z}}^{\times}/\boldsymbol{\Delta}^{\times}}[-1] \underset{p^{-1}\mathcal{O}_{\boldsymbol{\Delta}^{\times}}}{\otimes} p^{-1}\Omega^1_{\boldsymbol{\Delta}^{\times}} \arrow[d] \\ \left(\Omega^{\bullet}_{\boldsymbol{\mathcal{Z}}^{\times}}\left(\log D_{\boldsymbol{\mathcal{Z}}^{\times}}, \mathrm{rel}\ \boldsymbol{\Gamma}^{\times}\right), d\right) \arrow[d] \\ \mathsf{E}^{\bullet}_{\boldsymbol{\mathcal{Z}}^{\times}/\boldsymbol{\Delta}^{\times}} \arrow[d] \\ 0 \end{tikzcd} \tag{2.4.5}$$

In order to check that the parallel transport with respect to ∇^{GM} has a well defined limit when $\boldsymbol{\epsilon} \to 0$ it suffices to show that the complex $\left(\Omega^{\bullet}_{\boldsymbol{\mathcal{Z}}^{\times}}\left(\log D_{\boldsymbol{\mathcal{Z}}^{\times}}, \mathrm{rel}\ \boldsymbol{\Gamma}^{\times}\right), d\right)$ extends to a well defined subcomplex $\mathsf{E}^{\bullet}_{\boldsymbol{\mathcal{Z}}}$ in $(\Omega^{\bullet}_{\boldsymbol{\mathcal{Z}}}(\log D_{\boldsymbol{\mathcal{Z}}}), d)$, so that $\mathsf{E}^{\bullet}_{\boldsymbol{\mathcal{Z}}}$ is defined on all of $\boldsymbol{\mathcal{Z}}$ and fits in a short exact sequence of complexes

$$0 \longrightarrow \mathsf{E}^{\bullet}_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}}[-1] \underset{p^{-1}\mathcal{O}_{\boldsymbol{\Delta}}}{\otimes} p^{-1}\Omega^1_{\boldsymbol{\Delta}} \longrightarrow \mathsf{E}^{\bullet}_{\boldsymbol{\mathcal{Z}}} \longrightarrow \mathsf{E}^{\bullet}_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}} \longrightarrow 0 \tag{2.4.6}$$

extending (2.4.5) to all of $\boldsymbol{\mathcal{Z}}$.

The naive guess of taking $\mathsf{E}^{\bullet}_{\boldsymbol{\mathcal{Z}}}$ to be $(\Omega^{\bullet}_{\boldsymbol{\mathcal{Z}}}(\log D_{\boldsymbol{\mathcal{Z}}}, \mathrm{rel}\ \boldsymbol{\Gamma}), d)$ will not work since the natural maps $\Omega^{a}_{\boldsymbol{\mathcal{Z}}}(\log D_{\boldsymbol{\mathcal{Z}}}, \mathrm{rel}\ \boldsymbol{\Gamma}) \to \Omega^{a}_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}}(\log D_{\boldsymbol{\mathcal{Z}}}, \mathrm{rel}\ \boldsymbol{\Gamma})$ are not surjective. Because of this we will have to work with the logarithmic de Rham complexes directly. Consider the short exact sequence

$$\begin{array}{c} 0 \\ \downarrow \\ \Omega^{\bullet}_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}}(\log D_{\boldsymbol{\mathcal{Z}}})[-1] \otimes_{\mathcal{O}_{\boldsymbol{\mathcal{Z}}}} p^{*}\Omega^1_{\boldsymbol{\Delta}} \\ \downarrow \\ \Omega^{\bullet}_{\boldsymbol{\mathcal{Z}}}(\log D_{\boldsymbol{\mathcal{Z}}}) \\ \downarrow \\ \Omega^{\bullet}_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}}(\log D_{\boldsymbol{\mathcal{Z}}}) \\ \downarrow \\ 0 \end{array} \tag{2.4.7}$$

of logarithmic de Rham complexes[3] on $\boldsymbol{\mathcal{Z}}$.

View (2.4.7) as a morphism

$$\xi_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}} : \Omega^{\bullet}_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}}(\log D_{\boldsymbol{\mathcal{Z}}}) \to \Omega^{\bullet}_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}}(\log D_{\boldsymbol{\mathcal{Z}}}) \otimes_{\mathcal{O}_{\boldsymbol{\mathcal{Z}}}} p^{*}\Omega^1_{\boldsymbol{\Delta}},$$

in the derived category of sheaves of $\mathbb{C}$-vector spaces on $\boldsymbol{\mathcal{Z}}$. Write $\imath_{\boldsymbol{\Gamma}} : \boldsymbol{\Gamma} \hookrightarrow \boldsymbol{\mathcal{Z}}$ for the inclusion of the divisor $\boldsymbol{\Gamma}$ in $\boldsymbol{\mathcal{Z}}$, and let i and q denote the maps in the short exact sequence of complexes defining $\mathsf{E}^{\bullet}_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}}$:

$$0 \longrightarrow \mathsf{E}^{\bullet}_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}} \xrightarrow{i} \Omega^{\bullet}_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}}(\log D_{\boldsymbol{\mathcal{Z}}}) \xrightarrow{q} \imath_{\boldsymbol{\Gamma}*}\Omega^{\bullet}_{\boldsymbol{\Gamma}/\boldsymbol{\Delta}}(\log D_{\boldsymbol{\Gamma}}) \longrightarrow 0.$$

[3]Here all terms are equipped with the obvious absolute or relative de Rham differentials, so we have omitted them from the notation. The differential in the first term is defined via the identification $\Omega^{\bullet}_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}}(\log D_{\boldsymbol{\mathcal{Z}}})[-1] \otimes_{\mathcal{O}_{\boldsymbol{\mathcal{Z}}}} p^{*}\Omega^1_{\boldsymbol{\Delta}} \cong \Omega^{\bullet}_{\boldsymbol{\mathcal{Z}}/\boldsymbol{\Delta}}(\log D_{\boldsymbol{\mathcal{Z}}})[-1] \otimes_{p^{-1}\mathcal{O}_{\boldsymbol{\Delta}}} p^{-1}\Omega^1_{\boldsymbol{\Delta}}$ and the fact that de Rham differential on relative forms along the fibers of $p : \boldsymbol{\mathcal{Z}} \to \boldsymbol{\Delta}$ is $p^{-1}\mathcal{O}_{\boldsymbol{\Delta}}$-linear.

We claim that the composition

$$(q \otimes 1) \circ \xi_{\mathcal{Z}/\mathbf{\Delta}} \circ i : \mathsf{E}^{\bullet}_{\mathcal{Z}/\mathbf{\Delta}} \to \imath_{\mathbf{\Gamma}*}\left[\Omega^{\bullet}_{\mathbf{\Gamma}/\mathbf{\Delta}}(\log D_{\mathbf{\Gamma}}) \otimes_{\mathcal{O}_{\Gamma}} p^*\Omega^1_{\mathbf{\Delta}}\right]$$

is the zero morphism in $D^b(\mathbb{C}_{\mathcal{Z}})$. This follows immediately by noting that $\xi_{\mathcal{Z}/\mathbf{\Delta}}$ fits in a commutative diagram in $D^b(\mathbb{C}_{\mathcal{Z}})$:

$$\begin{tikzcd}
\mathsf{E}^{\bullet}_{\mathcal{Z}/\mathbf{\Delta}} \arrow[d, "i"] & \mathsf{E}^{\bullet}_{\mathcal{Z}/\mathbf{\Delta}} \otimes_{\mathcal{O}_{\mathcal{Z}}} p^*\Omega^1_{\mathbf{\Delta}} \arrow[d, "i\otimes 1"] \\
\Omega^{\bullet}_{\mathcal{Z}/\mathbf{\Delta}}(\log D_{\mathcal{Z}}) \arrow[r, "\xi_{\mathcal{Z}/\mathbf{\Delta}}"] \arrow[d, "q"] & \Omega^{\bullet}_{\mathcal{Z}/\mathbf{\Delta}}(\log D_{\mathcal{Z}}) \otimes_{\mathcal{O}_{\mathcal{Z}}} p^*\Omega^1_{\mathbf{\Delta}} \arrow[d, "q\otimes 1"] \\
\imath_{\mathbf{\Gamma}*}\Omega^{\bullet}_{\mathbf{\Gamma}/\mathbf{\Delta}}(\log D_{\mathbf{\Gamma}}) \arrow[r, "\imath_{\mathbf{\Gamma}*}\xi_{\mathbf{\Gamma}/\mathbf{\Delta}}"] & \imath_{\mathbf{\Gamma}*}\left[\Omega^{\bullet}_{\mathbf{\Gamma}/\mathbf{\Delta}}(\log D_{\mathbf{\Gamma}}) \otimes_{\mathcal{O}_{\mathbf{\Gamma}}} p^*\Omega^1_{\mathbf{\Delta}}\right]
\end{tikzcd} \tag{2.4.8}$$

in which the columns are parts of exact triangles and

$$\xi_{\mathbf{\Gamma}/\mathbf{\Delta}} : \Omega^{\bullet}_{\mathbf{\Gamma}/\mathbf{\Delta}}(\log D_{\mathbf{\Gamma}}) \to \Omega^{\bullet}_{\mathbf{\Gamma}/\mathbf{\Delta}}(\log D_{\mathbf{\Gamma}}) \otimes_{\mathcal{O}_{\mathbf{\Gamma}}} p^*\Omega^1_{\mathbf{\Delta}}$$

is the map in $D^b(\mathbb{C}_{\mathbf{\Gamma}})$ corresponding to the short exact sequence of complexes

$$\begin{tikzcd}
0 \arrow[d] \\
\Omega^{\bullet}_{\mathbf{\Gamma}/\mathbf{\Delta}}(\log D_{\mathbf{\Gamma}})[-1] \otimes_{\mathcal{O}_{\mathbf{\Gamma}}} p^*\Omega^1_{\mathbf{\Delta}} \arrow[d] \\
\Omega^{\bullet}_{\mathbf{\Gamma}}(\log D_{\mathbf{\Gamma}}) \arrow[d] \\
\Omega^{\bullet}_{\mathbf{\Gamma}/\mathbf{\Delta}}(\log D_{\mathbf{\Gamma}}) \arrow[d] \\
0
\end{tikzcd}$$

of logarithmic forms on $\mathbf{\Gamma}$. The vanishing of $(q \otimes 1) \circ \xi_{\mathcal{Z}/\mathbf{\Delta}} \circ i$ follows immediately now since from (2.4.8) we see that $(q \otimes 1) \circ \xi_{\mathcal{Z}/\mathbf{\Delta}} \circ i = \left(\imath_{\mathbf{\Gamma}*}\xi_{\mathbf{\Gamma}/\mathbf{\Delta}}\right) \circ q \circ i = 0$. This in turn implies that $\xi_{\mathcal{Z}/\mathbf{\Delta}}$ comes from a morphism $\operatorname{cone}(q) \to \operatorname{cone}(q \otimes 1)$ in $D^b(\mathbb{C}_{\mathcal{Z}})$. In other words we can find a map $\xi_{\mathsf{E}} : \mathsf{E}^{\bullet}_{\mathcal{Z}/\mathbf{\Delta}} \to \mathsf{E}^{\bullet}_{\mathcal{Z}/\mathbf{\Delta}} \otimes_{\mathcal{O}_{\mathcal{Z}}} p^*\Omega^1_{\mathbf{\Delta}}$ so that $(i \otimes 1) \circ \xi_{\mathsf{E}} = \xi_{\mathcal{Z}/\mathbf{\Delta}} \circ i$. Since i is an isomorphism over the open $\mathcal{Z}^{\times} \subset \mathcal{Z}$ it now follows that over $\mathcal{Z}^{\times}$ the map ξ_{E} coincides with the map

$$\xi^{\times}_{\mathsf{E}} : \mathsf{E}^{\bullet}_{\mathcal{Z}^{\times}/\mathbf{\Delta}^{\times}} \to \mathsf{E}^{\bullet}_{\mathcal{Z}^{\times}/\mathbf{\Delta}^{\times}} \otimes_{p^{-1}\mathcal{O}_{\mathbf{\Delta}^{\times}}} p^{-1}\Omega^1_{\mathbf{\Delta}^{\times}}$$

corresponding to the short exact sequence of complexes (2.4.5). Since

$\nabla^{\mathrm{GM}} = \mathbb{R}^a p_*(\xi_{\mathsf{E}}^{\times})$ it follows that ∇^{GM} extends to a holomorphic connection $\mathbb{R}^a p_*(\xi_{\mathsf{E}}) : \mathbb{R}^a p_* \mathsf{E}^{\bullet}_{\mathcal{Z}/\boldsymbol{\Delta}} \to \mathbb{R}^a p_* \mathsf{E}^{\bullet}_{\mathcal{Z}/\boldsymbol{\Delta}} \otimes_{\mathcal{O}_{\boldsymbol{\Delta}}} \Omega^1_{\boldsymbol{\Delta}}$. This completes the proof of the claim. q.e.d.

