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A CLASSICAL MODEL FOR DERIVED CRITICAL LOCI

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Abstract

Let $f : U \to \mathbb{A}^1$ be a regular function on a smooth scheme U over a field $\mathbb{K}$. Pantev, Toën, Vaquié and Vezzosi [30, 37] define the 'derived critical locus' $\mathbf{Crit}(f)$, an example of a new class of spaces in derived algebraic geometry, which they call '-1-shifted symplectic derived schemes'.

They show intersections of algebraic Lagrangians in a smooth symplectic $\mathbb{K}$-scheme, and stable moduli schemes of coherent sheaves on a Calabi–Yau 3-fold over $\mathbb{K}$, are also -1-shifted symplectic derived schemes. Thus, their theory may have applications in algebraic symplectic geometry, and in Donaldson–Thomas theory of Calabi–Yau 3-folds.

This paper defines and studies a new class of spaces we call 'algebraic d-critical loci', which should be regarded as classical truncations of the -1-shifted symplectic derived schemes of [30]. They are simpler than their derived analogues. We also give a complex analytic version of the theory, and an extension to Artin stacks.

In the sequels [4–8] we will apply d-critical loci to motivic and categorified Donaldson–Thomas theory, and to intersections of complex Lagrangians in complex symplectic manifolds. We will show that the important structures one wants to associate to a derived critical locus — virtual cycles, perverse sheaves, $\mathscr{D}$-modules, and mixed Hodge modules of vanishing cycles, and motivic Milnor fibres — can be defined for oriented d-critical loci.

Contents

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1. Introduction

Pantev, Toën, Vaquié and Vezzosi [30, 37] defined a new notion of *derived critical locus*. It is set in the context of Toën and Vezzosi's theory of derived algebraic geometry [34–36], and consists of a quasi-smooth derived scheme $\boldsymbol{X}$ equipped with a -1-shifted symplectic structure ω. In fact Pantev et al. [30] define k-shifted symplectic structures on derived stacks for $k \in \mathbb{Z}$, but the case relevant to this paper is $k = -1$, and derived schemes rather than derived stacks.

The following are examples of -1-shifted symplectic derived schemes:

(a) The critical locus $\mathrm{Crit}(f)$ of a regular function $f : U \to \mathbb{A}^1$ on a smooth $\mathbb{K}$-scheme U.
(b) The intersection $L \cap M$ of smooth Lagrangians L, M in an algebraic symplectic manifold (S, ω).
(c) A moduli scheme $\mathcal{M}$ of stable coherent sheaves on a Calabi–Yau 3-fold.

Parts (b),(c) are the beginning of applications of these structures in symplectic geometry and in Donaldson–Thomas theory of Calabi–Yau 3-folds.

This paper will define and study a new class of geometric objects we call *d-critical loci* (X, s). They are much simpler than -1-shifted symplectic derived schemes, and are entirely 'classical', by which we mean they are defined up to isomorphism in an ordinary category using classical algebraic geometry in the style of Hartshorne [14], rather than being defined up to equivalence in an ∞-category using homotopy theory and derived algebraic geometry as in [34–36].

In fact we give two versions of the theory, *complex analytic d-critical loci* (X, s) in which X is a complex analytic space, and *algebraic d-critical loci* (X, s) in which X is a scheme over a field $\mathbb{K}$. In both cases $s \in H^0(\mathcal{S}_X^0)$ is a global section of a certain sheaf $\mathcal{S}_X^0$ on X, satisfying some local conditions. When we can we give results and/or proofs for both complex analytic and algebraic versions simultaneously, or just briefly indicate the differences between the two.

In the algebraic case there are several topologies we could work with — the Zariski topology, the étale topology, and for an embedding $X \hookrightarrow U$ of a $\mathbb{K}$-scheme X into a smooth $\mathbb{K}$-scheme U, it may be natural to consider the formal completion $\hat{U}$ of U along X, and work Zariski or étale locally on $\hat{U}$. Whenever we can, we will use the Zariski topology. One reason is that Theorem 1.4 below, proved in [8], requires the Zariski rather than the étale topology, as distinct motives in $\overline{\mathcal{M}}_X^{\hat{\mu}}[\mathbb{L}^{-1/2}]$ can become equal on an étale open cover of X.

To persuade the reader that d-critical loci are a useful idea, we quote four results from the sequels [4–8] to this paper:

Theorem 1.1 (Bussi, Brav and Joyce [6])**.** *Suppose* $(\boldsymbol{X}, \omega)$ *is a* -1*-shifted symplectic derived scheme in the sense of Pantev et al.* [30] *over an algebraically closed field* $\mathbb{K}$ *of characteristic zero, and let* $X = t_0(\boldsymbol{X})$ *be the associated classical* $\mathbb{K}$*-scheme of* $\boldsymbol{X}$*. Then* X *extends naturally to an algebraic d-critical locus* (X, s)*. The canonical bundle* $K_{X,s}$ *from* §2.4 *is naturally isomorphic to the determinant line bundle* $\det(\mathbb{L}_{\boldsymbol{X}})|_{X^{\mathrm{red}}}$ *of the cotangent complex* $\mathbb{L}_{\boldsymbol{X}}$ *of* $\boldsymbol{X}$*. Zariski locally on* X *one can reconstruct* $(\boldsymbol{X}, \omega)$ *from* (X, s) *up to equivalence, but this may not be possible globally.*

That is, there is a (non-full) truncation functor from -1-shifted symplectic derived schemes to algebraic d-critical loci. The theorem implies that examples (a)–(c) above have the structure of d-critical loci.

Theorem 1.2. (a) (Bussi, Brav and Joyce [6, Cor. 6.8]) *Suppose (S,ω) is an algebraic symplectic manifold over $\mathbb{K}$, and L,M are smooth algebraic Lagrangians in S. Then the intersection $X = L\cap M$, as a $\mathbb{K}$-subscheme of S, extends naturally to an algebraic d-critical locus (X,s). The canonical bundle $K_{X,s}$ from §2.4 is isomorphic to $K_L|_{X^{\rm red}}\otimes K_M|_{X^{\rm red}}$.*

(b) (Bussi [7, Th. 3.1]) *Suppose (S,ω) is a complex symplectic manifold, and L,M are complex Lagrangian submanifolds in S. Then the intersection $X = L\cap M$, as a complex analytic subspace of S, extends naturally to a complex analytic d-critical locus (X,s), with canonical bundle $K_{X,s}\cong K_L|_{X^{\rm red}}\otimes K_M|_{X^{\rm red}}$.*

Theorem 1.2(a) is a corollary of Theorem 1.1 and [30, Th. 2.10], but Theorem 1.2(b) is proved directly, without going via derived algebraic geometry.

In §2.4, for a d-critical locus (X,s) we construct a line bundle $K_{X,s}$ on the reduced complex analytic subspace or subscheme $X^{\rm red}$, called the *canonical bundle* of (X,s), and in §2.5 we define an *orientation* on (X,s) to be a choice of square root $K_{X,s}^{1/2}$ of $K_{X,s}$ on $X^{\rm red}$. If $X = \mathrm{Crit}(f)$ for U a complex manifold and $f : U\to\mathbb{C}$ holomorphic then $K_{X,s}\cong K_U^{\otimes^2}|_{X^{\rm red}}$, so there is a natural square root $K_U|_{X^{\rm red}}$ for $K_{X,s}$. Examples in §2.5 show that orientations need not exist, or be unique. Here are two results on oriented d-critical loci:

Theorem 1.3 (Bussi, Brav, Dupont, Joyce and Szendrői [5]). *Suppose (X,s) is a complex analytic d-critical locus with orientation $K_{X,s}^{1/2}$. Then we construct a $\mathbb{Z}$-perverse sheaf $P^\bullet_{X,s}$, a $\mathscr{D}$-module $D_{X,s}$, and a mixed Hodge module $H^\bullet_{X,s}$ on X. If $(X,s), K_{X,s}^{1/2}$ are locally modelled on $\mathrm{Crit}(f), K_U|_{\mathrm{Crit}(f)^{\rm red}}$ for U a complex manifold and $f : U\to\mathbb{C}$ holomorphic, then $P^\bullet_{X,s}, D_{X,s}, H^\bullet_{X,s}$ are modelled on the perverse sheaf, $\mathscr{D}$-module and mixed Hodge module of vanishing cycles of U,f.*

Analogues hold for oriented algebraic d-critical loci (X,s), yielding an algebraic $\mathbb{Z}$-perverse sheaf $P^\bullet_{X,s}$, $\mathscr{D}$-module $D_{X,s}$, and mixed Hodge module $H^\bullet_{X,s}$ on X if X is a $\mathbb{C}$-scheme, and a $\mathbb{Z}_l$-perverse sheaf $P^\bullet_{X,s}$ and a $\mathscr{D}$-module $D_{X,s}$ on X if X is a $\mathbb{K}$-scheme and $l\neq \mathrm{char}\,\mathbb{K}$ a prime.

Theorem 1.4 (Bussi, Joyce and Meinhardt [8]). *Let (X,s) be an algebraic d-critical locus over $\mathbb{K}$ with an orientation $K_{X,s}^{1/2}$. Then we construct a motive $MF_{X,s}$ in a certain ring of motives $\overline{\mathcal{M}}_X^{\hat\mu}$ over X. If $(X,s), K_{X,s}^{1/2}$ are Zariski locally modelled on $\mathrm{Crit}(f), K_U|_{\mathrm{Crit}(f)^{\rm red}}$ for U a* *smooth $\mathbb{K}$-scheme and $f : U \to \mathbb{A}^1$ regular, then $MF_{X,s}$ is locally modelled on $\mathbb{L}^{-\dim U/2}\big([X] - MF^{\mathrm{mot}}_{U,f}\big)$, where $MF^{\mathrm{mot}}_{U,f}$ is the motivic Milnor fibre of f.*

In [4] we will generalize Theorems 1.1, 1.3 and 1.4 from $\mathbb{K}$-schemes to Artin $\mathbb{K}$-stacks. Theorems 1.3 and 1.4 (and their extension to stacks [4]) also have applications to extensions of Donaldson–Thomas theory of Calabi–Yau 3-folds, as in [17–19, 33]. Theorem 1.3 is important for *categorification* of Donaldson–Thomas invariants — defining a graded vector space (the hypercohomology $\mathbb{H}^*(P^\bullet_{X,s})$ of the perverse sheaf) whose dimension is the Donaldson–Thomas invariant, as proposed by Dimca and Szendrői [9] — and hence for constructing *cohomological Hall algebras*, following Kontsevich and Soibelman [19]. Theorem 1.4 is helpful for defining *motivic* Donaldson–Thomas invariants, as in [18].

We have explained that d-critical loci are classical truncations of −1-shifted symplectic derived schemes in [30]. There is another geometric structure which is a semiclassical truncation of −1-shifted symplectic derived schemes: Behrend's schemes with symmetric obstruction theories [2], which we now define.

Definition 1.5. Let X be a $\mathbb{K}$-scheme. A *perfect obstruction theory* on X in the sense of Behrend and Fantechi [3] is a morphism $\phi : \mathcal{E}^\bullet \to \mathbb{L}_X$ in the derived category $D(\mathrm{qcoh}(X))$, where $\mathbb{L}_X$ is the cotangent complex of X, satisfying:

- (i) $\mathcal{E}^\bullet$ is quasi-isomorphic locally on X to a complex $[\mathcal{F}^{-1} \to \mathcal{F}^0]$ of vector bundles in degrees $-1, 0$;
- (ii) $h^0(\phi) : h^0(\mathcal{E}^\bullet) \to h^0(\mathbb{L}_X)$ is an isomorphism; and
- (iii) $h^{-1}(\phi) : h^{-1}(\mathcal{E}^\bullet) \to h^{-1}(\mathbb{L}_X)$ is surjective.

Following Behrend [2], we call $\phi : \mathcal{E}^\bullet \to \mathbb{L}_X$ a *symmetric obstruction theory* if we also are given an isomorphism $\theta : \mathcal{E}^\bullet \to \mathcal{E}^{\bullet\vee}[1]$ with $\theta^\vee[1] = \theta$.

If U is a smooth $\mathbb{K}$-scheme and $f : U \to \mathbb{A}^1$ a regular function then $X = \mathrm{Crit}(f)$ has a natural symmetric obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_X$ with

$$\mathcal{E}^\bullet = \big[TU|_X \xrightarrow{\partial^2 f|_X} T^*U|_X\big]. \tag{1.1}$$

But Pandharipande and Thomas [29] give examples of schemes X with symmetric obstruction theories with X not locally isomorphic to a critical locus. Schemes with symmetric obstruction theories are the basis of Joyce and Song's theory of Donaldson–Thomas invariants of Calabi–Yau 3-folds [17]. If $(\boldsymbol{X}, \omega)$ is a −1-shifted symplectic derived scheme in the sense of Pantev et al. [30], then the classical scheme $X = t_0(\boldsymbol{X})$ has a symmetric obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_X$ with $\mathcal{E}^\bullet = i^*(\mathbb{L}_{\boldsymbol{X}})$ and $\theta = i^*(\omega_0)$, where $i : X \hookrightarrow \boldsymbol{X}$ is the inclusion.

We illustrate the relations between these structures in Figure 1.1. The two dotted arrows '$\dashrightarrow$' indicate a construction which works locally, but not globally. That is, given an algebraic d-critical locus (X, s), then Zariski locally on X we can construct both a -1-shifted symplectic derived scheme $(\boldsymbol{X}, \omega)$, and a symmetric obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_X$, θ, uniquely up to equivalence, but we cannot combine these local models to make $(\boldsymbol{X}, \omega)$ or $\mathcal{E}^\bullet, \phi, \theta$ globally on X because of difficulties with gluing 'derived' objects on open covers.

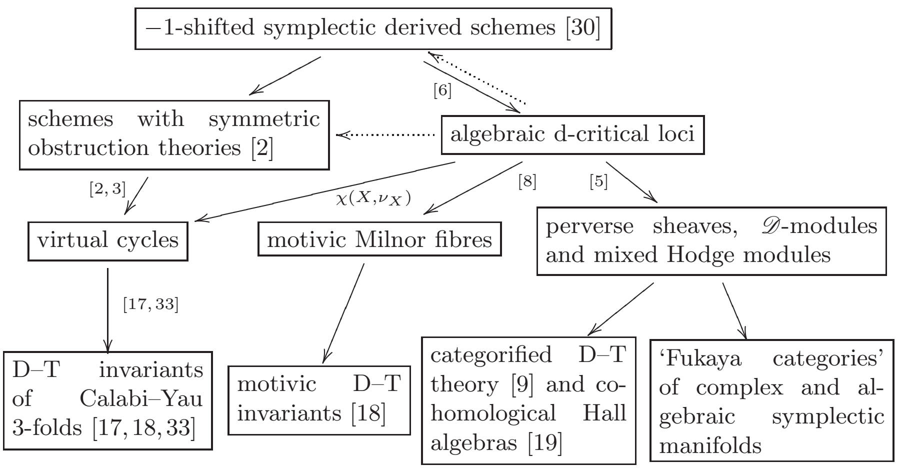

Figure 1.1. Relations between different structures, and applications

If X is a proper $\mathbb{K}$-scheme with obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_X$ then Behrend and Fantechi define a *virtual cycle* $[X]^{\mathrm{vir}}$ in Chow homology $A_*(X)$. If the obstruction theory is symmetric, and $\mathbb{K}$ algebraically closed of characteristic zero, then Behrend [2] (see also [17, §4]) shows that $[X]^{\mathrm{vir}} \in A_0(X)$, and

$$\int_{[X]^{\mathrm{vir}}} 1 = \chi(X, \nu_X), \tag{1.2}$$

where ν_X is a $\mathbb{Z}$-valued constructible function on X called the *Behrend function*, which depends only on X as a $\mathbb{K}$-scheme. In particular, $[X]^{\mathrm{vir}}$ is independent of the choice of symmetric obstruction theory on X.

If (X, s) is a proper algebraic d-critical locus, we define the *virtual cycle* of X to be $\chi(X, \nu_X) \in \mathbb{Z}$, as in (1.2). Although we will not do it in this paper, one can define a notion of family of d-critical loci over a base Y, and show that the virtual cycles of a proper family of d-critical loci are locally constant on Y.

Example 2.16 below shows that locally, schemes with symmetric obstruction theories can contain strictly less information than algebraic d-critical loci. On the other hand, Example 2.17 shows that schemes with (symmetric) obstruction theories can contain global, nonlocal information (in the form of a class in $\mathrm{Ext}^2(T^*X, T^*X^\vee)$) which is forgotten by algebraic d-critical loci.

The author and his collaborators tried for some time to construct perverse sheaves, and motivic Milnor fibres, from a scheme with symmetric obstruction theory, but failed, and the author now believes this is not possible. So, one moral of Figure 1.1 is that d-critical loci are more useful than schemes with symmetric obstruction theories for various applications.

Conventions. Throughout $\mathbb{K}$ will be an algebraically closed field with $\operatorname{char}\mathbb{K} \neq 2$. As in Theorem 1.1, the sequel [6] and those parts of [4,5,8] which depend on [6] also require $\operatorname{char}\mathbb{K} = 0$, but this paper does not need $\operatorname{char}\mathbb{K} = 0$. All complex analytic spaces, $\mathbb{K}$-schemes and Artin $\mathbb{K}$-stacks X will be assumed to be locally of finite type, as this is necessary for the existence of local embeddings $X \hookrightarrow U$ with U a complex manifold or smooth $\mathbb{K}$-scheme.

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2. The main results

This section, the heart of the paper, gives our central definitions, the main results, and some examples. The proofs of results stated in §§2.1, 2.2, 2.3, 2.4, 2.6, and 2.8 will be deferred until sections 3–8, respectively.

Sections 2.1–2.6 concern d-critical structures on complex analytic spaces and $\mathbb{K}$-schemes. Some good background references on complex analytic spaces and analytic coherent sheaves upon them are Gunning and Rossi [13] and Grauert and Remmert [12]. A good book on $\mathbb{K}$-schemes and sheaves in algebraic geometry is Hartshorne [14]. The relationship between $\mathbb{C}$-schemes and complex analytic spaces is discussed in Hartshorne [14, App. B] and Serre [31].

After some background material on Artin stacks and sheaves upon them in §2.7, section 2.8 extends parts of §2.1–§2.6 from $\mathbb{K}$-schemes to Artin $\mathbb{K}$-stacks. Given an Artin $\mathbb{K}$-stack X, the main idea is to consider smooth 1-morphisms $t : T \to X$ from $\mathbb{K}$-schemes T, and apply the results of §2.1–§2.6 on T.

2.1. The sheaves $\mathcal{S}_X, \mathcal{S}_X^0$ and their properties.

The next theorem, which will be proved in §3.1–§3.3, associates a sheaf $\mathcal{S}_X$ to each complex analytic space (or $\mathbb{K}$-scheme) X, such that (very roughly) sections of $\mathcal{S}_X$ parametrize different ways of writing X as $\mathrm{Crit}(f)$ for U a complex manifold (or smooth $\mathbb{K}$-scheme) and $f : U \to \mathbb{C}$ holomorphic (or $f : U \to \mathbb{A}^1$ regular). This will be needed in the definition of d-critical loci in §2.2.

Note our convention from §1 that all complex analytic spaces and $\mathbb{K}$-schemes X in this paper are *locally of finite type*, which is necessary for the existence of embeddings $i : X \hookrightarrow U$ for U a complex manifold or smooth $\mathbb{K}$-scheme.

Theorem 2.1. *Let X be a complex analytic space. Then there exists a sheaf $\mathcal{S}_X$ of commutative $\mathbb{C}$-algebras on X, unique up to canonical isomorphism, which is uniquely characterized by the following two properties:*

(i) *Suppose U is a complex manifold, R is an open subset in X, and $i : R \hookrightarrow U$ is an embedding of R as a closed complex analytic subspace of U. Then we have an exact sequence of sheaves of $\mathbb{C}$-vector spaces on R:*

$$0 \longrightarrow I_{R,U} \longrightarrow i^{-1}(\mathcal{O}_U) \xrightarrow{i^\sharp} \mathcal{O}_X|_R \longrightarrow 0, \tag{2.1}$$

where $\mathcal{O}_X, \mathcal{O}_U$ are the sheaves of holomorphic functions on X, U, and $i^\sharp$ is the morphism of sheaves of $\mathbb{C}$-algebras on R induced by i, which is surjective as i is an embedding, and $I_{R,U} = \mathrm{Ker}(i^\sharp)$ is the sheaf of ideals in $i^{-1}(\mathcal{O}_U)$ of functions on U near $i(R)$ which vanish on $i(R)$.

There is an exact sequence of sheaves of $\mathbb{C}$-vector spaces on R:

$$0 \longrightarrow \mathcal{S}_X|_R \xrightarrow{\iota_{R,U}} \frac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2} \xrightarrow{\mathrm{d}} \frac{i^{-1}(T^*U)}{I_{R,U} \cdot i^{-1}(T^*U)}, \tag{2.2}$$

where d maps $f + I_{R,U}^2 \mapsto \mathrm{d}f + I_{R,U} \cdot i^{-1}(T^*U)$, and $\iota_{R,U}$ is a morphism of sheaves of commutative $\mathbb{C}$-algebras.

(ii) *Let $R, U, i, \iota_{R,U}$ and $S, V, j, \iota_{S,V}$ be as in* **(i)** *with $R \subseteq S \subseteq X$, and suppose $\Phi : U \to V$ is holomorphic with $\Phi \circ i = j|_R$ as a morphism of complex analytic spaces $R \to V$. Then the following diagram of sheaves on R commutes:*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{S}_X|_R \arrow[r, "\iota_{S,V}|_R"] \arrow[d, "\mathrm{id}"] & \frac{j^{-1}(\mathcal{O}_V)}{I_{S,V}^2}\Big|_R \arrow[r, "\mathrm{d}"] \arrow[d, "i^{-1}(\Phi^\sharp)"] & \frac{j^{-1}(T^*V)}{I_{S,V} \cdot j^{-1}(T^*V)}\Big|_R \arrow[d, "i^{-1}(\mathrm{d}\Phi)"] \\ 0 \arrow[r] & \mathcal{S}_X|_R \arrow[r, "\iota_{R,U}"] & \frac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2} \arrow[r, "\mathrm{d}"] & \frac{i^{-1}(T^*U)}{I_{R,U} \cdot i^{-1}(T^*U)}. \end{tikzcd} \tag{2.3}$$

Here $\Phi : U \to V$ induces $\Phi^\sharp : \Phi^{-1}(\mathcal{O}_V) \to \mathcal{O}_U$ on U, so we have

$$i^{-1}(\Phi^\sharp) : j^{-1}(\mathcal{O}_V)|_R = i^{-1} \circ \Phi^{-1}(\mathcal{O}_V) \longrightarrow i^{-1}(\mathcal{O}_U), \tag{2.4}$$

a morphism of sheaves of $\mathbb{C}$-algebras on R. As $\Phi \circ i = j|_R$, equation (2.4) *maps $I_{S,V}|_R \to I_{R,U}$, and so maps $I_{S,V}^2|_R \to I_{R,U}^2$. Thus* (2.4) *induces the morphism of sheaves of $\mathbb{C}$-algebras in the second* *column of* (2.3). *Similarly,* $\mathrm{d}\Phi : \Phi^{-1}(T^*V) \to T^*U$ *induces the third column of* (2.3).

These sheaves $\mathcal{S}_X$ *also satisfy:*

(a) *There is a natural decomposition* $\mathcal{S}_X = \mathbb{C}_X \oplus \mathcal{S}_X^0$, *where* $\mathbb{C}_X$ *is the constant sheaf on* X *with fibre* $\mathbb{C}$, *as a sheaf of* $\mathbb{C}$-*subalgebras in* $\mathcal{S}_X$, *and* $\mathcal{S}_X^0 \subset \mathcal{S}_X$ *is a sheaf of ideals in* $\mathcal{S}_X$, *the kernel of the composition of morphisms of sheaves of commutative* $\mathbb{C}$-*algebras*

$$\mathcal{S}_X \xrightarrow{\beta_X} \mathcal{O}_X \xrightarrow{i_X^\sharp} \mathcal{O}_{X^{\mathrm{red}}}, \tag{2.5}$$

with X^{red} *the reduced complex analytic subspace of* X, *and* $i_X : X^{\mathrm{red}} \hookrightarrow X$ *the inclusion.*

(b) *There are natural exact sequences of sheaves of* $\mathbb{C}$-*vector spaces on* X:

$$0 \longrightarrow h^{-1}(\mathbb{L}_X) \xrightarrow[\alpha_X]{} \mathcal{S}_X \xrightarrow[\beta_X]{} \mathcal{O}_X \xrightarrow[\mathrm{d}]{} T^*X \cong h^0(\mathbb{L}_X), \tag{2.6}$$

$$0 \longrightarrow h^{-1}(\mathbb{L}_X) \xrightarrow{\alpha_X^0} \mathcal{S}_X^0 \xrightarrow{\beta_X^0} \mathcal{O}_X \xrightarrow{\mathrm{d}\oplus i_X^\sharp} T^*X \oplus \mathcal{O}_{X^{\mathrm{red}}}, \tag{2.7}$$

where $\mathbb{L}_X$ *is the cotangent complex and* T^*X *the cotangent sheaf of* X.

(c) *The sheaf* $\mathcal{S}_X^0$ *is canonically isomorphic to the cohomology of the complex*

$$I_{R,U}^2 \xrightarrow{\mathrm{d}} I_{R,U} \cdot i^{-1}(T^*U) \xrightarrow{\mathrm{d}} i^{-1}(\Lambda^2 T^*U). \tag{2.8}$$

With the exception of **(c)**, *the analogue of all the above also holds for schemes over a field* $\mathbb{K}$ *in algebraic geometry, taking* X *to be a* $\mathbb{K}$-*scheme with structure sheaf* $\mathcal{O}_X$ *and reduced* $\mathbb{K}$-*subscheme* X^{red}, *and* $\mathcal{S}_X$ *a sheaf of commutative* $\mathbb{K}$-*algebras on* X *in either the Zariski or the étale topology, and* $R \subseteq X$ *a Zariski open* $\mathbb{K}$-*subscheme, and* U *a smooth* $\mathbb{K}$-*scheme, and replacing* $\mathbb{C}_X$ *by* $\mathbb{K}_X$. *For* **(c)**, *we must replace* U *by the formal completion* $\hat{U}$ *of* U *along* $i(R)$, *so that the analogue of* (2.8) *is*

$$\begin{gathered} I_{R,\hat{U}}^2 \xrightarrow{\mathrm{d}} I_{R,\hat{U}} \cdot i^{-1}(T^*\hat{U}) \xrightarrow{\mathrm{d}} i^{-1}(\Lambda^2 T^*\hat{U}), \quad \text{where} \\ i^{-1}(\mathcal{O}_{\hat{U}}) = \lim_{n\to\infty} i^{-1}(\mathcal{O}_U)/I_{R,U}^n, \; I_{R,\hat{U}} = \lim_{n\to\infty} I_{R,U}/I_{R,U}^n \subset i^{-1}(\mathcal{O}_{\hat{U}}). \end{gathered} \tag{2.9}$$

Here in part (b), for (*co*)*tangent complexes* of $\mathbb{K}$-schemes see Illusie [15, 16], and of complex analytic spaces see Palamodov [25–28].

Remark 2.2. **(a)** In this paper and the sequels [4–8] we will make no use of the fact that $\mathcal{S}_X$ is a *sheaf of commutative* $\mathbb{C}$-*algebras*, rather than just a sheaf of sets. Material in [5, Th. 6.9] on Verdier duality and monodromy isomorphisms $\Sigma_{X,s}, \mathrm{T}_{X,s}$ for the perverse sheaves $P_{X,s}^\bullet$

in Theorem 1.3 above depends implicitly on being able to multiply $s \in H^0(\mathcal{S}_X^0)$ by -1 or by $\mathrm{e}^{i\theta}$, but we have not yet found an application for the additive or multiplicative structures on $\mathcal{S}_X, \mathcal{S}_X^0$. Although $(X, \mathcal{S}_X)$ is a locally ringed space, it is generally far from being a scheme. The ideals $\mathcal{S}_X^0$ in $\mathcal{S}_X$ need not be square-zero, so the multiplicative structure on $\mathcal{S}_X = \mathbb{C}_X \oplus \mathcal{S}_X^0$ can be nontrivial.

(b) Equation (2.6) suggests the following interpretation of the sheaf $\mathcal{S}_X$: on a complex analytic space or $\mathbb{K}$-scheme X we have the *de Rham differential* $\mathrm{d}_{\mathrm{dR}} : \mathcal{O}_X \to \mathbb{L}_X$, which is a morphism in $D\,\mathrm{mod}$-$\mathbb{C}_X$ or $D\,\mathrm{mod}$-$\mathbb{K}_X$, the derived category of complexes of sheaves of $\mathbb{C}$- or $\mathbb{K}$-vector spaces on X. Write $\mathcal{D}_X^\bullet$ for the cone on d_{dR}, so that we have a distinguished triangle

$$\mathcal{O}_X \xrightarrow{\mathrm{d}_{\mathrm{dR}}} \mathbb{L}_X \longrightarrow \mathcal{D}_X^\bullet \longrightarrow \mathcal{O}_X[1]. \tag{2.10}$$

Comparing (2.6) with the long exact sequence of cohomology sheaves of (2.10), we see that $\mathcal{S}_X \cong h^{-1}(\mathcal{D}_X^\bullet)$.

There are natural *pullback morphisms* $\phi^\star$ for the sheaves $\mathcal{S}_X, \mathcal{S}_X^0$:

Proposition 2.3. *Let $\phi : X \to Y$ be a morphism of complex analytic spaces, and $\mathcal{S}_X, \mathcal{S}_X^0, \iota_{R,U}, I_{R,U}, \mathcal{S}_Y, \mathcal{S}_Y^0, \iota_{S,V}, I_{S,V}$ be as in Theorem* 2.1*. Then there is a unique morphism $\phi^\star : \phi^{-1}(\mathcal{S}_Y) \to \mathcal{S}_X$ of sheaves of commutative $\mathbb{C}$-algebras on X, which maps $\phi^{-1}(\mathcal{S}_Y^0) \to \mathcal{S}_X^0$, such that if $R \subseteq X$, $S \subseteq Y$ are open with $\phi(R) \subseteq S$, U, V are complex manifolds, $i : R \hookrightarrow U$, $j : S \hookrightarrow V$ are closed embeddings, and $\Phi : U \to V$ is holomorphic with $\Phi \circ i = j \circ \phi|_R : R \to V$, then as for* (2.3) *the following diagram of sheaves on R commutes:*

(2.11)

$$\begin{array}{ccccccc} 0 \to \phi^{-1}(\mathcal{S}_Y)|_R & \xrightarrow[\phi^{-1}(\iota_{S,V})|_R]{} & \dfrac{\phi^{-1}\circ j^{-1}(\mathcal{O}_V)|_R}{\phi^{-1}(I_{S,V}^2)|_R} & \xrightarrow[\phi^{-1}(\mathrm{d})]{} & \dfrac{\phi^{-1}(j^{-1}(T^*V))|_R}{\phi^{-1}(I_{S,V}\cdot j^{-1}(T^*V))|_R} \\ \downarrow{\scriptstyle \phi^\star|_R} & & \downarrow{\scriptstyle i^{-1}(\Phi^\sharp)} & & \downarrow{\scriptstyle i^{-1}(\mathrm{d}\Phi)} \\ 0 \longrightarrow \mathcal{S}_X|_R & \xrightarrow{\iota_{R,U}} & \dfrac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2} & \xrightarrow{\mathrm{d}} & \dfrac{i^{-1}(T^*U)}{I_{R,U}\cdot i^{-1}(T^*U)}. \end{array}$$

If $\psi : Y \to Z$ is another morphism of complex analytic spaces, then

$$(\psi\circ\phi)^\star = \phi^\star \circ \phi^{-1}(\psi^\star) : (\psi\circ\phi)^{-1}(\mathcal{S}_Z) = \phi^{-1}\circ\psi^{-1}(\mathcal{S}_Z) \longrightarrow \mathcal{S}_X. \tag{2.12}$$

If $\phi : X \to Y$ is $\mathrm{id}_X : X \to X$ then $\mathrm{id}_X^\star = \mathrm{id}_{\mathcal{S}_X} : \mathrm{id}_X^{-1}(\mathcal{S}_X) = \mathcal{S}_X \to \mathcal{S}_X$.

If $\phi : X \to Y$ is an étale morphism of complex analytic spaces, then $\phi^\star : \phi^{-1}(\mathcal{S}_Y) \to \mathcal{S}_X$ is an isomorphism of sheaves of commutative $\mathbb{C}$-algebras.

With the exception of the last part, the analogue of all the above holds for schemes over a field $\mathbb{K}$ in algebraic geometry, taking $\phi : X \to Y$ to be a morphism of $\mathbb{K}$-schemes, $R \subseteq X$, $S \subseteq Y$ to be Zariski open, *and U,V to be smooth $\mathbb{K}$-schemes, and taking $\mathcal{S}_X,\mathcal{S}_Y$ to be sheaves of commutative $\mathbb{K}$-algebras on X,Y in either the Zariski or the étale topology, as in Theorem* 2.1.

For the last part, if $\mathcal{S}_X,\mathcal{S}_Y$ are sheaves in the Zariski topology, then $\phi^\star$ is an isomorphism if $\phi : X \to Y$ is a Zariski open inclusion, and if $\mathcal{S}_X,\mathcal{S}_Y$ are sheaves in the étale topology, then $\phi^\star$ is an isomorphism if $\phi : X \to Y$ is étale.

The next example, central to our theory, shows the point of $\mathcal{S}_X$.

Example 2.4. Let U be a complex manifold, $f : U \to \mathbb{C}$ be holomorphic, and $X = \mathrm{Crit}(f)$, as a closed complex analytic subspace of U. Write $i : X \hookrightarrow U$ for the inclusion, and $I_{X,U} \subseteq i^{-1}(\mathcal{O}_U)$ for the sheaf of ideals vanishing on $X \subseteq U$. Then $i^{-1}(f) \in H^0\big(i^{-1}(\mathcal{O}_U)\big)$ with $\mathrm{d}\big(i^{-1}(f)\big) \in H^0\big(I_{X,U}\cdot i^{-1}(T^*U)\big) \subseteq H^0\big(i^{-1}(T^*U)\big)$, so $i^{-1}(f)+I_{X,U}^2 \in H^0\big(i^{-1}(\mathcal{O}_U)/I_{X,U}^2\big)$ with $\mathrm{d}\big(i^{-1}(f)+I_{X,U}^2\big)=0$ in $H^0\big(i^{-1}(T^*U)/I_{X,U}\cdot i^{-1}(T^*U)\big)$. Thus by equation (2.2) with $R = X$, we see there is a unique section $s \in H^0(\mathcal{S}_X)$ with $\iota_{X,U}(s) = i^{-1}(f) + I_{X,U}^2$.