The statement of Claim 2.22 proves Lemma 2.21 in the case when the divisor D_{Z} does not have a horizontal part. In fact we can incorporate the horizontal divisor into the proof of Claim 2.22 without any modification. The local calculation for the limit, and the extension argument repeat verbatim, only the notation becomes more cumbersome. We will not spell this out here and leave it to the interested reader to fill in the details. q.e.d.

We can now complete the proof of the double degeneration property by combining Lemma 2.21 with the following well known facts:

Lemma 2.23. *For every $a \geq 0$ we have*

$$\begin{aligned}\dim_{\mathbb{C}} H^a\left(Y, Y_{-\infty}; \mathbb{C}\right) &= \dim_{\mathbb{C}} \mathbb{H}^a\left(Y_{\mathrm{Zar}}, (\Omega^{\bullet}_Y, d + d\mathsf{w}\wedge)\right) \\ &= \dim_{\mathbb{C}} \mathbb{H}^a\left(Y_{\mathrm{Zar}}, (\Omega^{\bullet}_Y, d\mathsf{w}\wedge)\right).\end{aligned}$$

Proof. The first equality is the usual identification of de Rham nearby cycles with twisted de Rham cohomology via the Fourier transform for regular holonomic D-modules on the affine line. The second is the degeneration theorem for twisted de Rham complexes proven in the work of Barannikov and Kontsevich (unpublished), Sabbah [**Sab99**], or Ogus–Vologodsky [**OV05**]. q.e.d.

Lemma 2.24. *For every $a \geq 0$ we have*

$$\begin{aligned}&\dim_{\mathbb{C}} \mathbb{H}^a\left(Y_{\mathrm{Zar}}, (\Omega^{\bullet}_Y, d + d\mathsf{w}\wedge)\right) \\ &\qquad = \dim_{\mathbb{C}} \mathbb{H}^a\left(\mathsf{Z}_{\mathrm{Zar}}, (\Omega^{\bullet}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}), d + d\mathsf{f}\wedge)\right).\end{aligned}$$

Proof. This follows from the usual Grothendieck argument [**Gro66**]. The local calculation comparing logarithmic forms with meromorphic forms transfers immediately to the f-adapted complex and combined with the local description of adapted forms given in Lemma 2.12 implies that the natural inclusion of complexes

$$(\Omega^{\bullet}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}), d + d\mathsf{f}\wedge) \hookrightarrow (\Omega^{\bullet}_{\mathsf{Z}}(*D_{\mathsf{Z}}), d + d\mathsf{f}\wedge) = R\jmath_{Y*}\left(\Omega^{\bullet}_Y, d + d\mathsf{w}\wedge\right)$$

is a quasi-isomorphism. q.e.d.

Taken together Lemmas 2.21, 2.23, and 2.24 imply that the hypercohomology of the complex $(\Omega^{\bullet}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}), c_1 \cdot d + c_2 \cdot d\mathsf{f}\wedge)$ is constant on the line $c_1 = c_2$. This completes the proof of Theorem 2.18. q.e.d.

3. Invariants of nc Hodge structures of geometric origin

In this section we use the deformation theory developed in Section 2 to elucidate the motivic and Hodge theoretic data naturally present on the cohomology of a compactifiable Landau–Ginzburg model. Using considerations from mirror symmetry we propose various new refined invariants of **nc** Hodge structures of Landau–Ginzburg type, and discuss, in particular, the subtleties involved in understanding Hodge numbers and decorations.

3.1. Hodge numbers of Landau–Ginzburg models. Suppose $((\mathsf{Z},\mathsf{f}), D_{\mathsf{Z}}, \mathsf{vol}_{\mathsf{Z}})$ is a tame compactified Landau–Ginzburg model in the sense of Definition 2.4 and assumption (T). Conjecturally (see [**KKP08**]) the cohomology $H^{\bullet}(Y, Y_{-\infty}; \mathbb{C})$ of the associated quasi-projective Landau–Ginzburg model $\mathsf{w} : Y \to \mathbb{A}^1$ carries a B-model pure **nc** Hodge structure. The de Rham data for this **nc** Hodge structure is described in [**KKP08**, Section 3.2], where it is also argued that this data satisfies the **nc** Hodge filtration axiom. The much trickier opposedness axiom has only been verified for models $((Y,\mathsf{w}), \mathsf{vol}_Y)$ which mirror a general symplectic toric weak Fano manifold [**RS15**].

A somewhat disappointing feature of **nc** Hodge structures in general is that their complexity is not readily captured by simple numerical invariants. The absence of easy to compute linear–algebraic invariants in this setting is a reflection of the nature of the **nc** de Rham datum. The **nc** Hodge filtration is encoded in a connection with irregular singularities, and the Stokes structures characterizing this connection cannot be encoded in simple linear algebraic quantities. The special nature of the Landau–Ginzburg context, however, allows one to discern additional sophisticated linear–algebraic data compatible with the **nc** Hodge structure on $H^{\bullet}(Y, Y_{-\infty}; \mathbb{C})$. Moreover, this data possesses computable numerical invariants, such as weights, level, amplitude, and Hodge numbers. Most naturally this additional data arises from the concept of an irregular Hodge filtration that can be associated with a Landau–Ginzburg potential. There are two variants of such irregular Hodge filtrations – the version of Deligne and Sabbah [**DMR07**, **Sab10a**], and the version of J.-D. Yu [**Yu12**, **ESY13**]. In [**ESY13**] these two variants of the irregular Hodge filtration are generalized, ultimately identified with each other, and under the assumption (T) identified with the Hodge filtration on the complex of f-adapted logarithmic forms. Here we will not discuss this identification but rather will look more closely at the resulting Hodge numbers and will compare those to other more classical definitions of Hodge numbers arising from vanishing cohomology and mirror data.

Given a Calabi–Yau Landau–Ginzburg model $\mathsf{w} : Y \to \mathbb{A}^1$ which admits a tame compactification $((\mathsf{Z},\mathsf{f}), D_{\mathsf{Z}}, \mathsf{vol}_{\mathsf{Z}})$, we will define geometrically three sets of Hodge numbers $i^{p,q}(Y,\mathsf{w})$, $h^{p,q}(Y,\mathsf{w})$, and $f^{p,q}(Y,\mathsf{w})$, each of which adds up to the rank of the algebraic de Rham cohomology $H^a_{DR}((Y_{\mathrm{Zar}},\mathsf{w}); \mathbb{C}) = \mathbb{H}^a(Y_{\mathrm{Zar}}, (\Omega^{\bullet}_Y, d + d\mathsf{w}\wedge))$ of the Landau–Ginzburg model. Since by Lemma 2.23 we have $\dim_{\mathbb{C}} H^a_{DR}(Y_{\mathrm{Zar}}, \mathsf{w}; \mathbb{C}) =$

$\dim_{\mathbb{C}} H^a(Y, Y_{-\infty}; \mathbb{C})$ this implies

$$\dim_{\mathbb{C}} H^a(Y, Y_{-\infty}; \mathbb{C}) = \sum_{p+q=a} i^{p,q}(Y, \mathsf{w}) = \sum_{p+q=a} h^{p,q}(Y, \mathsf{w}) = \sum_{p+q=a} f^{p,q}(Y, \mathsf{w}).$$

Each of these sets of Hodge numbers has a different origin. The numbers $i^{p,q}(Y, \mathsf{w})$ come from ordinary mixed Hodge theory, the numbers $h^{p,q}(Y, \mathsf{w})$ come from mirror considerations, and the numbers $f^{p,q}(Y, \mathsf{w})$ come from the sheaf cohomology of the f-adapted logarithmic forms. The specific definitions are as follows.

3.1.1. The numbers $f^{p,q}(Y, \mathsf{w})$. Let $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}}, \mathsf{vol}_{\mathsf{Z}})$ be a tame compactification of $\mathsf{w} : Y \to \mathbb{A}^1$.

Definition 3.1. The **Landau–Ginzburg Hodge numbers** $f^{p,q}(Y, \mathsf{w})$ are defined by

$$f^{p,q}(Y, \mathsf{w}) = \dim_{\mathbb{C}} H^p(\mathsf{Z}, \Omega^q_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})).$$

The fact that $\dim_{\mathbb{C}} H^a(Y, Y_{-\infty}; \mathbb{C}) = \sum_{p+q=a} f^{p,q}(Y, \mathsf{w})$ follows from Theorem 2.6 and Lemma 2.24.

3.1.2. The numbers $h^{p,q}(Y, \mathsf{w})$. Before we explain the definition we need to recall a basic construction from linear algebra. Let V be a finite dimensional complex vector space, $N : V \to V$ be a nilpotent linear operator satisfying $N^{m+1} = 0$ for some $m \geq 0$. The **(monodromy) weight filtration of N centered at m** is the unique increasing filtration $W = W_\bullet(N, m)$ of V:

$$0 \subset W_0(N, m) \subset W_1(N, m) \subset \cdots \subset W_{2m-1}(N, m) \subset W_{2m}(N, m) = V,$$

with the properties

- $N(W_i(N, m)) \subset W_{i-2}(N, m)$;
- the map $N^\ell : \mathrm{gr}^W_{m+\ell} V \to \mathrm{gr}^W_{m-\ell} V$ is an isomorphism for all $\ell \geq 0$.

The existence and uniquencess of this filtration can be deduced from the representation theory of $\mathfrak{sl}_2$-triples and the Jacobson–Morozov theorem. A direct elementary proof can also be found in [**Sch73**, Lemma 6.4]. Explicitly the monodromy weight filtration $W_\bullet(N, m)$ is defined as follows. Choose a Jordan basis for the nilpotent endomorphism $N : V \to V$ and assign integer weights to the basis vectors so that N lowers weights by 2, and so that the weights of each Jordan block are arranged symmetrically about m. Note that even though the Jordan canonical form is not canonical, the monodromy weight filtration will be canonical since $W_k(N, m)$ is the span of the basis vectors of weights less than or equal to k.

Now let $c_0 \in \mathbb{A}^1$ be a regular value of w near infinity. Consider the monodromy transformation $T : H^\bullet(Y, Y_{c_0}; \mathbb{C}) \to H^\bullet(Y, Y_{c_0}; \mathbb{C})$ corresponding to moving the smooth fiber Y_{c_0} once around infinity. By assumption (Y, w) admits a tame compactification and so, as explained in Remark 2.5**(ii)** the operator T is unipotent. Set $N = \log T$. With this notation we have the following:

Definition 3.2. The **Landau–Ginzburg Hodge numbers** $h^{p,q}(Y, \mathsf{w})$ are defined by

$$h^{p,q}(Y, \mathsf{w}) := \dim_{\mathbb{C}} \mathrm{gr}_p^{W(N,p+q)} H^{p+q}(Y, Y_{c_0}; \mathbb{C}).$$

The rationale behind this definition is the geometric mirror symmetry prediction explained in Remark 2.5**(ii)**. Specifically, if (Y, w) is part of a mirror pair

$$(X, \omega_X, s_X) \mid ((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y),$$

of Fano type with $\dim_{\mathbb{C}} X = \dim_{\mathbb{C}} Y = n$, then the homological mirror equivalence

$$\mathbf{D}^b(X) \cong \mathbf{FS}((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y)$$

induces an isomorphism on period cyclic and on Hochschild homologies of these categories. In particular, we expect a mirror isomorphism

$$HH_a(\mathbf{D}^b(X)) \cong HH_a(\mathbf{FS}((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y)), \tag{3.1.1}$$

for all a where the Hochschild homology can possibly be non-zero, i.e., for all a such that $-n \leq a \leq n$. It is also expected that these categorical homology groups have geometric incarnations:

$$\begin{aligned} HH_a(\mathbf{D}^b(X)) &\cong \bigoplus_{p-q=a} H^p(X, \Omega_X^q), \\ HH_a(\mathbf{FS}((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y)) &\cong H^{a+n}(Y, Y_{-\infty}; \mathbb{C}). \end{aligned} \tag{3.1.2}$$

The first of the above identification follows from the work [**Wei96**] of Weibel, while the second has been conjectured in general, and proven in special cases in the works of Seidel (see, e.g., [**Sei08**, **Sei09**]).

Thus combining the conjectural mirror isomorphism (3.1.1) with this geometric interpretation of Hochschild homology we will get a conjectural isomorphism

$$H^{a+n}(Y, Y_{-\infty}; \mathbb{C}) \cong \bigoplus_{p-q=a} H^p(X, \Omega_X^q). \tag{3.1.3}$$

Remark 3.3. Since the mirror identification (3.1.1) comes from the mirror equivalence of categories, it is clear that a similar equivalence can also be formulated for the periodic cyclic homologies of the d($\mathbb{Z}$)g categories $\mathbf{D}^b(X)$ and $\mathbf{FS}((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y)$. Respectively, the mirror identification (3.1.3) can be formulated for the de Rham cohomologies of X and (Y, w). In these cases we have natural **nc** Hodge filtrations on each group. In the categorical setting the Hodge filtrations are encoded in the negative cyclic homologies $HC_\bullet^-(\mathbf{D}^b(X))$ and $HC_\bullet^-((\mathbf{FS}((Y, \mathsf{w}), \omega_Y,$

vol_Y)) viewed as modules over $\mathbb{C}[[u]]$ (see [**KKP08**, Section 2.2.3]). Homological mirror symmetry implies the existence of an isomorphism of $\mathbb{C}[[u]]$-modules

$$HC^-_\bullet(\mathbf{D}^b(X)) \cong HC^-_\bullet(\mathbf{FS}((Y,\mathsf{w}),\omega_Y,\mathsf{vol}_Y)),$$

which after tensoring with $\mathbb{C}((u))$ induces an isomorphism of $\mathbb{C}((u))$-vector spaces

$$HP_\bullet(\mathbf{D}^b(X)) \cong HP_\bullet(\mathbf{FS}((Y,\mathsf{w}),\omega_Y,\mathsf{vol}_Y)).$$

From this point of view the isomorphism (3.1.1) is recovered as the induced isomorphism of specializations

$$\begin{aligned} &HC^-_\bullet(\mathbf{D}^b(X))/uHC^-_\bullet(\mathbf{D}^b(X)) \\ &\qquad \cong HP_\bullet(\mathbf{FS}((Y,\mathsf{w}),\omega_Y,\mathsf{vol}_Y))/uHP_\bullet(\mathbf{FS}((Y,\mathsf{w}),\omega_Y,\mathsf{vol}_Y)). \end{aligned}$$

These mirror isomorphisms translate readily into the geometric language. Recall that similarly to (3.1.2) we have identifications

$$\begin{aligned} HP_\bullet(\mathbf{D}^b(X)) &= H^\bullet_{DR}(X,\mathbb{C}) \otimes \mathbb{C}((u)), \\ HP_\bullet(\mathbf{FS}((Y,\mathsf{w}),\omega_Y,\mathsf{vol}_Y)), &= H^\bullet_{DR}(Y,Y_{-\infty};\mathbb{C}) \otimes \mathbb{C}((u)). \end{aligned} \tag{3.1.4}$$

Furthermore, in geometric terms the $\mathbb{C}[[u]]$-module $HC^-_\bullet(\mathbf{D}^b(X))$ is identified with the Rees module of the filtration

$$F^a_{\mathbf{nc}} H^\bullet_{DR}(X,\mathbb{C}) = \bigoplus_{p-q\geq a} H^p(X,\Omega^q_X),$$

on the complex vector space $H^\bullet_{DR}(X,\mathbb{C})$, while the $\mathbb{C}[[u]]$-module $HC^-_\bullet(\mathbf{FS}((Y,\mathsf{w}),\omega_Y,\mathsf{vol}_Y))$ is identified with the Rees module of the filtration

$$F^a_{\mathbf{nc}} H^\bullet_{DR}(Y,Y_{-\infty};\mathbb{C}) = \bigoplus_{b\geq n+a} H^b(Y,Y_{-\infty};\mathbb{C}),$$

on the complex vector space $H^\bullet(Y,Y_{-\infty};\mathbb{C})$.

The de Rham version of the Dolbeault mirror statement (3.1.3) then becomes the statement that mirror symmetry induces an isomorphism of the filtered complex vector spaces $F^\bullet_{\mathbf{nc}} H^\bullet_{DR}(X,\mathbb{C})$ and $F^\bullet_{\mathbf{nc}} H^\bullet_{DR}(Y,Y_{-\infty};\mathbb{C})$. In fact, more should be true. The induced isomorphism of the algebraic vector bundles on $\mathbb{A}^1$ associated with the respective Rees modules should also intertwine the irregular meromorphic connections describing the **nc** Hodge structures on both sides.

Going back to the mirror isomorphism (3.1.3), we are faced with the usual conundrum: simply identifying Hochschild (or cyclic) homologies of the two mirror categories does not give us matching of Hodge numbers. The comparison (3.1.3) identifies the homological de Rham grading on the Landau–Ginzburg side with the $(p-q)$-folding of the Dolbeault bigrading on the Fano side.

The key to reconstructing the bigradings and thus extracting Hodge numbers on both sides lies in the observation that, in the Fano case, the Dolbeault bigrading also has a categorical interpretation. Indeed, the nilpotent operator $c_1(K_X) \cup (\bullet)$ acts on each $HH_a(\mathbf{D}^b(X)) = \oplus_{p-q} H^p(X, \Omega_X^q)$, and so induces a monodromy weight filtration centered at a. Since the canonical class is anti-ample this filtration is given by the forms of degree $\leq (p+q)$. In particular, the dimensions of the graded pieces for this monodromy weight filtration are precisely the Hodge numbers $h^{p,q}(X)$ of the Fano variety X. Up to a sign, the nilpotent operator $c_1(K_X) \cup (\bullet) : H^\bullet_{DR}(X, \mathbb{C}) \to H^\bullet_{DR}(X, \mathbb{C})$ is just the logarithm of the action of the Serre functor $S_{\mathbf{D}^b(X)}$ on $HH_\bullet(\mathbf{D}^b(X)) \cong H^\bullet_{DR}(X, \mathbb{C})$. Thus, this monodromy weight filtration has a categorical interpretation. But, as we noted in Remark 2.5**(ii)**, the Serre functor of $\mathbf{FS}((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y)$ can be identified with the inverse of the monodromy autoequivalence T. The logarithm of the action of T on $HH_\bullet(\mathbf{FS}((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y)) \cong H^{\bullet+n}_{DR}(Y, Y_{c_0}; \mathbb{C})$ is just the nilpotent operator N we considered above. Therefore, the monodromy weight filtration corresponding to N is expected to have categorical origin, and homological mirror symmetry predicts, the mirror matching of Hodge numbers:

$$h^{p,q}(Y, \mathsf{w}) = h^{p,n-q}(X), \tag{3.1.5}$$

for all p, q. This prediction is still conjectural in general but the case $(p,q) = (1,1)$ was recently proven by Przyjalkowski and Shramov [**PS13**] for all smooth Fano varieties.

3.1.3. The numbers $i^{p,q}(Y, \mathsf{w})$. To simplify the discussion we will first assume that $D_{\mathsf{Z}}^{\mathsf{h}} = \varnothing$, i.e., that $\mathsf{w} : Y \to \mathbb{A}^1$ is proper.

It is well known (see, e.g., [**Sab99**]) that the dimension of the (Zariski) hypercohomology of the w-twisted de Rham complex on Y can be computed from the dimensions of the vanishing cohomology for w:

$$\dim_{\mathbb{C}} \mathbb{H}^a \left(Y_{\text{Zar}}, (\Omega_Y^\bullet, d + d\mathsf{w}\wedge)\right) = \sum_{\lambda \in \mathbb{A}^1} \dim_{\mathbb{C}} \mathbb{H}^{a-1}\left(Y_{\lambda,\text{an}}, \boldsymbol{\phi}_{\mathsf{w}-\lambda}\mathbb{C}_Y\right),$$

where as usual $\boldsymbol{\phi}_{\mathsf{w}-\lambda}\mathbb{C}_Y$ denotes the perverse sheaf of vanishing cocycles for the fiber Y_λ. From the works of Schmid and Steenbrink (see, e.g., [**Sch73**], [**PS08**, Section 11.2]) and Saito [**Sai90**] it is classically known that the constructible complex $\boldsymbol{\phi}_{\mathsf{w}-\lambda}\mathbb{C}_Y$ carries a structure of a mixed Hodge module and so its cohomology is furnished with a functorial mixed Hodge structure.

Given a mixed Hodge structure $\mathcal{V}$ we will write $i^{p,q}\mathcal{V}$ for the (p,q) Hodge number of the $p+q$ weight graded piece $\mathrm{gr}^W_{p+q}\mathcal{V}$. We now have the following:

Definition 3.4. For a proper potential $\mathsf{w} : Y \to \mathbb{A}^1$ on a quasi-projective variety Y the **Landau–Ginzburg Hodge numbers** $i^{p,q}(Y, \mathsf{w})$

are defined by

$$i^{p,q}(Y, \mathsf{w}) := \sum_{\lambda \in \mathbb{A}^1} \sum_{k} i^{p,q+k} \mathbb{H}^{p+q-1}\left(Y_\lambda, \boldsymbol{\phi}_{\mathsf{w}-\lambda} \mathbb{C}_Y\right),$$

where each vanishing cohomology $\mathbb{H}^a\left(Y_\lambda, \boldsymbol{\phi}_{\mathsf{w}-\lambda} \mathbb{C}_Y\right)$ is taken with its Schmid–Steenbrink mixed Hodge structure.

Remark 3.5. (i) The combination of Hodge numbers of different weight pieces in this definition is motivated by mirror symmetry. In the paper [**GKR12**] it was argued that for a Landau–Ginzburg mirror of a general type complete-intersection S in a toric variety, the above definition of Hodge numbers reproduces the rotated Hodge diamond of S.

(ii) The assumption that $D^{\mathsf{h}}_{\mathsf{Z}} = \varnothing$ above was introduced solely for technical convenience and is not really needed. If $D^{\mathsf{h}}_{\mathsf{Z}} \neq \varnothing$, we can still define $i^{p,q}(Y, \mathsf{w})$ by setting

$$i^{p,q}(Y, \mathsf{w}) := \sum_{\lambda \in \mathbb{A}^1} \sum_{k} i^{p,q+k} \mathbb{H}^{p+q-1}\left(\mathsf{Z}_\lambda, \boldsymbol{\phi}_{\mathsf{f}-\lambda} R\jmath_{Y*} \mathbb{C}_Y\right),$$

where $\jmath_Y : Y \hookrightarrow \mathsf{Z}$ is the natural inclusion.

(iii) It is very interesting to try and understand the categorical meaning of the numbers $i^{p,q}(Y, \mathsf{w})$. At a first glance, the definition of $i^{p,q}(Y, \mathsf{w})$ relies heavily on the geometry since the information of the variety Y and the potential w enter in an essential way in the construction of the pertinent mixed Hodge structures. On the other hand, from the works of Shklyarov [**Shk11**] and Efimov [**Efi12**] it is known that the space $H^\bullet_{DR}(Y, \mathsf{w}; \mathbb{C})$ together with its **nc** Hodge filtration admits a purely categorical interpretation. Specifically, in [**Efi12**] it is shown that $H^\bullet_{DR}\left(Y_{\mathrm{Zar}}, (\Omega^\bullet_Y((u)), ud - d\mathsf{w}\wedge)\right)$ is isomorphic to the periodic cyclic homology $HP_\bullet(\mathbf{MF}(Y, \mathsf{w}))$ of the d$(\mathbb{Z}/2)$g category of matrix factorizations of w, and that this isomorphism can be chosen so that the irregular connection $\nabla^{\mathrm{DR}}_{d/du} = d/du + u^{-1}\mathbf{Gr} + u^{-2}\mathsf{w} \cdot (\bullet)$ codifying the **nc** Hodge filtration on $H^\bullet_{DR}(Y, \mathsf{w}; \mathbb{C})$ gets identified with the connection $\nabla^{\mathrm{cat}}_{d/du}$ from [**KKP08**, Section 2.2.5] used to define the categorical **nc** Hodge filtration on $HP_\bullet(\mathbf{MF}(Y, \mathsf{w}))$.