Thus, if we can write $X = \mathrm{Crit}(f)$ for $f : U \to \mathbb{C}$ holomorphic, then we obtain a natural section $s \in H^0(\mathcal{S}_X)$. Essentially $s = f + I_{\mathrm{d}f}^2$, where $I_{\mathrm{d}f} \subseteq \mathcal{O}_U$ is the ideal generated by $\mathrm{d}f$. Note that $f|_X = f + I_{\mathrm{d}f}$, so s determines $f|_X$. Basically, s remembers all of the information about f which makes sense intrinsically on X, rather than on the ambient space U.

We can also explain the decomposition $\mathcal{S}_X = \mathbb{C}_X \oplus \mathcal{S}_X^0$ in this example. We will see in Example 2.13 that if $X = \mathrm{Crit}(f)$ then f need not be locally constant on X, but f is locally constant on the reduced complex analytic space X^{red}. Since locally constant functions on $X^{\mathrm{red}} \subseteq U$ extend uniquely to locally constant functions on U near X^{red}, after shrinking U we can uniquely write $f = c + f^0$, where $c : U \to \mathbb{C}$ is locally constant and $f^0 : U \to \mathbb{C}$ has $f^0|_{X^{\mathrm{red}}} = 0$. Then c, f^0 correspond to the components of s in $H^0(\mathbb{C}_X), H^0(\mathcal{S}_X^0)$, and $X = \mathrm{Crit}(f^0)$.

The analogue also holds in the algebraic case, with U a smooth $\mathbb{K}$-scheme and $f : U \to \mathbb{A}^1$ a regular function.

2.2. The definition of d-critical loci, and some examples. We can now define d-critical loci:

Definition 2.5. A (*complex analytic*) *d-critical locus* is a pair (X, s), where X is a complex analytic space, and $s \in H^0(\mathcal{S}_X^0)$ for $\mathcal{S}_X^0$ as in Theorem 2.1, satisfying the condition that for each $x \in X$, there exists an open neighbourhood R of x in X, a complex manifold U, a holomorphic function $f : U \to \mathbb{C}$, and an embedding $i : R \hookrightarrow U$ of R as a closed complex analytic subspace of U, such that $i(R) = \mathrm{Crit}(f)$ as complex analytic subspaces of U, and $\iota_{R,U}(s|_R) = i^{-1}(f) + I_{R,U}^2$.

Similarly, for $\mathbb{K}$-schemes we define an (*algebraic*) *d-critical locus* to be a pair (X,s), where X is a $\mathbb{K}$-scheme, and $s\in H^0(\mathcal{S}_X^0)$ for $\mathcal{S}_X$ as in Theorem 2.1, such that X may be covered by Zariski open sets $R\subseteq X$ with a closed embedding $i:R\hookrightarrow U$ into a smooth $\mathbb{K}$-scheme U and a regular function $f:U\to\mathbb{A}^1=\mathbb{K}$, such that $i(R)=\mathrm{Crit}(f)$ as $\mathbb{K}$-subschemes of U, and $\iota_{R,U}(s|_R)=i^{-1}(f)+I_{R,U}^2$.

In both cases we call (R,U,f,i) a *critical chart* on (X,s).

A *morphism* $\phi:(X,s)\to(Y,t)$ of d-critical loci $(X,s),(Y,t)$ (either complex analytic or algebraic) is a morphism $\phi:X\to Y$ (of complex analytic spaces or $\mathbb{K}$-schemes) such that $\phi^\star(t)=s$, for $\phi^\star$ as in Proposition 2.3. If $\phi:(X,s)\to(Y,t)$, $\psi:(Y,t)\to(Z,u)$ are morphisms then equation (2.12) implies that $\psi\circ\phi:(X,s)\to(Z,u)$ is a morphism, and the last part of Proposition 2.3 shows that $\mathrm{id}_X:(X,s)\to(X,s)$ is a morphism. Thus, (complex analytic or algebraic) *d-critical loci form a category.*

Remark 2.6. **(a)** In Definition 2.5, we could instead have defined a d-critical locus (X,s) to have $s\in H^0(\mathcal{S}_X)$ rather than $s\in H^0(\mathcal{S}_X^0)$, but with the rest of the definition the same. The difference is this: as in Example 2.4, if $X=\mathrm{Crit}(f)$ for holomorphic $f:U\to\mathbb{C}$, then $f|_{X^{\mathrm{red}}}$ is locally constant, and we can write $f=f^0+c$ uniquely near X in U for $f^0:U\to\mathbb{C}$ holomorphic with $\mathrm{Crit}(f^0)=X=\mathrm{Crit}(f)$, $f^0|_{X^{\mathrm{red}}}=0$, and $c:U\to\mathbb{C}$ locally constant with $c|_{X^{\mathrm{red}}}=f|_{X^{\mathrm{red}}}$.

Defining d-critical loci using $s\in H^0(\mathcal{S}_X^0)$, as we have done, corresponds to remembering only the function f^0 near X in U, and forgetting the locally constant function $f|_{X^{\mathrm{red}}}:X^{\mathrm{red}}\to\mathbb{C}$. Equivalently, it corresponds to remembering the closed 1-form $\mathrm{d}f=\mathrm{d}f^0$ on U near $X=(\mathrm{d}f)^{-1}(0)$. In the applications the author has in mind [4–8], taking s in $H^0(\mathcal{S}_X^0)$ rather than $H^0(\mathcal{S}_X)$ is more natural, as there is no canonical value for $f|_{X^{\mathrm{red}}}$ other than $f|_{X^{\mathrm{red}}}=0$. Also (2.8)–(2.9) give an alternative description for $\mathcal{S}_X^0$ rather than $\mathcal{S}_X$.

(b) As in Theorem 1.1, in [6, Th. 6.6] we define a truncation functor from -1-shifted symplectic derived $\mathbb{K}$-schemes $(\boldsymbol{X},\omega)$ in the sense of Pantev et al. [30] to algebraic d-critical loci (X,s), so that algebraic d-critical loci may be regarded as classical truncations of -1-shifted symplectic derived $\mathbb{K}$-schemes.

If we define a morphism $\boldsymbol{\phi}:(\boldsymbol{X},\omega)\to(\boldsymbol{Y},\omega')$ of -1-shifted symplectic derived $\mathbb{K}$-schemes to be a morphism $\boldsymbol{\phi}:\boldsymbol{X}\to\boldsymbol{Y}$ of derived $\mathbb{K}$-schemes with $\boldsymbol{\phi}^*(\omega')\simeq\omega$, this forces $\boldsymbol{\phi}$ to be étale. However, the notion of morphism $\phi:(X,s)\to(Y,t)$ of d-critical loci in Definition 2.5 is more general, e.g. $\phi:X\to Y$ can be smooth of positive dimension, as in Proposition 2.8.

(c) For (X,s) to be a (complex analytic or algebraic) d-critical locus places strong local restrictions on the singularities of X. For example, Behrend [2] notes that if X has reduced local complete intersection singularities then locally it cannot be the zeroes of an almost closed 1-form on a smooth space, and hence not locally a critical locus, and Pandharipande and Thomas [29] give examples which are zeroes of almost closed 1-forms, but are not locally critical loci.

On a d-critical locus (X, s), any closed embedding $X \supseteq R \xrightarrow{i} U$ with U smooth can be made into a critical chart (R', U', f', i'), after shrinking R, U.

Proposition 2.7. *Suppose (X, s) is a complex analytic d-critical locus, $R \subseteq X$ is open, and $i : R \hookrightarrow U$ is a closed embedding, where U is a complex manifold. Then for each $x \in R$, there exist open $x \in R' \subseteq R$ and $i(R') \subseteq U' \subseteq U$ and a holomorphic function $f' : U' \to \mathbb{C}$ such that (R', U', f', i') is a critical chart on (X, s), where $i' = i|_{R'} : R' \hookrightarrow U'$.*

Suppose also that $\dim U = \dim T_x X$, so that $\mathrm{d}i|_x : T_x X \to T_{i(x)} U$ is an isomorphism, and $f : U \to \mathbb{C}$ is holomorphic with $\iota_{R,U}(s|_R) = i^{-1}(f) + I_{R,U}^2$. Then we may take $f' = f|_{U'}$ in (R', U', f', i').

The analogue holds for algebraic d-critical loci, with U a smooth $\mathbb{K}$-scheme, $R' \subseteq R \subseteq X$, $U' \subseteq U$ Zariski open, and $f : U \to \mathbb{A}^1$, $f' : U' \to \mathbb{A}^1$ regular.

The next result will be useful in §2.8.

Proposition 2.8. *Let $\phi : X \to Y$ be a smooth morphism of complex analytic spaces or $\mathbb{K}$-schemes. Suppose $t \in H^0(\mathcal{S}_Y^0)$, and set $s := \phi^\star(t) \in H^0(\mathcal{S}_X^0)$, for $\phi^\star$ as in Proposition 2.3. If (Y, t) is a d-critical locus, then (X, s) is a d-critical locus, and $\phi : (X, s) \to (Y, t)$ is a morphism of d-critical loci.*

Conversely, if also $\phi : X \to Y$ is surjective, then (X, s) a d-critical locus implies (Y, t) is a d-critical locus.

As in Hartshorne [14, App. B] and Serre [31], there is an analytification functor from algebraic $\mathbb{C}$-schemes X to complex analytic spaces X^{an}, where the points of X^{an} are the $\mathbb{C}$-points of X. It is easy to show that this extends to d-critical loci, and we leave the details to the reader:

Proposition 2.9. *Let (X, s) be an algebraic d-critical locus over the field $\mathbb{K} = \mathbb{C}$. Then the complex analytic space X^{an} associated to the $\mathbb{C}$-scheme X extends naturally to a complex analytic d-critical locus $(X^{\mathrm{an}}, s^{\mathrm{an}})$.*

The proofs of the following lemma and proposition are also more-or-less immediate, and we leave them as exercises.

Lemma 2.10. *Let (X, s) be a d-critical locus, and $0 \neq c \in \mathbb{C}$ or $0 \neq c \in \mathbb{K}$. Then $(X, c \cdot s)$ is also a d-critical locus, and if (R, U, f, i) is a critical chart on (X, s) then $(R, U, c \cdot f, i)$ is a critical chart on $(X, c \cdot s)$.*

Proposition 2.11. *Let $(X,s),(Y,t)$ be d-critical loci. Write $\pi_X : X\times Y\to X$, $\pi_Y : X\times Y\to Y$ for the projections, and define $s\boxplus t := \pi_X^\star(s)+\pi_Y^\star(t)$ in $H^0(\mathcal{S}^0_{X\times Y})$, for $\pi_X^\star,\pi_Y^\star$ as in Proposition 2.3. Then $(X\times Y,s\boxplus t)$ is a d-critical locus, and if (R,U,f,i), (S,V,g,j) are critical charts on (X,s), (Y,t) respectively then $(R\times S,U\times V,f\boxplus g,i\times j)$ is a critical chart on $(X\times Y,s\boxplus t)$.*

Remark 2.12. Note that in Proposition 2.11, π_X,π_Y are in general *not* morphisms of d-critical loci $(X\times Y,s\boxplus t)\to(X,s)$, $(X\times Y,s\boxplus t)\to(Y,t)$ in the sense of Definition 2.5, since $\pi_X^\star(s)\neq s\boxplus t\neq\pi_Y^\star(t)$. Also $(X\times Y,s\boxplus t)$ is *not* a product $(X,s)\times(Y,t)$ in the category of d-critical loci, in the sense of category theory. Nonetheless, we will call $(X\times Y,s\boxplus t)$ the *product* of $(X,s),(Y,t)$.

Let U be a complex manifold, $f:U\to\mathbb{C}$ be holomorphic, and $X=\mathrm{Crit}(f)$, as a complex analytic space. Then $f|_X : X\to\mathbb{C}$ is holomorphic, and $\mathrm{d}(f|_X)=0$ in $H^0(T^*X)$. Experience with calculus on manifolds suggests that if $g:X\to\mathbb{C}$ is holomorphic with $\mathrm{d}g=0$ in $H^0(T^*X)$ then g is locally constant on X. However, this is true only for *reduced* complex analytic spaces or $\mathbb{K}$-schemes X. Here is an example of a non-reduced critical locus with $f|_X$ not locally constant:

Example 2.13. Define $f:\mathbb{C}^2\to\mathbb{C}$ by $f(x,y)=x^5+x^2y^2+y^5$, and let $X=\mathrm{Crit}(f)$, as a complex analytic space. Then $f|_X\in H^0(\mathcal{O}_X)$. We have $\mathrm{d}(f|_X)=0\in H^0(T^*X)$, since $X=\mathrm{d}f^{-1}(0)$. Suppose for a contradiction that f is constant on X near $(0,0)$. Then we may write

$$f(x,y)=A+\tfrac{\partial f}{\partial x}\,B(x,y)+\tfrac{\partial f}{\partial y}\,C(x,y)$$

on $\mathbb{C}^2$ near $(0,0)$, for some holomorphic functions B,C defined near $(0,0)$ in $\mathbb{C}^2$. That is, we have

$$x^5+x^2y^2+y^5=A+(5x^4+2xy^2)\sum_{i,j\geqslant 0}B_{i,j}x^iy^j+(2x^2y+5y^4)\sum_{i,j\geqslant 0}C_{i,j}x^iy^j.$$

Comparing coefficients of x^5,y^5,x^2y^2 give the equations

$$1=5B_{1,0},\quad 1=5C_{0,1},\quad 1=2B_{1,0}+2C_{0,1},$$

which have no solution. Thus in this case, $f|_X$ is not locally constant on X.

If $X=\mathrm{Crit}(f)$ and X^{red} is the reduced complex analytic subspace of X, then $f|_{X^{\mathrm{red}}}$ is always locally constant. This is why we defined $\mathcal{S}^0_X$ using restriction to X^{red} in Theorem 2.1(a). Combining Theorem 2.1(a),(b) we deduce:

Corollary 2.14. *Suppose X is a complex analytic space, and the following sequence of sheaves of $\mathbb{C}$-vector spaces on X is exact:*

$$0\longrightarrow\mathbb{C}_X\xrightarrow{\ \mathrm{inc}\ }\mathcal{O}_X\xrightarrow{\ \mathrm{d}\ }T^*X,\tag{2.13}$$

where $\mathrm{inc}:\mathbb{C}_X\hookrightarrow\mathcal{O}_X$ *is the inclusion of the constant functions into the holomorphic functions. Then* $\mathcal{S}_X^0\cong h^{-1}(\mathbb{L}_X)$, *so that* $\mathcal{S}_X^0$ *is a coherent sheaf on* X, *and* $\mathcal{S}_X\cong\mathbb{C}_X\oplus h^{-1}(\mathbb{L}_X)$. *The analogue also holds for* $\mathbb{K}$*-schemes.*

Now (2.13) is exact if for $g:X\to\mathbb{C}$ a locally defined holomorphic function, $\mathrm{d}g=0$ implies g is locally constant. In Example 2.13 this fails, so in this example (2.13) is not exact, and $\mathcal{S}_X^0\not\cong h^{-1}(\mathbb{L}_X)$, and $\mathcal{S}_X^0$ is not a coherent sheaf on X. Next we consider smooth complex analytic spaces and $\mathbb{K}$-schemes:

Example 2.15. Suppose X is a complex manifold, considered as a complex analytic space. Then in Theorem 2.1 we see that $\mathcal{S}_X\cong\mathbb{C}_X$, the constant sheaf, and $\mathcal{S}_X^0=0$, the zero sheaf. To see this, take $R=X=U$ and $i=\mathrm{id}_X:X\to X$ in Theorem 2.1. Then $I_{X,X}=0$ by (2.1), and $\mathcal{S}_X\cong\mathrm{Ker}\big(\mathrm{d}:\mathcal{O}_X\to T^*X\big)\cong\mathbb{C}_X$ by (2.2). As $\mathcal{S}_X^0=0$ there is a unique global section $s=0\in H^0(\mathcal{S}_X^0)$, and $(X,0)$ is a complex analytic d-critical locus, as it is $\mathrm{Crit}(0:X\to\mathbb{C})$.

Similarly, if X is a smooth $\mathbb{K}$-scheme then $\mathcal{S}_X\cong\mathbb{K}_X$, and $\mathcal{S}_X^0=0$, and $(X,0)$ is an algebraic d-critical locus.

Our next two examples compare algebraic d-critical loci with symmetric obstruction theories on $\mathbb{K}$-schemes, as defined in Definition 1.5.

Example 2.16. Let $\mathbb{K}$ be a field of characteristic zero, and define X to be the $\mathbb{K}$-scheme $X=\mathrm{Spec}\big(\mathbb{K}[z]/(z^n)\big)$ for $n\geqslant 2$. Then X has an obvious embedding $i:X\hookrightarrow\mathbb{A}^1=\mathrm{Spec}\big(\mathbb{K}[z]\big)$ as the subscheme $z^n=0$ in $\mathbb{A}^1$. It is a non-reduced point. Using this embedding $X\hookrightarrow\mathbb{A}^1$, from Theorem 2.1(i) we find that

$$\begin{aligned}H^0(\mathcal{S}_X)=\big\{&a_0+a_{n+1}z^{n+1}+\cdots+a_{2n-1}z^{2n-1}+(z^{2n}):\\ &a_0,a_{n+1},\ldots,a_{2n-1}\in\mathbb{K}\big\}\cong\mathbb{K}^n,\end{aligned}$$

and $H^0(\mathcal{S}_X^0)\subset H^0(\mathcal{S}_X)$ is the subspace with $a_0=0$, isomorphic to $\mathbb{K}^{n-1}$.

Now let $0\in U\subseteq\mathbb{A}^1$ be open, and suppose $f:U\to\mathbb{A}^1$ is regular with $f(0)=0$ and $\mathrm{Crit}(f)=i(X)$. Write $a_k=\frac{1}{k!}\frac{\partial^k f}{\partial z^k}(0)$ for $k=0,1,\ldots.$ Then $f(0)=0$ gives $a_0=0$, and $\mathrm{Crit}(f)=X$ is equivalent to $a_1=\cdots=a_n=0$ and $a_{n+1}\neq 0$. The section $s\in H^0(\mathcal{S}_X^0)$ corresponding to f is $f+(z^{2n})=a_{n+1}z^{n+1}+\cdots+a_{2n-1}z^{2n-1}+(z^{2n})$. From this we see that if $s=a_{n+1}z^{n+1}+\cdots+a_{2n-1}z^{2n-1}+(z^{2n})\in H^0(\mathcal{S}_X^0)$, then (X,s) is an algebraic d-critical locus if and only if $a_{n+1}\neq 0$.

By equation (1.1), the natural symmetric obstruction theory on $X=\mathrm{Crit}(f)$ is determined by $\frac{\partial^2 f}{\partial z^2}\big|_X$, that is, by $(n+1)na_{n+1}z^{n-1}+\cdots$ modulo (z^n). Thus, in this example, the d-critical locus (X,s) associated to $\mathrm{Crit}(f)$ records the first $n-1$ coefficients $a_{n+1},a_{n+2},\ldots,a_{2n-1}$ in the power series expansion of $f(z)=a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+\cdots$ at 0, but the symmetric obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_X$, θ records only the first coefficient a_{n+1}. Hence (at least in this case), *the algebraic d-critical locus remembers more information, locally, than the symmetric obstruction theory.*

Example 2.17. Let $t : U \to \mathbb{A}^1$ be a smooth morphism of $\mathbb{K}$-schemes of relative dimension 2, whose fibres U_t for t at and near 0 in $\mathbb{A}^1$ are $K3$ surfaces. Set $X = U_0 \subset U$, and regard U, t as a 1-parameter family of deformations of X.

We wish to compare $\mathrm{Crit}(t^2 : U \to \mathbb{A}^1)$ and $\mathrm{Crit}(0 : X \to \mathbb{A}^1)$ in the categories (or higher categories) of:

(i) classical $\mathbb{K}$-schemes;
(ii) algebraic d-critical loci;
(iii) schemes with perfect obstruction theories [3];
(iv) schemes with symmetric obstruction theories [2]; and
(v) -1-shifted symplectic derived schemes [30].

We will see that the two are isomorphic in (i),(ii), but not equivalent in (iii)–(v). For (i), $\mathrm{Crit}(t^2 : U \to \mathbb{A}^1)$ and $\mathrm{Crit}(0 : X \to \mathbb{A}^1)$ are both X as classical schemes. For (ii), as X is smooth $\mathcal{S}_X^0 = 0$, so $\mathrm{Crit}(t^2 : U \to \mathbb{A}^1)$ and $\mathrm{Crit}(0 : X \to \mathbb{A}^1)$ are both $(X, 0)$ as algebraic d-critical loci.

For (iii), write $\phi : \mathcal{E}^\bullet \to \mathbb{L}_X$ and $\psi : \mathcal{F}^\bullet \to \mathbb{L}_X$ for the obstruction theories from $\mathrm{Crit}(t^2 : U \to \mathbb{A}^1)$ and $\mathrm{Crit}(0 : X \to \mathbb{A}^1)$. Then (1.1) gives

$$\mathcal{E}^\bullet = \big[TU|_X \xrightarrow{\partial^2(t^2)|_X} T^*U|_X\big], \qquad \mathcal{F}^\bullet = \big[TX \xrightarrow{0} T^*X|_X\big].$$

We want to know whether $\mathcal{E}^\bullet \cong \mathcal{F}^\bullet$ in $D(\mathrm{qcoh}(X))$. Now $\tau_{\leqslant -1}(\mathcal{E}^\bullet) \cong TX[1]$ and $\tau_{\geqslant 0}(\mathcal{E}^\bullet) \cong T^*X$, so we have a distinguished triangle in $D(\mathrm{qcoh}(X))$:

$$\cdots \longrightarrow T^*X[-1] \xrightarrow{\alpha} TX[1] \longrightarrow \mathcal{E}^\bullet \longrightarrow T^*X \longrightarrow \cdots .$$

That is, $\mathcal{E}^\bullet$ is the cone on $\alpha : T^*X[-1] \to TX[1]$ in $D(\mathrm{qcoh}(X))$ for some α in $\mathrm{Ext}^2(T^*X, TX)$. Hence $\mathcal{E}^\bullet \cong \mathcal{F}^\bullet$ if and only if $\alpha = 0$.

The normal bundle ν of X in U, and its dual ν^*, are both isomorphic to $\mathcal{O}_X$ as $t : U \to \mathbb{A}^1$ induces isomorphisms $\nu \cong t^*(T_0\mathbb{A}^1)$, $\nu^* \cong t^*(T_0^*\mathbb{A}^1)$. Hence we have exact sequences

$$0 \longrightarrow TX \longrightarrow TU|_X \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_X \longrightarrow T^*U|_X \longrightarrow T^*X \longrightarrow 0.$$

Let these correspond to $\beta' \in \mathrm{Ext}^1(\mathcal{O}_X, TX)$, and $\beta'' \in \mathrm{Ext}^1(T^*X, \mathcal{O}_X)$. Then $\alpha = \beta' \circ \beta'' \in \mathrm{Ext}^2(T^*X, TX) \cong H^2(TX \otimes TX)$.

Under the isomorphisms $\mathrm{Ext}^1(\mathcal{O}_X, TX) \cong H^1(TX) \cong \mathrm{Ext}^1(T^*X, \mathcal{O}_X)$, we see that β', β'' are both identified with $\beta \in H^1(TX)$, which parametrizes the infinitesimal deformation of $\{U_t : t \in \mathbb{A}^1\}$ at $t = 0$, so that informally $\beta \sim \frac{\mathrm{d}}{\mathrm{d}t}U_t|_{t=0}$. The projection of α from $H^2(TX \otimes TX)$ to $H^2(\Lambda^2 TX) \cong \mathbb{K}$ is β^2.

Let us choose the deformation $t : U \to \mathbb{A}^1$ of $X = U_0$ so that the infinitesimal deformation $\beta \in H^1(TX)$ at $t = 0$ satisfies $\beta^2 \neq 0$ in $H^2(\Lambda^2 TX) \cong \mathbb{K}$, which is possible by well known facts about $K3$ surfaces. Then $\alpha \neq 0$ in $\mathrm{Ext}^2(T^*X, TX)$, and $\mathcal{E}^\bullet \ncong \mathcal{F}^\bullet$. Hence $\mathrm{Crit}(t^2 : U \to \mathbb{A}^1)$ and $\mathrm{Crit}(0 : X \to \mathbb{A}^1)$ are not equivalent as schemes with perfect obstruction theories, as in (iii), and so a fortiori they are also not equivalent as schemes with symmetric obstruction theories, as in (iv), or as -1-shifted symplectic derived schemes, as in (v).

Observe that $\alpha \in \mathrm{Ext}^2(T^*X, TX)$ which distinguishes the obstruction theories is *global* information, which is *locally trivial*: if $Y \subset X$ is any affine open subset then $\alpha|_Y = 0$ as $\mathrm{Ext}^2(T^*Y, TY) = 0$. Thus (at least in this case), *the (symmetric) obstruction theory remembers global, non-local information which is forgotten by the algebraic d-critical locus.*

This example shows that the dotted arrows '$\dashrightarrow$' in Figure 1.1, which indicate local constructions, cannot be made to work globally.

Using related ideas, the author expects that if (X, s) is an algebraic d-critical locus, then there is an obstruction in $\mathrm{Ext}^3\big(T^*X, (T^*X)^\vee\big)$ to finding a symmetric obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_X$, θ on X which is locally modelled on (1.1) when (X, s) is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{C})$. But the author does not know of an example in which this obstruction is nonzero. Finding such an example would show that the truncation functor from -1-shifted symplectic derived schemes to algebraic d-critical loci in [6] is not essentially surjective.

2.3. Comparing critical charts (R, U, f, i)**.** In §2.2 we defined a d-critical locus (X, s) to admit an open cover by *critical charts* (R, U, f, i), which write (X, s) as $\mathrm{Crit}(f)$ in an open set $R \subset X$. We will treat critical charts like coordinate charts on a manifold. Our analogues of transition functions between coordinate charts are called *embeddings*.

Definition 2.18. Let (X, s) be a d-critical locus (either complex analytic or algebraic), and (R, U, f, i) be a critical chart on (X, s). Let $U' \subseteq U$ be (Zariski) open, and set $R' = i^{-1}(U') \subseteq R$, so that $R' \subseteq R \subseteq X$ are (Zariski) open, and $i' = i|_{R'} : R' \hookrightarrow U'$, and $f' = f|_{U'} : U' \to \mathbb{C}$ or $\mathbb{A}^1$. Then (R', U', f', i') is also a critical chart on (X, s), and we call it a *subchart* of (R, U, f, i). As a shorthand we write $(R', U', f', i') \subseteq (R, U, f, i)$.

Let (R, U, f, i) and (S, V, g, j) be critical charts on (X, s), with $R \subseteq S \subseteq X$. An *embedding* of (R, U, f, i) in (S, V, g, j) is a locally closed embedding $\Phi : U \hookrightarrow V$ of complex manifolds or $\mathbb{K}$-schemes such that $\Phi \circ i = j|_R : R \to V$ and $f = g \circ \Phi : U \to \mathbb{C}$ or $\mathbb{A}^1$. As a shorthand we write $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ to mean Φ is an embedding of (R, U, f, i) in (S, V, g, j).

Clearly, if $\Phi : (R,U,f,i) \hookrightarrow (S,V,g,j)$, $\Psi : (S,V,g,j) \hookrightarrow (T,W,h,k)$ are embeddings, then $\Psi \circ \Phi : (R,U,f,i) \hookrightarrow (T,W,h,k)$ is also an embedding.

If $\Phi : (R,U,f,i) \hookrightarrow (S,V,g,k)$ is an embedding then $\dim U \leqslant \dim V$. Thus, embeddings between critical charts $(R,U,f,i),(S,V,g,j)$ usually go in only one direction, and do not have inverses which are embeddings. The author drew some inspiration for these ideas from the theory of Kuranishi spaces in the work of Fukaya, Oh, Ohta and Ono [10, §A] in symplectic geometry: critical charts (R,U,f,i) are like Kuranishi neighbourhoods on a topological space, and embeddings are like coordinate changes between Kuranishi neighbourhoods.

In the algebraic case, it is sometimes convenient to work with critical charts (S,V,g,j) in which $V \subseteq \mathbb{A}^n$ is Zariski open in an affine space $\mathbb{A}^n$. Every critical chart (R,U,f,i) locally admits embeddings into such a (S,V,g,j). The proof of the next proposition in §5.1 uses the assumption $\operatorname{char}\mathbb{K} \neq 2$ from §1.

Proposition 2.19. *Let (R,U,f,i) be a critical chart on an algebraic d-critical locus (X,s). Then for each $x \in R$ there exists a subchart $(R',U',f',i') \subseteq (R,U,f,i)$ with $x \in R'$ and an embedding $\Phi : (R',U',f',i') \hookrightarrow (S,V,g,j)$ into a critical chart (S,V,g,j) with $V \subseteq \mathbb{A}^n$ Zariski open for some $n \geqslant 0$.*

Given two critical charts $(R,U,f,i),(S,V,g,j)$ on (X,s), there need not exist embeddings between them (or their subcharts) in either direction. So to compare $(R,U,f,i),(S,V,g,j)$, we construct embeddings of subcharts $(R',U',f',i'),(S',V',g',j')$ into a third critical chart (T,W,h,k) with $\dim W \geqslant \dim U, \dim V$.

Theorem 2.20. *Let (X,s) be a d-critical locus (either complex analytic or algebraic), and $(R,U,f,i),(S,V,g,j)$ be critical charts on (X,s). Then for each $x \in R \cap S \subseteq X$ there exist subcharts $(R',U',f',i') \subseteq (R,U,f,i)$, $(S',V',g',j') \subseteq (S,V,g,j)$ with $x \in R' \cap S' \subseteq X$, a critical chart (T,W,h,k) on (X,s), and embeddings $\Phi : (R',U',f',i') \hookrightarrow (T,W,h,k)$, $\Psi : (S',V',g',j') \hookrightarrow (T,W,h,k)$.*

Remark 2.21. To see the point of the definition and theorem, we explain how they will be used. Often, given a d-critical locus (X,s), we want to construct some global object $\mathcal{G}$ on X by gluing together local data by isomorphisms. Examples include the canonical bundle $K_{X,s}$ in §2.4, perverse sheaves, $\mathscr{D}$-modules and mixed Hodge modules on oriented d-critical loci (X,s) in [5], and motivic Milnor fibres on oriented algebraic d-critical loci (X,s) in [8]. For each of these constructions, we use the following method:

(i) For each critical chart (R,U,f,i) on (X,s), we define a geometric structure $\mathcal{G}_{R,U,f,i}$ on R, with $\mathcal{G}_{R',U',f',i'} = \mathcal{G}_{R,U,f,i}|_{R'}$ for $(R',U',f',i') \subseteq (R,U,f,i)$.

(ii) For each embedding $\Phi : (R,U,f,i) \hookrightarrow (S,V,g,j)$, we define a canonical isomorphism $\Phi_* : \mathcal{G}_{R,U,f,i} \to \mathcal{G}_{S,V,g,j}|_R$. We show Φ_* is independent of Φ, that is, if $\Phi, \tilde{\Phi} : (R,U,f,i) \hookrightarrow (S,V,g,j)$ are embeddings then $\Phi_* = \tilde{\Phi}_*$.

(iii) For embeddings $\Phi : (R,U,f,i) \hookrightarrow (S,V,g,j)$, $\Psi : (S,V,g,j) \hookrightarrow (T,W,h,k)$, we show that $(\Psi \circ \Phi)_* = \Psi_*|_R \circ \Phi_* : \mathcal{G}_{R,U,f,i} \to \mathcal{G}_{T,W,h,k}|_R$.

(iv) Choose critical charts $\{(R_a,U_a,f_a,i_a) : a \in A\}$ with $\{R_a : a \in A\}$ an open cover of X. Using (ii),(iii) and Theorem 2.20 we obtain isomorphisms $\iota_{ab} : \mathcal{G}_{R_a,U_a,f_a,i_a}|_{R_a \cap R_b} \to \mathcal{G}_{R_b,U_b,f_b,i_b}|_{R_a \cap R_b}$ for $a,b \in A$, with $\iota_{aa} = \mathrm{id}$ and $\iota_{bc} \circ \iota_{ab}|_{R_a \cap R_b \cap R_c} = \iota_{ac}|_{R_a \cap R_b \cap R_c}$ for $a,b,c \in A$. Thus, provided the geometric structures concerned form a sheaf, there exists $\mathcal{G}$ on X, unique up to canonical isomorphism, with $\mathcal{G}|_{R_a} \cong \mathcal{G}_{R_a,U_a,f_a,i_a}$ for all $a \in A$.

Our next three results say roughly that if $\Phi : (R,U,f,i) \hookrightarrow (S,V,g,j)$ is an embedding of critical charts on (X,s), then locally near $j(R)$ in V we have $V \cong U \times \mathbb{C}^n$ or $V \cong U \times \mathbb{A}^n$ and $g \cong f \boxplus z_1^2 + \cdots + z_n^2$, where $n = \dim V - \dim U$. But in the algebraic case we have to be careful about which topology we mean when we say 'locally'. The complex analytic case is straightforward:

Proposition 2.22. *Let (X,s) be a complex analytic d-critical locus, and $\Phi : (R,U,f,i) \hookrightarrow (S,V,g,j)$ an embedding of critical charts on (X,s). Then for each $x \in R$ there exist open neighbourhoods U',V' of $i(x),j(x)$ in U,V with $\Phi(U') \subseteq V'$, and holomorphic $\alpha : V' \to U$, $\beta : V' \to \mathbb{C}^n$ for $n = \dim V - \dim U$, such that $\alpha \times \beta : V' \to U \times \mathbb{C}^n$ is a biholomorphism with an open subset of $U \times \mathbb{C}^n$, and $\alpha \circ \Phi|_{U'} = \mathrm{id}_{U'}$, $\beta \circ \Phi|_{U'} = 0$, $g|_{V'} = f \circ \alpha + (z_1^2 + \cdots + z_n^2) \circ \beta$.*

For the algebraic case, we give two statements. The first is a direct analogue of Proposition 2.22 for the étale topology, regarding $\iota : U' \to U$, $\jmath : V' \to V$ as étale neighbourhoods of $i(x),j(x)$ in U,V, and $\Phi' : U' \to V'$ with $\Phi \circ \iota = \jmath \circ \Phi'$ as the analogue of $\Phi(U') \subseteq V'$. It will be used in [5].