In other words, the **nc** Hodge filtration of a Landau–Ginzburg model (Y, w) admits a purely categorical interpretation. In the case when ∇^{cat} satisfies the **nc**-opposedness axiom of [**KKP08**] we can hope for more. In this case we expect that the pure complex **nc** Hodge structure on $HP_\bullet(\mathbf{MF}(Y, \mathsf{w}))$ is polarizable and that it admits a natural limit mixed twistor structure (in the sense of [**Sab05**]) which in turn is isomorphic to the $\mathbb{Z}/2$-folding of the ordinary mixed Hodge structure on the vanishing cohomology $\oplus_{\lambda \in \mathbb{A}^1} \mathbb{H}^{\bullet-1}\left(Y_\lambda, \boldsymbol{\phi}_{\mathsf{w}-\lambda} \mathbb{C}_Y\right)$. Concretely, we have a one parameter deformation $\{\mathcal{A}_t\}_{t \in \mathbb{A}^1}$ of the d$(\mathbb{Z}/2)$g category $\mathcal{A}_1 := \mathbf{MF}(Y, \mathsf{w})$, where $\mathcal{A}_t$ has the same objects and hom sets as $\mathsf{MF}(Y, \mathsf{w})$ but the composition, differential, and units in $\mathcal{A}_t$ are scaled as $m_{\mathcal{A}_t} = t \cdot m_{\mathsf{MF}(Y,\mathsf{w})}$, $d_{\mathcal{A}_t} = t \cdot d_{\mathsf{MF}(Y,\mathsf{w})}$, $1_{\mathcal{A}_t} = t^{-1} \cdot 1_{\mathsf{MF}(Y,\mathsf{w})}$. The periodic cyclic homology of these categories equipped with the connection ∇^{cat} in the u-direction and with the Getzler–Gauss–Manin connection [**Get93**] in the t-direction is a variation of twistor D-modules. When the opposedness and polarizability properties hold, e.g., for Landau–Ginzburg mirrors of toric Fano varieties, see [**RS15**], we can form the limit mixed twistor D-module for $t \to \infty$ and we conjecture that this mixed twistor D-module is the one corresponding to the ordinary mixed Hodge structure $\oplus_{\lambda \in \mathbb{A}^1} \mathbb{H}^{\bullet - 1}\left(Y_\lambda, \phi_{\mathsf{w}-\lambda}\mathbb{C}_Y\right)$. In the case of potentials given by tame Laurent polynomials this conjecture is verified in [**Sab10a**]. The conjecture gives a categorical interpretation of the mixed Hodge structure on vanishing cohomology (modulo Tate twists) and as a consequence gives a categorical interpretation of the $\mathbb{Z}/2$-folding of the numbers $i^{p,q}(Y, \mathsf{w})$.

3.1.4. Comparison conjectures. Because of their similar behavior under the mirror correspondence we expect that the various Landau–Ginzburg Hodge numbers are equal to each other:

Conjecture 3.6. If $\mathsf{w} : Y \to \mathbb{A}^1$ is an n-dimensional Landau–Ginzburg model which admits a tame compactification, then

$$f^{p,q}(Y, \mathsf{w}) = h^{p,q}(Y, \mathsf{w}) = i^{p,q}(Y, \mathsf{w}).$$

Combined with the mirror matching (3.1.5) the previous conjecture predicts:

Conjecture 3.7. If $(X, \omega_X, s_X) \mid ((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y)$ is a mirror pair of Fano type, and if $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}}, \mathsf{vol}_{\mathsf{Z}})$ is a tame compactification of $((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y)$, then we have

$$f^{p,q}(Y, \mathsf{w}) = h^{p,n-q}(X),$$

for all p, q.

3.2. Mirrors of compactified Landau–Ginzburg models. In this section we look more closely at the role that compactified Landau–Ginzburg models play in mirror symmetry. In the setting where a complex Landau–Ginzburg model (Y, w) is the mirror of a symplectic Fano variety (X, ω_X), we give a mirror A-model interpretation of the Hodge information encoded in a compactification $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}})$. This suggests that the mirror symmetry between (Y, w) and (X, ω_X) can be extended to a mirror symmetry between $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}})$ and a one parameter symplectic deformation of (X, ω_X) which interpolates between the Fukaya category of the symplectic Fano variety (X, ω_X) and the Fukaya category of the symplectic non-compact Calabi–Yau datum $(X - D_X, \omega_{X|X-D_X}, \mathsf{vol}_{X-D_X})$. We discuss such an extension and give some evidence for its validity. This picture is not new and has already been proposed and analyzed in one form or another in the works of Seidel [**Sei08**, **Sei11**, **Sei12**, **Sei14a**, **Sei14b**] and Abouzaid et al. [**AS10**, **Abo13**, **AAE**$^+$**13**, **AAE**$^+$**13**]. Our main contribution here is to formulate a new procedure for reconstructing the Hodge theory of f-adapted logarithmic forms from the **nc** Hodge structure on the cohomology of the Landau–Ginzburg model (Y, w) or, in the mirror picture, from the A-model **nc** Hodge structure on the cohomology of the Fano variety X.

3.2.1. One parameter families of symplectic Fano varieties. Let (X, ω_X) be a symplectic manifold underlying a smooth compact Fano variety of $\dim_{\mathbb{C}} X = n$. Let $\boldsymbol{k}_X$ be a closed 2-form representing the canonical class K_X and let $\kappa_X \in H^2(X, \mathbb{Z})$ denote the first Chern class of K_X, i.e., $\kappa_X = [\boldsymbol{k}_X] = c_1(K_X)$. Consider the (multivalued) family $\{\omega_q\}_{q\in\mathbb{C}}$ of complex 2-forms $\omega_q := \omega_X + \log(q)\boldsymbol{k}_X$ on X. In the regime when $|q| \to 1$ these are complexified Kähler forms.

This is an affine-linear one-parameter family of symplectic structures on X which gives rise to a one-parameter variation of pure **nc** Hodge structures parametrized by the q-line. As discussed in [**KKP08**, Section 3.1] the de Rham part of such variation is encoded in a pair $({}^{\mathfrak{a}}H, {}^{\mathfrak{a}}\nabla)$, where

- ${}^{\mathfrak{a}}H := H^\bullet(X, \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^2}$ is a trivial algebraic $\mathbb{Z}/2$-graded vector bundle on the affine plane with coordinates (u, q), with $\mathbb{Z}/2$-grading given by

$$ {}^{\mathfrak{a}}H^0 = \left(\bigoplus_{\substack{k=n \\ \mathrm{mod}\ 2}} H^k(X, \mathbb{C}) \right) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^2}, $$

$$ {}^{\mathfrak{a}}H^1 = \left(\bigoplus_{\substack{k=n+1 \\ \mathrm{mod}\ 2}} H^k(X, \mathbb{C}) \right) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^2}. $$

- ${}^{\mathfrak{a}}\nabla$ is a meromorphic connection on ${}^{\mathfrak{a}}H$, with poles along the divisor $uq = 0$, given by

$$ (3.2.1) \qquad \begin{aligned} {}^{\mathfrak{a}}\nabla_{\frac{\partial}{\partial u}} &:= \frac{\partial}{\partial u} + u^{-2} \left(\kappa_X *_q \bullet \right) + u^{-1}\mathsf{Gr} \\ {}^{\mathfrak{a}}\nabla_{\frac{\partial}{\partial q}} &:= \frac{\partial}{\partial q} - q^{-1}u^{-1} \left(\kappa_X *_q \bullet \right), \end{aligned} $$

  where

  $*_q$**:** denotes the quantum product corresponding to ω_q, and
  $\mathsf{Gr} : {}^{\mathfrak{a}}H \to {}^{\mathfrak{a}}H$**:** is the grading operator defined to be $\mathsf{Gr}_{|H^k(X,\mathbb{C})} := \frac{k-n}{2} \mathrm{id}_{H^k(X,\mathbb{C})}$.

Remark 3.8. More generally we have a variation of nc Hodge structures over the whole complexified Kähler cone. The meromorphic connection defining the de Rham part of the variation is the Dubrovin first structure connection [**Dub98**, **Man99**] which on each affine line $\omega + \log(q)\alpha$ is given by the formula

$$\begin{aligned} {}^{\mathfrak{a}}\nabla_{\frac{\partial}{\partial u}} &:= \frac{\partial}{\partial u} + u^{-2}\left(\kappa_X *_q \bullet\right) + u^{-1}\mathsf{Gr} \\ {}^{\mathfrak{a}}\nabla_{\frac{\partial}{\partial q}} &:= \frac{\partial}{\partial q} - q^{-1}u^{-1}\left([\alpha] *_q \bullet\right), \end{aligned}$$

with $*_q$ being the quantum product corresponding to $\omega + \log(q)\alpha$. Traditionally in mirror symmetry one works with the line of slope ω passing through the large volume limit, i.e., the line $0 + \log(q)\omega$. This is the situation considered in [**FOOO09a**, **FOOO09b**] and in [**KKP08**, Section 3.1]. In contrast, here we need to work with a line of slope $\boldsymbol{k}_X$, i.e., the line $\omega + \log(q)\boldsymbol{k}_X$ which leads to the formula (3.2.1).

The particular affine linear deformation of the symplectic structure $(X, \omega + \log(q)\boldsymbol{k}_X)$ that we are considering has many special properties even when compared to other affine linear families. For instance, it is expected that the affine one parameter deformation of the symplectic structure $(X, \omega+\log(q)\boldsymbol{k}_X)$ does not change the Fukaya category. From the point of view of nc Hodge theory this family is significant because of the following simple observation:

Lemma 3.9. *The restriction of* $({}^{\mathfrak{a}}H, {}^{\mathfrak{a}}\nabla)$ *to any non-vertical line* L *through the origin in* $\mathbb{A}^2$ *is a meromorphic connection on the trivial* $\mathbb{Z}/2$*-graded vector bundle* $H^\bullet(X,\mathbb{C}) \otimes \mathcal{O}_L$*, which has a first order pole at* 0*, and monodromy around* 0 *equal to* $(-1)^k$ *on the graded piece* $H^k(X,\mathbb{C}) \otimes \mathcal{O}$*.*

Proof. From the formulas (3.2.1) we see that the ${}^{\mathfrak{a}}\nabla$-covariant derivative in the direction of the Euler vector field $\dfrac{u\partial}{\partial u} + \dfrac{q\partial}{\partial q}$ is given by

$${}^{\mathfrak{a}}\nabla_{\frac{u\partial}{\partial u}+\frac{q\partial}{\partial q}} = \frac{u\partial}{\partial u} + \frac{q\partial}{\partial q} + \mathsf{Gr}.$$

Since $\dfrac{u\partial}{\partial u} + \dfrac{q\partial}{\partial q}$ is tangent to any line through the origin and is equal to the Euler vector field on any such line, we get the statement of the lemma.

To explicate, choose a slope $v \neq 0$ and let $L_v \subset \mathbb{A}^2$ be the line given by $u = vq$. The variable q is the natural coordinate on L_v, and so on L_v we have $du = vdq$. To shorten the notation, write $\mathsf{M} := \kappa_X *_q (\bullet)$ for the operator of quantum multiplication by κ_X. Then on $\mathbb{A}^2$ we have

$${}^{\mathfrak{a}}\nabla = d + \left(u^{-2}\,\mathsf{M} + u^{-1}\mathsf{Gr}\right) du + \left(-u^{-1}q^{-1}\,\mathsf{M}\right) dq,$$

and

$$\begin{aligned} {}^{\mathfrak{a}}\nabla_{|L_v} &= d + \left(v^{-2}q^{-2}\,\mathsf{M} + v^{-1}q^{-1}\mathbf{Gr}\right)\cdot v dq + \left(-v^{-1}q^{-2}\,\mathsf{M}\right)\cdot dq \\ &= d + \frac{\mathbf{Gr}}{q} dq. \end{aligned}$$

Thus ${}^{\mathfrak{a}}\nabla_{|L_v}$ is logarithmic at 0 and has half integer residues. This completes the proof of the lemma. q.e.d.

Recall from [**KKP08**, Section 2.2.7] that when viewed as **nc** Hodge structures, ordinary pure Hodge structures are given by meromorphic connections on algebraic vector bundles over $\mathbb{A}^1$ that have a first order pole at zero and monodromy ± 1 on graded pieces. Thus the pair $({}^{\mathfrak{a}}H, {}^{\mathfrak{a}}\nabla)$ can be viewed as a family of ordinary pure complex Hodge structures parametrized by $v \in \mathbb{A}^1 - \{0\}$. But this is exactly the type of data that our Theorem 2.18 associates with a compactified tame complex Landau–Ginzburg model.

3.2.2. One parameter families of complex Landau–Ginzburg models. Let $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}})$ be a compactified tame complex Landau–Ginzburg model. By Theorem 2.18 the one parameter family of potentials $((\mathsf{Z}, q\cdot\mathsf{f}), D_{\mathsf{Z}})$ gives rise to a variation of complex pure Hodge structures parametrized by $q \in \mathbb{A}^1$. The de Rham part of this variation is given by a pair $\left({}^{\mathfrak{b}}H, {}^{\mathfrak{b}}\nabla\right)$, where

- ${}^{\mathfrak{b}}H$ is the coherent sheaf over $\mathbb{A}^2$ corresponding to the $\mathbb{C}[u,q]$-module

$$\mathbb{H}^{\bullet}\left(\mathsf{Z}, (\Omega^{\bullet}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})[u,q], ud + qd\mathsf{f})\right).$$

  By Theorem 2.18 the cohomology $\mathbb{H}^{\bullet}\left(\mathsf{Z}, (\Omega^{\bullet}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}), ud + qd\mathsf{f})\right)$ has constant dimension for all $(u,q) \in \mathbb{A}^2$ and so the sheaf ${}^{\mathfrak{b}}H$ is locally free.
- ${}^{\mathfrak{b}}\nabla$ is the Gauss–Manin connection for the family of complexes of f-adapted logarithmic forms. This is an algebraic meromorphic connection. For $u, q \neq 0$ the locally constant sections for ${}^{\mathfrak{b}}\nabla$ are identified with the topological cohomology $H^{\bullet}(Y, Y_{-\infty}; \mathbb{C})$ via the identifications from Lemma 2.21 and Lemma 2.23.

By construction, the restriction of $\left({}^{\mathfrak{b}}H, {}^{\mathfrak{b}}\nabla\right)$ on a line of the form $q = c$ is the u-connection describing the Tate twist folding of the pure Hodge structure on the vector space $\mathbb{H}^{\bullet}\left(\mathsf{Z}, (\Omega^{\bullet}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}), ud + c\cdot d\mathsf{f})\right)$. As explained in [**KKP08**, Section 2.2.7], this means that the vector bundle ${}^{\mathfrak{b}}H_{|q=c}$ is the Rees module associated with the Hodge filtration

$$\begin{aligned} &F^p\mathbb{H}^{\bullet}\left(\mathsf{Z}, (\Omega^{\bullet}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}), ud + c\cdot d\mathsf{f})\right) \\ &\qquad := \mathbb{H}^{\bullet}\left(\mathsf{Z}, \left(\Omega^{\bullet \geq p}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}), ud + c\cdot d\mathsf{f}\right)\right), \end{aligned}$$

on $\mathbb{H}^{\bullet}\left(\mathsf{Z},\left(\Omega^{\bullet}_{\mathsf{Z}}(\log D_{\mathsf{Z}},\mathsf{f}), ud + c\cdot d\mathsf{f}\right)\right)$, and that ${}^{\mathfrak{b}}\nabla$ is a connection with logarithmic singularity at $u = 0$ and monodromy ± 1 on graded pieces. In particular, $\left({}^{\mathfrak{b}}H, {}^{\mathfrak{b}}\nabla\right)$ has a logarithmic pole at $u = 0$.

Remark 3.10. It will be useful to have an explicit formula for the Gauss–Manin connection ${}^{\mathfrak{b}}\nabla$, similar to the formula (3.2.1). We can write such a formula for the Gauss–Manin connection acting on the complex $\left(\Omega^{\bullet}_{\mathsf{Z}}\left(*D_{\mathsf{Z}}\right)[u,q], ud - q d\mathsf{f}\wedge\right)$:

$$\begin{aligned} {}^{\mathfrak{b}}\nabla_{\frac{\partial}{\partial u}} &:= \frac{\partial}{\partial u} + u^{-2}\left(\mathsf{f}\cdot(\bullet)\right) + u^{-1}\mathbf{G} \\ {}^{\mathfrak{b}}\nabla_{\frac{\partial}{\partial q}} &:= \frac{\partial}{\partial q} - q^{-1}u^{-1}\left(\mathsf{f}\cdot(\bullet)\right), \end{aligned} \tag{3.2.2}$$

where $\mathbf{G}$ is the grading operator defined to be $\mathbf{G} := -\frac{p}{2}$ on $\Omega^{p}_{\mathsf{Z}}\left(*D_{\mathsf{Z}}\right)$.

This is the same formula that appears in the works of Shklyarov [**Shk11**] and Efimov [**Efi12**]. Note, however, that this formula does not preserve the subcomplex of f-adapted logarithmic forms since if $\alpha \in \Omega^{\bullet}_{\mathsf{Z}}\left(\log D_{\mathsf{Z}},\mathsf{f}\right)$, the form $f\alpha$ will not necessarily be in $\Omega^{\bullet}_{\mathsf{Z}}\left(\log D_{\mathsf{Z}},\mathsf{f}\right)$. Therefore, we cannot use these formulas directly to describe the action of ${}^{\mathfrak{b}}\nabla$ on ${}^{\mathfrak{b}}H$. This latter action is a combination of the formulas (3.2.2) and the complicated limiting quasi-isomorphism in the proof of Lemma 2.21.

3.2.3. Mirror symmetry for one parameter families. The formal similarity between the two connections $\left({}^{\mathfrak{a}}H, {}^{\mathfrak{a}}\nabla\right)$ and $\left({}^{\mathfrak{b}}H, {}^{\mathfrak{b}}\nabla\right)$ is very suggestive. We expect that when the geometric data defining these connections is part of a mirror pair, we should be able to go beyond a mere similarity and identify the pairs $\left({}^{\mathfrak{a}}H, {}^{\mathfrak{a}}\nabla\right)$ and $\left({}^{\mathfrak{b}}H, {}^{\mathfrak{b}}\nabla\right)$. More precisely we propose the following conjecture:

Conjecture 3.11. Suppose $(X, \omega_X, s_X) \mid ((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y)$ is a mirror pair of Fano type. Then

- **(i)** The one parameter symplectic family $(X, \omega_X + \log(q)\boldsymbol{k}_X)$ is mirrored into a one parameter complex family $(Y, q\cdot\mathsf{w})$ of deformations of (Y,w).
- **(ii)** The homological mirror correspondence induces an isomorphism

$$\left({}^{\mathfrak{a}}H, {}^{\mathfrak{a}}\nabla\right) \cong \left({}^{\mathfrak{b}}H, {}^{\mathfrak{b}}\nabla\right),$$

of meromorphic connections on $\mathbb{A}^2$.

The attentive reader will notice that the part **(i)** of this conjecture relies on a geometric one parameter perturbation of a Fano mirror pair but does not involve a compactification of the Landau–Ginzburg side of the pair. On the other hand, at least the B-side of Conjecture 3.11**(ii)** depends on a tame compactification $\left((\mathsf{Z},\mathsf{f}), D_{\mathsf{Z}}\right)$ of the Landau–Ginzburg model (Y,w).

Nevertheless, in Section 3.2.5 we will argue that part **(ii)** of Conjecture 3.11 is in fact a consequence of the homological mirror symmetry conjecture for the Fano pair itself. In other words: the existence of the tame compactification matters, while the choice of a particular compactification is not important. Indeed, as explained in [**KKP08**, Sections 2.2.2, 3.1, and 3.2], the pure **nc** Hodge structures on $H^\bullet(X,\mathbb{C})$ and $H^\bullet(Y,Y_{\mathrm{sm}};\mathbb{C})$ can be defined intrinsically in terms of the categories $\mathsf{Fuk}(X,\omega_X)$ and $\mathsf{MF}(Y,\mathsf{w})$ respectively. Since homological mirror symmetry identifies these two categories, it follows that the A-model **nc** Hodge structure on $H^\bullet(X,\mathbb{C})$ will be isomorphic to the B-model **nc** Hodge structure on $H^\bullet(Y,Y_{\mathrm{sm}};\mathbb{C})$. Via these identifications Conjecture 3.11**(ii)** reduces to checking that the two parameter meromorphic connections ${}^{\mathfrak{a}}\nabla$ and ${}^{\mathfrak{b}}\nabla$ can be reconstructed from the **nc** Hodge structures on $H^\bullet(X,\mathbb{C})$ and $H^\bullet(Y,Y_{\mathrm{sm}};\mathbb{C})$ respectively. To that end, in Section 3.2.5 we describe a general method for constructing a two parameter meromorphic connection from a pure **nc** Hodge structure.

3.2.4. Mirrors of tame compactifications of Landau–Ginzburg models. Before we proceed with the construction in Section 3.2.5, it is instructive to examine more closely the apparent mismatch in the information contained in the one parameter mirror symmetry

$$(X,\omega_X+\log(q)\boldsymbol{k}_X) \mid (Y,q\cdot\mathsf{w}),$$

and in the tame compactification of the Landau–Ginzburg model. This mismatch is ultimately a reflection of the fact that the one parameter deformations $(X,\omega_X+\log(q)\boldsymbol{k}_X)$ and $(Y,q\cdot\mathsf{w})$ only perturb one direction of the mirror symmetry: going from the A-model on the Fano side to the B-model on the Landau–Ginzburg side.

Since the choice of a tame compactification $((\mathsf{Z},\mathsf{f}),D_{\mathsf{Z}})$ is a choice additional data on the Landau–Ginzburg side, its mirror partner will necessarily depend on the choice of some additional data on the Fano side. A clue of what this additional data should be, appears in the works of Seidel [**Sei08**, **Sei11**] where the one parameter deformation $(X,\omega_X+\log(q)\boldsymbol{k}_X)$ is interpreted intrinsically in categorical terms. The relevant key fact from [**Sei08**, **Sei11**] is the statement that the family of Fukaya categories $\mathsf{Fuk}\,(X,\omega_X+\log(q)\boldsymbol{k}_X)$ has a well defined limit as $q\to 0$, namely the Fukaya category of the symplectic manifold underlying the non-compact Calabi–Yau $X-D_X$.

To simplify notation write $U := X-D_X$ for the complement of D_X, $\omega_U := \omega_{X|U}$ for the restriction of the symplectic structure to U, and $\mathsf{vol}_U = 1/s_X$ for the holomorphic volume form corresponding to the anticanonical section s_X. As explained in [**Sei08**, **Sei11**] (see also [**Aur07**]) the $\mathbb{Z}$-graded A_∞ category $\mathscr{F}_0 = \mathsf{Fuk}(U,\omega_U,\mathsf{vol}_U)$ admits a natural one-parameter deformation $\{\mathscr{F}_q\}$ as a $\mathbb{Z}/2$-graded A_∞ category. By construction $\mathscr{F}_q$ has the same objects and morphisms as $\mathscr{F}_0$ but the

A_∞ operations m_k^q in $\mathscr{F}_q$ are q-perturbations of the A_∞ operations m_k^0 in $\mathscr{F}_0$ where the correction term q^a comes with a coefficient counting not pseudo-holomorphic discs in U but rather disks in X that intersect the boundary divisor D_X at a points[4] . Now, a comparison with the standard construction [**FOOO09a**, **FOOO09b**] of the Fukaya category identifies $\mathscr{F}_q$ for $q \neq 0$ the with the $\mathbb{Z}/2$ category $\mathsf{Fuk}(X, \omega_X + \log(q)\boldsymbol{k}_X)$.

Thus we get a streamlined categorical (or **nc** geometric) interpretation of the A-model data associated with the Fano geometry (X, ω_X, s_X). In summary Seidel's analysis shows that:

- from the point of view of **nc** geometry, the primordial object is the $\mathbb{Z}$-graded Fukaya category $\mathscr{F}_0 = \mathsf{Fuk}\,(U, \omega_U, \mathsf{vol}_U)$;
- the data of a symplectic compactification $(U, \omega_U) \subset (X, \omega_X)$ with anti-canonical boundary $D_X = X - U$ corresponds to a q-deformation $\mathscr{F}_q = \mathsf{Fuk}(X, \omega_X + \log(q)\boldsymbol{k}_X)$ of $\mathscr{F}_0$ as a $\mathbb{Z}/2$-graded Calabi–Yau category.

To put it differently, the symplectic anti-canonical compactification $(U, \omega_U) \subset (X, \omega_X)$ is encoded in a one parameter degeneration of the Fukaya category $\mathscr{F}_1 = \mathsf{Fuk}(X, \omega_X)$ of the compact symplectic Fano (X, ω_X) to the Fukaya category $\mathscr{F}_0 = \mathsf{Fuk}\,(U, \omega_U, \mathsf{vol}_U)$ of the symplectic non-compact Calabi–Yau (U, ω_U).

This categorical interpretation of the compactification of U has a natural mirror incarnation. The non-compact symplectic Calabi–Yau $(U, \omega_U, \mathsf{vol}_U)$ has a complex non-compact Calabi–Yau mirror Y, constructed say by the SYZ prescription as in [**Aur07**]. Homological mirror symmetry predicts that $\mathscr{F}_0$ is equivalent to the category $\mathbf{D}_c^b(Y)$. The one parameter deformation $\mathscr{F}_q$ of $\mathscr{F}_0 = \mathbf{D}_c^b(Y)$ then corresponds to a class in the Hochschild cohomology $HH^\bullet(\mathscr{F}_0) = HH^\bullet(\mathbf{D}_c^b(Y))$. Since $\mathscr{F}_q$ is only a $\mathbb{Z}/2$-graded deformation, this Hochschild cohomology class will have a non-trivial component in HH^0, i.e., will give us a well defined element $\mathsf{w} \in H^0(Y, \mathcal{O}_Y)$. If we assume for symplicity that the boundary divisor D_X is smooth, then the Fano/Landau–Ginzburg homological mirror symmetry conjecture will identify $\mathscr{F}_q$ with $\mathsf{MF}(Y, q \cdot \mathsf{w})$ for $q \neq 0$. If we interpret $\mathsf{MF}(Y, q \cdot \mathsf{w})$ as a coproduct of the derived categories of singularities of the singular fibers of $q \cdot \mathsf{w}$, we see that this identification will specialize correctly when $q \to 0$. The category $\mathscr{F}_q$ specializes to $\mathscr{F}_0$ while $\mathsf{MF}(Y, q \cdot \mathsf{w})$ specializes to the compactly supported derived category of singularities of the derived fiber of the zero function on Y, which is readily identified with $\mathbf{D}_c^b(Y)$.