Recall our convention in §1 that the base field $\mathbb{K}$ of X is algebraically closed with $\mathrm{char}\,\mathbb{K} \neq 2$. Both these assumptions will be used in the proofs of Propositions 2.23 and 2.24, since to define V' we need to take square roots in $\mathbb{K}$, and we need $\mathrm{char}\,\mathbb{K} \neq 2$ to diagonalize quadratic forms over $\mathbb{K}$.

Proposition 2.23. *Let (X,s) be an algebraic d-critical locus, and $\Phi : (R,U,f,i) \hookrightarrow (S,V,g,j)$ an embedding of critical charts on (X,s). Then for each $x \in R$ there exist smooth $\mathbb{K}$-schemes U',V', a point $u' \in U'$, and morphisms $\iota : U' \to U$, $\jmath : V' \to V$, $\Phi' : U' \to V'$, $\alpha : V' \to U$, and $\beta : V' \to \mathbb{A}^n$ for $n = \dim V - \dim U$, such that*

$\imath(u') = i(x)$, and $\imath : U' \to U$, $\jmath : V' \to V$, $\alpha\times\beta : V' \to U\times\mathbb{A}^n$ are étale, and $\Phi\circ\imath = \jmath\circ\Phi'$, $\alpha\circ\Phi' = \imath$, $\beta\circ\Phi' = 0$, and $g\circ\jmath = f\circ\alpha + (z_1^2+\cdots+z_n^2)\circ\beta$.

The second holds with U' a Zariski open neighbourhood of x in U, at the cost of giving a more general form for $g\circ\jmath : V' \to \mathbb{A}^1$. It will be used in [8].

Proposition 2.24. *Let (X, s) be an algebraic d-critical locus, and $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ an embedding of critical charts on (X, s). Then for each $x \in R$ there exist a Zariski open neighbourhood U' of $i(x)$ in U, a smooth $\mathbb{K}$-scheme V', and morphisms $\jmath : V' \to V$, $\Phi' : U' \to V'$, $\alpha : V' \to U'$, $\beta : V' \to \mathbb{A}^n$ and $q_1, \ldots, q_n : U' \to \mathbb{A}^1 \setminus \{0\}$ for $n = \dim V - \dim U$, such that $\jmath : V' \to V$ and $\alpha\times\beta : V' \to U'\times\mathbb{A}^n$ are étale, $\Phi|_{U'} = \jmath\circ\Phi'$, $\alpha\circ\Phi' = \mathrm{id}_{U'}$, $\beta\circ\Phi' = 0$, and*

$$g\circ\jmath = f\circ\alpha + (q_1\circ\alpha)\cdot(z_1^2\circ\beta) + \cdots + (q_n\circ\alpha)\cdot(z_n^2\circ\beta). \tag{2.14}$$

2.4. Canonical bundles of d-critical loci. Propositions 2.22–2.24 locally describe embeddings $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ of critical charts on (X, s). We can also associate a piece of global data to Φ, a nondegenerate quadratic form q_{UV} on the pullback $i^*(N_{UV})$ of the normal bundle N_{UV} of $\Phi(U)$ in V.

Proposition 2.25. *Let (X, s) be a d-critical locus (either complex analytic or algebraic), and $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ be an embedding of critical charts on (X, s). Write N_{UV} for the normal bundle of $\Phi(U)$ in V, regarded as a (holomorphic or algebraic) vector bundle on U in the exact sequence*

$$0 \longrightarrow TU \xrightarrow{\mathrm{d}\Phi} \Phi^*(TV) \xrightarrow{\Pi_{UV}} N_{UV} \longrightarrow 0, \tag{2.15}$$

so that $i^*(N_{UV})$ is a vector bundle on $R \subseteq X$. Then there exists a unique $q_{UV} \in H^0(S^2 i^*(N_{UV}^*))$ which is a nondegenerate quadratic form on $i^*(N_{UV})$, with the following property in each case:

(a) *If (X, s) is a complex analytic d-critical locus and $x, U', V', n, \alpha, \beta$ are as in Proposition 2.22, writing $\langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n\rangle_{U'}$ for the trivial vector bundle on U' with basis $\mathrm{d}z_1, \ldots, \mathrm{d}z_n$ and $R' = i^{-1}(U') \subseteq R \subseteq X$, there is a natural isomorphism $\hat\beta : \langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n\rangle_{U'} \to N_{UV}^*|_{U'}$ such that*

$$\begin{aligned}\Phi|_{U'}^*(\mathrm{d}\beta^*) = \Pi_{UV}|_{U'}^*\circ\hat\beta :&\\ \Phi|_{U'}^*\circ\beta^*(T_0^*\mathbb{C}^n) = \langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n\rangle_{U'} &\longrightarrow \Phi|_{U'}^*(T^*V),\end{aligned} \tag{2.16}$$

$$\text{with}\quad q_{UV}|_{R'} = i|_{R'}^*\big[(S^2\hat\beta)(\mathrm{d}z_1\otimes\mathrm{d}z_1 + \cdots + \mathrm{d}z_n\otimes\mathrm{d}z_n)\big]. \tag{2.17}$$

(b) *If (X, s) is an algebraic d-critical locus and $x, U', V', \imath, \jmath, \Phi', \alpha, \beta, n$ are as in Proposition 2.23, then there is an isomorphism $\hat\beta$:*

$\langle \mathrm{d}z_1,\ldots,\mathrm{d}z_n\rangle_{U'} \to \iota^*(N_{UV}^*)$ making the following diagram of vector bundles on U' commute:

$$\begin{tikzcd}[column sep=large]
\iota^*(N_{UV}^*) \arrow[rr, "\iota^*(\Pi_{UV}^*)"'] & & \iota^*\circ\Phi^*(T^*V)=\Phi'^*\circ\jmath^*(T^*V) \arrow[d, "\Phi'^*(\mathrm{d}\jmath^*)"'] \\
\langle \mathrm{d}z_1,\ldots,\mathrm{d}z_n\rangle_{U'}=\Phi'^*\circ\beta^*(T_0^*\mathbb{C}^n) \arrow[u, dotted, "\hat\beta"'] \arrow[rr, "\Phi'^*(\mathrm{d}\beta^*)"] & & \Phi'^*(T^*V'),
\end{tikzcd}\tag{2.18}$$

and if $R'=R\times_{i,U,\iota}U'$ with projections $\rho: R'\to R$, $i': R'\to U'$, then

$$\rho^*(q_{UV})=i'^*\big[(S^2\hat\beta)(\mathrm{d}z_1\otimes\mathrm{d}z_1+\cdots+\mathrm{d}z_n\otimes\mathrm{d}z_n)\big].\tag{2.19}$$

(c) *If (X,s) is an algebraic d-critical locus and $x,U',V',\jmath,\Phi',\alpha,\beta,n,q_a$ are as in Proposition 2.24, then there is an isomorphism $\hat\beta:\langle \mathrm{d}z_1,\ldots,\mathrm{d}z_n\rangle_{U'}\to\iota^*(N_{UV}^*)$ making the following commute:*

$$\begin{tikzcd}[column sep=large]
N_{UV}^*|_{U'} \arrow[rr, "\Pi_{UV}^*|_{U'}"'] & & \Phi|_{U'}^*(T^*V)=\Phi'^*\circ\jmath^*(T^*V) \arrow[d, "\Phi'^*(\mathrm{d}\jmath^*)"'] \\
\langle \mathrm{d}z_1,\ldots,\mathrm{d}z_n\rangle_{U'}=\Phi'^*\circ\beta^*(T_0^*\mathbb{C}^n) \arrow[u, dotted, "\hat\beta"'] \arrow[rr, "\Phi'^*(\mathrm{d}\beta^*)"] & & \Phi'^*(T^*V'),
\end{tikzcd}\tag{2.20}$$

and if $R'=i^{-1}(U')\subseteq R\subseteq X$, then

$$q_{UV}|_{R'}=i|_{R'}^*\big[q_1\cdot(S^2\hat\beta)(\mathrm{d}z_1\otimes\mathrm{d}z_1)+\cdots+q_n\cdot(S^2\hat\beta)(\mathrm{d}z_n\otimes\mathrm{d}z_n)\big].\tag{2.21}$$

Now suppose $\Psi:(S,V,g,j)\hookrightarrow(T,W,h,k)$ is another embedding of critical charts, so that $\Psi\circ\Phi:(R,U,f,i)\hookrightarrow(T,W,h,k)$ is also an embedding, and define N_{VW},q_{VW} and N_{UW},q_{UW} using $\Psi,\Psi\circ\Phi$ as above. Then there are unique morphisms $\gamma_{UVW},\delta_{UVW}$ which make the following diagram of vector bundles on U commute, with straight lines exact:

$$\begin{tikzcd}
 & & & & & 0 \arrow[d] & \\
 & & 0 \arrow[dr] & & & TU \arrow[dll, "\mathrm{d}\Phi"'] \arrow[ddl, "\mathrm{d}(\Psi\circ\Phi)"] & 0 \arrow[l] \\
 & & & \Phi^*(TV) \arrow[dll, "\Pi_{UV}"'] \arrow[dr, "\Phi^*(\mathrm{d}\Psi)"] & & & \\
0 \arrow[r] & N_{UV} \arrow[dl] \arrow[drr, dotted, "\gamma_{UVW}"'] & & & (\Psi\circ\Phi)^*(TW) \arrow[dl, "\Pi_{UW}"] \arrow[ddr, "\Phi^*(\Pi_{VW})"] & & \\
0 & & & N_{UW} \arrow[dl] \arrow[drr, dotted, "\delta_{UVW}"'] & & & \\
 & & 0 & & & \Phi^*(N_{VW}) \arrow[r] \arrow[d] & 0 \\
 & & & & & 0. &
\end{tikzcd}\tag{2.22}$$

Pulling back by i^* gives an exact sequence of vector bundles on $R\subseteq X$:

$$0\longrightarrow i^*(N_{UV})\xrightarrow{i^*(\gamma_{UVW})} i^*(N_{UW})\xrightarrow{i^*(\delta_{UVW})} \jmath^*(N_{VW})|_R\longrightarrow 0.\tag{2.23}$$

Then there is a natural isomorphism of vector bundles on R

$$i^*(N_{UW}) \cong i^*(N_{UV}) \oplus j^*(N_{VW})|_R, \tag{2.24}$$

compatible with the exact sequence (2.23), *which identifies*

$$q_{UW} \cong q_{UV} \oplus q_{VW} \oplus 0 \qquad \textit{under the splitting} \tag{2.25}$$
$$S^2(i^*(N_{UW}^*)) \cong S^2(i^*(N_{UV}^*)) \oplus S^2(j^*(N_{VW}^*))|_R \oplus i^*(N_{UV}^*) \otimes j^*(N_{VW}^*)|_R.$$

Using N_{UV}, q_{UV} we define an isomorphism of line bundles J_Φ on $R^{\rm red}$:

Definition 2.26. Let $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ be an embedding of critical charts on a d-critical locus (X, s). Define N_{UV}, q_{UV} as in Proposition 2.25, and set $n = \dim V - \dim U$. Write $R^{\rm red}$ for the reduced complex analytic subspace or reduced $\mathbb{K}$-subscheme of R. Taking top exterior powers in the dual of (2.15) and pulling back to $R^{\rm red}$ using i^* gives an isomorphism of line bundles

$$\rho_{UV} : \big(i^*(K_U) \otimes i^*(\Lambda^n N_{UV}^*)\big)|_{R^{\rm red}} \xrightarrow{\cong} j^*(K_V)|_{R^{\rm red}}. \tag{2.26}$$

As q_{UV} is a nondegenerate quadratic form on $i^*(N_{UV})$, its determinant $\det(q_{UV})$ is a nonvanishing section of $i^*(\Lambda^n N_{UV}^*)^{\otimes^2}$. Define an isomorphism of line bundles $J_\Phi : i^*\big(K_U^{\otimes^2}\big)|_{R^{\rm red}} \to j^*\big(K_V^{\otimes^2}\big)|_{R^{\rm red}}$ on $R^{\rm red}$ by the commutative diagram

$$\begin{tikzcd}
i^*\big(K_U^{\otimes^2}\big)|_{R^{\rm red}} \arrow[r, "\mathrm{id}_{i^*(K_U^2)}\otimes\det(q_{UV})|_{R^{\rm red}}"'] \arrow[dr, bend right=10, "J_\Phi"'] & \big(i^*\big(K_U^{\otimes^2}\big)\otimes i^*\big(\Lambda^{\rm top}N_{UV}^*\big)^{\otimes^2}\big)\big|_{R^{\rm red}} \arrow[d, "\rho_{UV}^{\otimes^2}|_{R^{\rm red}}"] \\
& j^*\big(K_V^{\otimes^2}\big)|_{R^{\rm red}}.
\end{tikzcd} \tag{2.27}$$

Here are some useful properties of the J_Φ. The proof that J_Φ is independent of Φ needs $R^{\rm red}$ reduced, which is why we restricted to $R^{\rm red}$ in Definition 2.26.

Proposition 2.27. *In Definition* 2.26, *the isomorphism* J_Φ *is independent of the choice of* Φ. *That is, if* $\Phi, \tilde\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ *are embeddings of critical charts then* $J_\Phi = J_{\tilde\Phi}$.

If $\Psi : (S, V, g, j) \hookrightarrow (T, W, h, k)$ *is another embedding of critical charts then*

$$J_\Psi|_{R^{\rm red}} \circ J_\Phi = J_{\Psi\circ\Phi}. \tag{2.28}$$

We can now define the *canonical bundle* of a d-critical locus.

Theorem 2.28. *Let* (X, s) *be a d-critical locus (either complex analytic or algebraic), and* $X^{\rm red} \subseteq X$ *the associated reduced complex analytic space or reduced* $\mathbb{K}$*-scheme. Then there exists a (holomorphic or algebraic) line bundle* $K_{X,s}$ *on* $X^{\rm red}$ *which we call the* ***canonical bundle*** *of* (X, s), *which is natural up to canonical isomorphism, and is characterized by the following properties:*

(i) *If (R,U,f,i) is a critical chart on (X,s), there is a natural isomorphism*

$$\iota_{R,U,f,i} : K_{X,s}|_{R^{\mathrm{red}}} \longrightarrow i^*\big(K_U^{\otimes^2}\big)|_{R^{\mathrm{red}}}, \tag{2.29}$$

where $K_U = \Lambda^{\dim U} T^* U$ is the canonical bundle of U in the usual sense.

(ii) *Let $\Phi : (R,U,f,i) \hookrightarrow (S,V,g,j)$ be an embedding of critical charts on (X,s), and let J_Φ be as in* (2.27). *Then*

$$\iota_{S,V,g,j}|_{R^{\mathrm{red}}} = J_\Phi \circ \iota_{R,U,f,i} : K_{X,s}|_{R^{\mathrm{red}}} \longrightarrow j^*\big(K_V^{\otimes^2}\big)\big|_{R^{\mathrm{red}}}. \tag{2.30}$$

(iii) *For each $x \in X^{\mathrm{red}}$, there is a canonical isomorphism*

$$\kappa_x : K_{X,s}|_x \xrightarrow{\cong} \big(\Lambda^{\mathrm{top}} T_x^* X\big)^{\otimes^2}, \tag{2.31}$$

where $T_x X$ is the Zariski tangent space of X at x.

(iv) *Suppose (R,U,f,i) is a critical chart on (X,s) and $x \in R$, and let $\iota_{R,U,f,i}, \kappa_x$ be as in* **(i)**,**(iii)***. Then we have an exact sequence*

$$0 \longrightarrow T_x X \xrightarrow{\mathrm{d}i|_x} T_{i(x)} U \xrightarrow{\mathrm{Hess}_{i(x)} f} T^*_{i(x)} U \xrightarrow{\mathrm{d}i|_x^*} T_x^* X \longrightarrow 0, \tag{2.32}$$

and the following diagram commutes:

$$\begin{tikzcd} K_{X,s}|_x \arrow[rr, "\kappa_x"'] \arrow[drr, "\iota_{R,U,f,i}|_x"'] & & \big(\Lambda^{\mathrm{top}} T_x^* X\big)^{\otimes^2} \arrow[d, "\alpha_{x,R,U,f,i}"'] \\ & & K_U|_{i(x)}^{\otimes^2}, \end{tikzcd} \tag{2.33}$$

where $\alpha_{x,R,U,f,i}$ is induced by taking top exterior powers in (2.32).

Remark 2.29. **(a)** As in Theorem 1.1 proved in [6], if (X,s) is the truncation of a -1-shifted symplectic derived scheme $(\boldsymbol{X},\omega)$ in the sense of Pantev et al. [30], then $K_{X,s} \cong \det(\mathbb{L}_{\boldsymbol{X}})|_{X^{\mathrm{red}}}$. So $K_{X,s}$ is isomorphic to the canonical bundle of the derived scheme $\boldsymbol{X}$ in this case, which is why we call it a canonical bundle.

(b) The line bundle $K_{X,s}$ in Theorem 2.28 is characterized uniquely up to isomorphism either by parts (i),(ii), or by parts (i),(iii),(iv).

Here is a formula for pullback of canonical bundles under smooth morphisms of d-critical loci, which will be useful in §2.8. By saying that $T^*_{X/Y}$ is a *vector bundle of mixed rank*, and by the *top exterior power* $\Lambda^{\mathrm{top}} T^*_{X/Y}$, we mean the following: as $\phi : X \to Y$ is smooth, there is a decomposition $\coprod_{n \geqslant 0} X_n$ with $X_n \subseteq X$ open and closed, such that $\phi|_{X_n} : X_n \to Y$ is smooth of relative dimension n. Then $T^*_{X/Y}|_{X_n}$ is a vector bundle on X_n of rank n, and the line bundle $\Lambda^{\mathrm{top}} T^*_{X/Y}$ on X is defined by $\Lambda^{\mathrm{top}} T^*_{X/Y}|_{X_n} = \Lambda^n T^*_{X/Y}|_{X_n}$ for each n.

Proposition 2.30. *Suppose $\phi : (X,s) \to (Y,t)$ is a morphism of d-critical loci with $\phi : X \to Y$ smooth, as in Proposition* 2.8*. The **relative cotangent bundle** $T^*_{X/Y}$ is a vector bundle of mixed rank on X in the exact sequence of coherent sheaves on X:*

$$0 \longrightarrow \phi^*(T^*Y) \xrightarrow{\mathrm{d}\phi^*} T^*X \longrightarrow T^*_{X/Y} \longrightarrow 0. \tag{2.34}$$

There is a natural isomorphism of line bundles on X^{red}:

$$\Upsilon_\phi : \phi|^*_{X^{\mathrm{red}}}(K_{Y,t}) \otimes \bigl(\Lambda^{\mathrm{top}} T^*_{X/Y}\bigr)\big|^{\otimes^2}_{X^{\mathrm{red}}} \xrightarrow{\cong} K_{X,s}, \tag{2.35}$$

such that for each $x \in X^{\mathrm{red}}$ the following diagram of isomorphisms commutes:

$$\begin{tikzcd} K_{Y,t}|_{\phi(x)} \otimes \bigl(\Lambda^{\mathrm{top}} T^*_{X/Y}|_x\bigr)^{\otimes^2} \arrow[rr, "\Upsilon_\phi|_x"'] \arrow[d, "\kappa_{\phi(x)}\otimes \mathrm{id}"] && K_{X,s}|_x \arrow[d, "\kappa_x"'] \\ \bigl(\Lambda^{\mathrm{top}} T^*_{\phi(x)}Y\bigr)^{\otimes^2} \otimes \bigl(\Lambda^{\mathrm{top}} T^*_{X/Y}|_x\bigr)^{\otimes^2} \arrow[rr, "\upsilon_x^{\otimes^2}"] && \bigl(\Lambda^{\mathrm{top}} T^*_x X\bigr)^{\otimes^2}, \end{tikzcd} \tag{2.36}$$

where $\kappa_x, \kappa_{\phi(x)}$ are as in (2.31)*, and $\upsilon_x : \Lambda^{\mathrm{top}} T^*_{\phi(x)}Y \otimes \Lambda^{\mathrm{top}} T^*_{X/Y}|_x \to \Lambda^{\mathrm{top}} T^*_x X$ is obtained by restricting* (2.34) *to x and taking top exterior powers.*

2.5. Orientations on d-critical loci. The next two definitions will be important in the sequels [4–8]. For examples of results on oriented d-critical loci, see Theorems 1.3 and 1.4 above.

Definition 2.31. Let (X,s) be a d-critical locus (either complex analytic or algebraic), and $K_{X,s}$ its canonical bundle from Theorem 2.28. An *orientation* on (X,s) is a choice of square root line bundle $K^{1/2}_{X,s}$ for $K_{X,s}$ on X^{red}. That is, an orientation is a (holomorphic or algebraic) line bundle L on X^{red}, together with an isomorphism $L^{\otimes^2} = L \otimes L \cong K_{X,s}$. A d-critical locus with an orientation will be called an *oriented d-critical locus.*

An *oriented critical chart* on an oriented d-critical locus is a critical chart (R,U,f,i) on (X,s) with an isomorphism $\jmath_{R,U,f,i} : K^{1/2}_{X,s}|_{R^{\mathrm{red}}} \to i^*(K_U)|_{R^{\mathrm{red}}}$ satisfying $\jmath^2_{R,U,f,i} = \iota_{R,U,f,i}$, for $\iota_{R,U,f,i}$ as in (2.29).

Remark 2.32. In view of equation (2.31), one might hope to define a canonical orientation $K^{1/2}_{X,s}$ for a d-critical locus (X,s) by $K^{1/2}_{X,s}\big|_x = \Lambda^{\mathrm{top}} T^*_x X$ for $x \in X^{\mathrm{red}}$. However, *this does not work*, as the spaces $\Lambda^{\mathrm{top}} T^*_x X$ do not vary continuously with $x \in X^{\mathrm{red}}$ if X is not smooth. Example 2.39 shows that d-critical loci need not admit orientations.

In the situation of Proposition 2.30, the factor $(\Lambda^{\mathrm{top}} T^*_{X/Y})|^{\otimes^2}_{X^{\mathrm{red}}}$ in (2.35) has a natural square root $(\Lambda^{\mathrm{top}} T^*_{X/Y})|_{X^{\mathrm{red}}}$. Thus we deduce:

Corollary 2.33. *Let $\phi : (X,s) \to (Y,t)$ be a morphism of d-critical loci with $\phi : X \to Y$ smooth. Then each orientation $K_{Y,t}^{1/2}$ for (Y,t) lifts to a natural orientation $K_{X,s}^{1/2} = \phi|_{X^{\mathrm{red}}}^*(K_{Y,t}^{1/2}) \otimes (\Lambda^{\mathrm{top}} T^*_{X/Y})|_{X^{\mathrm{red}}}$ for (X,s).*

We can express orientations in terms of *principal $\mathbb{Z}_2$-bundles*.

Definition 2.34. Let (X,s) be a d-critical locus. For each embedding of critical charts $\Phi : (R,U,f,i) \hookrightarrow (S,V,g,j)$ on (X,s), define a principal $\mathbb{Z}_2$-bundle $\pi_\Phi : P_\Phi \to R$ over R to be the bundle of square roots of the isomorphism J_Φ in (2.27). That is, local sections $s_\alpha : R \to P_\Phi$ correspond to local isomorphisms $\alpha : i^*(K_U)|_{R^{\mathrm{red}}} \to j^*(K_V)|_{R^{\mathrm{red}}}$ with $\alpha \otimes \alpha = J_\Phi$. Note that Proposition 2.27 implies that P_Φ is independent of the choice of Φ, that is, if $\Phi, \tilde{\Phi} : (R,U,f,i) \hookrightarrow (S,V,g,j)$ are embeddings then $P_\Phi = P_{\tilde{\Phi}}$.

If $\Psi : (S,V,g,j) \hookrightarrow (T,W,h,k)$ is another embedding of critical charts then (2.28) implies that there is a canonical isomorphism

$$\Xi_{\Psi,\Phi} : P_{\Psi\circ\Phi} \xrightarrow{\cong} P_\Psi|_R \otimes_{\mathbb{Z}_2} P_\Phi,$$

such that if local isomorphisms $\alpha : i^*(K_U)|_{R^{\mathrm{red}}} \to j^*(K_V)|_{R^{\mathrm{red}}}$, $\beta : j^*(K_V)|_{R^{\mathrm{red}}} \to k^*(K_W)|_{R^{\mathrm{red}}}$, $\gamma : i^*(K_U)|_{R^{\mathrm{red}}} \to k^*(K_W)|_{R^{\mathrm{red}}}$ with $\alpha \otimes \alpha = J_\Phi$, $\beta \otimes \beta = J_\Psi|_{R^{\mathrm{red}}}$, $\gamma \otimes \gamma = J_{\Psi\circ\Phi}$ correspond to local sections $s_\alpha : R \to P_\Phi$, $s_\beta : R \to P_\Psi|_R$, $s_\gamma : R \to P_{\Phi\circ\Phi}$, then $\Xi_{\Psi,\Phi}(s_\gamma) = s_\beta \otimes_{\mathbb{Z}_2} s_\alpha$ if and only if $\gamma = \beta \circ \alpha$, where $\gamma = \beta \circ \alpha$ is possible by (2.28).

Now let $K_{X,s}^{1/2}$ be a choice of orientation on (X,s), as in Definition 2.31. For each critical chart (R,U,f,i) on (X,s), define a principal $\mathbb{Z}_2$-bundle $\pi_{R,U,f,i} : Q_{R,U,f,i} \to R$ to be the bundle of square roots of the isomorphism $\iota_{R,U,f,i}$ in (2.29). That is, local sections $s_\beta : R \to Q_{R,U,f,i}$ correspond to local isomorphisms $\beta : K_{X,s}^{1/2}|_{R^{\mathrm{red}}} \to i^*(K_U)|_{R^{\mathrm{red}}}$ with $\beta \otimes \beta = \iota_{R,U,f,i}$.

Given an orientation $K_{X,s}^{1/2}$ and an embedding $\Phi : (R,U,f,i) \hookrightarrow (S,V,g,j)$, we have principal $\mathbb{Z}_2$-bundles $\pi_\Phi : P_\Phi \to R$, $\pi_{R,U,f,i} : Q_{R,U,f,i} \to R$ and $\pi_{S,V,g,j} : Q_{S,V,g,j} \to S$. Then there is a natural isomorphism of principal $\mathbb{Z}_2$-bundles

$$\Lambda_\Phi : Q_{S,V,g,j}|_R \xrightarrow{\cong} P_\Phi \otimes_{\mathbb{Z}_2} Q_{R,U,f,i} \tag{2.37}$$

on R, defined as follows: local isomorphisms

$$\alpha : i^*(K_U)|_{R^{\mathrm{red}}} \longrightarrow j^*(K_V)|_{R^{\mathrm{red}}}, \quad \beta : K_{X,s}^{1/2}|_{R^{\mathrm{red}}} \longrightarrow i^*(K_U)|_{R^{\mathrm{red}}},$$
$$\text{and} \qquad \gamma : K_{X,s}^{1/2}|_{R^{\mathrm{red}}} \longrightarrow j^*(K_V)|_{R^{\mathrm{red}}}$$

with $\alpha \otimes \alpha = i|_{R^{\mathrm{red}}}^*(J_\Phi)$, $\beta \otimes \beta = \iota_{R,U,f,i}$ and $\gamma \otimes \gamma = \iota_{S,V,g,j}|_{R^{\mathrm{red}}}$ correspond to local sections $s_\alpha : R \to i^*(P_\Phi)$, $s_\beta : R \to Q_{R,U,f,i}$ and $s_\gamma : R \to Q_{S,V,g,j}|_R$. Equation (2.30) shows that $\gamma = \alpha \circ \beta$ is a possible solution for γ, and we define Λ_Φ in (2.37) such that $\Lambda_\Phi(s_\gamma) = s_\alpha \otimes_{\mathbb{Z}_2} s_\beta$ if and only if $\gamma = \alpha \circ \beta$. Note that Λ_Φ is independent of the choice of Φ, as J_Φ, P_Φ are.

If $\Psi : (S, V, g, j) \hookrightarrow (T, W, h, k)$ is another embedding of critical charts then it is easy to check that the following diagram commutes:

$$\begin{tikzcd} Q_{T,W,h,k}|_R \arrow[rr, "\Lambda_{\Psi\circ\Phi}"'] \arrow[d, "\Lambda_\Psi|_R"] && P_{\Psi\circ\Phi} \otimes_{\mathbb{Z}_2} Q_{R,U,f,i} \arrow[d, "\Xi_{\Psi,\Phi}\otimes\mathrm{id}_{Q_{R,U,f,i}}"'] \\ \bigl(P_\Psi \otimes_{\mathbb{Z}_2} Q_{S,V,g,j}\bigr)\big|_R \arrow[rr, "\mathrm{id}_{P_\Psi|_R}\otimes\Lambda_\Phi"] && P_\Psi|_R \otimes_{\mathbb{Z}_2} P_\Phi \otimes_{\mathbb{Z}_2} Q_{R,U,f,i}. \end{tikzcd} \tag{2.38}$$

Proposition 2.35. *Let (X, s) be a d-critical locus. Then Definition* 2.34 *induces an isomorphism between isomorphism classes of orientations $K_{X,s}^{1/2}$ on (X, s), and isomorphism classes of the following collections of data:*

- **(a)** *For each critical chart (R, U, f, i) on (X, s), a choice of principal $\mathbb{Z}_2$-bundle $\pi_{R,U,f,i} : Q_{R,U,f,i} \to R$ on R, and*
- **(b)** *For each embedding of critical charts $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$, a choice of isomorphism $\Lambda_\Phi : Q_{S,V,g,j}|_R \to P_\Phi \otimes_{\mathbb{Z}_2} Q_{R,U,f,i}$ as in* (2.37),

such that (2.38) *commutes for all embeddings $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$, $\Psi : (S, V, g, j) \hookrightarrow (T, W, h, k)$, where $P_\Phi, P_\Psi, P_{\Psi\circ\Phi}, \Xi_{\Psi,\Phi}$ are as in the first part of Definition* 2.34.

The proof of Proposition 2.35 is straightforward. Definition 2.34 shows how to go from an orientation $K_{X,s}^{1/2}$ to a collection of data $Q_{R,U,f,i}$, Λ_Φ. For the converse, given a collection of data $Q_{R,U,f,i}, \Lambda_\Phi$, note that each $Q_{R,U,f,i}$ determines a square root $L_{R,U,f,i}$ for $K_{X,s}|_{R^{\mathrm{red}}}$ uniquely up to isomorphism for each critical chart (R, U, f, i), and for an embedding $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ the isomorphism Λ_Φ determines an isomorphism $i_\Phi : L_{R,U,f,i} \to L_{S,V,g,j}|_{R^{\mathrm{red}}}$, and for $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$, $\Psi : (S, V, g, j) \hookrightarrow (T, W, h, k)$, equation (2.38) commuting implies that $i_{\Psi\circ\Phi} = i_\Psi|_{R^{\mathrm{red}}} \circ i_\Phi$. By the sheaf property of line bundles, we can then show there exists $K_{X,s}^{1/2}$ unique up to canonical isomorphism, with isomorphisms $K_{X,s}^{1/2}|_{R^{\mathrm{red}}} \cong L_{R,U,f,i}$ for all (R, U, f, i), which are compatible with i_Φ for all Φ. We leave the details to the reader.

Remark 2.36. Let $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ be an embedding of critical charts on a d-critical locus (X, s). Define N_{UV}, q_{UV} as in Proposition 2.25, and $\pi_\Phi : P_\Phi \to R$ as in Definition 2.34. Then an alternative interpretation of P_Φ is as *the principal $\mathbb{Z}_2$-bundle of orientations of the nondegenerate quadratic form q_{UV} on the vector bundle $i^*(N_{UV})$ over R.*

Thus, Proposition 2.35 shows that an orientation $K_{X,s}^{1/2}$ on (X, s) is equivalent to giving principal $\mathbb{Z}_2$-bundles $Q_{R,U,f,i} \to R$ for each chart

(R,U,f,i) on (X,s), such that $Q_{R,U,f,i}$ and $Q_{S,V,g,j}|_R$ differ by the principal $\mathbb{Z}_2$-bundle of orientations of q_{UV} for each embedding $\Phi:(R,U,f,i) \hookrightarrow (S,V,g,j)$. This is why we chose the term *orientation* for $K_{X,s}^{1/2}$. It is closely relation to the notion of *orientation data* in Kontsevich and Soibelman [18, §5].

Here are some examples of canonical bundles and orientations:

Example 2.37. Let U be a complex manifold, $f:U\to\mathbb{C}$ be holomorphic, and (X,s) be the complex analytic d-critical locus from Example 2.4 with $X=\mathrm{Crit}(f)$. Then Theorem 2.28(i) with $(R,U,f,i)=(X,U,f,\mathrm{inc})$ implies that $K_{X,s}\cong K_U^{\otimes^2}|_{X^{\mathrm{red}}}$. Hence $K_{X,s}$ has a natural square root $K_{X,s}^{1/2}=K_U|_{X^{\mathrm{red}}}$, and (X,s) a natural orientation. The analogue holds for algebraic critical loci.

Example 2.38. Let X be a complex manifold, so that $(X,0)$ is a d-critical locus as in Example 2.15. Then Theorem 2.28(i) with $(R,U,f,i)=(X,X,0,\mathrm{id}_X)$ shows that $K_{X,0}\cong K_X^{\otimes^2}$, where K_X is the usual canonical bundle of X. Again, $(X,0)$ has a natural orientation $K_{X,0}^{1/2}=K_X$.

As we call $K_{X,0}$ the canonical bundle of $(X,0)$, one might have expected $K_{X,0}\cong K_X$. The explanation is that as a derived scheme, $\mathrm{Crit}(0:X\to\mathbb{C})$ is not X, but the shifted cotangent bundle $T^*X[1]$, and the degree -1 fibres of the projection $T^*X[1]\to X$ include an extra factor of K_X in $K_{X,0}$.

Example 2.39. Let X be the non-reduced projective $\mathbb{C}$-scheme

$$\big\{[x,y,z]\in\mathbb{CP}^2 : z^2=2yz=0\big\}.$$

The reduced $\mathbb{C}$-subscheme $X^{\mathrm{red}}\subset X\subset\mathbb{CP}^2$ is the $\mathbb{CP}^1$ defined by $z=0$, and X has only one non-reduced point $[1,0,0]$, with $X\setminus\{[1,0,0]\}\cong\mathbb{C}$ smooth. The open neighbourhood $X\setminus\{[0,1,0]\}$ of [1,0,0] in X is isomorphic as a classical $\mathbb{C}$-scheme to $\mathrm{Crit}(yz^2:\mathbb{C}^2\to\mathbb{C})$, where (y,z) are the coordinates on $\mathbb{C}^2$.