[4]Making this precise is quite subtle (see [**WW10**, **Sei11**]) and requires a version of the Fukaya category which is linear over $\mathbb{C}$ (rather than a Novikov field). In [**Sei11**] such a version is built out of balanced (rather than arbitrary) Lagrangians in U. We thank Denis Auroux for illuminating explanations of this subtlety.

The upshot of the previous discussion is that the mirror one parameter families

$$(X, \omega_X + \log(q)\boldsymbol{k}_X) \mid (Y, q \cdot \mathsf{w})$$

arising from the Fano mirror pair $(X, \omega_X, s_X) \mid ((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y)$ have a natural homological interpretation as families of (term by term equivalent) categories

$$\{\mathscr{F}_q\} = \{\mathbf{MF}(Y, q \cdot \mathsf{w})\},$$

where the family on the left hand side is the Seidel $\mathbb{Z}/2$-graded deformation of $\mathbf{Fuk}(U, \omega_U, \mathsf{vol}_U)$ corresponding to the compactification $(U, \omega_U) \subset (X, \omega_X)$.

This interpretation allows us to reverse the process and identify the mirror information corresponding to a tame compactification of Y. If we choose a tame compactification $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}})$ and also choose an extension ω_{Z} of the symplectic form ω_Y, then we can apply Seidel's analysis to the symplectic anti-canonical compactification $(Y, \omega_Y) \subset (\mathsf{Z}, \omega_{\mathsf{Z}})$. Since by the tameness assumption D_{Z} is an anti-canonical divisor, the same reasoning shows that this compactification is encoded in a one parameter deformation of the $\mathbb{Z}$-graded category $\mathbf{Fuk}(Y, \omega_Y, \mathsf{vol}_Y)$ to the $\mathbb{Z}/2$-graded category $\mathbf{Fuk}\,(\mathsf{Z}, \omega_{\mathsf{Z}} + \log(r)\boldsymbol{k}_{\mathsf{Z}})$. Again the degree zero piece of the Hochschild cohomology class governing this deformation will give us a holomorphic function $\mathbf{v} : U \to \mathbb{A}^1$. In fact the description of $\mathbf{v}$ in terms of the weighted disk counting on Z relative to the boundary D_{Z} also predicts that $\mathbf{v}$ has first order poles along D_X and so $\mathbf{v} = s/s_X$ for some anti-canonical section $s \in H^0(X, K_X^{-1})$. This can be packaged in the following conjecture:

Conjecture 3.12. Suppose $(X, \omega_X, s_X) \mid ((Y, \mathsf{w}), \omega_Y, \mathsf{vol}_Y)$ is a mirror pair of Fano type. Then

(i) a choice of a tame compactification $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}}, \omega_{\mathsf{Z}})$ of the Landau–Ginzburg side gives rise to a compactified Fano mirror pair

$$(X, \omega_X, s_X, \mathbf{v}) \mid ((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}}, \omega_{\mathsf{Z}}),$$

where $\mathbf{v}$ is a meromorphic function on X with a first order pole along D_X.

(ii) The Fano/Landau–Ginzburg homological mirror correspondence induces equivalences

$$\mathbf{Fuk}(X, \omega_X + \log(q)\boldsymbol{k}_X) \cong \mathbf{D}^b_c(Y, q \cdot \mathsf{w}),$$
$$\mathbf{D}^b_c(U, r \cdot \mathbf{v}) \cong \mathbf{Fuk}(\mathsf{Z}, \omega_{\mathsf{Z}} + \log(r)\boldsymbol{k}_{\mathsf{Z}}),$$

of one parameter families of categories.

Note that the geometric ingredients of the compactified mirror pair

$$(X, \omega_X, s_X, \mathbf{v}) \mid ((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}}, \omega_{\mathsf{Z}}),$$

from Conjecture 3.12**(i)** appear now in a symmetric fashion in the two sides of the pair. In particular, we expect A-model data on one side to be mirror to the B-model data on the other.

It is clear also that the statement of Conjecture 3.12**(ii)** is but one facet of the homological mirror correspondence one should associate with the compactified mirror pair. The full homological mirror conjecture will involve various equivalences of categories generalizing the equivalences described in Table 1 for an ordinary (non-compactified) Fano mirror pair. It is possible to list all these equivalences but the list is somewhat cumbersome. Very recently Seidel found [**Sei14b**] a uniform conceptual way for capturing the homological content of either side of the compactified mirror pair and gave a clean formulation of the complete homological mirror conjecture for the pair Conjecture 3.12**(i)** in terms of an equivalence of categories equipped with **nc** anti-canonical pencils.

It is very interesting to understand how the two meromorphic connections from Conjecture 3.11**(ii)** arise directly as Hodge theoretic data associated with these **nc** pencils but we will leave this for future investigations.

3.2.5. Construction of meromorphic connections over $\mathbb{A}^2$. Suppose $(H^\bullet, \nabla)$ is the de Rham part of a pure **nc** Hodge structure. In this section we explain how, under some mild technical assumptions, the pair $(H^\bullet, \nabla)$ gives rise to a meromorphic connection over the affine plane $\mathbb{A}^2$.

To keep track of the various copies of the affine line and the affine plane appearing in the construction, we will indicate the coordinates on these lines and planes as subscripts. Thus $\mathbb{A}^1_u$ will denote the affine line with coordinate u, $\mathbb{A}^2_{(u,q)}$ will denote the affine plane with coordinates (u, q), etc. By definition (see [**KKP08**, Section 2.1.4]) the pair (H, ∇) is the de Rham part of a pure **nc** Hodge structure if it satisfies:

- **$H^\bullet$:** is a $\mathbb{Z}/2$-graded algebraic vector bundle on the affine line $\mathbb{A}^1_u$, and
- **∇:** is a meromorphic connection on $H^\bullet$, which has at most a regular singularity at $u = \infty$, at most a second order pole at $u = 0$, and no other singularities in $\mathbb{A}^1_u$.

View $\mathbb{A}^1_u$ as the u-axis in the plane $\mathbb{A}^2_{(u,q)}$. Our goal is to extend $H^\bullet$ to a holomorphic bundle ${}^\ddagger H^\bullet$ on all of $\mathbb{A}^2_{(u,q)}$, and ∇ to a meromorphic connection ${}^\ddagger\nabla$ on ${}^\ddagger H^\bullet$ over $\mathbb{A}^2_{(u,q)}$ so that ${}^\ddagger\nabla$ has poles only at $uq = 0$ and has logarithmic singularities along $q = 0$. We will carry this out in two steps:

Step 1. Start with the connection $(H^\bullet, \nabla)$ on $\mathbb{A}^1_u$. Write (H, ∇) for the underlying ungraded algebraic vector bundle with connection. Since by assumption ∇ has a regular singularity at $u = \infty$ we can consider the

Deligne extension $(\mathcal{H}, \nabla)$ of (H, ∇) (see, e.g., [**Del70**, Chapter II.5] or [**Sab02**, Corollary II.2.21]). The bundle $\mathcal{H}$ is the algebraic vector bundle on $\mathbb{P}^1_u = \mathbb{A}^1_u \cup \{\infty\}$ which is uniquely characterized by the properties that at ∞ the connection ∇ has a logarithmic pole on $\mathcal{H}$, and that the residue $\operatorname{Res}^{\mathcal{H}}_{\infty}(\nabla) : \mathcal{H}_\infty \to \mathcal{H}_\infty$ is a nilpotent, grading preserving endomorphism of the fiber of $\mathcal{H}$ at ∞.

The bundle $\mathcal{H}$ decomposes into a direct sum of line bundles $\mathcal{H} = \oplus_{k=1}^{r} \mathcal{O}_{\mathbb{P}^1}(d_k)$ and so it admits a natural decreasing biregular filtration by subbundles

$$F^i\mathcal{H} = \bigoplus_{d_k \geq i} \mathcal{O}_{\mathbb{P}^1}(d_k), \qquad i \in \mathbb{Z}.$$

The restrictions $F^iH := F^i\mathcal{H}_{|\mathbb{A}^1_u}$ give a $\mathbb{Z}$-labeled filtration of H by holomorphic subbundles.

For any complex number $v \in \mathbb{C}$ consider the Rees bundle $\xi(H_v, F^\bullet H_v) \to \mathbb{A}^1_q$ associated with this filtration, [**Sim97**]. The bundle $\xi(H_v, F^\bullet H_v)$ is defined as the locally free sheaf on $\mathbb{A}^1_q$ associated with the $\mathbb{C}[q]$-submodule $\sum_i q^{-i} F^i H_v \subset H_v[q, q^{-1}]$. By construction $\xi(H_v, F^\bullet H_v)$ is a $\mathbb{C}^\times$-equivariant vector bundle on $\mathbb{A}^2_q$ for the scaling action of $\mathbb{C}^\times$ on the q-line. Allowing v to vary we get a Rees bundle $\xi(H, F^\bullet H) \to \mathbb{A}^2_{(v,q)}$ which is algebraic and equivariant for the $\mathbb{C}^\times$-action $\lambda \cdot (v, q) := (v, \lambda q)$. By construction we have canonical identifications

$$\xi(H, F^\bullet H)_{(v,1)} \cong H_v,$$

$$\xi(H, F^\bullet H)_{(v,0)} \cong \operatorname{gr}_{F^\bullet H_v} H_v.$$

Since the filtration $F^\bullet H$ was compatible with the grading on H, we get a natural $\mathbb{Z}/2$-grading on $\xi(H, F^\bullet H)$. Similarly, since $F^\bullet H$ arose from the Deligne extension of (H, ∇), the meromorphic connection ∇ on $H = \xi(H, F^\bullet H)_{|\mathbb{A}^1_v \times \{1\}}$ extends to a well defined meromorphic connection on $\xi(H, F^\bullet H)$ over the plane $\mathbb{A}^2_{(v,q)}$ which has poles on $vq = 0$ and on each line $\{v\} \times ||$, $v \neq 0$ has monodromy $+1$ on the even graded piece of $\xi(H_v, F^\bullet H)$ and -1 on the odd graded piece. To simplify notation we will write ${}^\dagger H^\bullet$ for the $\mathbb{Z}/2$-graded $\xi(H, F^\bullet H)$ on $\mathbb{A}^2_{(v,q)}$, and will write ${}^\dagger\nabla$ for the extension of the connection ∇.

Step 2. In this step we will modify the $\mathbb{Z}/2$-graded bundle with connection $\left({}^\dagger H, {}^\dagger\nabla\right)$ to ensure that its monodromy is ± 1 not on vertical lines but rather on lines through the origin.

Consider an affine plane $\mathbb{A}^2_{(u,q)}$ with coordinates (u, q). Let $\mathfrak{s} : S \to \mathbb{A}^2_{(u,q)}$ be the blow-up of $\mathbb{A}^2_{(u,q)}$ at the origin $(u, q) = (0, 0)$. The surface S is glued out of two affine charts $\mathbb{A}^2_{(v,q)}$ and $\mathbb{A}^2_{(u,w)}$ via the gluing map $u = vq$, $w = 1/v$. In particular, $\mathbb{A}^2_{(v,q)}$ embeds as a Zariski open subset in S and we have a commutative diagram of surfaces

$$\begin{tikzcd} \mathbb{A}^2_{(v,q)} \arrow[r, hook, "i"] \arrow[dr, "\mathfrak{p}"'] & S \arrow[d, "\mathfrak{s}"] \\ & \mathbb{A}^2_{(u,q)} \end{tikzcd}$$

where $i : \mathbb{A}^2_{(v,q)} \hookrightarrow S$ denotes the inclusion, and $\mathfrak{p}$ is the map

$$\mathfrak{p} : \mathbb{A}^2_{(v,q)} \to \mathbb{A}^1_{(u,q)}, \qquad (u,q) = \mathfrak{p}(v,q) = (vq,q).$$

Note that $S - \mathbb{A}^2_{(v,q)} = \mathbb{A}^1_u$ and that a point $u \neq 0 \in \mathbb{A}^1_u \subset S$ is a limiting point in S completing the hyperbola $\{(v,q)\ |vq = u\} \subset \mathbb{A}^2_{(v,q)}$ to a copy C^u of $\mathbb{A}^1$ embedded in S.