Extend X to an algebraic d-critical locus as follows: on $X\setminus\{[0,1,0]\}$, define s as in Example 2.4 using $X\setminus\{[0,1,0]\}\cong\mathrm{Crit}(yz^2:\mathbb{C}^2\to\mathbb{C})$. But $\mathcal{S}_X^0\equiv 0$ on $X\setminus\{[1,0,0]\}$ by Example 2.15 as $X\setminus\{[1,0,0]\}$ is smooth, so s extends uniquely by zero to all of X. Since (X,s) is modelled on $\mathrm{Crit}(yz^2:\mathbb{C}^2\to\mathbb{C})$ on $X\setminus\{[0,1,0]\}$ and on $\mathrm{Crit}(0:\mathbb{C}\to\mathbb{C})$ on $X\setminus\{[1,0,0]\}$, it is an algebraic d-critical locus.

Theorem 2.28 defines a line bundle $K_{X,s}$ on $X^{\mathrm{red}}\cong\mathbb{CP}^1$. Calculation shows that $K_{X,s}\cong\mathcal{O}_{\mathbb{CP}^1}(-5)$. For the smooth algebraic d-critical locus $(\mathbb{CP}^1,0)$ we have $K_{\mathbb{CP}^1,0}\cong K_{\mathbb{CP}^1}^{\otimes^2}\cong\mathcal{O}_{\mathbb{CP}^1}(-4)$ as in Example 2.38, so the effect of the nonreduced point $[1,0,0]$ in X is to modify $K_{X,s}$ from $\mathcal{O}_{\mathbb{CP}^1}(-4)$ to $\mathcal{O}_{\mathbb{CP}^1}(-5)$. Since -5 is odd, $K_{X,s}$ admits no square root.

Thus, (X, s) *is an example of a non-orientable algebraic d-critical locus.* We can also consider (X, s) as a complex analytic d-critical locus, where again it is not orientable.

2.6. Equivariant d-critical loci. We now discuss group actions on algebraic d-critical loci.

Definition 2.40. Let (X, s) be an algebraic d-critical locus over $\mathbb{K}$, and $\mu : G \times X \to X$ an action of an algebraic $\mathbb{K}$-group G on the $\mathbb{K}$-scheme X. We also write the action as $\mu(\gamma) : X \to X$ for $\gamma \in G$. We say that (X, s) is G-*invariant* if $\mu(\gamma)^\star(s) = s$ for all $\gamma \in G$, or equivalently, if $\mu^\star(s) = \pi_X^\star(s)$ in $H^0(\mathcal{S}^0_{G\times X})$, where $\pi_X : G \times X \to X$ is the projection.

Let $\chi : G \to \mathbb{G}_m$ be a morphism of algebraic $\mathbb{K}$-groups, that is, a character of G, where $\mathbb{G}_m = \mathbb{K} \setminus \{0\}$ is the multiplicative group. We say that (X, s) is G-*equivariant, with character* χ, if $\mu(\gamma)^\star(s) = \chi(\gamma) \cdot s$ for all $\gamma \in G$, or equivalently, if $\mu^\star(s) = (\chi \circ \pi_G) \cdot (\pi_X^\star(s))$ in $H^0(\mathcal{S}^0_{G\times X})$, where $H^0(\mathcal{O}_G) \ni \chi$ acts on $H^0(\mathcal{S}^0_{G\times X})$ by multiplication, as G is a smooth $\mathbb{K}$-scheme.

Suppose (X, s) is G-invariant or G-equivariant, with $\chi = 1$ in the G-invariant case. We call a critical chart (R, U, f, i) on (X, s) with a G-action $\rho : G \times U \to U$ a G-*equivariant critical chart* if $R \subseteq X$ is a G-invariant open subscheme, and $i : R \hookrightarrow U$, $f : U \to \mathbb{A}^1$ are equivariant with respect to the actions $\mu|_{G\times R}, \rho, \chi$ of G on $R, U, \mathbb{A}^1$, respectively.

We call a subchart $(R', U', f', i') \subseteq (R, U, f, i)$ a G-*equivariant subchart* if $R' \subseteq R$ and $U' \subseteq U$ are G-invariant open subschemes. Then $(R', U', f', i'), \rho'$ is a G-equivariant critical chart, where $\rho' = \rho|_{G\times U'}$.

Let $(R, U, f, i), \rho$ and $(S, V, g, j), \sigma$ be G-equivariant critical charts on (X, s), and $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ an embedding. We call Φ *equivariant* if $\Phi : U \hookrightarrow V$ is equivariant with respect to the actions ρ, σ of G on U, V.

When we have a G-equivariant d-critical locus (X, s), we would like to be able to work only with G-equivariant critical charts and subcharts (so in particular, we would like X to be covered by such charts) and G-equivariant embeddings. However, as Example 2.46 below shows, X may not be covered by G-equivariant critical charts without extra assumptions on X, G.

We will restrict to the case when G is a torus, with a 'good' action on X:

Definition 2.41. Let X be a $\mathbb{K}$-scheme, G an algebraic $\mathbb{K}$-torus, and $\mu : G \times X \to X$ an action of G on X. We call μ a *good action* if X admits a Zariski open cover by G-invariant affine open $\mathbb{K}$-subschemes $U \subseteq X$.

Sumihiro [32, Cor. 2] proves that every torus action on a *normal* $\mathbb{K}$-variety is good. Applying this to the reduced $\mathbb{K}$-subscheme X^{red} of a $\mathbb{K}$-scheme X, and noting that open $U \subseteq X$ is affine if and only if $U^{\rm red} \subseteq X^{\rm red}$ is affine, yields:

Lemma 2.42. *Suppose X is a $\mathbb{K}$-scheme whose reduced $\mathbb{K}$-subscheme $X^{\rm red}$ is normal. Then any action μ of an algebraic $\mathbb{K}$-torus G on X is good.*

A torus-equivariant d-critical locus (X,s) admits an open cover by equivariant critical charts if and only if the torus action is good:

Proposition 2.43. *Let (X,s) be an algebraic d-critical locus which is invariant or equivariant under the action $\mu : G\times X\rightarrow X$ of an algebraic torus G.*

(a) *If μ is good then for all $x\in X$ there exists a G-equivariant critical chart $(R,U,f,i),\rho$ on (X,s) with $x\in R$, and we may take* $\dim U=\dim T_xX$.

(b) *Conversely, if for all $x\in X$ there exists a G-equivariant critical chart $(R,U,f,i),\rho$ on (X,s) with $x\in R$, then μ is good.*

We can also prove a torus-equivariant analogue of Theorem 2.20:

Proposition 2.44. *Let (X,s) be an algebraic d-critical locus equivariant under an algebraic torus G, and $(R,U,f,i),\rho$, $(S,V,g,j),\sigma$ be G-equivariant critical charts on (X,s). Then for each $x\in R\cap S$ there exist G-equivariant subcharts $(R',U',f',i')\subseteq(R,U,f,i)$, $(S',V',g',j')\subseteq(S,V,g,j)$ with $x\in R'\cap S'$, a G-equivariant critical chart $(T,W,h,k),\tau$ on (X,s), and G-equivariant embeddings $\Phi:(R',U',f',i')\hookrightarrow(T,W,h,k)$ and $\Psi:(S',V',g',j')\hookrightarrow(T,W,h,k)$.*

Suppose now that (X,s) is an algebraic d-critical locus invariant under a good action μ of an algebraic torus G. Write X^G for the G-fixed subscheme of X, so that X^G is a closed $\mathbb{K}$-subscheme of X with inclusion $\iota:X^G\hookrightarrow X$. Set $s^G=\iota^\star(s)\in H^0(\mathcal{S}^0_{X^G})$, for $\iota^\star$ as in Proposition 2.3. Let $(R,U,f,i),\rho$ be a G-equivariant critical chart on X. Write R^G,U^G for the G-fixed subschemes of R,U, and $f^G=f|_{U^G}$, $i^G=i|_{R^G}$. It is easy to see that (R^G,U^G,f^G,i^G) is a critical chart on (X^G,s^G). Since we can cover X by such $(R,U,f,i),\rho$ by Proposition 2.43, we can cover X^G by such (R^G,U^G,f^G,i^G). This proves:

Corollary 2.45. *Suppose (X,s) is an algebraic d-critical locus invariant under a good action μ of an algebraic torus G. Write X^G for the G-fixed subscheme of X, with inclusion $\iota:X^G\hookrightarrow X$, and $s^G=\iota^\star(s)\in H^0(\mathcal{S}^0_{X^G})$. Then (X^G,s^G) is an algebraic d-critical locus.*

Maulik [23] will use the last three results when $G=\mathbb{G}_m$ to prove a torus localization formula for the motives $MF_{X,s}$ associated to oriented algebraic d-critical loci (X,s) by Bussi, Joyce and Meinhardt [8], as in

Theorem 1.4, writing $\pi_*(MF_{X,s})$ in terms of $\pi_*(MF_{X^{\mathbb{G}_m},s^{\mathbb{G}_m}})$ for $\pi : X, X^{\mathbb{G}_m} \to \operatorname{Spec}\mathbb{K}$.

Here is an example of a non-good torus action on a d-critical locus:

Example 2.46. Let $\mathbb{A}^2$ have coordinates (x, y), and define $f : \mathbb{A}^2 \to \mathbb{A}^1$ by $f(x, y) = x^2y^2$. Write (X, s) for the corresponding affine d-critical locus with $X = \operatorname{Crit}(f)$. It is the union of the x- and y-axes in $\mathbb{A}^2$, with a non-reduced point at $(0, 0)$. Let $G = \mathbb{G}_m$ act on $\mathbb{A}^2$ by $u : (x, y) \mapsto (ux, u^{-1}y)$. Then f is $\mathbb{G}_m$-invariant, so (X, s) is also $\mathbb{G}_m$-invariant.

Define an étale equivalence relation $\sim$ on X by $(x, 0) \sim (0, x^{-1})$ for $0 \neq x \in \mathbb{A}^1$, and let $\tilde{X} = X/{\sim}$ be the quotient $\mathbb{K}$-scheme. As $\sim$ is $\mathbb{G}_m$-equivariant and preserves s, the $\mathbb{G}_m$-action and d-critical structure s on X both descend to $\tilde{X}$, so $(\tilde{X}, \tilde{s})$ is a $\mathbb{G}_m$-invariant d-critical locus.

Now $\tilde{X}$ is a projective scheme (it can be embedded in $\mathbb{KP}^2$, with reduced subscheme the nodal cubic $u^2w = v^2w + v^3$ in homogeneous coordinates $[u, v, w]$ on $\mathbb{KP}^2$), but it is not affine. The $\mathbb{G}_m$-action on $\tilde{X}$ has only two orbits, $(0, 0)$ and $\tilde{X} \setminus \{(0, 0)\}$. Thus, the only $\mathbb{G}_m$-invariant open neighbourhood of $(0, 0)$ in $\tilde{X}$ is $\tilde{X}$ itself, which is not affine, so the $\mathbb{G}_m$-action on $\tilde{X}$ is not good.

Proposition 2.43(b) shows that there does not exist a $\mathbb{G}_m$-equivariant critical chart $(R, U, f, i), \rho$ on $(\tilde{X}, \tilde{s})$ with $(0, 0) \in R$.

Remark 2.47. For actions of *reductive* groups, we can prove the following weaker analogues of Propositions 2.43 and 2.44 by similar methods:

(i) Let (X, s) be an algebraic d-critical locus which is invariant or equivariant under the action $\mu : T \times X \to X$ of a reductive algebraic $\mathbb{K}$-group G. Suppose $x \in X$ is a fixed point of G, and there exists a G-invariant affine open neighbourhood of x in X (this is automatic if X^{red} is normal). Then there exists a G-equivariant critical chart $(R, U, f, i), \rho$ on (X, s) with $x \in R$, and we may take $\dim U = \dim T_xX$.

(ii) Let (X, s) be an algebraic d-critical locus equivariant under a reductive $\mathbb{K}$-group G, $x \in X$ a fixed point of G, and $(R, U, f, i), \rho$, $(S, V, g, j), \sigma$ be G-equivariant critical charts on (X, s) with x in $R \cap S$. Then there exist G-equivariant subcharts $(R', U', f', i') \subseteq (R, U, f, i)$, $(S', V', g', j') \subseteq (S, V, g, j)$ with $x \in R' \cap S'$, a G-equivariant critical chart $(T, W, h, k), \tau$ on (X, s), and G-equivariant embeddings $\Phi : (R', U', f', i') \hookrightarrow (T, W, h, k)$ and $\Psi : (S', V', g', j') \hookrightarrow (T, W, h, k)$.

We make no claims about points $x \in X$ not fixed by G.

2.7. Background material on sheaves on Artin stacks. Section 2.8 will extend §2.1–§2.5 from $\mathbb{K}$-schemes to Artin $\mathbb{K}$-stacks. As a preliminary, to establish notation, we discuss Artin stacks and sheaves upon them.

Artin stacks are a class of geometric spaces, generalizing schemes and algebraic spaces. For a good introduction to Artin stacks see Gómez [11], and for a thorough treatment see Laumon and Moret-Bailly [20]. Artin stacks over a field $\mathbb{K}$ form a *2-category* $\mathrm{Art}_{\mathbb{K}}$, with objects the Artin stacks $X, Y, \dots$, 1-morphisms $f, g : X \to Y$, and 2-morphisms $\eta : f \Rightarrow g$, which are all 2-isomorphisms.

There is a natural full and faithful strict (2-)functor $F_{\mathrm{Sch}}^{\mathrm{Art}} : \mathrm{Sch}_{\mathbb{K}} \to \mathrm{Art}_{\mathbb{K}}$ from the category $\mathrm{Sch}_{\mathbb{K}}$ of $\mathbb{K}$-schemes (regarded as a 2-category with only identity 2-morphisms) to the 2-category $\mathrm{Art}_{\mathbb{K}}$ of Artin $\mathbb{K}$-stacks. By a common abuse of notation, we will identify $\mathrm{Sch}_{\mathbb{K}}$ with its image in $\mathrm{Art}_{\mathbb{K}}$, and consider schemes as special examples of Artin stacks. By definition, every Artin $\mathbb{K}$-stack X admits a *smooth atlas*, which is a smooth, surjective 1-morphism $t : T \to X$ in $\mathrm{Art}_{\mathbb{K}}$, for some $\mathbb{K}$-scheme T.

Let X be a $\mathbb{K}$-scheme, and $K \supseteq \mathbb{K}$ a field containing $\mathbb{K}$. A *K-point* x of X is a morphism $x : \operatorname{Spec} K \to X$ in $\mathrm{Sch}_{\mathbb{K}}$, and a *point* x of X, written $x \in X$, is a K-point for any K. Similarly, if X is an Artin $\mathbb{K}$-stack, a *K-point* of X is a 1-morphism $x : \operatorname{Spec} K \to X$ in $\mathrm{Art}_{\mathbb{K}}$. Two K-points x, x' are *equivalent*, written $x \cong x'$, if there is a 2-isomorphism $\theta : x \Rightarrow x'$. A *point* x of X, written $x \in X$, is a K-point for any K.

If x is a K-point in X, the *isotropy group* or *stabilizer group* $\mathrm{Iso}_x(X)$ is the group of 2-isomorphisms $\theta : x \Rightarrow x$. It has the structure of an algebraic K-group, and we write $\mathfrak{Iso}_x(X)$ for its Lie algebra, a K-vector space. The *Zariski cotangent space* T_x^*X of X at x is also a K-vector space; we have $h^0(\mathbb{L}_X)|_x \cong T_x^*X$ and $h^1(\mathbb{L}_X)|_x \cong \mathfrak{Iso}_x(X)^*$, where $\mathbb{L}_X$ is the *cotangent complex* of X, as in Remark 2.51 below.

Laumon and Moret-Bailly [20, §§12, 13, 15, 18] develop a theory of *sheaves on Artin stacks*, including *quasi-coherent*, *coherent*, and *constructible* sheaves, and their derived categories. Unfortunately, Laumon and Moret-Bailly wrongly assume that 1-morphisms of algebraic stacks induce morphisms of lisse-étale topoi, so parts of their theory concerning pullbacks, etc., are unsatisfactory. Olsson [24] rewrites the theory, correcting this mistake. Laszlo and Olsson [21, 22] study derived categories of constructible sheaves, and perverse sheaves, on Artin stacks, in more detail. All of [20–22, 24] work with sheaves on Artin stacks in the *lisse-étale topology*, which we now define.

Definition 2.48. Recall that a *site* is a category with a Grothendieck topology, as in Artin [1]. Let X be an Artin $\mathbb{K}$-stack. Define the *lisse-étale site* $\text{Lis-ét}(X)$ of X as follows. The category of $\text{Lis-ét}(X)$ has objects pairs (T, t), where T is a $\mathbb{K}$-scheme and $t : T \to X$ a smooth

1-morphism in $\mathrm{Art}_{\mathbb{K}}$, and morphisms $(\phi,\eta):(T,t)\to(U,u)$, for $\phi: T\to U$ a morphism in $\mathrm{Sch}_{\mathbb{K}}$ and $\eta: t\Rightarrow u\circ\phi$ a 2-morphism in $\mathrm{Art}_{\mathbb{K}}$. Composition of morphisms $(T,t)\xrightarrow{(\phi,\eta)}(U,u)\xrightarrow{(\psi,\zeta)}(V,v)$ is

$$(\psi,\zeta)\circ(\phi,\eta):=\bigl(\psi\circ\phi,(\zeta*\mathrm{id}_\phi)\odot\eta\bigr).$$

Identity morphisms are $\mathrm{id}_{(T,t)}=(\mathrm{id}_T,\mathrm{id}_t)$. The coverings of an object (T,t) in the Grothendieck topology on Lis-ét(X) are those collections of morphisms $\bigl\{(\phi_i,\eta_i):(T_i,t_i)\to(T,t)\bigr\}_{i\in I}$ for which $\bigl\{\phi_i:T_i\to T\bigr\}_{i\in I}$ is an open cover of T in the étale topology on $\mathrm{Sch}_{\mathbb{K}}$.

Definition 2.48 differs from Laumon and Moret-Bailly [20, Def. 12.1] in taking objects (T,t) with T a $\mathbb{K}$-scheme rather than an algebraic $\mathbb{K}$-space. But as in [20, Lem. 12.1.2(i)], the two definitions yield the same notion of sheaf on X.

We can now define *sheaves* (of sets, or $\mathbb{K}$-vector spaces, or $\mathbb{K}$-algebras, or ...) *on* X to be sheaves on the site Lis-ét(X), using the notion of *sheaves on a site* from Artin [1]. The *structure sheaf* $\mathcal{O}_X$ is a sheaf of $\mathbb{K}$-algebras on Lis-ét(X), and by considering sheaves of $\mathcal{O}_X$-modules on Lis-ét(X) we can define *quasi-coherent* and *coherent sheaves* on X, as in [20, §13] and [24, §6].

Laumon and Moret-Bailly [20, Lem. 12.2.1] give an alternative, explicit description of the categories of sheaves on an Artin $\mathbb{K}$-stack X. In §2.8 we will use the category $\mathrm{Sh}(X)$ in Proposition 2.49 as our definition of sheaves on X.

Proposition 2.49 (Laumon and Moret-Bailly [20])**.** *Let X be an Artin $\mathbb{K}$-stack. The category of sheaves of sets on X in the lisse-étale topology is equivalent to the category* $\mathrm{Sh}(X)$ *defined as follows:*

(A) *Objects $\mathcal{A}$ of* $\mathrm{Sh}(X)$ *comprise the following data:*

- **(a)** *For each $\mathbb{K}$-scheme T and smooth 1-morphism $t:T\to X$ in* $\mathrm{Art}_{\mathbb{K}}$, *we are given a sheaf of sets $\mathcal{A}(T,t)$ on T, in the étale topology.*
- **(b)** *For each 2-commutative diagram in* $\mathrm{Art}_{\mathbb{K}}$:

(2.39)
$$\begin{tikzcd} & U \arrow[dr, "u"] \arrow[d, phantom, "\eta\Uparrow"] & \\ T \arrow[ur, "\phi"] \arrow[rr, "t"'] & {} & X, \end{tikzcd}$$

where T,U are schemes and $t:T\to X$, $u:U\to X$ are smooth 1-morphisms, we are given a morphism $\mathcal{A}(\phi,\eta):\phi^{-1}(\mathcal{A}(U,u))\to\mathcal{A}(T,t)$ of étale sheaves of sets on T.

This data must satisfy the following conditions:

- **(i)** *If $\phi:T\to U$ in* **(b)** *is étale, then $\mathcal{A}(\phi,\eta)$ is an isomorphism.*

(ii) *For each 2-commutative diagram in* $\mathrm{Art}_{\mathbb{K}}$*:*

$$\begin{tikzcd} & V \arrow[drr, "v"] \arrow[d, phantom, "\zeta\Uparrow"] & & \\ U \arrow[ur, "\psi"] \arrow[rrr, "u"'] \arrow[rrr, phantom, shift right=3, "\eta\Uparrow"] & & & X, \\ T \arrow[u, "\phi"] \arrow[urrr, bend right=10, "t"'] & & & \end{tikzcd}$$

with T, U, V *schemes and* t, u, v *smooth, we must have*

$$\mathcal{A}\big(\psi\circ\phi, (\zeta * \mathrm{id}_\phi)\odot\eta\big) = \mathcal{A}(\phi,\eta)\circ\phi^{-1}(\mathcal{A}(\psi,\zeta)) \quad \textit{as morphisms}$$
$$(\psi\circ\phi)^{-1}(\mathcal{A}(V,v)) = \phi^{-1}\circ\psi^{-1}(\mathcal{A}(V,v)) \longrightarrow \mathcal{A}(T,t).$$

(B) *Morphisms* $\alpha : \mathcal{A}\to\mathcal{B}$ *of* $\mathrm{Sh}(X)$ *comprise a morphism* $\alpha(T,t) : \mathcal{A}(T,t)\to\mathcal{B}(T,t)$ *of étale sheaves of sets on a scheme* T *for all smooth* 1*-morphisms* $t : T\to X$*, such that for each diagram* (2.39) *in* **(b)** *the following commutes:*

$$\begin{tikzcd} \phi^{-1}(\mathcal{A}(U,u)) \arrow[rr, "\mathcal{A}(\phi,\eta)"'] \arrow[d, "\phi^{-1}(\alpha(U,u))"] & & \mathcal{A}(T,t) \arrow[d, "\alpha(T,t)"'] \\ \phi^{-1}(\mathcal{B}(U,u)) \arrow[rr, "\mathcal{B}(\phi,\eta)"] & & \mathcal{B}(T,t). \end{tikzcd}$$

(C) *Composition of morphisms* $\mathcal{A}\xrightarrow{\alpha}\mathcal{B}\xrightarrow{\beta}\mathcal{C}$ *in* $\mathrm{Sh}(X)$ *is* $(\beta\circ\alpha)(T,t) = \beta(T,t)\circ\alpha(T,t)$*. Identity morphisms* $\mathrm{id}_{\mathcal{A}} : \mathcal{A}\to\mathcal{A}$ *are* $\mathrm{id}_{\mathcal{A}}(T,t) = \mathrm{id}_{\mathcal{A}(T,t)}$*.*

The analogue of all the above also holds for (étale) sheaves of $\mathbb{K}$*-vector spaces, sheaves of* $\mathbb{K}$*-algebras, and so on, in place of (étale) sheaves of sets.*

Furthermore, the analogue of all the above holds for quasi-coherent sheaves, (or coherent sheaves, or vector bundles, or line bundles) on X*, where in* **(a)** $\mathcal{A}(T,t)$ *becomes a quasi-coherent sheaf (or coherent sheaf, or vector bundle, or line bundle) on* T*, in* **(b)** *we replace* $\phi^{-1}(\mathcal{A}(U,u))$ *by the pullback* $\phi^*(\mathcal{A}(U,u))$ *of quasi-coherent sheaves (etc.), and* $\mathcal{A}(\phi,\eta)$*,* $\alpha(T,t)$ *become morphisms of quasi-coherent sheaves (etc.) on* T*.*

We can also describe ***global sections*** *of sheaves on Artin* $\mathbb{K}$*-stacks in the above framework: a global section* $s\in H^0(\mathcal{A})$ *of* $\mathcal{A}$ *in part* **(A)** *assigns a global section* $s(T,t)\in H^0(\mathcal{A}(T,t))$ *of* $\mathcal{A}(T,t)$ *on* T *for all smooth* $t : T\to X$ *from a scheme* T*, such that* $\mathcal{A}(\phi,\eta)^*(s(U,u)) = s(T,t)$ *in* $H^0(\mathcal{A}(T,t))$ *for all 2-commutative diagrams* (2.39) *with* t,u *smooth.*

Remark 2.50. As in Laumon and Moret-Bailly [20, §13.1], if T is a $\mathbb{K}$-scheme, there is a difference between the categories $\mathrm{Sh}(T)_{\mathrm{Zar}}$ and $\mathrm{Sh}(T)_{\text{ét}}$ of sheaves of sets (say) on T in the Zariski and étale topologies. There are adjoint functors $\epsilon_* : \mathrm{Sh}(T)_{\text{ét}}\to\mathrm{Sh}(T)_{\mathrm{Zar}}$ and $\epsilon^{-1} : \mathrm{Sh}(T)_{\mathrm{Zar}}\to\mathrm{Sh}(T)_{\text{ét}}$, with ϵ^{-1} fully faithful, but in general $\mathrm{Sh}(T)_{\text{ét}}$

may be larger than $\mathrm{Sh}(T)_{\mathrm{Zar}}$. So one should distinguish between sheaves in the Zariski and the étale topologies.

However, as in [20, p. 120], the categories of quasi-coherent sheaves on T (and hence their full subcategories of coherent sheaves, vector bundles, and line bundles) in the Zariski and étale topologies are equivalent, essentially by definition. So for quasi-coherent sheaves we need not distinguish between the Zariski and étale topologies, and in the last part of Proposition 2.49 we can take the $\mathcal{A}(T,t)$ to be quasi-coherent sheaves on T in the usual (Zariski) sense.

For Theorem 2.56 we will need the following properties of cotangent complexes, as in Illusie [15, 16] and Laumon and Moret-Bailly [20, §17].

Remark 2.51. **(i)** If $f : X \to Y$ is a 1-morphism of Artin $\mathbb{K}$-stacks, we have a distinguished triangle in $D(\mathrm{qcoh}(X))$:

$$f^*(\mathbb{L}_Y) \xrightarrow{\mathbb{L}_f} \mathbb{L}_X \longrightarrow \mathbb{L}_{X/Y} \longrightarrow f^*(\mathbb{L}_X)[1], \tag{2.40}$$

where $\mathbb{L}_X, \mathbb{L}_Y$ are the *cotangent complexes* of X, Y and $\mathbb{L}_{X/Y}$ (also written $\mathbb{L}^f_{X/Y}$) is the *relative cotangent complex* of f.

(ii) If $f, g : X \to Y$ are 1-morphisms of Artin $\mathbb{K}$-stacks and $\eta : f \Rightarrow g$ is a 2-morphism, then we have a commutative diagram

$$\begin{tikzcd} f^*(\mathbb{L}_Y) \arrow[r, "\mathbb{L}_f"'] \arrow[d, "\eta^*(\mathbb{L}_Y)"] & \mathbb{L}_X \arrow[r] \arrow[d, "\mathrm{id}"] & \mathbb{L}^f_{X/Y} \arrow[r] \arrow[d, "\cong"] & f^*(\mathbb{L}_X)[1] \arrow[d, "\eta^*(\mathbb{L}_Y)"] \\ g^*(\mathbb{L}_Y) \arrow[r, "\mathbb{L}_g"] & \mathbb{L}_X \arrow[r] & \mathbb{L}^g_{X/Y} \arrow[r] & g^*(\mathbb{L}_X)[1]. \end{tikzcd}$$

(iii) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be 1-morphisms of Artin $\mathbb{K}$-stacks. Then there is a distinguished triangle in $D(\mathrm{qcoh}(X))$:

$$f^*(\mathbb{L}_{Y/Z}) \longrightarrow \mathbb{L}_{X/Z} \longrightarrow \mathbb{L}_{X/Y} \longrightarrow f^*(\mathbb{L}_{Y/Z})[1].$$

(iv) If $f : X \to Y$ is smooth then $\mathbb{L}_{X/Y}$ is equivalent to a *vector bundle of mixed rank* on X in degree 0, in the sense of Proposition 2.30. In this case we write $T^*_{X/Y}$ or $T^{f\,*}_{X/Y}$ for $\mathbb{L}_{X/Y}$ considered as a vector bundle of mixed rank. The top exterior power $\Lambda^{\mathrm{top}}\mathbb{L}_{X/Y} = \Lambda^{\mathrm{top}}T^*_{X/Y}$ is a line bundle on X.

(v) If $\phi : T \to U$ is a smooth morphism in $\mathrm{Sch}_{\mathbb{K}}$ then the relative cotangent bundle $T^*_{T/U}$ in Proposition 2.30 is canonically isomorphic to $\mathbb{L}_{T/U}$.

2.8. Extension of §2.1–§2.5 to Artin stacks. We now extend parts of §2.1–§2.5 from $\mathbb{K}$-schemes to Artin $\mathbb{K}$-stacks. In [4] we will use the ideas of this section to extend the results of [5, 6, 8] (summarized in Theorems 1.1, 1.3 and 1.4 above) to Artin stacks. Note that by the same methods we can also extend §2.1–§2.5 to *Deligne–Mumford* $\mathbb{K}$*-stacks* or *algebraic* $\mathbb{K}$*-spaces*, and the proofs simplify as the étale topology is easier to work with than the lisse-étale topology. We leave the details to the interested reader.

Combining Theorem 2.1, Proposition 2.3, and the material of §2.7, we deduce an analogue of Theorem 2.1 for Artin $\mathbb{K}$-stacks:

Corollary 2.52. *Let X be an Artin $\mathbb{K}$-stack, and write $\mathrm{Sh}(X)_{\mathbb{K}\text{-alg}}$, $\mathrm{Sh}(X)_{\mathbb{K}\text{-vect}}$ for the categories of sheaves of $\mathbb{K}$-algebras and $\mathbb{K}$-vector spaces on X defined in Proposition 2.49. Then:*

(a) *We may define canonical objects $\mathcal{S}_X$ in both categories $\mathrm{Sh}(X)_{\mathbb{K}\text{-alg}}$ and $\mathrm{Sh}(X)_{\mathbb{K}\text{-vect}}$ by $\mathcal{S}_X(T,t) := \mathcal{S}_T$ for all smooth morphisms $t : T \to X$ for $T \in \mathrm{Sch}_{\mathbb{K}}$, for $\mathcal{S}_T$ as in Theorem 2.1 taken to be a sheaf of $\mathbb{K}$-algebras (or $\mathbb{K}$-vector spaces) on T in the étale topology, and $\mathcal{S}_X(\phi,\eta) := \phi^\star : \phi^{-1}(\mathcal{S}_X(U,u)) = \phi^{-1}(\mathcal{S}_U) \to \mathcal{S}_T = \mathcal{S}_X(T,t)$ for all 2-commutative diagrams (2.39) in $\mathrm{Art}_{\mathbb{K}}$ with t,u smooth, where $\phi^\star$ is as in Proposition 2.3.*

(b) *There is a natural decomposition $\mathcal{S}_X = \mathbb{K}_X \oplus \mathcal{S}_X^0$ in $\mathrm{Sh}(X)_{\mathbb{K}\text{-vect}}$ induced by the splitting $\mathcal{S}_X(T,t) = \mathcal{S}_T = \mathbb{K}_T \oplus \mathcal{S}_T^0$ in Theorem 2.1(a), where $\mathbb{K}_X$ is a sheaf of $\mathbb{K}$-subalgebras of $\mathcal{S}_X$ in $\mathrm{Sh}(X)_{\mathbb{K}\text{-alg}}$, and $\mathcal{S}_X^0$ a sheaf of ideals in $\mathcal{S}_X$.*

Here the conditions (i),(ii) on the data $\mathcal{S}_X(T,t), \mathcal{S}_X(\phi,\eta)$ in Proposition 2.49(A) follow from the last part of Proposition 2.3 and equation (2.12). We can now generalize algebraic d-critical loci to Artin stacks.

Definition 2.53. A *d-critical stack* (X,s) is an Artin $\mathbb{K}$-stack X and a global section $s \in H^0(\mathcal{S}_X^0)$, where $\mathcal{S}_X^0$ is as in Corollary 2.52 and global sections as in Proposition 2.49, such that $\bigl(T, s(T,t)\bigr)$ is an algebraic d-critical locus in the sense of Definition 2.5 for all smooth morphisms $t : T \to X$ with $T \in \mathrm{Sch}_{\mathbb{K}}$.

The next proposition gives a convenient way to understand global sections of $\mathcal{S}_X, \mathcal{S}_X^0$ and d-critical structures on X by working on the scheme T for an atlas $t : T \to X$ for X. Then $T \times_{t,X,t} T$ is an algebraic $\mathbb{K}$-space, so choosing a surjective étale morphism $U \to T \times_{t,X,t} T$ for U a $\mathbb{K}$-scheme gives a diagram (2.41) with the properties required.

Proposition 2.54. *Suppose we are given a 2-commutative diagram*

$$\begin{tikzcd} U \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\eta\Uparrow"] & T \arrow[d, "t"] \\ T \arrow[r, "t"] & X, \end{tikzcd} \tag{2.41}$$

in $\mathrm{Art}_{\mathbb{K}}$, where X is an Artin $\mathbb{K}$-stack, T,U are $\mathbb{K}$-schemes, t,π_1,π_2 are smooth 1-morphisms, $t : T \to X$ is surjective, and the 1-morphism $U \to T \times_{t,X,t} T$ induced by (2.41) is surjective. For instance, this happens *if $U \rightrightarrows T$ is a groupoid in $\mathbb{K}$-schemes, and $X = [U \rightrightarrows T]$ the associated groupoid stack. Then:*

(i) *Let $\mathcal{S}_X$ be as in Corollary 2.52, and $\mathcal{S}_T, \mathcal{S}_U$ be as in Theorem 2.1, regarded as sheaves on T, U in the étale topology, and define $\pi_i^\star : \pi_i^{-1}(\mathcal{S}_T) \to \mathcal{S}_U$ as in Proposition 2.3 for $i = 1, 2$. Consider the map $t^* : H^0(\mathcal{S}_X) \to H^0(\mathcal{S}_T)$ mapping $t^* : s \mapsto s(T, t)$. This is injective, and induces a bijection*

$$t^* : H^0(\mathcal{S}_X) \xrightarrow{\cong} \bigl\{ s' \in H^0(\mathcal{S}_T) : \pi_1^\star(s') = \pi_2^\star(s') \text{ in } H^0(\mathcal{S}_U) \bigr\}. \tag{2.42}$$

The analogue holds for $\mathcal{S}_X^0, \mathcal{S}_T^0, \mathcal{S}_U^0$.