Now observe that if we restrict $({}^\dagger H, {}^\dagger\nabla)$ to the hyperbola $vq = u$, the restricted connection has a regular singularity as $v \to \infty$. Therefore, we have a canonical Deligne extension of ${}^\dagger H_{|vq=u}$ to an algebraic vector bundle on C^u. This process depends algebraically on u and so gives an extension of ${}^\dagger H$ to an algebraic vector bundle Ξ on the punctured surface $S - \{x\}$, where $x \in \mathbb{A}^2_{(u,w)}$ is the point with coordinates $u = 0$, $w = 0$.

Next observe that since the surface S is smooth, any vector bundle V on $S - \{x\}$ will extend to a (necessarily unique) vector bundle on S. Indeed, choose a torsion free coherent sheaf F on S which restricts to V on $S - \{x\}$. For instance, if $\jmath$ denotes the inclusion of $S - \{x\}$ in S, we can take F to be the intersection $\cap K$ of all coherent subsheaves $K \subset \jmath_* V$ such that $V \subset \jmath^* K$. Since V is locally free, the double dual $F^{\vee\vee}$ will also restrict to V. Being the dual of a coherent sheaf $F^{\vee\vee}$ is automatically reflexive, and by Auslander–Buchsbaum theorem can only fail to be locally free in codimension three. Thus $F^{\vee\vee}$ is a locally free sheaf which extends V to S. The uniqueness of the extension follows again from the fact that x is a smooth point and so the local ring $\mathcal{O}_{S,x}$ satisfies the Serre condition S_2.

Let $\widetilde{\Xi}$ be the unique extension of Ξ to S. To complete the construction we will need to know that $\widetilde{\Xi}$ satisfies a descent property for the morphism $\mathfrak{s} : S \to \mathbb{A}^2_{(u,q)}$. Let $E \subset S$ denote the exceptional $\mathbb{P}^1$ of the blow-up morphism $\mathfrak{s} : S \to \mathbb{A}^2_{(u,q)}$. With this notation we have the following:

Definition 3.13. We will say that an **nc** Hodge filtration $(H^\bullet, \nabla)$ is **extendable** if the restriction of the algebraic vector bundle $\widetilde{\Xi}$ to E is holomorphically trivial.

Note that if $(H^\bullet, \nabla)$ is extendable, then by the projection formula this extension $\widetilde{\Xi}$ is canonically a pullback of a vector bundle on $\mathbb{A}^2_{(u,q)}$, namely $\mathfrak{s}_*\widetilde{\Xi}$ is a vector bundle and $\widetilde{\Xi} \cong \mathfrak{s}^*\mathfrak{s}_*\widetilde{\Xi}$.

In particular, if the **nc** Hodge filtration $(H^\bullet, \nabla)$ is extendable, we get a well defined holomorphic bundle ${}^\ddagger H := \mathfrak{s}_* \widetilde{\Xi}$ on $\mathbb{A}^2_{(u,q)}$. The meromorphic connection ${}^\dagger\nabla$ on ${}^\dagger H$ is holomorphic on the open set $vq \neq 0$ and so can be viewed as a meromorphic connection ${}^\ddagger\nabla$ on ${}^\ddagger H$ with poles on $uq = 0$. Altogether we have proven the following:

Lemma 3.14. *Let $(H^\bullet, \nabla)$ be an extendable **nc** Hodge filtration, then $(H^\bullet, \nabla)$ gives rise to a $\mathbb{Z}/2$-graded meromorphic connection $\left({}^\ddagger H^\bullet, {}^\ddagger\nabla\right)$ on $\mathbb{A}^2_{(u,q)}$, such that*

- *${}^\ddagger\nabla$ is holomorphic away from $uq = 0$;*
- *${}^\ddagger\nabla$ has at most a logarithmic pole along $u = 0$, and a pole of order ≤ 2 along $q = 0$;*
- *The restriction of $\left({}^\ddagger H^0, {}^\ddagger\nabla\right)$ to a line through the origin has trivial monodromy, while the restriction of $\left({}^\ddagger H^1, {}^\ddagger\nabla\right)$ to a line through the origin has monodromy (-1).*

The discussion in Section 3.2.1 shows that the extendability assumption in the previous lemma holds for the de Rham part of the **nc** Hodge structure associated with a symplectic Fano variety:

Corollary 3.15. *Let (X, ω_X) be a symplectic manifold underlying a smooth Fano variety of complex dimension n. Let $*_1$ denote the quantum product corresponding to the symplectic form ω_X. Then the A-model **nc** Hodge filtration*

$$\left({}^{\mathsf{A}}H^\bullet, {}^{\mathsf{A}}\nabla\right) := \left(H^\bullet(X, \mathbb{C}) \otimes \mathcal{O}_{\mathbb{A}^1_u},\ d + (u^{-2}(\kappa_X \otimes_1 (\bullet)) + u^{-1}\mathbf{Gr})du\right),$$

*for the **nc** Hodge structure on the cohomology of (X, ω_X) is extendable and $\left({}^\ddagger H^\bullet, {}^\ddagger\nabla\right)$ reconstructs the standard q-variation of **nc** Hodge structures for the symplectic manifold (X, ω_X). That is, we have a canonical identification*

$$\left({}^\ddagger\left({}^{\mathsf{A}}H\right)^\bullet, {}^\ddagger\left({}^{\mathsf{A}}\nabla\right)\right) = \left({}^{\mathfrak{a}}H^\bullet, {}^{\mathfrak{a}}\nabla\right),$$

where $\left({}^{\mathfrak{a}}H^\bullet, {}^{\mathfrak{a}}\nabla\right)$ is the connection defined in (3.2.1).

Proof. Follows immediately from the two step construction above and by Lemma 3.9 and Lemma 3.14. q.e.d.

In particular, Corollary 3.15 shows that the one parameter mirror symmetry Conjecture 3.11**(ii)** is equivalent to the extendability property for the B-model **nc** Hodge filtration. More precisely, suppose (Y, w) is a complex Landau–Ginzburg model. Consider the B-model **nc** Hodge filtration for the cohomology of (Y, w):

$$\left({}^{\mathsf{B}}H^\bullet, {}^{\mathsf{B}}\nabla\right) = \left(\mathbb{H}^\bullet\left(\Omega^\bullet_Y[u], ud - d\mathsf{w}\wedge\right),\ d + \left(u^{-2}\left(\mathsf{w}\cdot(\bullet)\right) + u^{-1}\mathbf{G}\right)\right).$$

Here $\mathbf{G}$ is the grading operator of multiplication by $-p/2$ on Ω^p_Y. When (Y, w) is the mirror of a symplectic Fano variety (X, ω_X), homological mirror symmetry identifies the **nc** Hodge filtrations on the cohomology, i.e., gives an isomorphism

$$\left({}^{\mathsf{A}}H^{\bullet}, {}^{\mathsf{A}}\nabla\right) \cong \left({}^{\mathsf{B}}H^{\bullet}, {}^{\mathsf{B}}\nabla\right).$$

Combined with Corollary 3.15 this identification reduces Conjecture 3.11**(ii)** to the following purely algebro–geometric conjecture:

Conjecture 3.16. Suppose (Y, w) is a complex Landau–Ginzburg model which admits a tame compactification $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}})$ of log Calabi–Yau type. Then the associated **nc** Hodge filtration $\left({}^{\mathsf{B}}H^{\bullet}, {}^{\mathsf{B}}\nabla\right)$ is extendable, and

$$\left({}^{\ddagger}\left({}^{\mathsf{B}}H\right)^{\bullet}, {}^{\ddagger}\left({}^{\mathsf{B}}\nabla\right)\right) = \left({}^{\mathfrak{b}}H^{\bullet}, {}^{\mathfrak{b}}\nabla\right).$$

Remark 3.17. In a very interesting recent work Sabbah and Yu [**SY14**] consider a different but related notion of extendability arising from a nilpotent orbit for the pure complex Hodge structure attached to a compactification of a Landau–Ginzburg model. Moreover, they prove that the scaling variation of this pure Hodge structure is polarizable and satisfies their extendability condition. This result seems closely related to Conjecture 3.16 but we have not investigated the precise relation between the two statements.

3.3. Canonical decorations. In this section we take a closer look at the data needed to write special coordinates on the versal deformation space $\mathscr{M}$ of tame compactified Landau–Ginzburg models $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}})$ of log Calabi–Yau type. Recall from [**Tia87**] and [**Tod89**], that when Y is a smooth compact Calabi–Yau manifold of dimension $\dim_{\mathbb{C}} Y = d$, then any choice of a splitting of the Hodge filtration on $H^d_{DR}(Y, \mathbb{C})$ defines an analytic affine structure (= an integrable torsion free connection on the tangent bundle on) on the versal deformation space of Y. In [**KKP08**, Section 4.1.3] we analyzed the **nc** counterpart of this statement. In the **nc** setting the splitting of the **nc** Hodge filtration is encoded in the notion of a ***decoration*** (see [**KKP08**, Definition 4.5]) and in [**KKP08**, Claim 4.6] we argue that for decorated variations of pure **nc** Hodge structures of Calabi–Yau type there is a natural affine structure on the base of the variation.

This analysis applies directly to $\mathscr{M}$ and the B-model variation of **nc** Hodge structures over it. Concretely this variation is given by a $\mathbb{Z}/2$-graded holomorphic bundle with connection $\left({}^{\mathfrak{B}}H^{\bullet}, {}^{\mathfrak{B}}\nabla\right)$ over $\mathbb{A}^1_u \times \mathscr{M}$, where the fiber of ${}^{\mathfrak{B}}H^{\bullet}$ over a point $\{u\} \times \{(\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}})\}$ is the hypercohomology $\mathbb{H}^{\bullet}\left(\mathsf{Z}; (\Omega^{\bullet}_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}), ud + d\mathsf{f}\wedge)\right)$ and ${}^{\mathfrak{B}}\nabla$ is the Gauss–Manin connection. Consider the projective line $\mathbb{P}^1_u = \mathbb{A}^1_u \cup \{\infty\}$ compactifying $\mathbb{A}^1_u$. As explained in [**KKP08**, Section 4.1.3] the special coordinates on $\mathscr{M}$ arise from decoration data for $\left({}^{\mathfrak{B}}H^{\bullet}, {}^{\mathfrak{B}}\nabla\right)$. By definition a decoration is a pair $\left({}^{\mathfrak{B}}\widetilde{H}^{\bullet}, \psi\right)$, where

- ${}^{\mathfrak{B}}\widetilde{H}^{\bullet}$ is an extension of ${}^{\mathfrak{B}}H^{\bullet}$ to a $\mathbb{Z}/2$-graded holomorphic vector bundle on $\mathbb{P}^1_u \times \mathscr{M}$ for which ${}^{\mathfrak{B}}\nabla$ has a logarithmic pole along $\{\infty\} \times \mathscr{M}$.
- ψ is a holomorphic section of ${}^{\mathfrak{B}}\widetilde{H}^{\bullet}_{|\{\infty\}\times\mathscr{M}}$ which is horizontal with respect to the holomorphic connection ${}^{\mathfrak{B}}\widetilde{H}\left({}^{\mathfrak{B}}\nabla\right)$ induced from ${}^{\mathfrak{B}}\nabla$.

Remark 3.18. The variation $\left({}^{\mathfrak{B}}H^{\bullet}, {}^{\mathfrak{B}}\nabla\right)$ is a multi parameter variant of the one parameter variation $\left({}^{\mathfrak{b}}H^{\bullet}, {}^{\mathfrak{b}}\nabla\right)$ we considered in Section 3.2.2 and Conjecture 3.16. In fact, the one parameter variation $\left({}^{\mathfrak{b}}H^{\bullet}, {}^{\mathfrak{b}}\nabla\right)$ is the restriction of the multi parameter variation $\left({}^{\mathfrak{B}}H^{\bullet}, {}^{\mathfrak{B}}\nabla\right)$ to the straight line in $\mathscr{M}$ given by the scaling of a fixed potential by a complex number q.

In the remainder of this section we will describe a conjectural construction which will produce a natural decoration in this setting, i.e., will lead to canonical special coordinates that do not depend on random choices. The construction is based on mirror symmetry considerations and a description of decorations for the A-model variation of **nc** Hodge structures. We begin by recalling the relationship between filtrations and logarithmic extensions that we used repeatedly in the previous section and in [**KKP08**, Section 4.1.3].

3.3.1. Extensions and filtrations. Let $\boldsymbol{D} = \{t \in \mathbb{C} \mid |t| < R \ll 1\}$ be a small one dimensional complex disk centered at zero, and let $\boldsymbol{D}^{\times} = \boldsymbol{D} - \{0\}$ denote the corresponding punctured disk. Let (V, ∇) be a holomorphic bundle with holomorphic connection on $\boldsymbol{D}^{\times}$, and suppose ∇ is meromorphic and has a regular singularity at 0. By Deligne's extension theorem [**Del70**, Chapter II.5], [**Sab02**, Corollary II.2.21] we can always find a holomorphic bundle on $\boldsymbol{D}$ which extends V, and on which ∇ has a logarithmic pole. Fixing one such extension $\mathcal{V}$ as a reference point we can use the Deligne–Malgrange classification theorem [**Sab02**, Theorem III.1.1] to enumerate all other logarithmic extensions of (V, ∇) by their relative position to $\mathcal{V}$ (see, e.g., [**Sab02**, Chapter III] or [**KKP08**, Section 4.1.3]). In particular, the choice of $\mathcal{V}$ gives a bijection

$$\left(\begin{array}{l}\text{Holomorphic extensions of } V \text{ to}\\ \boldsymbol{D} \text{ on which } \nabla \text{ has logarithmic}\\ \text{pole at } 0\end{array}\right)$$

$$\updownarrow$$

$$\left(\begin{array}{l}\text{Increasing biregular filtrations of } V \text{ by}\\ \text{covariantly constant holomorphic sub-}\\ \text{bundles } V^{\leq i} \subset V \text{ on } \boldsymbol{D}^{\times}\end{array}\right).$$

If we choose for concreteness $\mathcal{V}$ to be the unique Deligne extension on which ∇ has a logarithmic pole at 0 and a residue with eigenvalues whose real parts are in $(-1,0]$, then the above bijection can be described explicitly as follows. Let $\widetilde{V}$ be another extension of V on which ∇ has a logarithmic pole. Fix an analytic trivialization of $\widetilde{V}$ near $t=0$ and let $||\bullet||$ denote the Hermitian norm of a section of V computed in this trivialization. For any $t \in \boldsymbol{D}$ and any $v \in V_t$ we have a well defined ∇-horizontal section $s_v(r)$ of V over the segment $(0,1]\cdot t$ uniquely determined by the initial condition $s_v(1)=v$. With this notation we have

$$V_t^{\leq i} = \left\{ v \in V_t \;\middle|\; \begin{array}{l} \text{The } \nabla\text{-horizontal section } s_v(r) \\ \text{satisfies } ||s_v(r)|| = \boldsymbol{O}\left(r^{-i}\right) \end{array} \right\}.$$

Remark 3.19. • The growth condition defining $V^{\leq i}$ depends on the extension $\widetilde{V}$ but not on the choice of a local holomorphic frame of $\widetilde{V}$ near 0.

• In [**KKP08**, Section 4.1.3] we discussed the classification of logarithmic extensions of (V,∇) in terms of biregular decreasing filtrations of V. The above description of $V^{\leq i}$ is just a relabeling of the filtrations described in [**KKP08**, Section 4.1.3].

3.3.2. The A-model decoration. Let (X,ω_X) be a compact symplectic manifold of real dimension $2n$. Under the convergence assumption from [**KKP08**, Section 3.1] for the quantum multiplication $*_q$, the **nc** Hodge filtration on the de Rham cohomology of X is encoded in the meromorphic connection on $\mathbb{A}^1_u$:

$$\left({}^{\mathsf{A}}H, {}^{\mathsf{A}}\nabla\right) := \left(H^\bullet(X,\mathbb{C}) \otimes \mathcal{O}_{\mathbb{A}^1_u},\ d + (u^{-2}(\kappa_X \otimes_1 (\bullet)) + u^{-1}\mathsf{Gr})du\right),$$

where $*_1$ denotes the quantum product for ω_X. Note that by definition ${}^{\mathsf{A}}\nabla$ has a regular singularity at $u=\infty$.

Remark 3.20. Conjecturally the convergence assumption on $*_q$ is closely related to the properties of the **nc** geometry attached to the pair (X,ω_X). In particular, if convergence for $q=1$ holds we expect that

(i) the Fukaya category $\mathsf{Fuk}(X,\omega_X)$ is smooth and compact;
(ii) the geometrically defined **nc** Hodge filtration $\left({}^{\mathsf{A}}H, {}^{\mathsf{A}}\nabla\right)$ coincides with the **nc** Hodge filtration on $HP_\bullet(\mathsf{Fuk}(X,\omega_X))$ defined in [**KKP08**, Section 2.2.5];
(iii) the monodromy of ${}^{\mathsf{A}}\nabla$ around $u=\infty$ is unipotent and conjugate to the classical multiplication $\kappa_X \wedge (\bullet)$ by the canonical class.

Trough a combination of various results from [**Abo09**, **RS15**, **Shk11**, **Efi12**] properties (i)–(iii) are known to hold when (X,ω_X) underlies a smooth toric Fano variety.

As we explained in Section 3.3.1, logarithmic extensions of $\left({}^{\mathsf{A}}H, {}^{\mathsf{A}}\nabla\right)$ across $u=\infty$ will correspond to ${}^{\mathsf{A}}\nabla$-horizontal filtrations of ${}^{\mathsf{A}}H$. In particular, from the definition of Gr we see that extensions of ${}^{\mathsf{A}}H$ as a trivial bundle over $\mathbb{P}^1_u = \mathbb{A}^1_u \cup \{\infty\}$, will correspond to filtrations ${}^{\mathsf{A}}H^{\leq \bullet}$ whose associated graded is isomorphic to $H^{\bullet+n}(X,\mathbb{C}) \otimes \mathcal{O}_{\mathbb{A}^1_u}$ as a $\mathbb{Z}$-graded bundle on $\mathbb{A}^1_u$. Therefore, in order to get a decoration for the A-model data $\left({}^{\mathsf{A}}H, {}^{\mathsf{A}}\nabla\right)$ we need to specify a canonical ${}^{\mathsf{A}}\nabla$-covariant filtration on ${}^{\mathsf{A}}H$ whose associated graded pieces have dimensions $h^{\bullet+n}(X,\mathbb{C})$. If in addition this filtration depends holomorphically on ω_X, then it will automatically give a decoration not only for the fixed nc Hodge filtration $\left({}^{\mathsf{A}}H, {}^{\mathsf{A}}\nabla\right)$ but also for the universal variation $\left({}^{\mathfrak{A}}H, {}^{\mathfrak{A}}\nabla\right)$ over the cone of complexified symplectic structures.

Such a canonical decoration arises naturally in the Fano case. Indeed, suppose that (X, ω_X) underlies a complex Fano manifold of complex dimension n, and that property (iii) from Remark 3.20 holds for (X, ω_X). In this case the operator $\kappa_X \wedge (\bullet)$ satisfies the Lefschetz property on $H^\bullet(X,\mathbb{C})$ and, in particular, has Jordan blocks which are symmetrically situated around the middle dimension n. In particular, the Lefschetz filtration (= the monodromy weight filtration for the nilpotent operator $\kappa_X \wedge (\bullet)$) will have associated graded pieces with dimensions $h^{\bullet+n}(X,\mathbb{C})$. Thus the extension of ${}^{\mathsf{A}}H$ across $u=\infty$ corresponding to this filtration will be holomorphically trivial on $\mathbb{P}^1_u$. This shows that for a symplectic Fano the universal Calabi–Yau variation of nc Hodge structures $\left({}^{\mathfrak{A}}H, {}^{\mathfrak{A}}\nabla\right) \to \mathbb{A}^1_u \times \mathscr{K}$ over the complexified Kähler cone $\mathscr{K}$ will have a canonical decoration data:

- ${}^{\mathfrak{A}}\widetilde{H}$**:** is the holomorphic extension of ${}^{\mathfrak{A}}H$ to $\mathbb{P}^1_u \times \mathscr{K}$ which corresponds to the monodromy weight filtration for the monodromy around $u = \infty$.
- ψ**:** is the covariantly constant section of ${}^{\mathfrak{A}}\widetilde{H}_{|\{\infty\}\times\mathscr{K}}$ defined by $\psi(\infty,\beta) = s(\infty)$, where $s \in \Gamma\left(\mathbb{P}^1_u \times \{\beta\}, {}^{\mathfrak{A}}\widetilde{H}\right)$ is the unique holomorphic section in the trivial bundle ${}^{\mathfrak{A}}\widetilde{H}_{|\mathbb{P}^1_u\times\{\beta\}} \cong H^\bullet(X,\mathbb{C}) \otimes \mathcal{O}$ whose value at $(0,\beta)$ is $1 \in H^0(X,\mathbb{C})$.

For ease of reference it will be useful to introduce terminology that describes this extendability behavior. Again fix a small complex disk $\boldsymbol{D}$ and a meromorphic connection (V,∇) on $\boldsymbol{D}^\times$ with a regular singularity and unipotent monoromy around zero. Fix the unique Deligne extension $\mathscr{V} \to \boldsymbol{D}$ of V on which ∇ has a logarithmic pole with nilpotent residue. As we saw in Section 3.3.1 this data establishes a 1-to-1 correspondence between logarithmic extensions of V to $\mathbb{P}^1_u$ and covariantly constant biregular increasing filtrations of $V \to \mathbb{A}^1_u$.

Definition 3.21. • The **skewed canonical extension** of V is the holomorphic vector bundle $\widetilde{V}$ which corresponds to the monodromy weight filtration for the monodromy operator around $u = \infty$.

• An abstract **nc** Hodge filtration $(H^\bullet \to \mathbb{A}^1_u, \nabla)$ will be called **special** if ∇ has unipotent monodromy around $u = \infty$ and the corresponding skewed canonical extension $\widetilde{H}$ of H is holomorphically trivial.

In these terms the discussion about the extendability behavior of the A-model **nc** Hodge structure above can be rephrased as the statement that the A-model **nc** Hodge filtration $\left({}^{\mathsf{A}}H, {}^{\mathsf{A}}\nabla\right)$ associated with a symplectic Fano variety is special. Specialty is the main property needed to define the canonical decoration for the universal A-model variation $\left({}^{\mathfrak{A}}H, {}^{\mathfrak{A}}\nabla\right)$.

Since the skewed extension and the specialty property are intrinsically determined by the monodromy, it is straightforward to transfer them through the mirror correspondence and to formulate the B-model extendability that will give rise to a canonical decoration of $\left({}^{\mathfrak{B}}H, {}^{\mathfrak{B}}\nabla\right)$ and canonical special coordinates on the moduli $\mathscr{M}$ of compactified Landau–Ginzburg models.

3.3.3. The B-model decoration. The A-model picture in the previous section and the mirror identification of the A and B model universal variations of **nc** Hodge structures suggest that a canonical decoration for $\left({}^{\mathfrak{B}}H, {}^{\mathfrak{B}}\nabla\right)$ and canonical special coordinates on the moduli $\mathscr{M}$ arise from the skewed extension of the B-model **nc** Hodge structure. Specifically we get the following purely algebro-geometric conjecture:

Conjecture 3.22. (a) Let (Y, w) be a complex Landau–Ginzburg model, and let

$$\left({}^{\mathsf{B}}H^\bullet, {}^{\mathsf{B}}\nabla\right) = \left(\mathbb{H}^\bullet\left(\Omega^\bullet_Y[u], ud - d\mathsf{w}\wedge\right),\ d + \left(u^{-2}\left(\mathsf{w}\cdot(\bullet)\right) + u^{-1}\mathbf{G}\right)\right)$$

be the **nc** Hodge filtration on the de Rham cohomology of (Y, w). If (Y, w) admits a tame compactification $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}})$ of Calabi–Yau type, then $\left({}^{\mathsf{B}}H^\bullet, {}^{\mathsf{B}}\nabla\right)$ is special.

(b) Let $\mathscr{M}$ be the versal deformation space of $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}})$. The universal B-model variation $\left({}^{\mathfrak{B}}H^\bullet, {}^{\mathfrak{B}}\nabla\right)$ over $\mathbb{A}^1_u \times \mathscr{M}$ has a canonical decoration data:

${}^{\mathfrak{B}}\widetilde{H}$**:** is the skewed extension of ${}^{\mathfrak{B}}H$ to $\mathbb{P}^1_u \times \mathscr{M}$.

ψ**:** is the covariantly constant section of ${}^{\mathfrak{B}}\widetilde{H}_{|\{\infty\}\times\mathscr{M}}$ defined by $\psi(\infty, ((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}})) = s(\infty)$, where $s \in \Gamma\Big(\mathbb{P}^1_u \times \{((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}})\}, {}^{\mathfrak{B}}\widetilde{H}\Big)$ is the unique holomorphic section in the trivial bundle

$${}^{\mathfrak{B}}\widetilde{H}_{|\mathbb{P}^1_u\times\{((\mathsf{Z},\mathsf{f}),D_{\mathsf{Z}})\}} \cong H^\bullet_{DR}((Y, \mathsf{w}); \mathbb{C}) \otimes \mathcal{O},$$

whose value at $(0, ((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}}))$ is $1 \in H^0(\mathsf{Z}, \Omega^n_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}))$.

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