(ii) *Suppose $s \in H^0(\mathcal{S}_X^0)$, so that $t^*(s) \in H^0(\mathcal{S}_T^0)$ with $\pi_1^\star \circ t^*(s) = \pi_2^\star \circ t^*(s)$. Then (X, s) is a d-critical stack if and only if $\bigl(T, t^*(s)\bigr)$ is an algebraic d-critical locus, and then $\bigl(U, \pi_1^\star \circ t^*(s)\bigr)$ is also an algebraic d-critical locus.*

Example 2.55. Suppose an algebraic $\mathbb{K}$-group G acts on a $\mathbb{K}$-scheme T with action $\mu : G \times T \to T$, and write X for the quotient Artin $\mathbb{K}$-stack $[T/G]$. Then as in (2.41) there is a natural 2-Cartesian diagram

$$\begin{tikzcd} G \times T \arrow[rr, "\mu"'] \arrow[d, "\pi_T"] & \arrow[d, phantom, "\eta \Uparrow"] & T \arrow[d, "t"'] \\ T \arrow[r, "t"] & X = [T/G], & \end{tikzcd}$$

where $t : T \to X$ is a smooth atlas for X. If $s' \in H^0(\mathcal{S}_T^0)$ then $\pi_1^\star(s') = \pi_2^\star(s')$ in (2.42) becomes $\pi_T^\star(s') = \mu^\star(s')$ on $G \times T$, that is, s' is G-invariant in the sense of §2.6. Hence, Proposition 2.54 shows that d-critical structures s on $X = [T/G]$ are in 1-1 correspondence with G-invariant d-critical structures s' on T.

Next we state an analogue of Theorem 2.28, constructing the canonical bundle $K_{X,s}$ of a d-critical stack (X, s).

Theorem 2.56. *Let (X, s) be a d-critical stack. Using the description of quasi-coherent sheaves on X^{red} in Proposition 2.49 and the notation of Remark 2.51, there is a line bundle $K_{X,s}$ on the reduced $\mathbb{K}$-substack X^{red} of X called the* ***canonical bundle*** *of (X, s), unique up to canonical isomorphism, such that:*

(a) *For each point $x \in X^{\mathrm{red}} \subseteq X$ we have a canonical isomorphism*

$$\kappa_x : K_{X,s}|_x \xrightarrow{\cong} \bigl(\Lambda^{\mathrm{top}} T_x^* X\bigr)^{\otimes^2} \otimes \bigl(\Lambda^{\mathrm{top}} \mathfrak{Iso}_x(X)\bigr)^{\otimes^2}, \tag{2.43}$$

where $T_x^* X$ is the Zariski cotangent space of X at x, and $\mathfrak{Iso}_x(X)$ the Lie algebra of the isotropy group (stabilizer group) $\mathrm{Iso}_x(X)$ of X at x.

(b) *If T is a $\mathbb{K}$-scheme and $t : T \to X$ a smooth 1-morphism, so that $t^{\mathrm{red}} : T^{\mathrm{red}} \to X^{\mathrm{red}}$ is also smooth, then there is a natural* *isomorphism of line bundles on* T^{red}*:*

$$\Gamma_{T,t} : K_{X,s}(T^{\mathrm{red}}, t^{\mathrm{red}}) \xrightarrow{\cong} K_{T,s(T,t)} \otimes \bigl(\Lambda^{\mathrm{top}} T^*_{T/X}\bigr)\big|^{\otimes^{-2}}_{T^{\mathrm{red}}}. \tag{2.44}$$

Here $\bigl(T, s(T,t)\bigr)$ *is an algebraic d-critical locus by Definition* 2.53*, and* $K_{T,s(T,t)} \to T^{\mathrm{red}}$ *is its canonical bundle from Theorem* 2.28*.*

(c) *In the situation of* **(b)***, let* $p \in T^{\mathrm{red}} \subseteq T$*, so that* $t(p) \in X$*. Taking the long exact cohomology sequence of* (2.40) *for* $t : T \to X$ *and restricting to* $p \in T$ *gives an exact sequence*

$$0 \longrightarrow T^*_{t(p)}X \longrightarrow T^*_p T \longrightarrow T^*_{T/X}|_p \longrightarrow \mathfrak{Iso}_{t(p)}(X)^* \longrightarrow 0. \tag{2.45}$$

Then the following diagram commutes:

$$\begin{tikzcd}[column sep=large] K_{X,s}|_{t(p)} = K_{X,s}(T^{\mathrm{red}}, t^{\mathrm{red}})|_p \arrow[r, "\Gamma_{T,t}|_p"'] \arrow[d, "\kappa_{t(p)}"] & K_{T,s(T,t)}|_p \otimes \bigl(\Lambda^{\mathrm{top}} T^*_{T/X}\bigr)\big|^{\otimes^{-2}}_p \arrow[d, "\kappa_p \otimes \mathrm{id}"'] \\ \bigl(\Lambda^{\mathrm{top}} T^*_{t(p)}X\bigr)^{\otimes^2} \otimes \bigl(\Lambda^{\mathrm{top}} \mathfrak{Iso}_{t(p)}(X)\bigr)^{\otimes^2} \arrow[r, "\alpha_p^2"] & \bigl(\Lambda^{\mathrm{top}} T^*_p T\bigr)^{\otimes^2} \otimes \bigl(\Lambda^{\mathrm{top}} T^*_{T/X}\bigr)\big|^{\otimes^{-2}}_p, \end{tikzcd} \tag{2.46}$$

where $\kappa_p, \kappa_{t(p)}, \Gamma_{T,t}$ *are as in* (2.31), (2.43) *and* (2.44)*, respectively, and* $\alpha_p : \Lambda^{\mathrm{top}} T^*_{t(p)}X \otimes \Lambda^{\mathrm{top}} \mathfrak{Iso}_{t(p)}(X) \xrightarrow{\cong} \Lambda^{\mathrm{top}} T^*_p T \otimes \Lambda^{\mathrm{top}} T^*_{T/X}|_p^{-1}$ *is induced by taking top exterior powers in* (2.45)*.*

Here is the analogue of Definition 2.31:

Definition 2.57. Let (X,s) be a d-critical stack, and $K_{X,s}$ its canonical bundle from Theorem 2.56. An *orientation* on (X,s) is a choice of square root line bundle $K^{1/2}_{X,s}$ for $K_{X,s}$ on X^{red}. That is, an orientation is a line bundle L on X^{red}, together with an isomorphism $L^{\otimes^2} = L \otimes L \cong K_{X,s}$. A d-critical stack with an orientation will be called an *oriented d-critical stack*.

Suppose (X,s) is an oriented d-critical stack. Then for each smooth $t : T \to X$ we have a square root $K^{1/2}_{X,s}(T^{\mathrm{red}}, t^{\mathrm{red}})$ for $K_{X,s}(T^{\mathrm{red}}, t^{\mathrm{red}})$. Thus by (2.44), $K^{1/2}_{X,s}(T^{\mathrm{red}}, t^{\mathrm{red}}) \otimes (\Lambda^{\mathrm{top}} \mathbb{L}_{T/X})|_{T^{\mathrm{red}}}$ is a square root for $K_{T,s(T,t)}$. This proves:

Lemma 2.58. *Let* (X,s) *be a d-critical stack. Then an orientation* $K^{1/2}_{X,s}$ *for* (X,s) *determines a canonical orientation* $K^{1/2}_{T,s(T,t)}$ *for the algebraic d-critical locus* $\bigl(T, s(T,t)\bigr)$*, for all smooth* $t : T \to X$ *with* T *a* $\mathbb{K}$*-scheme.*

In [4] we will prove that an oriented d-critical stack (X,s) has a natural perverse sheaf $P^\bullet_{X,s}$, $\mathscr{D}$-module $D_{X,s}$, mixed Hodge module $H^\bullet_{X,s}$ (over $\mathbb{K} = \mathbb{C}$) and motive $MF_{X,s}$, as in Theorems 1.3 and 1.4 for $\mathbb{K}$-schemes.

3. The sheaves $\mathcal{S}_X, \mathcal{S}_X^0$

Sections 3.1–3.3 prove Theorem 2.1 from §2.1, and §3.4 proves Proposition 2.3.

3.1. Construction of the sheaf $\mathcal{S}_X$ in Theorem 2.1. Let X be a fixed complex analytic space. In this section we construct the sheaf $\mathcal{S}_X$ in Theorem 2.1, satisfying Theorem 2.1(i),(ii). We will use the following notation. Define a *triple* (R,U,i) to be an open subset $R \subseteq X$, a complex manifold U, and an embedding $i : R \hookrightarrow U$ of R as a closed complex analytic subspace of U, as in Theorem 2.1(i). For such a triple (R,U,i), define the sheaf of ideals $I_{R,U} \subseteq i^{-1}(\mathcal{O}_U)$ as in (2.1). We will also write $I'_{R,U} \subseteq \mathcal{O}_U$ for the sheaf of ideals vanishing on the closed complex analytic subspace $i(R) \subseteq U$.

If (R,U,i) is a triple and $U' \subseteq U$ is open, set $R' := i^{-1}(U') \subseteq R$ and $i' := i|_{R'} : R' \hookrightarrow U'$. Then (R',U',i') is another triple, which we call a *subtriple* of (R,U,i), and write as $(R',U',i') \subseteq (R,U,i)$.

For each triple (R,U,i), as in (2.2) define $\mathcal{K}_{R,U,i}, \kappa_{R,U}$ by the exact sequence of sheaves of $\mathbb{C}$-vector spaces on R:

$$0 \longrightarrow \mathcal{K}_{R,U,i} \xrightarrow{\ \kappa_{R,U}\ } \frac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2} \xrightarrow{\ \mathrm{d}\ } \frac{i^{-1}(T^*U)}{I_{R,U} \cdot i^{-1}(T^*U)}. \tag{3.1}$$

That is, $\kappa_{R,U} : \mathcal{K}_{R,U,i} \to i^{-1}(\mathcal{O}_U)/I_{R,U}^2$ is the kernel of $\mathrm{d} : i^{-1}(\mathcal{O}_U)/I_{R,U}^2 \to i^{-1}(T^*U)/I_{R,U} \cdot i^{-1}(T^*U)$. The difference between (2.2) and (3.1) is that (2.2) includes an isomorphism $\mathcal{S}_X|_R \cong \mathcal{K}_{R,U,i}$, but we have not yet defined $\mathcal{S}_X$. If $(R',U',i') \subseteq (R,U,i)$ then $\mathcal{K}_{R',U',i'} = \mathcal{K}_{R,U,i}|_{R'}$ and $\kappa_{R',U'} = \kappa_{R,U}|_{R'}$.

Note that $i^{-1}(\mathcal{O}_U)/I_{R,U}^2$ in (3.1) is a sheaf of commutative $\mathbb{C}$-algebras on R, since $i^{-1}(\mathcal{O}_U)$ is and $I_{R,U}^2 \subset i^{-1}(\mathcal{O}_U)$ is a sheaf of ideals. Now $\kappa_{R,U}(\mathcal{K}_{R,U,i})$ is the subsheaf of local sections $f + I_{R,U}^2$ in $i^{-1}(\mathcal{O}_U)/I_{R,U}^2$ such that $\mathrm{d}f \in I_{R,U} \cdot i^{-1}(T^*U)$. If $f + I_{R,U}^2, g + I_{R,U}^2$ are two such sections then $\mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f \in I_{R,U} \cdot i^{-1}(T^*U)$, so $(f + I_{R,U}^2) \cdot (g + I_{R,U}^2) \in \kappa_{R,U}(\mathcal{K}_{R,U,i})$. Also $1 + I_{R,U}^2 \in \kappa_{R,U}(\mathcal{K}_{R,U,i})$ as $\mathrm{d}(1) = 0$. Hence $\kappa_{R,U}(\mathcal{K}_{R,U,i})$ is a subsheaf of $\mathbb{C}$-vector spaces in $i^{-1}(\mathcal{O}_U)/I_{R,U}^2$ which is closed under multiplication and contains the identity, so $\kappa_{R,U}(\mathcal{K}_{R,U,i})$ is a sheaf of $\mathbb{C}$-subalgebras in $i^{-1}(\mathcal{O}_U)/I_{R,U}^2$. Therefore $\mathcal{K}_{R,U,i}$ has the structure of a sheaf of commutative $\mathbb{C}$-algebras on R in a unique way, such that $\kappa_{R,U}$ in (3.1) is a morphism of sheaves of $\mathbb{C}$-algebras.

Call $\Phi : (R,U,i) \to (S,V,j)$ a *morphism of triples* if $R \subseteq S \subseteq X$, and $\Phi : U \to V$ is holomorphic with $\Phi \circ i = j|_R : R \to V$. As for (2.3), form the commutative diagram of sheaves of $\mathbb{C}$-vector spaces on R:

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{K}_{S,V,j}|_R \arrow[r, "\kappa_{S,V}|_R"] \arrow[d, dotted, "\Phi^*"] & \frac{j^{-1}(\mathcal{O}_V)}{I_{S,V}^2}\Big|_R \arrow[r, "\mathrm{d}"] \arrow[d, "i^{-1}(\Phi^\sharp)"] & \frac{j^{-1}(T^*V)}{I_{S,V}\cdot j^{-1}(T^*V)}\Big|_R \arrow[d, "i^{-1}(\mathrm{d}\Phi)"] \\
0 \arrow[r] & \mathcal{K}_{R,U,i} \arrow[r, "\kappa_{R,U}"] & \frac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2} \arrow[r, "\mathrm{d}"] & \frac{i^{-1}(T^*U)}{I_{R,U}\cdot i^{-1}(T^*U)}.
\end{tikzcd}
\tag{3.2}
$$

Since the right hand square of (3.2) commutes as in (2.3), exactness of the rows implies there is a unique $\Phi^* : \mathcal{K}_{S,V,j}|_R \to \mathcal{K}_{R,U,i}$ making (3.2) commute. As $\kappa_{R,U}, \kappa_{S,V}|_R, i^{-1}(\Phi^\sharp)$ are morphisms of sheaves of $\mathbb{C}$-algebras, so is Φ^*.

If $(R',U',i') \subseteq (R,U,i)$ and $(S',V',j') \subseteq (S,V,j)$ with $\Phi(U') \subseteq V' \subseteq V$ then setting $\Phi' := \Phi|_{U'} : U' \to V'$, we have $\Phi'^* = \Phi^*|_{R'}$.

If $\Psi : (S,V,j) \to (T,W,k)$ is another morphism then so is $\Psi \circ \Phi : (R,U,i) \to (T,W,k)$, and by considering the diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{K}_{T,W,k}|_R \arrow[r, "\kappa_{T,W}|_R"] \arrow[d, "\Psi^*|_R"] \arrow[dd, bend right, "(\Psi\circ\Phi)^*"'] & \frac{k^{-1}(\mathcal{O}_W)}{I_{T,W}^2}\Big|_R \arrow[r, "\mathrm{d}"] \arrow[d, "j^{-1}(\Psi^\sharp)|_R"] \arrow[dd, bend right, "i^{-1}((\Psi\circ\Phi)^\sharp)"'] & \frac{k^{-1}(T^*W)}{I_{T,W}\cdot k^{-1}(T^*W)}\Big|_R \arrow[d, "j^{-1}(\mathrm{d}\Psi)|_R"] \arrow[dd, bend right, "i^{-1}(\mathrm{d}(\Psi\circ\Phi))"'] \\
0 \arrow[r] & \mathcal{K}_{S,V,j}|_R \arrow[r, "\kappa_{S,V}|_R"] \arrow[d, "\Phi^*"] & \frac{j^{-1}(\mathcal{O}_V)}{I_{S,V}^2}\Big|_R \arrow[r, "\mathrm{d}"] \arrow[d, "i^{-1}(\Phi^\sharp)"] & \frac{j^{-1}(T^*V)}{I_{S,V}\cdot j^{-1}(T^*V)}\Big|_R \arrow[d, "i^{-1}(\mathrm{d}\Phi)"] \\
0 \arrow[r] & \mathcal{K}_{R,U,i} \arrow[r, "\kappa_{R,U}"] & \frac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2} \arrow[r, "\mathrm{d}"] & \frac{i^{-1}(T^*U)}{I_{R,U}\cdot i^{-1}(T^*U)},
\end{tikzcd}
$$

we see that $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*|_R$, that is, the morphisms Φ^* in (3.2) are contravariantly functorial. If $(R,U,i) = (S,V,j)$ and $\Phi = \mathrm{id}_U$ then $\Phi^* = \mathrm{id}$.

We begin with three lemmas. The first is the main point of the proof:

Lemma 3.1. *The morphism* Φ^* *in* (3.2) *is independent of the choice of* Φ. *That is, if* $\Phi, \tilde{\Phi} : (R,U,i) \to (S,V,j)$ *are morphisms of triples then*

$$
\Phi^* = \tilde{\Phi}^* : \mathcal{K}_{S,V,j}|_R \longrightarrow \mathcal{K}_{R,U,i}. \tag{3.3}
$$

Proof. If $x \in R \subseteq S \subseteq X$ and α is a local section of $\mathcal{K}_{S,V,j}|_R$ defined near x in R, then $\alpha = f + (I'_{S,V})^2$ for f a local section of $\mathcal{O}_V$ defined near $j(x)$ in V such that $\mathrm{d}f$ is a section of $I'_{S,V} \cdot T^*V \subseteq T^*V$ near $j(x)$ in V. Then $\Phi^*(\alpha) = f \circ \Phi + (I'_{R,U})^2$ and $\tilde{\Phi}^*(\alpha) = f \circ \tilde{\Phi} + (I'_{R,U})^2$ near x in R.

Choose holomorphic coordinates $(z_1, \ldots, z_n)$ on V near $j(x) = \Phi(i(x)) = \tilde{\Phi}(i(x))$. Then by a holomorphic version of Taylor's Theorem we have

$$
\begin{aligned}
f \circ \tilde{\Phi} - f \circ \Phi = {} & \textstyle\sum_{a=1}^n \left(\frac{\partial f}{\partial z_a} \circ \Phi\right) \cdot \left(z_a \circ \tilde{\Phi} - z_a \circ \Phi\right) \\
& + \textstyle\sum_{a,b=1}^n A_{ab}\left(z_a \circ \tilde{\Phi} - z_a \circ \Phi\right)\left(z_b \circ \tilde{\Phi} - z_b \circ \Phi\right)
\end{aligned} \tag{3.4}
$$

near $j(x)$, for some holomorphic $A_{ab} : V \to \mathbb{C}$ defined near $j(x)$.

Since $\Phi \circ i = \tilde{\Phi} \circ i$ and $I'_{R,U}$ is the ideal in $\mathcal{O}_U$ vanishing on $i(R)$ near $i(x)$, we see that $z_a \circ \tilde{\Phi} - z_a \circ \Phi \in I'_{R,U}$ on U near $i(x)$ for each $a = 1, \ldots, n$. Also $\frac{\partial f}{\partial z_a} \in I'_{S,V}$ near $j(x)$ by choice of α, f, and $g \mapsto g \circ \Phi$ maps $I'_{S,V} \to I'_{R,U}$ near $j(x), i(x)$ as $\Phi \circ i = j|_R$, so $\frac{\partial f}{\partial z_a} \circ \Phi \in I'_{R,U}$ near $i(x)$. Thus each factor $(\cdots)$ on the right hand side of (3.4) lies in $I'_{R,U}$ near $i(x)$, so $f \circ \tilde{\Phi} - f \circ \Phi \in (I'_{R,U})^2$ near $i(x)$. Therefore

$$
\Phi^*(\alpha) = f \circ \Phi + (I'_{R,U})^2 = f \circ \tilde{\Phi} + (I'_{R,U})^2 = \tilde{\Phi}^*(\alpha)
$$

near x in R for any local section α of $\mathcal{K}_{S,V,j}|_R$, which proves (3.3). q.e.d.

Lemma 3.2. *Let $(R, U, i), (S, V, j)$ be triples. Then for each $x \in R \cap S \subseteq X$, there exists a subtriple $(R', U', i') \subseteq (R, U, i)$ with $x \in R' \subseteq R \cap S$ and a morphism $\Phi : (R', U', i') \to (S, V, j)$.*

Proof. Choose holomorphic coordinates $(z_1, \ldots, z_n)$ on an open neighbourhood $\tilde{V}$ of $j(x)$ in V, so that $(z_1, \ldots, z_n) : \tilde{V} \to \mathbb{C}^n$ is a biholomorphism with an open set $\tilde{W} \subseteq \mathbb{C}^n$. Let U' be an open neighbourhood of $i(x)$ in U small enough that $R' := i^{-1}(U') \subseteq R \cap j^{-1}(\tilde{V}) \subseteq R \cap S \subseteq X$. Then $z_a \circ j|_{R'}$ for $a = 1, \ldots, n$ are morphisms $R' \to \mathbb{C}$. Since $i(R')$ is a closed complex analytic subspace of U' isomorphic to R', and any holomorphic function on $i(R')$ extends locally to U' near $i(x)$, by making R', U' smaller we can suppose there exist holomorphic $f_a : U' \to \mathbb{C}$ with $f_a \circ i|_{R'} = z_a \circ j|_{R'}$ for $a = 1, \ldots, n$.

Making R', U' smaller again, we can suppose that $(f_1, \ldots, f_n) : U' \to \mathbb{C}^n$ maps into $\tilde{V} \subseteq \mathbb{C}^n$. Then there is a unique holomorphic map $\Phi : U' \to \tilde{V} \subseteq V$ with $z_a \circ \Phi = f_a$ for $a = 1, \ldots, n$. Hence $z_a \circ \Phi \circ i|_{R'} = f_a \circ i|_{R'} = z_a \circ j|_{R'}$ for $a = 1, \ldots, n$, which implies that $\Phi \circ i|_{R'} = j|_{R'}$ as $(z_1, \ldots, z_n) : \tilde{V} \to \mathbb{C}^n$ is injective. Thus $\Phi : (R', U', i') \to (S, V, j)$ is a morphism of triples. q.e.d.

Lemma 3.3. *Let $(R, U, i), (S, V, j)$ be triples. Then there exists a unique isomorphism of sheaves of commutative $\mathbb{C}$-algebras on $R \cap S$:*

$$
I_{S,V,j}^{R,U,i} : \mathcal{K}_{S,V,j}|_{R \cap S} \longrightarrow \mathcal{K}_{R,U,i}|_{R \cap S} \tag{3.5}
$$

such that if $(R', U', i'), \Phi$ are as in Lemma 3.2 then $I_{S,V,j}^{R,U,i}\big|_{R'} = \Phi^*$. Also

$$
I_{R,U,i}^{R,U,i} = \mathrm{id}_{\mathcal{K}_{R,U,i}}, \qquad I_{R,U,i}^{S,V,j} = \left(I_{S,V,j}^{R,U,i}\right)^{-1}, \tag{3.6}
$$

and if (T, W, k) *is another triple then*

$$I^{R,U,i}_{S,V,j}\big|_{R\cap S\cap T} \circ I^{S,V,j}_{T,W,k}\big|_{R\cap S\cap T} = I^{R,U,i}_{T,W,k}\big|_{R\cap S\cap T} : \mathcal{K}_{T,W,k}|_{R\cap S\cap T} \longrightarrow \mathcal{K}_{R,U,i}|_{R\cap S\cap T}. \tag{3.7}$$

Proof. Suppose $x, (R', U', i'), \Phi$ and $\hat{x}, (\hat{R}', \hat{U}', \hat{i}'), \hat{\Phi}$ are two possible choices in Lemma 3.2. Then $\Phi : U' \to V$, $\hat{\Phi} : \hat{U}' \to V$ induce morphisms

$$\Phi^* : \mathcal{K}_{S,V,j}|_{R'} \longrightarrow \mathcal{K}_{R,U,i}|_{R'} \text{ and } \hat{\Phi}^* : \mathcal{K}_{S,V,j}|_{\hat{R}'} \longrightarrow \mathcal{K}_{R,U,i}|_{\hat{R}'}.$$

The restrictions $\Phi|_{U'\cap\hat{U}'}$ and $\hat{\Phi}|_{U'\cap\hat{U}'}$ thus induce morphisms

$$\Phi^*|_{R'\cap\hat{R}'}, \hat{\Phi}^*|_{R'\cap\hat{R}'} : \mathcal{K}_{S,V,j}|_{R'\cap\hat{R}'} \longrightarrow \mathcal{K}_{R,U,i}|_{R'\cap\hat{R}'}.$$

Lemma 3.1 now shows that $\Phi^*|_{R'\cap\hat{R}'} = \hat{\Phi}^*|_{R'\cap\hat{R}'}$.

Thus, Lemma 3.2 shows that for each $x \in R \cap S$ we can choose an open neighbourhood R' of x in $R\cap S$, and a morphism $\Phi^* : \mathcal{K}_{S,V,j}|_{R'} \longrightarrow \mathcal{K}_{R,U,i}|_{R'}$. These open neighbourhoods R' form an open cover of X, and on overlaps $R' \cap \hat{R}'$ the corresponding morphisms $\Phi^*, \hat{\Phi}^*$ agree. Therefore by properties of sheaves there is a unique morphism $I^{R,U,i}_{S,V,j}$ of sheaves of commutative $\mathbb{C}$-algebras in (3.5) such that $I^{R,U,i}_{S,V,j}\big|_{R'} = \Phi^*$ for all $(R', U', i'), \Phi$ as in Lemma 3.2.

To see that $I^{R,U,i}_{R,U,i} = \mathrm{id}_{\mathcal{K}_{R,U,i}}$ as in (3.6), take $(R', U', i') = (R, U, i) = (S, V, j)$ and $\Phi = \mathrm{id}_U$, so that $\Phi^* = \mathrm{id}$.

To prove (3.7), let $(R, U, i), (S, V, j), (T, W, k)$ be triples and $x \in R \cap S \cap T$. Apply Lemma 3.2 twice to get open $i(x) \in U' \subseteq U$ and $j(x) \in V' \subseteq V$ and morphisms $\Phi : (R', U', i') \to (S, V, j)$ and $\Psi : (S', V', j') \to (T, W, k)$. Making U', R' smaller we can suppose that $j(x) \in \Phi(U') \subseteq V' \subseteq V$, so that Φ is also a morphism $(R', U', i') \to (S', V', j')$. Functoriality of the Φ^* now gives $\Phi^* \circ \Psi^*|_{R'} = (\Psi \circ \Phi)^*$. So the defining property of the $I^{R,U,i}_{S,V,j}$ gives

$$I^{R,U,i}_{S,V,j}\big|_{R'} \circ I^{S,V,j}_{T,W,k}\big|_{R'} = I^{R,U,i}_{T,W,k}\big|_{R'}.$$

As we can cover $R \cap S \cap T$ by such open R', equation (3.7) follows.

Finally, applying (3.7) with $(T, W, k) = (R, U, i)$ and using the first equation of (3.6) yields

$$I^{R,U,i}_{S,V,j} \circ I^{S,V,j}_{R,U,i} = I^{R,U,i}_{R,U,i}\big|_{R\cap S} = \mathrm{id}_{\mathcal{K}_{R,U,i}}\big|_{R\cap S}.$$

Exchanging $(R, U, i), (S, V, j)$ proves the second equation of (3.6), and also shows that $I^{R,U,i}_{S,V,j}$ is an isomorphism. The lemma is complete. q.e.d.

We can now construct the sheaf $\mathcal{S}_X$ in Theorem 2.1. Since X is locally of finite type by our convention in §1, near each $x \in X$ it admits a local embedding $i : X \hookrightarrow U$ into a complex manifold U. Therefore we can choose a family $\big\{(R_a, U_a, i_a) : a \in A\big\}$ of triples such that $\{R_a : a \in A\}$ is an open cover of X. For each $a \in A$ we have a sheaf of commutative

$\mathbb{C}$-algebras $\mathcal{K}_{R_a,U_a,i_a}$ on R_a from (3.1), and for all $a,b\in A$ we have an isomorphism

$$I^{R_a,U_a,i_a}_{R_b,U_b,i_b} : \mathcal{K}_{R_b,U_b,i_b}|_{R_a\cap R_b} \longrightarrow \mathcal{K}_{R_a,U_a,i_a}|_{R_a\cap R_b}$$

on $R_a\cap R_b$ by (3.5), which satisfy the usual conditions (3.6)–(3.7) on identities, symmetry, and triple overlaps.

Hence by properties of sheaves there exists a sheaf $\mathcal{S}_X$ of commutative $\mathbb{C}$-algebras on X, unique up to canonical isomorphism, with isomorphisms $\iota_{R_a,U_a}:\mathcal{S}_X|_{R_a}\to\mathcal{K}_{R_a,U_a,i_a}$ for $a\in A$, such that $I^{R_a,U_a,i_a}_{R_b,U_b,i_b}\circ \iota_{R_b,U_b}|_{R_a\cap R_b}=\iota_{R_a,U_a}|_{R_a\cap R_b}$ for all $a,b\in A$. One way to do this is to define an explicit presheaf $\mathcal{PS}_X$ on X using $R_a,\mathcal{K}_{R_a,U_a,i_a},I^{R_a,U_a,i_a}_{R_b,U_b,i_b}$, and take $\mathcal{S}_X$ to be the sheafification of $\mathcal{PS}_X$.

Now suppose (R,U,i) is any triple, as in Theorem 2.1(i). We can construct $\mathcal{S}_X$ using the family $\bigl\{(R_a,U_a,i_a):a\in A\bigr\}\cup\bigl\{(R,U,i)\bigr\}$ instead of $\bigl\{(R_a,U_a,i_a):a\in A\bigr\}$, and get the same (not just isomorphic) sheaf $\mathcal{S}_X$ and the same isomorphisms $\iota_{R_a,U_a}:\mathcal{S}_X|_{R_a}\to\mathcal{K}_{R_a,U_a,i_a}$, but now we also have an isomorphism $\iota_{R,U}:\mathcal{S}_X|_R\to\mathcal{K}_{R,U,i}$. Combining this with (3.1) gives the exact sequence (2.2), proving Theorem 2.1(i).

Suppose $\Phi:(R,U,i)\to(S,V,j)$ is any morphism of triples, as in Theorem 2.1(ii), so that $R\subseteq S$. We can construct $\mathcal{S}_X$ using the family $\bigl\{(R_a,U_a,i_a):a\in A\bigr\}\cup\bigl\{(R,U,i),(S,V,j)\bigr\}$ instead of $\bigl\{(R_a,U_a,i_a):a\in A\bigr\}$, and get the same sheaf $\mathcal{S}_X$ and isomorphisms ι_{R_a,U_a}, but now we also have isomorphisms $\iota_{R,U}:\mathcal{S}_X|_R\to\mathcal{K}_{R,U,i}$ and $\iota_{S,V}:\mathcal{S}_X|_S\to\mathcal{K}_{S,V,j}$ satisfying $I^{R,U,i}_{S,V,j}\circ\iota_{S,V}|_R=\iota_{R,U}$. Consider the diagram:

$$\begin{tikzcd}
\mathcal{S}_X|_R \arrow[r, "\iota_{S,V}|_R"] \arrow[d, "\mathrm{id}"] & \mathcal{K}_{S,V,j}|_R \arrow[r, "\kappa_{S,V}|_R"] \arrow[d, "I^{R,U,i}_{S,V,j}=\Phi^*"] & \frac{j^{-1}(\mathcal{O}_V)}{I^2_{S,V}}\Big|_R \arrow[r, "\mathrm{d}"] \arrow[d, "i^{-1}(\Phi^\sharp)"] & \frac{j^{-1}(T^*V)}{I_{S,V}\cdot j^{-1}(T^*V)}\Big|_R \arrow[d, "i^{-1}(\mathrm{d}\Phi)"] \\
\mathcal{S}_X|_R \arrow[r, "\iota_{R,U}"] & \mathcal{K}_{R,U,i} \arrow[r, "\kappa_{R,U}"] & \frac{i^{-1}(\mathcal{O}_U)}{I^2_{R,U}} \arrow[r, "\mathrm{d}"] & \frac{i^{-1}(T^*U)}{I_{R,U}\cdot i^{-1}(T^*U)}.
\end{tikzcd} \tag{3.8}$$

Here $I^{R,U,i}_{S,V,j}=\Phi^*$ by Lemma 3.3. The left hand square of (3.8) commutes as $I^{R,U,i}_{S,V,j}\circ\iota_{S,V}|_R=\iota_{R,U}$, and the right hand two squares commute by (3.2). Composing the first two horizontal morphisms in the rows of (3.8) gives (2.3). This proves Theorem 2.1(ii).

3.2. Theorem 2.1(a)–(c): properties of $\mathcal{S}_X$. Next we prove that the sheaf $\mathcal{S}_X$ constructed in §3.1 satisfies Theorem 2.1(a)–(c). We continue to use the notation of §3.1. For part (a), define $\mathcal{S}^0_X\subset\mathcal{S}_X$ to be the kernel of the morphism $\mathcal{S}_X\to\mathcal{O}_{X^{\mathrm{red}}}$ in (2.5), where $\beta_X:\mathcal{S}_X\to\mathcal{O}_X$ is defined using (3.1)–(3.2) and $\mathcal{O}_X|_R\cong i^{-1}(\mathcal{O}_U)/I_{R,U}$. As (2.5) are morphisms of sheaves of $\mathbb{C}$-algebras, $\mathcal{S}^0_X$ is a sheaf of ideals in $\mathcal{S}_X$.

There is also a natural inclusion $\mathbb{C}_X\hookrightarrow\mathcal{S}_X$ of sheaves of $\mathbb{C}$-algebras, where $\mathbb{C}_X$ is the sheaf of locally constant functions $X\to\mathbb{C}$: for any triple

(R,U,i), the sheaf $\mathbb{C}_U$ is a subsheaf of $\mathcal{O}_U$, so $\bigl(i^{-1}(\mathbb{C}_U)+I_{R,U}^2\bigr)/I_{R,U}^2$ is a subsheaf of $i^{-1}(\mathcal{O}_U)/I_{R,U}^2$ which lies in the kernel of d in (2.2), and so lifts to a subsheaf of $\mathcal{S}_X|_R$, which is isomorphic to $\mathbb{C}_X|_R$, as locally constant functions on X lift locally uniquely to locally constant functions on U near $i(X)$.

It remains to show that $\mathcal{S}_X=\mathbb{C}_X\oplus\mathcal{S}_X^0$. To see this, suppose (R,U,i) is a triple and s is a local section of $\mathcal{S}_X$ near $x\in R$, so that $\iota_{R,U}(s|_R)=f+(I'_{R,U})^2$ for f a local section of $\mathcal{O}_U$ defined near $i(x)$ in U with $\mathrm{d}f$ a local section of $I'_{R,U}\cdot T^*U$. Then $f|_{i(X^{\mathrm{red}})}$ is locally constant on $i(X^{\mathrm{red}})$ near $i(x)$, so $f|_{i(X^{\mathrm{red}})}$ extends locally uniquely to a locally constant function $c:U\to\mathbb{C}$ defined near $i(x)$. Writing $f_0=f-c$, we have $f_0|_{i(X^{\mathrm{red}})}=0$ near $i(x)$. The local section s now splits uniquely as $s=s_0+t$ with $\iota_{R,U}(s_0)=f_0+(I'_{R,U})^2$ and $\iota_{R,U}(t)=c+(I'_{R,U})^2$. But s_0 is a local section of $\mathcal{S}_X^0$ as $f_0|_{i(X^{\mathrm{red}})}=0$, and t a local section of $\mathbb{C}_X$ as c is locally constant. Hence $\mathcal{S}_X=\mathbb{C}_X\oplus\mathcal{S}_X^0$, proving (a).

Part (b) involves the *cotangent complex* $\mathbb{L}_X\in D(\mathrm{qcoh}(X))$. For background on (co)tangent complexes, see Illusie [15], [16, §1] for $\mathbb{K}$-schemes, and Palamodov [25–28] for complex analytic spaces. We need only two facts: that $h^0(\mathbb{L}_X)\cong T^*X$, and if $R\subseteq X$ is open and $i:R\hookrightarrow U$ is a closed embedding of R into a complex manifold U, then the truncation $\tau_{\geqslant -1}(\mathbb{L}_X)$ satisfies

$$\tau_{\geqslant -1}(\mathbb{L}_X)|_R\cong\bigl[\,I_{R,U}/I_{R,U}^2\xrightarrow{\ \gamma\ }i^*(T^*U)\,\bigr] \tag{3.9}$$

in $D(\mathrm{qcoh}(R))$, where $I_{R,U}/I_{R,U}^2$ is in degree -1 and $i^*(T^*U)$ in degree 0, and the morphism γ maps $\gamma:f+I_{R,U}^2\mapsto i^*(\mathrm{d}f)$.

Consider the diagram of sheaves on R:

(3.10)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & h^{-1}(\mathbb{L}_X)|_R \arrow[r] \arrow[d, dotted, "\alpha_X|_R"] & I_{R,U}/I_{R,U}^2 \arrow[r, "\gamma"] \arrow[d] & i^*(T^*U) \arrow[r] \arrow[d, "\cong"] & {\begin{array}{c} h^0(\mathbb{L}_X)|_R \\ \cong T^*X|_R \end{array}} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{S}_X|_R \arrow[r, "\iota_{R,U}"] \arrow[d, dotted, "\beta_X|_R"] & \dfrac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2} \arrow[r, "\mathrm{d}"] \arrow[d, "\pi"] & \dfrac{i^{-1}(T^*U)}{I_{R,U}\cdot i^{-1}(T^*U)} \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_X|_R \arrow[r, "\delta", "\cong"'] \arrow[d, "\mathrm{d}"] & \dfrac{i^{-1}(\mathcal{O}_U)}{I_{R,U}} \arrow[r] \arrow[d] & 0 \arrow[d] & & \\
 & T^*X|_R & 0 & 0. & &
\end{tikzcd}$$

Here the first row is exact by (3.9), the second row by (2.2), and the third row and second and third columns are obviously exact. Also the middle two squares commute by definition of γ in (3.9). Properties of exact sequences now imply there are unique morphisms $\alpha_X|_R, \beta_X|_R$ as shown making (3.10) commute. Using functoriality of the isomorphism (3.9), one can show that these $\alpha_X|_R, \beta_X|_R$ are independent of R, U, i locally, and so glue on an open cover to give unique global morphisms α_X, β_X in (2.6), noting that the first column of (3.10) is the restriction of (2.6) to R.

Since taking kernels of morphisms of sheaves of $\mathbb{C}$-vector spaces is a left exact functor from the category of morphisms of such sheaves to the category of such sheaves, equation (3.10) also implies that the first column of (3.10) is exact at $h^{-1}(\mathbb{L}_X)|_R$ and $\mathcal{S}_X|_R$. To prove it is exact at $\mathcal{O}_X|_R$, we work at the level of stalks. Let $x \in R \subseteq X$, and write $\mathcal{S}_{X,x}, \mathcal{O}_{X,x}, T^*X_x, \ldots$ for the stalks of $\mathcal{S}_X, \mathcal{O}_X, T^*X, \ldots$ at $x \in X$, so that $\mathcal{S}_{X,x}, \ldots$ are $\mathbb{C}$-vector spaces whose elements are germs at x of sections of $\mathcal{S}_X, \ldots$. A sequence of sheaves of $\mathbb{C}$-vector spaces on R is exact if and only if it is exact on stalks at every $x \in R$.

Let $\eta \in \mathcal{O}_{X,x}$ with $\mathrm{d}\eta = 0 \in T^*X_x$. Then $\delta_x(\eta) \in i^{-1}(\mathcal{O}_U)_x/I_{R,U,x}$, so we may write $\delta_x(\eta) = \zeta + I_{R,U,x}$ for some $\zeta \in i^{-1}(\mathcal{O}_U)_x$. The exact sequences in (3.10) induce an isomorphism

$$T^*X_x \cong i^{-1}(T^*U)_x / \big(I_{R,U,x} \cdot i^{-1}(T^*U)_x + \mathrm{d}(I_{R,U,x})\big),$$

which identifies $\mathrm{d}\eta \in T^*X_x$ with $\mathrm{d}\zeta + \big(I_{R,U,x} \cdot i^{-1}(T^*U)_x + \mathrm{d}(I_{R,U,x})\big)$, so $\mathrm{d}\zeta \in I_{R,U,x} \cdot i^{-1}(T^*U)_x + \mathrm{d}(I_{R,U,x}) \subseteq i^{-1}(T^*U)_x$ as $\mathrm{d}\eta = 0$. As ζ is unique up to addition of an element of $I_{R,U,x}$, by changing our choice of ζ we can eliminate the $\mathrm{d}(I_{R,U,x})$ component in $\mathrm{d}\zeta$, so that $\mathrm{d}\zeta \in I_{R,U,x} \cdot i^{-1}(T^*U)_x$. Then $\zeta + I_{R,U,x}^2$ lies in $i^{-1}(\mathcal{O}_U)_x / I_{R,U,x}^2$ with $\mathrm{d}(\zeta + I_{R,U,x}^2) = 0$ in $i^{-1}(T^*U)_x / I_{R,U,x} \cdot i^{-1}(T^*U)_x$. Hence by the exactness of the middle row of (3.10) in stalks at x, there exists a unique $\theta \in \mathcal{S}_{X,x}$ with $\iota_{R,U,x}(\theta) = \zeta + I_{R,U,x}^2$. Therefore

$$\delta_x \circ \beta_{X,x}(\theta) = \pi_x \circ \iota_{R,U,x}(\theta) = \pi_x(\zeta + I_{R,U,x}^2) = \zeta + I_{R,U,x} = \delta_x(\eta),$$

as the bottom left square in (3.10) commutes.

Since δ is an isomorphism, this forces $\beta_{X,x}(\theta) = \eta$. Similarly, if $\theta \in \mathcal{S}_{X,x}$ then $\mathrm{d} \circ \beta_{X,x}(\theta) = 0$. Thus the first column of (3.10) is exact on stalks at $\mathcal{O}_X|_R$, so it is exact. As such open $R \subseteq X$ cover X, we see that (2.6) is exact. Exactness of (2.7) follows from (2.5)–(2.6). This proves Theorem 2.1(b).

For part (c), consider the morphism of sheaves on R

$$\mathrm{d} : i^{-1}(\mathcal{O}_U)/I_{R,U}^2 \longrightarrow i^{-1}(T^*U)/\mathrm{d}(I_{R,U}^2).$$

Composing with $\iota_{R,U}$ maps $\mathcal{S}_X|_R \to i^{-1}(T^*U)/\mathrm{d}(I_{R,U}^2)$. From (2.2), the image of $\mathrm{d} \circ \iota_{R,U}$ lies in $\big(I_{R,U} \cdot i^{-1}(T^*U)/\mathrm{d}(I_{R,U}^2)\big) \subseteq i^{-1}(T^*U)/\mathrm{d}(I_{R,U}^2)$. Also, as $\mathrm{d}^2 = 0$, the image lies in $\operatorname{Ker} \mathrm{d} + \mathrm{d}(I_{R,U}^2)$. This defines a morphism

$$\mathrm{d}\circ\iota_{R,U}:\mathcal{S}_X|_R\longrightarrow\frac{\mathrm{Ker}\big(\mathrm{d}:I_{R,U}\cdot i^{-1}(T^*U)\longrightarrow i^{-1}(\Lambda^2T^*U)\big)}{\mathrm{Im}\big(\mathrm{d}:I_{R,U}^2\longrightarrow I_{R,U}\cdot i^{-1}(T^*U)\big)},\tag{3.11}$$

where the right hand side is the cohomology of (2.8).

In the splitting $\mathcal{S}_X|_R=\mathbb{C}_X|_R\oplus\mathcal{S}_X^0|_R$, clearly $\mathbb{C}_X|_R$ lies in the kernel of $\mathrm{d}\circ\iota_{R,U}$ in (3.11), since d of a locally constant function is zero. We claim that the restriction of (3.11) to $\mathcal{S}_X^0|_R\subset\mathcal{S}_R$ is an isomorphism. As for part (b), it is enough to prove this on the stalks at each $x\in R$. By the Poincaré Lemma on the complex manifold U, written using morphisms $\mathrm{d}:\Lambda^kT^*U_{i(x)}\rightarrow\Lambda^{k+1}T^*U_{i(x)}$ of stalks at $i(x)$, and pulled back to X using i, we have an exact sequence

$$0\longrightarrow\mathbb{C}\xrightarrow{\ 1\ }i^{-1}(\mathcal{O}_X)_x\xrightarrow{\ \mathrm{d}\ }i^{-1}(T^*U)_x\xrightarrow{\ \mathrm{d}\ }i^{-1}(\Lambda^2T^*U)_x.\tag{3.12}$$

Let ϕ lie in the stalk at x of the r.h.s. of (3.11). Then $\phi=\psi+\mathrm{d}(I_{R,U,x}^2)$, where $\psi\in I_{R,U,x}\cdot i^{-1}(T^*U)_x\subseteq i^{-1}(T^*U)_x$ with $\mathrm{d}\psi=0$ in $i^{-1}(\Lambda^2T^*U)_x$. By exactness of (3.12), we may write $\psi=\mathrm{d}\zeta$ for $\zeta\in i^{-1}(\mathcal{O}_U)_x$, where ζ is unique up to addition of a constant $1(c)$ for $c\in\mathbb{C}$. We fix ζ uniquely by requiring that $\zeta(i(x))=0$. Then $\zeta+I_{R,U,x}^2$ lies in $i^{-1}(\mathcal{O}_U)_x/I_{R,U,x}^2$ with $\mathrm{d}(\zeta+I_{R,U,x}^2)=0$ in $i^{-1}(T^*U)_x/(I_{R,U,x}\cdot i^{-1}(T^*U)_x)$, since $\mathrm{d}\zeta=\psi\in I_{R,U,x}\cdot i^{-1}(T^*U)_x$. Hence $\zeta+I_{R,U,x}^2=\iota_{R,U,x}(\theta)$ for unique $\theta\in\mathcal{S}_{X,x}$ as (2.2) is exact, and $\zeta(i(x))=0$ implies that $\theta\in\mathcal{S}_{X,x}^0$. Therefore $\mathrm{d}\circ\iota_{R,U}|_{\mathcal{S}_X|_R}$ is an isomorphism on stalks, and so is an isomorphism, proving Theorem 2.1(c).

3.3. Modifications to the proof for the algebraic case. Next we explain how to modify §3.1–§3.2 to work with $\mathbb{K}$-schemes in algebraic geometry, rather than complex analytic spaces and complex manifolds.

In §3.1 we replace complex analytic spaces X,R by $\mathbb{K}$-schemes, and complex manifolds U by smooth $\mathbb{K}$-schemes, and open subsets $R\subseteq X$, $U'\subseteq U$, etc., are taken to be open in the Zariski topology. Then the proofs work in the algebraic case without modification, working throughout with sheaves in either the Zariski or the étale topology, with the exception of Lemma 3.2, which is false: one can write down examples of $\mathbb{K}$-schemes X with embeddings $i:X\hookrightarrow U$, $j:X\hookrightarrow V$ into smooth $\mathbb{K}$-schemes U,V, such that for Zariski or étale open $\emptyset\neq U'\subseteq U$, there exist no morphisms $\Phi:U'\rightarrow V$ with $\Phi\circ i'=j|_{U'}$.

Here are two different ways to fix this:

(A) The analogue of Lemma 3.2 is true if we require that the smooth $\mathbb{K}$-scheme V is isomorphic to a Zariski open subset of an affine space $\mathbb{A}^n$, as then we can take the coordinates $(z_1,\dots,z_n)$ in the proof of Lemma 3.2 to be the embedding $V\hookrightarrow\mathbb{A}^n$.

(B) Suppose $R,S\subseteq X$ are Zariski open and $i:R\hookrightarrow U$, $j:S\hookrightarrow V$ are closed embeddings into smooth $\mathbb{K}$-schemes U,V. Write $\hat U,\hat V$ for the formal completions of U,V along $i(R),j(S)$, with inclusions $\hat\imath:R\hookrightarrow\hat U$, $\hat\jmath:S\hookrightarrow\hat V$. Then one can prove the following formal analogue of Lemma 3.2: for each $x\in R\cap S\subseteq X$, there exists a Zariski open $\hat U'\subseteq\hat U$ with $x\in R'\subseteq R\cap S$, and a morphism of formal $\mathbb{K}$-schemes $\hat\Phi:\hat U'\rightarrow\hat V$ with $\hat\Phi\circ\hat\imath'=\hat\jmath|_{R'}$, where $R'=\hat\imath^{-1}(\hat U')$ and $\hat\imath'=\hat\imath|_{R'}$.

Using approach (A), the whole of §3.1 works provided we restrict to triples (R,U,i) with U isomorphic to a Zariski open in $\mathbb{A}^n$. In particular, we can construct the sheaf $\mathcal{S}_X$ using a family $\bigl\{(R_a,U_a,i_a):a\in A\bigr\}$ with $U_a\subseteq\mathbb{A}^{n_a}$ Zariski open. Note that immediately after the proof of Lemma 3.3, Zariski locally near each $x\in X$ there exists an embedding $i:X\hookrightarrow U$ with $U\subseteq\mathbb{A}^n$.

The disadvantage of this is that it proves Theorem 2.1(i),(ii) only for U,V Zariski open in affine spaces $\mathbb{A}^n$. To prove them for U,V general smooth $\mathbb{K}$-schemes, we have to do some more work.

To prove the algebraic version of §3.1 using (B), observe that (2.2)–(2.3) defining $\mathcal{S}_X$ depend only on the formal completion $\hat U$ of U along $i(R)$, since

$$\frac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2}\cong\frac{\hat\imath^{-1}(\mathcal{O}_{\hat U})}{I_{R,\hat U}^2}\quad\text{and}\quad\frac{i^{-1}(T^*U)}{I_{R,U}\cdot i^{-1}(T^*U)}\cong\frac{\hat\imath^{-1}(T^*\hat U)}{I_{R,\hat U}\cdot\hat\imath^{-1}(T^*\hat U)}.$$

Thus we may replace U,V by $\hat U,\hat V$ throughout the proof. For triples $(R,U,i),(S,V,j)$ with U,V smooth $\mathbb{K}$-schemes and $R\subseteq S$, a morphism $\Phi:U\rightarrow V$ with $\Phi\circ i=j|_R$ induces a morphism $\hat\Phi:\hat U\rightarrow\hat V$ with $\hat\Phi\circ\hat\imath=\hat\jmath|_R$. However, the converse is false: there may be $\hat\Phi:\hat U\rightarrow\hat V$ with $\hat\Phi\circ\hat\imath=\hat\jmath|_R$ which are not induced by any $\Phi:U\rightarrow V$ Zariski or étale locally on U. Because there are more formal morphisms $\hat\Phi$ than morphisms Φ, the formal analogue of Lemma 3.2 holds, which makes the proof work.

For the extension of §3.2 to $\mathbb{K}$-schemes, the proofs of Theorem 2.1(a), (b) need no modification. But for (c), equation (3.12) is not exact at $i^{-1}(T^*U)_x$ in the algebraic case: algebraic closed 1-forms need not be locally exact, in either the Zariski or the étale topology. For example, the closed 1-form $z^{-1}\mathrm{d}z$ on $\mathbb{A}^1\setminus\{0\}$ is not algebraically locally exact, since $\log z$ is not an algebraic function.

Because of this, in the algebraic case the morphism from $\mathcal{S}_X^0$ to the cohomology of (2.8) constructed in §3.2 is injective, but generally not surjective. The solution is to modify (2.8), replacing U by the formal completion $\hat U$ of U along $i(R)$, as in (2.9). In fact the Poincaré Lemma may not hold on $\hat U$ either (consider the case $X=U=\hat U$ and $i=\mathrm{id}_X$), but what matters is that one can show that if $\hat\alpha\in H^0(I_{R,\hat U}\cdot T^*\hat U)\subset$

$H^0(T^*\hat{U})$ with $\mathrm{d}\hat{\alpha} = 0$ then $\hat{\alpha} = \mathrm{d}f$ for a unique $f \in H^0(\mathcal{O}_{\hat{U}})$ with $f \circ i|_{X^{\mathrm{red}}} = 0$, that is, the particular class of formal closed 1-forms we are interested in can be integrated to formal functions. This completes the proof of Theorem 2.1.

3.4. Proof of Proposition 2.3. Let $\phi : X \to Y$ be a morphism of complex analytic spaces. To construct the morphism $\phi^\star : \phi^{-1}(\mathcal{S}_Y) \to \mathcal{S}_X$ in Proposition 2.3, we will generalize the construction of $\mathcal{S}_X$ in §3.1. Modifying the notation of §3.1, define an *X-triple* $(R, U, i)_X$ to be an open subset $R \subseteq X$, a complex manifold U, and a closed embedding $i : R \hookrightarrow U$, and a *Y-triple* $(S, V, j)_Y$ to be an open subset $S \subseteq Y$, a complex manifold V, and a closed embedding $j : S \hookrightarrow V$. Define *X-subtriples* $(R', U', i')_X \subseteq (R, U, i)_X$ and *Y-subtriples* $(S', V', j')_Y \subseteq (S, V, j)_Y$ as in §3.1.

Call $\Upsilon : (R_1, U_1, i_1)_X \to (R_2, U_2, i_2)_X$ a *morphism of X-triples* if $R_1 \subseteq R_2 \subseteq X$, and $\Upsilon : U_1 \to U_2$ is holomorphic with $\Upsilon \circ i_1 = i_2|_{R_1} : R_1 \to U_2$. Call $\Phi : (R, U, i)_X \to (S, V, j)_Y$ a *morphism of X, Y-triples* if $\phi(R) \subseteq S \subseteq Y$, and $\Phi : U \to V$ is holomorphic with $\Upsilon \circ i = j \circ \phi|_R : R \to V$. Call $\Psi : (S_1, V_1, j_1)_Y \to (S_2, V_2, j_2)_Y$ a *morphism of Y-triples* if $S_1 \subseteq S_2 \subseteq Y$, and $\Psi : V_1 \to V_2$ is holomorphic with $\Psi \circ j_1 = j_2|_{S_1} : S_1 \to V_2$.

Let $\Phi : (R, U, i)_X \to (S, V, j)_Y$ be a morphism of X, Y-triples. Consider the diagram of sheaves of $\mathbb{C}$-vector spaces on R :

(3.13)

$$\begin{tikzcd}
0 \arrow[r] & \phi^{-1}(\mathcal{S}_Y)|_R \arrow[r, "\phi^{-1}(\iota_{S,V})|_R"'] \arrow[d, dotted, "\phi^\star_\Phi"] & \dfrac{\phi^{-1}\circ j^{-1}(\mathcal{O}_V)|_R}{\phi^{-1}(I^2_{S,V})|_R} \arrow[r, "\phi^{-1}(\mathrm{d})"'] \arrow[d, "i^{-1}(\Phi^\sharp)"] & \dfrac{\phi^{-1}(j^{-1}(T^*V))|_R}{\phi^{-1}(I_{S,V}\cdot j^{-1}(T^*V))|_R} \arrow[d, "i^{-1}(\mathrm{d}\Phi)"'] \\
0 \arrow[r] & \mathcal{S}_X|_R \arrow[r, "\iota_{R,U}"] & \dfrac{i^{-1}(\mathcal{O}_U)}{I^2_{R,U}} \arrow[r, "\mathrm{d}"] & \dfrac{i^{-1}(T^*U)}{I_{R,U}\cdot i^{-1}(T^*U)},
\end{tikzcd}$$

which is (2.11) with $\phi^\star_\Phi$ in place of $\phi^\star|_R$. The rows of (3.13) are exact, as (2.2) is exact, and the right hand square of (3.13) commutes by definition of $\mathrm{d}\Phi$. So by exactness, there is a unique morphism $\phi^\star_\Phi$ of sheaves of $\mathbb{C}$-vector spaces making (3.13) commute. Since $\iota_{R,U}, \phi^{-1}(\iota_{S,V})|_R, i^{-1}(\Phi^\sharp)$ are morphisms of sheaves of commutative $\mathbb{C}$-algebras, so is $\phi^\star_\Phi$.

Suppose $\Upsilon : (R_1, U_1, i_1)_X \to (R_2, U_2, i_2)_X$ is a morphism of X-triples, and $\Phi_2 : (R_2, U_2, i_2)_X \to (S_2, V_2, j_2)_Y$ a morphism of X, Y-triples. Then $\Phi_2 \circ \Upsilon : (R_1, U_1, i_1)_X \to (S_2, V_2, j_2)_Y$ is a morphism of X, Y-triples, and composing (2.3) for Υ with the restriction of (3.13) for Φ_2 to R_1 yields

$$\phi^\star_{\Phi_2\circ\Upsilon} = \phi^\star_{\Phi_2}|_{R_1} : \phi^{-1}(\mathcal{S}_Y)|_{R_1} \longrightarrow \mathcal{S}_X|_{R_1}. \tag{3.14}$$

Similarly, suppose $\Phi_1 : (R_1, U_1, i_1)_X \to (S_1, V_1, j_1)_Y$ is a morphism of X, Y-triples, and $\Psi : (S_1, V_1, j_1)_Y \to (S_2, V_2, j_2)_Y$ a morphism of Y-triples. Then $\Psi \circ \Phi_1 : (R_1, U_1, i_1)_X \to (S_2, V_2, j_2)_Y$ is a morphism of

X,Y-triples, and composing (3.13) for Φ_1 with $\phi|_{R_1}^{-1}$ applied to (2.3) for Ψ, we see that

$$\phi^\star_{\Psi\circ\Phi_1} = \phi^\star_{\Phi_1} : \phi^{-1}(\mathcal{S}_Y)|_{R_1} \longrightarrow \mathcal{S}_X|_{R_1}. \tag{3.15}$$

Easy generalizations of the proofs of Lemmas 3.1 and 3.2 show:

Lemma 3.4. *The morphism* $\phi^\star_\Phi$ *in* (3.13) *is independent of the choice of* Φ. *That is, if* $\Phi, \tilde{\Phi} : (R,U,i)_X \to (S,V,j)_Y$ *are morphisms of* X,Y*-triples then*

$$\phi^\star_\Phi = \phi^\star_{\tilde{\Phi}} : \phi^{-1}(\mathcal{S}_Y)|_R \longrightarrow \mathcal{S}_X|_R. \tag{3.16}$$

Lemma 3.5. *Let* $(R,U,i)_X$ *be an* X*-triple and* $(S,V,j)_Y$ *a* Y*-triple. Then for each* $x \in R \cap \phi^{-1}(S) \subseteq X$, *there exist an* X*-subtriple* $(R',U',i')_X \subseteq (R,U,i)_X$ *with* $x \in R' \subseteq R \cap \phi^{-1}(S)$ *and a morphism of* X,Y*-triples* $\Phi : (R',U',i')_X \to (S,V,j)_Y$.

We now claim that there exists a unique morphism $\phi^\star : \phi^{-1}(\mathcal{S}_Y) \to \mathcal{S}_X$ of sheaves of commutative $\mathbb{C}$-algebras on X, such that $\phi^\star|_R = \phi^\star_\Phi$ for all morphisms of X,Y-triples $\Phi : (R,U,i)_X \to (S,V,j)_Y$. To show this, it is enough to prove:

(a) For all $x \in X$, there exists a morphism of X,Y-triples $\Phi : (R,U,i)_X \to (S,V,j)_Y$ with $x \in R$; and
(b) If $\Phi_1 : (R_1,U_1,i_1)_X \to (S_1,V_1,j_1)_Y$ and $\Phi_2 : (R_2,U_2,i_2)_X \to (S_2,V_2,j_2)_Y$ are morphisms of X,Y-triples, then $\phi^\star_{\Phi_1}|_{R_1\cap R_2} = \phi^\star_{\Phi_2}|_{R_1\cap R_2}$.

To see part (a) holds, choose an X-triple $(R,U,i)_X$ with $x \in R$ and a Y-triple $(S,V,j)_Y$ with $\phi(x) \in S$, and apply Lemma 3.5. For (b), let $x \in R_1 \cap R_2$. Applying Lemma 3.2 twice gives an X-subtriple $(R_1',U_1',i_1')_X \subseteq (R_1,U_1,i_1)_X$ with $x \in R_1' \subseteq R_1 \cap R_2$ and a morphism of X-triples $\Upsilon : (R_1',U_1',i_1')_X \to (R_2,U_2,i_2)_X$, and a Y-subtriple $(S_1',V_1',j_1')_Y \subseteq (S_1,V_1,j_1)_Y$ with $\phi(x) \in S_1'$ and a morphism of Y-triples $\Psi : (S_1',V_1',j_1')_Y \to (S_2,V_2,j_2)_Y$. Making R_1', U_1' smaller if necessary we may suppose that $\Phi_1(U_1') \subseteq V_1' \subseteq V_1$. Then we have

$$\phi^\star_{\Phi_1}|_{R_1'} = \phi^\star_{\Phi_1|_{U_1'}} = \phi^\star_{\Psi\circ\Phi_1|_{U_1'}} = \phi^\star_{\Phi_2\circ\Upsilon} = \phi^\star_{\Phi_2}|_{R_1'},$$

using (3.15) in the second step, Lemma 3.4 for $\Psi \circ \Phi_1|_{U_1'}, \Phi_2 \circ \Upsilon : (R_1',U_1',i_1')_X \to (S_2,V_2,j_2)_Y$ in the third, and (3.14) in the fourth. Thus for each $x \in R_1 \cap R_2$ we can find an open $x \in R_1' \subseteq R_1 \cap R_2$ with $\phi^\star_{\Phi_1}|_{R_1'} = \phi^\star_{\Phi_2}|_{R_1'}$. This implies (b).

Thus, comparing (2.11) and (3.13) shows that there exists a unique morphism $\phi^\star$ such that (2.11) commutes for all R,S,U,V,i,j,Φ as in

Proposition 2.3. Consider the diagram

$$
\begin{tikzcd}
0 \arrow[r] & \phi^{-1}(\mathcal{S}_Y^0) \arrow[r] \arrow[d, dotted, "\phi^\star|_{\phi^{-1}(\mathcal{S}_Y^0)}"'] & \phi^{-1}(\mathcal{S}_Y) \arrow[r, "\phi^{-1}(i_Y^\sharp\circ\beta_Y)"'] \arrow[d, "\phi^\star"'] & \phi^{-1}(\mathcal{O}_{Y^{\mathrm{red}}}) \arrow[d, "\phi|^\sharp_{X^{\mathrm{red}}}"'] \\
0 \arrow[r] & \mathcal{S}_X^0 \arrow[r] & \mathcal{S}_X \arrow[r, "i_X^\sharp\circ\beta_X"] & \mathcal{O}_{X^{\mathrm{red}}}.
\end{tikzcd}
\tag{3.17}
$$

The rows are exact by Theorem 2.1(a), and one can see the right hand square commutes by composing the left hand square of (2.11) with the projections to $\phi^{-1}(\mathcal{O}_{Y^{\mathrm{red}}})|_R, \mathcal{O}_{X^{\mathrm{red}}}|_R$. Thus by exactness, $\phi^\star$ maps $\phi^{-1}(\mathcal{S}_Y^0) \to \mathcal{S}_X^0$ as in (3.17).

Now let $\psi : Y \to Z$ be another morphism of complex analytic spaces. Using the obvious notation, suppose $\Phi : (R,U,i)_X \to (S,V,j)_Y$, $\Psi : (S,V,j)_Y \to (T,W,k)_Z$ are morphisms of X,Y-triples and Y,Z-triples, so that $\Psi\circ\Phi : (R,U,i)_X \to (T,W,k)_Z$ is a morphism of X,Z-triples. Then comparing (2.11) for $\Psi\circ\Phi$ with the composition of $\phi|_R^{-1}$ applied to (2.11) for Ψ with (2.11) for Φ shows that $(\psi\circ\phi)^\star|_R = \phi^\star\circ\phi^{-1}(\psi^\star)|_R$. As we can cover X by such open $R \subseteq X$, equation (2.12) follows.

To show that $\mathrm{id}_X^\star = \mathrm{id}_{\mathcal{S}_X}$, compare (2.3) with (2.11) with R,U,i,id_U in place of S,V,j,Φ. Finally, if $\phi : X \to Y$ is an étale morphism of complex analytic spaces, then ϕ is a local isomorphism in the complex analytic topology, so $\phi^\star$ is an isomorphism. This proves Proposition 2.3 for complex analytic spaces.

The extension to $\mathbb{K}$-schemes works as in §3.3. For the last part, for $\mathcal{S}_X, \mathcal{S}_Y$ sheaves in the Zariski topology, if $\phi : X \to Y$ is a Zariski inclusion then it is an isomorphism locally in the Zariski topology, so $\phi^\star$ is an isomorphism. Similarly, for $\mathcal{S}_X, \mathcal{S}_Y$ sheaves in the étale topology, if $\phi : X \to Y$ is étale then it is an isomorphism locally in the étale topology, so again $\phi^\star$ is an isomorphism.

4. D-critical loci

We now prove Propositions 2.7 and 2.8 from §2.2.

4.1. Proof of Proposition 2.7. We will prove the $\mathbb{K}$-scheme case, as it is more complicated, and we tackle the second part of the proposition first.

Suppose (X,s) is an algebraic d-critical locus, and $x \in X$. Let (T,W,h,k) be a critical chart on (X,s) with $x \in T$, and set $\dim T_xX = m$ and $\dim W = n$. Then $\mathrm{Hess}_{k(x)}\, h$ has rank $n-m$. Choose étale coordinates $(z_1,\ldots,z_n)$ on a Zariski open neighbourhood W' of $k(x)$ in W, such that

$$
\frac{\partial h}{\partial z_i \partial z_j}\Big|_{k(x)} = \begin{cases} 1, & i=j\in\{m+1,m+2,\ldots,n\}, \\ 0, & \text{otherwise.} \end{cases}
\tag{4.1}
$$

Define $V=\bigl\{w\in W':\frac{\partial h}{\partial z_{m+1}}(w)=\cdots=\frac{\partial h}{\partial z_n}(w)=0\bigr\}$. Equation (4.1) implies that the equations $\frac{\partial h}{\partial z_j}(w)=0$ for $j=m+1,\ldots,n$ are transverse at $k(x)$, so V is smooth of dimension $n-m$ near $k(x)$. Making W' smaller, we can suppose V is smooth of dimension $n-m$.

Define $S=k^{-1}(W')$, so that $x\in S\subseteq T\subseteq X$ is Zariski open and $k|_S:S\hookrightarrow W'$ is a closed embedding with $k(S)$ the $\mathbb{K}$-subscheme $\mathrm{d}h|_{W'}^{-1}(0)$ in W'. But $\mathrm{d}h|_{W'}=0$ implies that $\frac{\partial h}{\partial z_{m+1}}(w)=\cdots=\frac{\partial h}{\partial z_n}(w)=0$, so $k(S)\subseteq V\subseteq W'$. Thus $j:=k|_S:S\hookrightarrow V$ is a closed embedding. Write $g:=h|_V:V\to\mathbb{A}^1$. It is now easy to check that (S,V,g,j) is a critical chart on (X,s) with $x\in S$ and $\dim V=\dim T_xX$.

Suppose now that $x\in R\subseteq X$ is Zariski open, and $i:R\hookrightarrow U$ is a closed embedding into a smooth $\mathbb{K}$-scheme U with $\dim U=\dim T_xX=m$, and $f:U\to\mathbb{A}^1$ is regular with $\iota_{R,U}(s|_R)=i^{-1}(f)+I_{R,U}^2$. Let (S,V,g,j) be as above. Then $i|_{R\cap S}\times j|_{R\cap S}:R\cap S\to U\times V$ is a closed embedding. Choose a Zariski open neighbourhood $\tilde S$ of x in $R\cap S$ and a smooth locally closed $\mathbb{K}$-subscheme $\tilde V$ of $U\times V$ such that $\dim\tilde V=m$ and $(i\times j)(\tilde S)\subseteq\tilde V\subseteq U\times V$ as $\mathbb{K}$-subschemes of $U\times V$, with $(i\times j)(\tilde S)$ closed in $\tilde V$.

Write $\tilde\jmath=(i\times j)|_{\tilde S}:\tilde S\to\tilde V$ and $\pi_U:\tilde V\to U$, $\pi_V:\tilde V\to V$ for the projections. As $T_{(i(x),j(x))}\tilde V=\mathrm{d}\tilde\jmath|_x(T_xX)=(\mathrm{d}i|_x\times\mathrm{d}j|_x)(T_xX)$ and $\mathrm{d}i|_x:T_xX\to T_{i(x)}U$, $\mathrm{d}j|_x:T_xX\to T_{j(x)}V$ are isomorphisms, we see that $\mathrm{d}\pi_U|_{(i(x),j(x))}:T_{(i(x),j(x))}\tilde V\to T_{i(x)}U$ and $\mathrm{d}\pi_V|_{(i(x),j(x))}:T_{(i(x),j(x))}\tilde V\to T_{j(x)}V$ are isomorphisms, so π_U,π_V are étale near $(i(x),j(x))$. Making $\tilde S,\tilde V$ smaller, we can suppose π_U,π_V are étale. Since $\pi_V\circ\tilde\jmath=j|_{\tilde S}$, and $j:S\hookrightarrow V$ is a closed embedding, we see that $\tilde\jmath(\tilde S)$ is open and closed in $\pi_V^{-1}(j(S))$. Thus, making $\tilde V$ smaller, we can suppose that $\tilde\jmath(\tilde S)=\pi_V^{-1}(j(S))$.

Define $\tilde g=g\circ\pi_V:\tilde V\to\mathbb{A}^1$. As π_V is étale, $\mathrm{Crit}(\tilde g)=\pi_V^{-1}(\mathrm{Crit}(g))=\pi_V^{-1}(j(S))=\tilde\jmath(\tilde S)$. Since (S,V,g,j) is a critical chart, we have $\iota_{S,V}(s|_S)=j^{-1}(g)+I_{S,V}^2$. Combining this with $\tilde g=g\circ\pi_V$, $\pi_V\circ\tilde\jmath=j|_{\tilde S}$, and Theorem 2.1(ii), proves that $\iota_{\tilde S,\tilde V}(s|_{\tilde S})=\tilde\jmath^{-1}(\tilde g)+I_{\tilde S,\tilde V}^2$. Therefore $(\tilde S,\tilde V,\tilde g,\tilde\jmath)$ is a critical chart on (X,s).

Now $\pi_U:\tilde S\to U$ is étale, and using $\iota_{R,U}(s|_R)=i^{-1}(f)+I_{R,U}^2$ and Theorem 2.1(ii) again shows that $\tilde\jmath^{-1}(\tilde g)+I_{\tilde S,\tilde V}^2=\tilde\jmath^{-1}(\pi_U^*(f))+I_{\tilde S,\tilde V}^2$, so that

$$\pi_U^*(f)-\tilde g\in I_{\mathrm{d}\tilde g}^2\subseteq\mathcal{O}_{\tilde V},\tag{4.2}$$

where $I_{\mathrm{d}\tilde g}\subseteq\mathcal{O}_{\tilde V}$ is the ideal generated by $\mathrm{d}\tilde g$. Differentiating (4.2) gives

$$\mathrm{d}\bigl(\pi_U^*(f)-\tilde g\bigr)\in I_{\mathrm{d}\tilde g}\cdot I_{\mathrm{d}\tilde g,\partial^2\tilde g}\cdot T^*V,$$

where $I_{\mathrm{d}\tilde g,\partial^2\tilde g}\subseteq \mathcal{O}_{\tilde V}$ is the ideal generated by the first and second derivatives of $\tilde g$. Therefore locally on $\tilde V$ we may write

$$\mathrm{d}\big(\pi_U^*(f)\big)=(\mathrm{id}+\alpha)\cdot \mathrm{d}\tilde g,\quad \text{where}\quad \alpha\in I_{\mathrm{d}\tilde g,\partial^2\tilde g}\cdot TV\otimes T^*V. \tag{4.3}$$

As $\tilde\jmath(x)\in \mathrm{Crit}(\tilde g)$ and $T_{\tilde\jmath(x)}\mathrm{Crit}(\tilde g)=T_{\tilde\jmath(x)}\tilde V$ we have $\mathrm{d}\tilde g|_{\tilde\jmath(x)}=\partial^2\tilde g|_{\tilde\jmath(x)}=0$, so $\alpha|_{\tilde\jmath(x)}=0$, and $\mathrm{id}+\alpha$ is invertible near $\tilde\jmath(x)$. Making $\tilde S,\tilde V$ smaller, we can suppose $\mathrm{id}+\alpha$ is invertible on $\tilde V$. So (4.3) implies that $\mathrm{Crit}(\pi_U^*(f))=\mathrm{Crit}(\tilde g)$, as $\mathbb{K}$-subschemes of $\tilde V$. Hence $(\tilde S,\tilde V,\pi_U^*(f),\tilde\jmath)$ is a critical chart on (X,s).

As $\pi_U:\tilde V\to U$ is étale, it maps $\mathrm{Crit}(\pi_U^*(f))\to\mathrm{Crit}(f)$ on the image of π_U. But $\mathrm{Crit}(\pi_U^*(f))=\tilde\jmath(\tilde S)$, so $\pi_U\big(\mathrm{Crit}(\pi_U^*(f))\big)=\pi_U\circ\jmath(\tilde S)=i(\tilde S)$. Thus $\mathrm{Crit}(f)\cap\pi_U(\tilde V)=i(\tilde S)$, and $\mathrm{Crit}(f)$ coincides with $i(R)$ near $i(x)$. Choose open $i(x)\in U'\subseteq\pi_U(\tilde V)\subseteq U$, and set $R'=i^{-1}(U')$, $f'=f|_{U'}$ and $i'=i|_{R'}$. Since π_U is étale, $\pi_U^{-1}(\mathrm{Crit}(f'))=\mathrm{Crit}(\pi_U^*(f))\cap\pi_U^{-1}(U')=\tilde\jmath(\tilde S)\cap\pi_U^{-1}(U')=\pi_U^{-1}(i'(R'))$, and this forces $\mathrm{Crit}(f')=i'(R')$, since the étale morphism π_U is a Zariski isomorphism $\jmath(\tilde S)\to i(\tilde S)$. Therefore (R',U',f',i') is a critical chart on (X,s). This proves the second part of Proposition 2.7 in the $\mathbb{K}$-scheme case.

For the first part, suppose (X,s) is an algebraic d-critical locus, $R\subseteq X$ is Zariski open, $i:R\hookrightarrow U$ is a closed embedding into a smooth $\mathbb{K}$-scheme U, and $x\in R$. Let $n=\dim U$ and $m=\dim T_xX$. Choose a Zariski open neighbourhood U' of x in X and étale coordinates $(z_1,\ldots,z_n):U'\to\mathbb{A}^n$ such that $z_{m+1}|_{i'(R')}=\cdots=z_n|_{i'(R')}=0$, where $R'=i^{-1}(U')$ and $i'=i|_{R'}:R'\hookrightarrow U'$, a closed embedding. Making R',U' smaller if necessary, we may choose regular $f:U'\to\mathbb{A}^1$ with $\iota_{R',U'}(s|_{R'})=i'^{-1}(f)+I^2_{R',U'}$.

Set $V'=\{v\in U': z_{m+1}(v)=\cdots=z_n(v)=0\}$, so that V' is a smooth $\mathbb{K}$-subscheme of U' with $\dim V'=m=\dim T_xX$ and $i'(R')\subseteq V'\subseteq U'$. The proof above shows that $\mathrm{Crit}(f|_{V'})=i'(R')$ near $i'(x)$, so making R',U',V' smaller we may suppose that $\mathrm{Crit}(f|_{V'})=i'(R')$. Define $f':U'\to\mathbb{A}^1$ by

$$f'=f-\sum_{j=m+1}^n\frac{\partial f}{\partial z_j}\,z_j+\frac12\sum_{j,k=m+1}^n\left(\delta_{jk}+\frac{\partial^2 f}{\partial z_j\partial z_k}\right)z_jz_k. \tag{4.4}$$

Differentiating (4.4) shows that for $j=m+1,\ldots,n$ we have

$$\frac{\partial f'}{\partial z_j}=z_j+\frac12\sum_{k,l=m+1}^n\frac{\partial^3 f}{\partial z_j\partial z_k\partial z_l}\,z_kz_l.$$

Rewriting this in matrix form gives

$$(4.5)\qquad \begin{pmatrix} \frac{\partial f'}{\partial z_{m+1}} \\ \vdots \\ \frac{\partial f'}{\partial z_n} \end{pmatrix} = \Bigl(\delta_{jk} + \tfrac{1}{2}\textstyle\sum_{l=m+1}^{n} \frac{\partial^3 f}{\partial z_j \partial z_k \partial z_l}\, z_l\Bigr)_{j,k=m+1}^{n} \begin{pmatrix} z_{m+1} \\ \vdots \\ z_n \end{pmatrix}.$$

As $z_l = 0$ on $V' \subseteq U'$ the first matrix on the r.h.s. of (4.5) is invertible near V' in U', so making U' smaller while fixing R', V', we can suppose it is invertible. Then the $\mathbb{K}$-subscheme $\frac{\partial f'}{\partial z_{m+1}} = \cdots = \frac{\partial f'}{\partial z_n} = 0$ in U' is $z_{m+1} = \cdots = z_n = 0$, that is, V'. Therefore as $f'|_{V'} = f|_{V'}$ by (4.4) we have

$$(4.6)\qquad \mathrm{Crit}(f') = \mathrm{Crit}(f'|_{V'}) = \mathrm{Crit}(f|_{V'}) = i'(R').$$

Since $z_j|_{i'(R')} = \frac{\partial f}{\partial z_j}|_{i'(R')} = 0$ for $j = m+1, \ldots, n$ we have $i'^{-1}(z_j) \in I_{R',U'}$ and $i'^{-1}\bigl(\frac{\partial f}{\partial z_j}\bigr) \in I_{R',U'}$ for $j = m+1, \ldots, n$. Thus (4.4) implies that

$$(4.7)\qquad i'^{-1}(f') + I_{R',U'}^2 = i'^{-1}(f) + I_{R',U'}^2 = \iota_{R',U'}(s|_{R'}).$$

Equations (4.6)–(4.7) imply that (R', U', f', i') is a critical chart on (X, s). This completes the proof of Proposition 2.7 for $\mathbb{K}$-schemes.

For complex analytic spaces, the proof above also works more-or-less without change, but it can be simplified, as étale morphisms of complex analytic spaces are local isomorphisms, and so are invertible on suitable open sets.

4.2. Proof of Proposition 2.8. Let $\phi : X \to Y$ be a smooth morphism of complex analytic spaces (or $\mathbb{K}$-schemes) and $t \in H^0(\mathcal{S}_Y^0)$, and set $s = \phi^\star(t) \in H^0(\mathcal{S}_X^0)$. Fix $x \in X$ with $y = \phi(x)$. Write $\dim T_x X = m$ and $\dim T_y Y = n$, so that $m \geqslant n$ and ϕ is smooth of relative dimension $m - n$ near x. We may choose (Zariski) open $y \in S \subseteq Y$ and $x \in R \subseteq \phi^{-1}(S) \subseteq X$, closed embeddings $i : R \hookrightarrow U$, $j : S \hookrightarrow V$ for U, V complex manifolds (or smooth $\mathbb{K}$-schemes) with $\dim U = m$, $\dim V = n$, and a morphism $\Phi : U \to V$ smooth of relative dimension $m - n$ with $\Phi \circ i = j \circ \phi|_R : R \to V$. Making S, V (and hence R, U) smaller, we may choose holomorphic $g : V \to \mathbb{C}$ (or regular $g : V \to \mathbb{A}^1$) with $\iota_{S,V}(t|_S) = j^{-1}(g) + I_{S,V}^2$. Define $f = g \circ \Phi$. Then $s = \phi^\star(t)$ and Proposition 2.3 imply that $\iota_{R,U}(s|_R) = i^{-1}(f) + I_{R,U}^2$.

Since $\dim T_x X = \dim U$, Proposition 2.7 implies that (X, s) is a d-critical locus near x if and only if $\mathrm{Crit}(f) = i(R)$ near $i(x)$ as complex analytic subspaces (or $\mathbb{K}$-subschemes) of X. Similarly, (Y, t) is a d-critical locus near y if and only if $\mathrm{Crit}(g) = j(R)$ near $j(y)$. But $\Phi : U \to V$ is smooth and $f = g \circ \Phi$ implies that $\mathrm{Crit}(f) = \Phi^{-1}(\mathrm{Crit}(g))$. Also $\Phi \circ i = j \circ \phi|_R$ and ϕ, Φ smooth of relative dimension $m - n$ imply that $i(R) = \Phi^{-1}(j(S))$ near $i(R)$.

Therefore $\mathrm{Crit}(f) = i(R)$ near $i(x)$ if and only if $\mathrm{Crit}(g) = j(R)$ near $j(y)$, and so (X, s) is a d-critical locus near x if and only if (Y, t) is a d-critical locus near y. Hence (Y, t) a d-critical locus implies (X, s) is a d-critical locus, and if $\phi : X \to Y$ is surjective then (X, s) a d-critical locus implies (Y, t) is a d-critical locus. If $(X, s), (Y, t)$ are d-critical loci then $\phi : (X, s) \to (Y, t)$ is a morphism of d-critical loci by Definition 2.5. This proves Proposition 2.8.

5. Comparing critical charts (R, U, f, i)

Next we prove Theorem 2.20 and Propositions 2.19 and 2.22–2.24 from §2.3.

5.1. Proof of Proposition 2.19. Let (R, U, f, i) be a critical chart on an algebraic d-critical locus (X, s) over $\mathbb{K}$, and $x \in R$. Then U is a smooth $\mathbb{K}$-scheme, so there exist an affine open neighbourhood $\tilde{U}$ of $i(x)$ in U and a closed embedding of $\mathbb{K}$-schemes $\tilde{\Phi} : \tilde{U} \hookrightarrow \mathbb{A}^n$ for some $n \geqslant 0$. Choose a Zariski open neighbourhood V of $\tilde{\Phi}(i(x))$ in $\mathbb{A}^n$ and étale coordinates $(z_1, \ldots, z_n)$ on V such that $\tilde{\Phi}(\tilde{U}) \cap V$ is the $\mathbb{K}$-subscheme defined by $z_{m+1} = \cdots = z_n = 0$ in V, where $m = \dim U$. Set $U' = \tilde{\Phi}^{-1}(V)$, $R' = S = i^{-1}(U')$, $i' = i|_{R'} : R' \hookrightarrow U'$, $f' = f|_{U'} : U' \to \mathbb{A}^1$, $\Phi = \tilde{\Phi}|_{U'} : U' \hookrightarrow V$, and $j = \Phi \circ i' : S = R' \hookrightarrow V$.

Now $\Phi(U')$ is a closed $\mathbb{K}$-subscheme of V, and $f' \circ \Phi^{-1} : \Phi(U') \to \mathbb{A}^1$ a regular function. Zariski locally on V, we may extend $f' \circ \Phi^{-1}$ from $\Phi(U')$ to V. Thus, making V, U', R', S smaller, there exists a regular $h : V \to \mathbb{A}^1$ with $h \circ \Phi = f' : U' \to \mathbb{A}^1$. Then we may form the partial derivatives $\frac{\partial h}{\partial z_a}, \frac{\partial^2 h}{\partial z_a \partial z_b}$ for $a, b = 1, \ldots, n$. Define $g : V \to \mathbb{A}^1$ by

$$g = h - \textstyle\sum_{a=m+1}^n z_a \cdot \frac{\partial h}{\partial z_a} + \frac{1}{2} \sum_{a,b=m+1}^n z_a z_b \cdot \frac{\partial^2 h}{\partial z_a \partial z_b} + \sum_{a=m+1}^n z_a^2. \tag{5.1}$$

Here we use $\mathrm{char}\, \mathbb{K} \neq 2$ from §1. Since $z_a|_{\Phi(U')} = 0$ for $a = m+1, \ldots, n$, we have $g \circ \Phi = h \circ \Phi = f' : U' \to \mathbb{A}^1$. Also equation (5.1) implies that

$$\begin{aligned} \tfrac{\partial g}{\partial z_a}|_{\Phi(U')} &= 0, \qquad & a &= m+1, \ldots, n, \\ \tfrac{\partial^2 g}{\partial z_a \partial z_b}|_{\Phi(U')} &= 2\delta_{ab}, & a, b &= m+1, \ldots, n. \end{aligned} \tag{5.2}$$

Consider the $\mathbb{K}$-subscheme of V defined by the equations $\frac{\partial g}{\partial z_a} = 0$ for $a = m+1, \ldots, n$. The first equation of (5.2) shows this $\mathbb{K}$-subscheme contains $\Phi(U')$, and the second that this $\mathbb{K}$-subscheme coincides with $\Phi(U')$ near $\Phi(U')$ in V. Hence, by making V smaller while keeping U', R', S fixed, we can take this $\mathbb{K}$-subscheme to be $\Phi(U')$. We now claim that

$$\mathrm{Crit}(g) = \Phi(\mathrm{Crit}(f')) = \Phi(i(R')) = j(S) \tag{5.3}$$

as $\mathbb{K}$-subschemes of V. To see this, note that the equations $\frac{\partial g}{\partial z_a} = 0$ for $a = 1, \ldots, m$ defining $\mathrm{Crit}(g)$ divide into $\frac{\partial g}{\partial z_a} = 0$ for $a = m+1, \ldots, n$, which define the smooth $\mathbb{K}$-subscheme $\Phi(U')$ in V, and $\frac{\partial g}{\partial z_a}=0$ for $a=1,\ldots,m$, which under $\Phi(U')\cong U'$ correspond to $\mathrm{d}f'=0$ as $g\circ\Phi=f'$.

As (R',U',f',i') is a critical chart on (X,s) we have $\iota_{R',U'}(s|_{R'})=i'^{-1}(f')+I^2_{R',U'}$. Applying Theorem 2.1(ii) with (R',U',i) in place of (R,U,i) and using $g\circ\Phi=f'$, $j=\Phi\circ i'$ and $R'=S$ shows that $\iota_{S,V}(s|_S)=j^{-1}(g)+I^2_{S,V}$. Together with (5.3), this implies that (S,V,g,j) is a critical chart, and $\Phi:U'\hookrightarrow V$ an embedding with $j=\Phi\circ i'$, $g\circ\Phi=f'$ show $\Phi:(R,U,f,i)\hookrightarrow(S,V,g,j)$ is an embedding of critical charts. This proves Proposition 2.19.

5.2. Proof of Theorem 2.20. We begin with the complex analytic case. Let (X,s) be a complex analytic d-critical locus, $(R,U,f,i),(S,V,g,j)$ be critical charts on (X,s), and $x\in R\cap S$. Then $i(R\cap S)$ is a locally closed complex analytic subspace of U, and $j\circ i^{-1}|_{i(R\cap S)}:i(R\cap S)\to V$ a morphism to a complex manifold V. So we can extend $j\circ i^{-1}$ locally to a holomorphic map $U\to V$. That is, we can choose an open neighbourhood U' of $i(x)$ in U with $R':=i^{-1}(U')\subseteq R\cap S$, and a holomorphic map $\Theta:U'\to V$ such that $\Theta\circ i'=j|_{R'}:R'\to V$, for $i':=i|_{R'}$.

Theorem 2.1(ii) with $(R',U',i'),\Theta$ in place of $(R,U,i),\Phi$ gives a commutative diagram (2.3). Applying this to $s|_{R'}$ shows that

$$\iota_{R',U'}(s|_{R'})=i'^{-1}(\Theta^\sharp)\bigl[\iota_{S,V}(s|_S)\bigr]. \tag{5.4}$$

Write $I'_{R',U'}\subseteq\mathcal{O}_{U'}$ for the ideal vanishing on $i(R')\subseteq U'$, and $f'=f|_{U'}$. Then

$$\begin{aligned} i'^{-1}\bigl[f'+(I'_{R',U'})^2\bigr]&=i'^{-1}(f')+I^2_{R',U'}=\iota_{R',U'}(s|_{R'})=i'^{-1}(\Theta^\sharp)\bigl[\iota_{S,V}(s|_S)\bigr]\\ &=i'^{-1}(\Theta^\sharp)\bigl[j^{-1}(g)+I^2_{S,V}\bigr]=i'^{-1}(\Theta^\sharp)\bigl[i'^{-1}\circ\Theta^{-1}(g)+I^2_{S,V}\bigr]\\ &=i'^{-1}\bigl[\Theta^\sharp\bigl(\Theta^{-1}(g)\bigr)\bigr]+I^2_{R',U'}=i'^{-1}\bigl[g\circ\Theta+(I'_{R',U'})^2\bigr], \end{aligned} \tag{5.5}$$

using $i'^{-1}(I'_{R',U'})=I_{R',U'}$ in the first and seventh steps, $\iota_{R',U'}(s|_{R'})=i'^{-1}(f')+I^2_{R',U'}$ in the second, (5.4) in the third, $\iota_{S,V}(s|_S)=j^{-1}(g)+I^2_{S,V}$ in the fourth, $\Theta\circ i'=j|_{R'}$ in the fifth, and $i'^{-1}(\Theta^\sharp)(I_{S,V})=I_{R',U'}$ in the sixth.

Equation (5.5) implies that $f'-g\circ\Theta\in(I'_{R',U'})^2$. Therefore, making U',R' smaller if necessary, we can choose holomorphic functions $r_a,s_a:U'\to\mathbb{C}$ for $a=1,\ldots,n$, some $n\geqslant 0$, such that $r_a,s_a\in H^0(I'_{R',U'})$ for $a=1,\ldots,n$ and

$$f'=g\circ\Theta+r_1s_1+\cdots+r_ns_n:U'\longrightarrow\mathbb{C}. \tag{5.6}$$

Define $W=V\times\mathbb{C}^{2n}$, $T=S$, $k=j\times(0,\ldots,0):T=S\hookrightarrow V\times\mathbb{C}^{2n}=W$, $(S',V',g',j')=(S,V,g,j)$, and $\Phi:U'\to W$, $\Psi:V'\to W$, $h:W\to\mathbb{C}$ by

$$\Phi(u)=\bigl(\Theta(u),(r_1(u),\ldots,r_n(u),s_1(u),\ldots,s_n(u))\bigr),\;\;\Psi(v)=\bigl(v,(0,\ldots,0)\bigr),$$

$$\text{and}\quad h\bigl(v,(y_1,\ldots,y_n,z_1,\ldots,z_n)\bigr)=g(v)+y_1z_1+\cdots+y_nz_n. \tag{5.7}$$

By increasing n and adding extra functions r_a, s_a with $s_a = 0$, we can suppose Φ is an embedding near $i(x) \in U'$. So making U', R' smaller we can take $\Phi : U' \hookrightarrow W$ to be an embedding.

Since (S, V, g, j) is a critical chart, (T, W, h, k) is one too. Also $\Phi : U' \hookrightarrow W$, $\Psi : V' \hookrightarrow W$ are embeddings, and $f' = h\circ\Phi$, $g' = h\circ\Psi$, $\Phi\circ i' = k|_{R'}$, $\Psi \circ j' = k|_{S'}$ follow from (5.6)–(5.7) and $\Theta \circ i' = j|_{R'}$. Therefore $\Phi : (R', U', f', i') \hookrightarrow (T, W, h, k)$, $\Psi : (S', V', g', j') \hookrightarrow (T, W, h, k)$ are embeddings of critical charts with $x \in R' \cap S'$, proving Theorem 2.20 in the complex analytic case.

For the algebraic case, with (X, s) an algebraic d-critical locus over $\mathbb{K}$, and $(R, U, f, i), (S, V, g, j)$ critical charts, we would like to follow the above method, but there is a problem with the first step: if V is a general smooth $\mathbb{K}$-scheme, we may not be able to choose a Zariski open $i(x) \in U' \subseteq U$ and a morphism $\Theta : U' \to V$ such that $\Theta \circ i' = j|_{R'} : R' \to V$. However, this is valid if V is Zariski open in an affine space $\mathbb{A}^m$.

So we modify the method above as follows: first we apply Proposition 2.19, proved in §5.1, to get a subchart $(S', V', g', j') \subseteq (S, V, g, j)$ with $x \in S'$ and an embedding $\Xi : (S', V', g', j') \hookrightarrow (\tilde S, \tilde V, \tilde g, \tilde\jmath)$ for $(\tilde S, \tilde V, \tilde g, \tilde\jmath)$ a critical chart on (X, s) with $\tilde V \subseteq \mathbb{A}^m$ Zariski open. Thus we may choose Zariski open $i(x) \in U' \subseteq U$ and a morphism $\Theta : U' \to \tilde V$ with $\Theta \circ i' = \tilde\jmath|_{R'} : R' \to W$. Then we follow the rest of the proof above with $(\tilde S, \tilde V, \tilde g, \tilde\jmath), \mathbb{A}^n$ in place of $(S, V, g, j), \mathbb{C}^n$, so that $W = \tilde V \times \mathbb{A}^{2n}$, except that we define $\Psi : V' \hookrightarrow W$ by $\Psi(v) = \bigl(\Xi(v), (0, \ldots, 0)\bigr)$. We leave the details to the reader.

5.3. Proof of Proposition 2.22. Choose holomorphic coordinates $(\dot y_1, \ldots, \dot y_m, \dot z_1, \ldots, \dot z_n)$ on an open neighbourhood $\dot V$ of $j(x)$ in V, where $\dim U = m$ and $\dim V = m + n$, such that $j(x) = (0, \ldots, 0)$ and $\Phi(U) \cap \dot V$ is the submanifold $\dot z_1 = \cdots = \dot z_n = 0$ in $\dot V$. Set $\dot U = \Phi^{-1}(\dot V)$ and $\dot x_a = \dot y_a \circ \Phi|_{\dot U}$ for $a = 1, \ldots, m$. Then $\dot U$ is an open neighbourhood of $i(x)$ in U, and $(\dot x_1, \ldots, \dot x_m)$ are holomorphic coordinates on $\dot U$ with $i(x) = (0, \ldots, 0)$. Write $f|_{\dot U} = \dot f(\dot x_1, \ldots, \dot x_m)$ and $g|_{\dot V} = \dot g(\dot y_1, \ldots, \dot y_m, \dot z_1, \ldots, \dot z_n)$ as functions of these coordinates, so that $f = g \circ \Phi$ implies that $\dot f(\dot y_1, \ldots, \dot y_m) = \dot g(\dot y_1, \ldots, \dot y_m, 0, \ldots, 0)$ for $(\dot y_1, \ldots, \dot y_m) \in \dot U$.

Then the ideal $I'_{R,U} = I_{(\mathrm{d}f)}$ is on $\dot U$ the ideal of holomorphic functions in $(\dot x_1, \ldots, \dot x_m)$ generated by $\frac{\partial \dot f}{\partial \dot x_a}$ for $a = 1, \ldots, m$, and the ideal $I'_{S,V} = I_{(\mathrm{d}g)}$ is on $\dot V$ the ideal of holomorphic functions in $(\dot y_1, \ldots, \dot y_m, \dot z_1, \ldots, \dot z_n)$ generated by $\frac{\partial \dot g}{\partial \dot y_a}$ for $a = 1, \ldots, m$ and $\frac{\partial \dot g}{\partial \dot z_b}$ for $b = 1, \ldots, n$. Since Φ maps U to $\dot z_1 = \cdots = \dot z_n = 0$ and $i(R)$ to $j(R) \subseteq j(S)$, we have

$I_{(\mathrm{d}f)} \cong I_{(\mathrm{d}g)}|_{\dot z_1=\cdots=\dot z_n=0}$, that is,

$$\begin{aligned}&\bigl(\tfrac{\partial \dot f}{\partial \dot x_a}(\dot y_1,\ldots,\dot y_m):a=1,\ldots,m\bigr)=\\ &\bigl(\tfrac{\partial \dot g}{\partial \dot y_a}(\dot y_1,\ldots,\dot y_m,0,\ldots,0):a=1,\ldots,m,\\ &\qquad \tfrac{\partial \dot g}{\partial \dot z_b}(\dot y_1,\ldots,\dot y_m,0,\ldots,0):b=1,\ldots,n\bigr).\end{aligned}$$

As $\frac{\partial \dot g}{\partial \dot y_a}(\dot y_1,\ldots,\dot y_m,0,\ldots,0)=\frac{\partial \dot f}{\partial \dot x_a}(\dot y_1,\ldots,\dot y_m)$, this holds provided each $\frac{\partial \dot g}{\partial \dot z_b}(\dot y_1,\ldots,\dot y_m,0,\ldots,0)$ lies in $\bigl(\frac{\partial \dot g}{\partial \dot y_a}(\dot y_1,\ldots,\dot y_m,0,\ldots,0):a=1,\ldots,m\bigr)$. Thus, making $\dot U,\dot V$ smaller if necessary, we can suppose there exist holomorphic functions $A_{ab}(\dot y_1,\ldots,\dot y_m)$ on $\dot U$ for $a=1,\ldots,m$, $b=1,\ldots,n$ such that for each b

$$\tfrac{\partial \dot g}{\partial \dot z_b}(\dot y_1,\ldots,\dot y_m,0,\ldots,0)=\sum_{a=1}^m A_{ab}(\dot y_1,\ldots,\dot y_m)\cdot\tfrac{\partial \dot g}{\partial \dot y_a}(\dot y_1,\ldots,\dot y_m,0,\ldots,0).$$

Define holomorphic coordinates $(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n)$ on an open neighbourhood $\tilde V$ of $j(x)$ in $\dot V$ by $\tilde y_a=\dot y_a+\sum_{b=1}^n A_{ab}(\dot y_1,\ldots,\dot y_m)\,\dot z_b$ and $\tilde z_b=\dot z_b$. Here $\tilde y_a,\tilde z_b$ are defined on all of $\dot V$, but they need only be a coordinate system near $j(x)$ in $\dot V$ where $\dot z_1,\ldots,\dot z_n$ are small, so we shrink $\dot V$ to $\tilde V\subseteq\dot V$. We also write $\tilde U=\Phi^{-1}(\tilde V)\subseteq\dot U$ and $\tilde x_a=\dot x_a|_{\tilde U}$. Then $\tilde y_a\circ\Phi|_{\tilde U}=\tilde x_a$ as $\dot z_b\circ\Phi|_{\tilde U}=0$. Making $\tilde V$ smaller if necessary we can suppose that if $(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n)\in\tilde V$ then $(\tilde y_1,\ldots,\tilde y_m)\in\dot U$, using the coordinates $(\dot x_1,\ldots,\dot x_m)$ on $\dot U$.

Write $g|_{\tilde V}=\tilde g(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n)$ using these coordinates. Then

$$\begin{aligned}&\tfrac{\partial \tilde g}{\partial \tilde z_b}(\tilde y_1,\ldots,\tilde y_m,0,\ldots,0)\\ &=\sum_{a=1}^m\tfrac{\partial \dot g}{\partial \dot y_a}(\dot y_1,\ldots,\dot y_m,0,\ldots,0)\cdot\tfrac{\partial \dot y_a}{\partial \tilde z_b}+\sum_{c=1}^n\tfrac{\partial \dot g}{\partial \dot z_c}(\dot y_1,\ldots,\dot y_m,0,\ldots,0)\cdot\tfrac{\partial \dot z_c}{\partial \tilde z_b}\\ &=\sum_{a=1}^m\tfrac{\partial \dot g}{\partial \dot y_a}(\dot y_1,\ldots,\dot y_m,0)\cdot(-A_{ab}(\dot y_1,\ldots,\dot y_m))\\ &\quad+\sum_{c=1}^n\bigl(\textstyle\sum_{a=1}^m A_{ac}(\dot y_1,\ldots,\dot y_m)\cdot\tfrac{\partial \dot g}{\partial \dot y_a}(\dot y_1,\ldots,\dot y_m,0,\ldots,0)\bigr)\cdot\delta_{bc}=0.\end{aligned}$$

So $\tilde g(\tilde y_1,\ldots,\tilde y_m,0,\ldots,0)=\dot f(\tilde y_1,\ldots,\tilde y_m)$ and $\frac{\partial\tilde g}{\partial\tilde z_b}(\tilde y_1,\ldots,\tilde y_m,0,\ldots,0)=0$ for $b=1,\ldots,n$ in the new coordinates $(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n)$.

Consider the holomorphic function $h:\tilde V\to\mathbb{C}$ given by

$$h(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n)=\tilde g(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n)-\dot f(\tilde y_1,\ldots,\tilde y_m). \tag{5.8}$$

It satisfies $h(\tilde y_1,\ldots,\tilde y_m,0,\ldots,0)=0$ and $\frac{\partial h}{\partial\tilde z_b}(\tilde y_1,\ldots,\tilde y_m,0,\ldots,0)=0$ for all $b=1,\ldots,n$ and $(\tilde y_1,\ldots,\tilde y_m)\in\tilde U$, so h lies in the ideal $(\tilde z_1,\ldots,\tilde z_n)^2$. Thus, making $\tilde V,\tilde U$ smaller if necessary, we may write

$$h(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n)=\sum_{b,c=1}^n\tilde z_b\tilde z_c\,Q_{bc}(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n), \tag{5.9}$$

for some (nonunique) holomorphic functions $Q_{bc}:\tilde V\to\mathbb{C}$ with $Q_{bc}=Q_{cb}$.

Now $\mathrm{Crit}(\tilde g)=\mathrm{Crit}(g)\cap\tilde V=j(S)\cap\tilde V$ lies in $\Phi(\tilde U)=\{\tilde z_1=\cdots=\tilde z_n=0\}$ as complex analytic subspaces of $\tilde V$. Therefore $\tilde z_1,\ldots,\tilde z_n$ lie in the ideal $I_{(\mathrm{d}\tilde g)}$ generated by $\frac{\partial\tilde g}{\partial\tilde y_a},\frac{\partial\tilde g}{\partial\tilde z_b}$, so making $\tilde U,\tilde V$ smaller, for each $d=1,\ldots,n$ there exist holomorphic functions $B_{ad}(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n)$, $C_{bd}(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n)$ on $\tilde V$ such that using (5.8)–(5.9) we have

$$
\begin{aligned}
\tilde z_d&=\sum_{a=1}^m B_{ad}\cdot\frac{\partial\tilde g}{\partial\tilde y_a}+\sum_{b=1}^n C_{bd}\cdot\frac{\partial\tilde g}{\partial\tilde z_b}\\
&=\sum_{a=1}^m B_{ad}\cdot\Big[\frac{\partial\dot f}{\partial\dot x_a}(\tilde y_1,\ldots,\tilde y_m)+\sum_{a=1}^m\sum_{b,c=1}^n\tilde z_b\tilde z_c\frac{\partial Q_{bc}}{\partial\tilde y_a}(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n)\Big]\\
&\quad+2\sum_{b,c=1}^n C_{bd}\cdot\tilde z_cQ_{bc}(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n)\\
&\quad+\sum_{b,c,e=1}^n C_{ed}\cdot\tilde z_b\tilde z_c\frac{\partial Q_{bc}}{\partial\tilde z_e}(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n).
\end{aligned}
\tag{5.10}
$$

Apply $\frac{\partial}{\partial\tilde z_c}$ to (5.10) and restrict it to the point $(0,\ldots,0)=j(x)$, noting that $\frac{\partial\dot f}{\partial\dot x_a}(0,\ldots,0)=0$. This yields

$$\delta_{cd}=2\textstyle\sum_{b=1}^n C_{bd}(0,\ldots,0)\cdot Q_{bc}(0,\ldots,0),\quad\text{for all } c,d=1,\ldots,n.$$

Hence the symmetric matrix $\big(Q_{bc}(0,\ldots,0)\big)_{b,c=1}^n$ is invertible. Thus, by applying an element of $\mathrm{GL}(n,\mathbb{C})$ to the coordinates $(\tilde z_1,\ldots,\tilde z_n)$ we can suppose that $Q_{bc}(0,\ldots,0)=\delta_{bc}$ for $b,c=1,\ldots,n$.

We now define new holomorphic coordinates $(y_1,\ldots,y_m,z_1,\ldots,z_n)$ on an open neighbourhood V' of $j(x)$ in $\tilde V$, and write $g'(y_1,\ldots,y_m,z_1,\ldots,z_n)=g|_{V'}$ for g as a function of these new coordinates and $U'=\Phi^{-1}(V')$, such that:

(a) $y_a=\tilde y_a$ for $a=1,\ldots,m$.
(b) $\frac{\partial z_b}{\partial\tilde z_c}=\delta_{bc}$ at $j(x)$ for $b,c=1,\ldots,n$.
(c) $\Phi(U')$ is the submanifold $z_1=\cdots=z_n=0$ in V'.
(d) $g'(y_1,\ldots,y_m,z_1,\ldots,z_n)=\dot f(y_1,\ldots,y_m)+z_1^2+\cdots+z_n^2$.

We define z_b by reverse induction on $b=n,n-1,\ldots,1$.

For the first step, as $Q_{nn}=1$ at $j(x)=(0,\ldots,0)$, we may restrict to a small open neighbourhood V' of $j(x)$ in $\tilde V$ on which $Q_{nn}(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n)$ is invertible and has a square root $Q_{nn}^{1/2}$. Rewrite (5.9) on

V' as

$$\tilde g(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n)=\dot f(\tilde y_1,\ldots,\tilde y_m)+\sum_{b,c=1}^{n-1}\tilde z_b\tilde z_c\Big[Q_{bc}-Q_{nn}^{-1}Q_{bn}Q_{cn}\Big]$$
$$+\Big[Q_{nn}^{1/2}\tilde z_n+\sum_{b=1}^{n-1}Q_{nn}^{-1/2}Q_{bn}\tilde z_b\Big]^2=\dot f(\tilde y_1,\ldots,\tilde y_m)+\sum_{b,c=1}^{n-1}\tilde z_b\tilde z_c\hat Q_{bc}+z_n^2,$$
$$\text{where } \hat Q_{bc}=Q_{bc}-Q_{nn}^{-1}Q_{bn}Q_{cn},\;\; z_n=Q_{nn}^{1/2}\tilde z_n+\sum_{b=1}^{n-1}Q_{nn}^{-1/2}Q_{bn}\tilde z_b. \tag{5.11}$$

Note that $\hat Q_{bc}(0,\ldots,0)=\delta_{bc}$ for $b,c=1,\ldots,n-1$, as $Q_{bc}(0,\ldots,0)=\delta_{bc}$. For the second step, making V' smaller so that $\hat Q_{n-1\,n-1}$ is invertible and has a square root $\hat Q_{n-1\,n-1}^{1/2}$ on V', we have

$$\begin{aligned}\tilde g(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n) &= f(\tilde y_1,\ldots,\tilde y_m)+\textstyle\sum_{b,c=1}^{n-2}\tilde z_b\tilde z_c\check Q_{bc}+z_{n-1}^2+z_n^2,\\ \text{where}\qquad \check Q_{bc} &= \hat Q_{bc}-\hat Q_{n-1\,n-1}^{-1}\hat Q_{b\,n-1}\hat Q_{c\,n-1}\\ \text{and}\qquad z_{n-1} &= \hat Q_{n-1\,n-1}^{1/2}\tilde z_{n-1}+\textstyle\sum_{b=1}^{n-2}\hat Q_{n-1\,n-1}^{-1/2}\hat Q_{b\,n-1}\tilde z_b.\end{aligned}$$

Continuing in this way, we define $j(x)\in V'\subseteq\tilde V$ and holomorphic functions $y_1,\ldots,y_m,z_1,\ldots,z_n:V'\to\mathbb{C}$ satisfying (a)–(d) above.

Parts (a)–(c) imply that $(y_1,\ldots,y_m,z_1,\ldots,z_n)$ are a coordinate system near $j(x)$ in V', so making V' smaller, we can suppose they are coordinates on V'. Define $\alpha:V'\to U$ and $\beta:V'\to\mathbb{C}^n$ by $\alpha:(y_1,\ldots,y_m,z_1,\ldots,z_n)\mapsto(y_1,\ldots,y_m)$ and $\beta:(y_1,\ldots,y_m,z_1,\ldots,z_n)\mapsto(z_1,\ldots,z_n)$, using coordinates $(y_1,\ldots,y_m,z_1,\ldots,z_n)$ on V' and $(\dot x_1,\ldots,\dot x_m)$ on $\dot U$. Proposition 2.22 then follows from (a),(c),(d) above.

5.4. Proof of Proposition 2.23. We will adapt §5.3 to the algebraic context. The first part, until just before (5.8), works with $i(x)\in\tilde U\subseteq\dot U\subseteq U$ and $j(x)\in\tilde V\subseteq\dot V\subseteq V$ Zariski open and $(\dot x_1,\ldots,\dot x_m):\dot U\to\mathbb{A}^n$, $(\dot y_1,\ldots,\dot y_m,\dot z_1,\ldots,\dot z_n):\dot V\to\mathbb{A}^{m+n}$, $(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n):\tilde V\to\mathbb{A}^{m+n}$ étale coordinates on $\dot U,\dot V,\tilde V$. Note that f,g are not functions of $\dot x_a$ or $\dot y_a,\dot z_b$ or $\tilde y_a,\tilde z_b$, except in an étale sense, so we cannot rigorously write $f|_{\dot U}=\dot f(\dot x_1,\ldots,\dot x_m)$, and so on. Nonetheless, the partial derivatives $\frac{\partial f}{\partial\dot x_a},\frac{\partial g}{\partial\dot y_a},\frac{\partial g}{\partial\dot z_b},\frac{\partial g}{\partial\tilde y_a},\frac{\partial g}{\partial\tilde z_b}$ are all well defined on $\dot U,\dot V,\tilde V$.

In this way, we construct étale coordinates $(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n)$ on $\tilde V\subseteq V$ such that $\Phi(\tilde U)$ is the smooth $\mathbb{K}$-subscheme $\tilde z_1=\cdots=\tilde z_n=0$ in $\tilde V$, and $\frac{\partial g}{\partial\tilde z_b}|_{\Phi(\tilde U)}=0$ for $b=1,\ldots,n$. Now (5.8) does not make sense on $\tilde V$, since we cannot extend f from $\tilde U$ to $\tilde V$ as f is not a function of $(\tilde y_1,\dots,\tilde y_m)$. Instead, we form the Cartesian square of smooth $\mathbb{K}$-schemes and étale morphisms

$$\begin{tikzcd} \check V=(\tilde U\times\mathbb{A}^n)\times_{\mathbb{A}^{m+n}}\tilde V \arrow[rr, "\check\jmath"'] \arrow[d, "\check\alpha\times\check\beta"] && \tilde V \arrow[d, "(\tilde y_1,\dots,\tilde y_m,\tilde z_1,\dots,\tilde z_n)"] \\ \tilde U\times\mathbb{A}^n \arrow[rr, "(\tilde x_1,\dots,\tilde x_m)\times\mathrm{id}_{\mathbb{A}^n}"] && \mathbb{A}^{m+n}=\mathbb{A}^m\times\mathbb{A}^n. \end{tikzcd} \tag{5.12}$$

There is a unique $\check v\in\check V$ with $\check\alpha(\check v)=i(x)$, $\check\beta(\check v)=(0,\dots,0)$ and $\check\jmath(\check v)=j(x)$. Then $\check\alpha\times\check\beta$ and $\check\jmath$ are étale, so we can regard $\check\jmath:\check V\to\tilde V\subseteq V$ as an étale open set in V. Define étale coordinates $(\check y_1,\dots,\check y_m,\check z_1,\dots,\check z_n):\check V\to\mathbb{A}^{m+n}$ by $\check y_a=\tilde y_a\circ\check\jmath$, $\check z_b=\tilde z_b\circ\check\jmath$, and define $\check f,\check g:\check V\to\mathbb{A}^1$ by $\check f=f\circ\check\alpha$ and $\check g=g\circ\check\jmath$. The analogue of (5.8) is now $\check h=\check g-\check f:\check V\to\mathbb{A}^1$. The previous argument now shows that on the smooth $\mathbb{K}$-subscheme $\check U\in\check V$ defined by $\check z_1=\cdots=\check z_n=0$ we have $\check h|_{\check U}=0$ and $\frac{\partial\check h}{\partial\check z_b}|_{\check U}=0$ for $b=1,\dots,n$. Therefore $\check h$ lies in the ideal $(\check z_1,\dots,\check z_n)^2$ on $\check V$. So making $\tilde U,\check U,\tilde V,\check V$ smaller, we may write $\check h=\sum_{b,c=1}^n\check z_b\check z_cQ_{bc}$, the analogue of (5.9), for some $Q_{bc}:\check V\to\mathbb{A}^1$ with $Q_{bc}=Q_{cb}$, where the last part requires $\operatorname{char}\mathbb{K}\neq2$, and $Q_{bc}(0,\dots,0)=\delta_{bc}$ becomes $Q_{bc}(\check v)=\delta_{bc}$.

The last part of §5.3, defining the coordinates $(y_1,\dots,y_m,z_1,\dots,z_n)$, involves taking square roots $Q_{nn}^{1/2},\hat Q_{n-1\,n-1}^{-1/2},\dots$. These generally will not exist on $\check V$ or Zariski open subsets of $\check V$, but they will exist on étale open subsets of $\check V$, noting that $\mathbb{K}$ is algebraically closed, so that square roots exist in $\mathbb{K}$. So we can construct an étale open neighbourhood $\jmath':V'\to\check V$ of $\check v$ in $\check V$ on which we define étale coordinates $(y_1,\dots,y_m,z_1,\dots,z_n):V'\to\mathbb{A}^{m+n}$ satisfying the analogues of (a)–(d) in §5.3, where (d) becomes $\check g\circ\jmath'=\check f\circ\jmath'+z_1^2+\cdots+z_n^2$.

Set $U'=\{v'\in V':z_1(v')=\cdots=z_n(v')=0\}$, which is a smooth $\mathbb{K}$-subscheme of V' as $(y_1,\dots,y_m,z_1,\dots,z_n)$ are étale coordinates on V'. Define $\iota:U'\to U$, $\jmath:V'\to V$, $\Phi':U'\to V'$, $\alpha:V'\to U$, and $\beta:V'\to\mathbb{A}^n$ by $\iota=\check\alpha\circ\jmath'|_{U'}$, $\jmath=\check\jmath\circ\jmath'$, $\Phi'=\mathrm{id}_{U'}$, $\alpha=\check\alpha\circ\jmath'$ and $\beta=(z_1,\dots,z_n)$. As $\jmath':V'\to\check V$ is an étale open neighbourhood of $\check v$ in $\check V$, there exists $u'\in V'$ with $\jmath'(u')=\check v$, and $z_b(u')=0$ for $b=1,\dots,n$ as $\check z_b(\check v)=\tilde z_b\circ\check\jmath(\check v)=\tilde z_b(j(x))=0$, so $u'\in U$ with $\iota(u')=\check\alpha\circ\jmath'(u')=\check\alpha(\check v)=i(x)$. Also $\iota,\jmath,\alpha\times\beta$ are étale as $\check\alpha\times\check\beta,\check\jmath,\jmath'$ are.

To see that $\Phi\circ\iota=\jmath\circ\Phi'$, note that

$$\begin{aligned}&(\tilde y_1,\dots,\tilde y_m,\tilde z_1,\dots,\tilde z_n)\circ\Phi\circ\iota=(\tilde x_1,\dots,\tilde x_m,0,\dots,0)\circ\check\alpha\circ\jmath'|_{U'}\\&=\bigl((\tilde x_1,\dots,\tilde x_m)\times\mathrm{id}_{\mathbb{A}^n}\bigr)\circ(\check\alpha\times\check\beta)\circ\jmath'|_{U'}\\&=(\tilde y_1,\dots,\tilde y_m,\tilde z_1,\dots,\tilde z_n)\circ\check\jmath\circ\jmath'|_{U'}\circ\mathrm{id}_{U'}=(\tilde y_1,\dots,\tilde y_m,\tilde z_1,\dots,\tilde z_n)\circ\jmath\circ\Phi',\end{aligned}$$

using the definitions and (5.12). Since $\Phi\circ\iota(u')=j(x)=\jmath\circ\Phi'(u')$ and $(\tilde y_1,\dots,\tilde y_m,\tilde z_1,\dots,\tilde z_n)$ are étale coordinates on $\tilde V\subseteq V$, this implies that $\Phi\circ\iota=\jmath\circ\Phi'$ near u' in U', so making U' smaller if necessary we have $\Phi\circ\iota=\jmath\circ\Phi'$. The equations $\alpha\circ\Phi'=\iota$, $\beta\circ\Phi'=0$ are immediate, and

$g\circ\jmath=f\circ\alpha+(z_1^2+\cdots+z_n^2)\circ\beta$ follows from $\check g\circ\jmath'=\check f\circ\jmath'+z_1^2+\cdots+z_n^2$. This proves Proposition 2.23.

5.5. Proof of Proposition 2.24. For Proposition 2.24, we first follow the proof of Proposition 2.23 in §5.4 until immediately before the choice of $\jmath':V'\to\check V$, so we have étale coordinates $(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n)$ on $j(x)\in\tilde V\subseteq V$ and $(\tilde x_1,\ldots,\tilde x_m)$ on $i(x)\in\tilde U=\Phi^{-1}(\tilde V)\subseteq U$ with $\tilde x_a=\tilde y_a\circ\Phi|_{\tilde U}$, $\frac{\partial g}{\partial\tilde z_b}|_{\Phi(\tilde U)}=0$, and a Cartesian square (5.12) with étale coordinates $(\check y_1,\ldots,\check y_m,\check z_1,\ldots,\check z_n)$ on $\check V$ and $\check v\in\check V$ with $\check\alpha(\check v)=i(x)$, $\check\beta(\check v)=(0,\ldots,0)$ and $\check\jmath(\check v)=j(x)$, and functions $\check f,\check g,\check h,Q_{bc}:\check V\to\mathbb{A}^1$ with $\check f=f\circ\check\alpha$, $\check g=g\circ\check\jmath$, and $\check h=\check g-\check f=\sum_{b,c=1}^n\check z_b\check z_cQ_{bc}$, with $Q_{bc}(\check v)=\delta_{bc}$.

We have morphisms $\mathrm{id}\times 0:\tilde U\to\tilde U\times\mathbb{A}^n$, $\Phi|_{\tilde U}:\tilde U\to\tilde V$ with $\bigl((\tilde x_1,\ldots,\tilde x_m)\times\mathrm{id}_{\mathbb{A}^n}\bigr)\circ(\mathrm{id}\times 0)=(\tilde y_1,\ldots,\tilde y_m,\tilde z_1,\ldots,\tilde z_n)\circ\Phi|_{\tilde U}:\tilde U\to\mathbb{A}^{m+n}$. Thus by the Cartesian property of (5.12), there is a unique morphism $\check\Phi:\tilde U\to\check V$ with $(\check\alpha\times\check\beta)\circ\check\Phi=\mathrm{id}\times 0$ and $\check\jmath\circ\check\Phi=\Phi|_{\tilde U}$. Also $\check\Phi(i(x))=\check v$, since $(\mathrm{id}\times 0)(i(x))=(i(x),0)=(\check\alpha\times\check\beta)(\check v)$ and $\Phi|_{\tilde U}(i(x))=j(x)=\check\jmath(\check v)$.

We now modify the inductive procedure in §5.3–§5.4, to construct a Zariski open neighbourhood U' of $i(x)$ in $\tilde U\subseteq U$, an étale open neighbourhood $\jmath':V'\to\check V$ of $\check v$ in $\check V$ with $\check\alpha\circ\jmath'(V')\subseteq U'\subseteq\tilde U$, a morphism $\Phi':U'\to V'$ with $\jmath'\circ\Phi'=\check\Phi|_{U'}$, étale coordinates $(y_1,\ldots,y_m,z_1,\ldots,z_n):V'\to\mathbb{A}^{m+n}$, and regular functions $q_1,\ldots,q_n:U'\to\mathbb{A}^1\setminus\{0\}$, such that:

(a) $y_a=\check y_a\circ\jmath'=\tilde y_a\circ\check\jmath\circ\jmath'$ for $a=1,\ldots,m$.
(b) $\frac{\partial z_b}{\partial z'_c}=\delta_{bc}$ at $\Phi'(i(x))$ for $b,c=1,\ldots,n$, where $(y'_1,\ldots,y'_m,z'_1,\ldots,z'_n)=(\check y_1,\ldots,\check y_m,\check z_1,\ldots,\check z_n)\circ\jmath'$ are étale coordinates on V'.
(c) $\Phi'(U')$ is the submanifold $z_1=\cdots=z_n=0$ in V'.
(d) $\check h\circ\jmath'=(q_1\circ\check\alpha\circ\jmath')\cdot z_1^2+\cdots+(q_n\circ\check\alpha\circ\jmath')\cdot z_n^2$.

In the first step of the induction, as $Q_{nn}(\check v)=1$ and $\check\Phi(i(x))=\check v$, we can choose a Zariski open neighbourhood U' of $i(x)$ in $\tilde U$ such that $q_n:=Q_{nn}\circ\check\Phi$ is nonzero on U', so that $q_n:U'\to\mathbb{A}^1\setminus\{0\}$. Then $\check\alpha^{-1}(U')$ is a Zariski open neighbourhood of $\check v$ in $\check V$, with $\check\Phi(U')\subseteq\check\alpha^{-1}(U')$ as $\check\alpha\circ\check\Phi=\mathrm{id}_{\tilde U}$, and

$$\check P_n:=\bigl(Q_{nn}|_{\check\alpha^{-1}(U')}\bigr)\cdot\bigl(q_n^{-1}\circ\check\alpha|_{\check\alpha^{-1}(U')}\bigr):\check\alpha^{-1}(U')\longrightarrow\mathbb{A}^1$$

is a regular function, with $\check P_n\circ\check\Phi|_{U'}=1$.

Define $\hat\jmath:\hat V\to\check P_n^{-1}(\mathbb{A}^1\setminus\{0\})\subseteq\check V$ to be the étale double cover parametrizing square roots of $\check P_n$ wherever $\check P_n$ is nonzero. Then $\check P_n\circ\hat\jmath$ has a natural square root $(\check P_n\circ\hat\jmath)^{1/2}:\hat V\to\mathbb{A}^1\setminus\{0\}$. Since $\check P_n\circ\check\Phi|_{U'}=1$, there is a unique lift $\hat\Phi:U'\to\hat V$ such that $\hat\jmath\circ\hat\Phi=\check\Phi|_{U'}$ and $(\check P_n\circ\hat\jmath)^{1/2}\circ\hat\Phi=1$.

In an analogue of (5.11), we may now write

$$\begin{aligned}
\check{h}\circ\hat{\jmath} &= \textstyle\sum_{b,c=1}^{n}(\check{z}_b\check{z}_cQ_{bc})\circ\hat{\jmath} = \sum_{b,c=1}^{n-1}\bigl(\check{z}_b\check{z}_c\bigl[Q_{bc}-Q_{nn}^{-1}Q_{bn}Q_{cn}\bigr]\bigr)\circ\hat{\jmath}\\
&\quad+(q_n\circ\check{\alpha}\circ\hat{\jmath})\cdot\Bigl[(\check{P}_n\circ\hat{\jmath})^{1/2}\bigl(\check{z}_n+\textstyle\sum_{b=1}^{n-1}Q_{nn}^{-1}Q_{bn}\check{z}_b\bigr)\circ\hat{\jmath}\Bigr]^2\\
&= \textstyle\sum_{b,c=1}^{n-1}\hat{z}_b\hat{z}_c\hat{Q}_{bc}+(q_n\circ\hat{\alpha})\cdot\hat{z}_n^2,
\end{aligned}\tag{5.13}$$

where $\hat{z}_b=\check{z}_b\circ\hat{\jmath}$ for $b<n$, $\hat{z}_n=(\check{P}_n\circ\hat{\jmath})^{1/2}\bigl(\check{z}_n+\sum_{b=1}^{n-1}Q_{nn}^{-1}Q_{bn}\check{z}_b\bigr)\circ\hat{\jmath}$, $\hat{Q}_{bc}=[Q_{bc}-Q_{nn}^{-1}Q_{bn}Q_{cn}]\circ\hat{\jmath}$ for $b,c=1,\ldots,n-1$, and $\hat{\alpha}=\check{\alpha}\circ\hat{\jmath}$.

In the second step we define $q_{n-1}:=\hat{Q}_{n-1\,n-1}\circ\hat{\Phi}$, and making U' smaller we can suppose that q_{n-1} maps $U'\to\mathbb{A}^1\setminus\{0\}$. Then we define

$$\hat{P}_{n-1}:=\bigl(\hat{Q}_{n-1\,n-1}|_{\hat{\alpha}^{-1}(U')}\bigr)\cdot\bigl(q_{n-1}^{-1}\circ\hat{\alpha}|_{\hat{\alpha}^{-1}(U')}\bigr):\hat{\alpha}^{-1}(U')\longrightarrow\mathbb{A}^1,$$

so that $\hat{P}_{n-1}\circ\hat{\Phi}=1$. Let $\ddot{\jmath}:\ddot{V}\to\hat{P}_{n-1}^{-1}(\mathbb{A}^1\setminus\{0\})\subseteq\hat{V}$ be the étale double cover parametrizing square roots of $\hat{P}_{n-1}$ where $\hat{P}_{n-1}$ is nonzero. Then $\hat{P}_{n-1}\circ\ddot{\jmath}$ has a square root $(\hat{P}_{n-1}\circ\ddot{\jmath})^{1/2}:\ddot{V}\to\mathbb{A}^1\setminus\{0\}$, and there is a unique lift $\ddot{\Phi}:U'\to\ddot{V}$ such that $\ddot{\jmath}\circ\ddot{\Phi}=\hat{\Phi}$ and $(\ddot{P}_{n-1}\circ\ddot{\jmath})^{1/2}\circ\ddot{\Phi}=1$. As for (5.13) we have

$$\check{h}\circ\hat{\jmath}\circ\ddot{\jmath}=\textstyle\sum_{b,c=1}^{n-2}\ddot{z}_b\ddot{z}_c\ddot{Q}_{bc}+(q_{n-1}\circ\ddot{\alpha})\cdot\ddot{z}_{n-1}^2+(q_n\circ\ddot{\alpha})\cdot\ddot{z}_n^2,$$

where $\ddot{z}_{n-1}=(\hat{P}_{n-1}\circ\ddot{\jmath})^{1/2}\bigl(\hat{z}_{n-1}+\sum_{b=1}^{n-2}\hat{Q}_{n-1n-1}^{-1}\hat{Q}_{bn-1}\hat{z}_b\bigr)\circ\ddot{\jmath}$, $\ddot{z}_b=\hat{z}_b\circ\ddot{\jmath}$ for $b=1,\ldots,n-2,n$, $\ddot{Q}_{bc}=[\hat{Q}_{bc}-\hat{Q}_{n-1n-1}^{-1}\hat{Q}_{bn-1}\hat{Q}_{cn-1}]\circ\ddot{\jmath}$ for $b,c=1,\ldots,n-2$, and $\ddot{\alpha}=\hat{\alpha}\circ\ddot{\jmath}$. Continuing in this way, after n inductive steps we define data $\jmath',V',\Phi',y_a,z_b,q_b$ satisfying (a)–(d) above. The important difference with §5.4 is that each time we pass to a further étale cover $\hat{V},\ddot{V},\ldots$ of $\check{V}$ to take a square root, we also lift $\check{\Phi}|_{U'}:U'\to\check{V}$ to morphisms $\hat{\Phi}:U'\to\hat{V}$, $\ddot{\Phi}:U'\to\ddot{V}$, …, to these étale covers, for $U'\subseteq U$ Zariski open. Proposition 2.24 now follows as in §5.3–§5.4, with (2.14) coming from (d) above.

6. Canonical bundles of d-critical loci

We prove the results of §2.4, Propositions 2.25, 2.27, 2.30 and Theorem 2.28.

6.1. Proof of Proposition 2.25. We will prove the complex analytic case of Proposition 2.25, involving (a) and Proposition 2.22. The algebraic cases with (b),(c) and Propositions 2.23 and 2.24 are similar.

Let (X,s) be a complex analytic d-critical locus and x,U',V',n,α,β be as in Proposition 2.22. To see there exists a unique isomorphism $\hat{\beta}$ satisfying (2.16), consider the diagram of vector bundles on U':

$$\begin{tikzcd}
0 \arrow[r] & TU|_{U'} \arrow[r, "\mathrm{d}\Phi|_{U'}"'] \arrow[d, "\cong"', "\mathrm{id}"] & \Phi^*(TV)|_{U'} \arrow[r, "\Pi|_{U'}"'] \arrow[d, "\cong"', "\Phi|_{U'}^*(\mathrm{d}(\alpha\times\beta))"] & N_{UV}|_{U'} \arrow[r] \arrow[d, dotted, "\cong"', "\hat{\beta}^*"] & 0 \\
0 \arrow[r] & TU|_{U'} \arrow[r, "\mathrm{d}(\mathrm{id}\times 0)"] & T(U\times\mathbb{C}^n)|_{U'\times 0} \arrow[r] & \bigl\langle\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_n}\bigr\rangle_{U'} \arrow[r] & 0.
\end{tikzcd}\tag{6.1}$$

Here (6.1) has exact rows, the first two columns are isomorphisms as $\alpha\times\beta$ is étale, and the left hand square commutes since $\alpha\circ\Phi|_{U'}=\mathrm{id}$ and $\beta\circ\Phi|_{U'}=0$. Therefore by exactness there is a unique isomorphism $\hat\beta^*$ making (6.1) commute, and taking duals shows that (2.16) holds. So (2.17) prescribes $q_{UV}|_{R'}$, for $R'=i^{-1}(U')\subseteq R\subseteq X$. This $q_{UV}|_{R'}$ is a nondegenerate holomorphic quadratic form on $i^*(N_{UV})|_{R'}$, since $\mathrm{d}z_1\otimes\mathrm{d}z_1+\cdots+\mathrm{d}z_n\otimes\mathrm{d}z_n$ is a nondegenerate holomorphic quadratic form on $\bigl\langle\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_n}\bigr\rangle_{U'}$.

Thus, for each $x\in R$ we can find an open $x\in R'\subseteq R$ and a given value for the restriction $q_{UV}|_{R'}$. Since such R' form an open cover for R, and $S^2i^*(N_{UV}^*)$ is a sheaf on R, these values for $q_{UV}|_{R'}$ come from a unique $q_{UV}\in H^0(S^2i^*(N_{UV}^*))$ if and only if they agree on overlaps $R'\cap R''$ between different subsets R',R'' in the open cover.

Let x,U',V',α,β,R' and $x',U'',V'',\alpha',\beta',R''$ be alternative choices in Proposition 2.22. Then (2.17) gives values for $q_{UV}|_{R'}$ and $q_{UV}|_{R''}$, which agree on the overlap $R'\cap R''$ if

$$\begin{aligned}(6.2)\qquad & i|_{R'\cap R''}^*\bigl[(S^2\hat\beta)(\mathrm{d}z_1\otimes\mathrm{d}z_1+\cdots+\mathrm{d}z_n\otimes\mathrm{d}z_n)\bigr]\\ &=i|_{R'\cap R''}^*\bigl[(S^2\hat\beta')(\mathrm{d}z_1\otimes\mathrm{d}z_1+\cdots+\mathrm{d}z_n\otimes\mathrm{d}z_n)\bigr].\end{aligned}$$

Combining (6.1) for U',V',α,β and U'',V'',α',β' gives a commutative diagram of vector bundles on $U'\cap U''$, with exact rows:

$$(6.3)\qquad\begin{tikzcd}[column sep=large] 0 \arrow[r] & TU|_{U'} \arrow[r, "\mathrm{d}(\mathrm{id}\times 0)"'] \arrow[d, "\mathrm{id}", "\cong"'] & T(U\times\mathbb{C}^n)|_{U'\times 0} \arrow[r] \arrow[d, "{\Phi|_{U'}^*(\mathrm{d}((\alpha'\times\beta')\circ(\alpha\times\beta)^{-1}))}", "\cong"'] & \bigl\langle\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_n}\bigr\rangle_{U'} \arrow[r] \arrow[d, "\hat\beta'^*\circ\hat\beta^{*-1}", "\cong"'] & 0 \\ 0 \arrow[r] & TU|_{U'} \arrow[r, "\mathrm{d}(\mathrm{id}\times 0)"] & T(U\times\mathbb{C}^n)|_{U'\times 0} \arrow[r] & \bigl\langle\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_n}\bigr\rangle_{U'} \arrow[r] & 0. \end{tikzcd}$$

Here $(\alpha'\times\beta')\circ(\alpha\times\beta)^{-1}$ is a local biholomorphism $U\times\mathbb{C}^n\to U\times\mathbb{C}^n$ defined near $(U'\cap U'')\times\{0\}$, which is the identity on $U\times\{0\}$, and preserves the function $f\boxplus z_1^2+\cdots+z_n^2:U\times\mathbb{C}^n\to\mathbb{C}$. So restricting to $\mathrm{Crit}(f\boxplus z_1^2+\cdots+z_n^2)=\mathrm{Crit}(f)\times\{0\}$ where $\mathrm{Hess}\bigl(f\boxplus z_1^2+\cdots+z_n^2\bigr)$ is defined, we see that $\mathrm{d}((\alpha'\times\beta')\circ(\alpha\times\beta)^{-1})|_{\mathrm{Crit}(f|_{U'\cap U''})\times\{0\}}$ preserves $\mathrm{Hess}\bigl(f\boxplus z_1^2+\cdots+z_n^2\bigr)$ in $H^0\bigl(S^2T^*(U\times\mathbb{C}^n)|_{\mathrm{Crit}(f|_{U'\cap U''})\times\{0\}}\bigr)$. As

$$\mathrm{Hess}\bigl(f\boxplus z_1^2+\cdots+z_n^2\bigr)=\mathrm{Hess}\,f+\mathrm{d}z_1\otimes\mathrm{d}z_1+\cdots+\mathrm{d}z_n\otimes\mathrm{d}z_n,$$

from (6.3) we see that $\hat\beta'^*\circ\hat\beta^{*-1}$ preserves $\mathrm{d}z_1\otimes\mathrm{d}z_1+\cdots+\mathrm{d}z_n\otimes\mathrm{d}z_n$ on $\mathrm{Crit}(f|_{U'\cap U''})$. That is,

$$\begin{aligned}&(S^2(\hat\beta^{-1}\circ\hat\beta'))(\mathrm{d}z_1\otimes\mathrm{d}z_1+\cdots+\mathrm{d}z_n\otimes\mathrm{d}z_n)|_{\mathrm{Crit}(f|_{U'\cap U''})}\\ &\qquad=(\mathrm{d}z_1\otimes\mathrm{d}z_1+\cdots+\mathrm{d}z_n\otimes\mathrm{d}z_n)|_{\mathrm{Crit}(f|_{U'\cap U''})}.\end{aligned}$$

Composing with $S^2\hat\beta$ and applying $i|_{R'\cap R''}^*$ gives (6.2). Hence, there exists a unique, nondegenerate q_{UV} satisfying Proposition 2.25(a).

For the final part of the proposition, suppose $\Psi:(S,V,g,j)\hookrightarrow(T,W,h,k)$ is another embedding, so $\Psi\circ\Phi:(R,U,f,i)\hookrightarrow(T,W,h,k)$ is also an embedding, and define N_{UV}, q_{UV}, N_{VW}, q_{VW} and N_{UW}, q_{UW} from Φ, Ψ and $\Psi \circ \Phi$ as above. Then existence of unique $\gamma_{UVW}, \delta_{UVW}$ making (2.22) commute is immediate by exactness, and exactness of the line in (2.22) including $\gamma_{UVW}, \delta_{UVW}$ also follows easily. So (2.23) is exact on R.

We will first prove the required isomorphism (2.24) exists locally. Let $x \in R$. Applying Proposition 2.22 to $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ at x gives $i(x) \in U' \subseteq U$, $j(x) \in V' \subseteq V$, $\alpha : V' \to U$ and $\beta : V' \to \mathbb{C}^m$ for $m = \dim V - \dim U$, satisfying conditions including $g|_{V'} = f \circ \alpha + (y_1^2 + \cdots + y_m^2) \circ \beta$, writing $(y_1, \ldots, y_m)$ for the coordinates on $\mathbb{C}^m$. Then Proposition 2.25(a) gives

$$\tag{6.4} \begin{aligned} \hat{\beta} : \langle \mathrm{d}y_1, \ldots, \mathrm{d}y_m \rangle_{U'} \xrightarrow{\cong} N_{UV}^*|_{U'}, \qquad \text{and} \\ q_{UV}|_{R'} = i|_{R'}^* \big[(S^2 \hat{\beta})(\mathrm{d}y_1 \otimes \mathrm{d}y_1 + \cdots + \mathrm{d}y_m \otimes \mathrm{d}y_m) \big], \end{aligned}$$

where $R' = i^{-1}(U') \subseteq R \subseteq X$. Similarly, applying Proposition 2.22 to $\Psi : (S, V, g, j) \hookrightarrow (T, W, h, k)$ at x gives $j(x) \in V'' \subseteq V$, $k(x) \in W'' \subseteq V$, $\alpha' : W'' \to V$ and $\beta' : W'' \to \mathbb{C}^n$ for $n = \dim W - \dim V$, satisfying conditions including $h|_{W''} = g \circ \alpha' + (z_1^2 + \cdots + z_n^2) \circ \beta'$, writing $(z_1, \ldots, z_n)$ for the coordinates on $\mathbb{C}^n$, and with $S'' = j^{-1}(V'')$, Proposition 2.25(a) gives

$$\tag{6.5} \begin{aligned} \hat{\beta}' : \langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n \rangle_{V''} \xrightarrow{\cong} N_{VW}^*|_{V''}, \qquad \text{and} \\ q_{VW}|_{S''} = j|_{S''}^* \big[(S^2 \hat{\beta}')(\mathrm{d}z_1 \otimes \mathrm{d}z_1 + \cdots + \mathrm{d}z_n \otimes \mathrm{d}z_n) \big]. \end{aligned}$$

Set $U''' = U' \cap \Phi^{-1}(V'')$ and $W''' = \alpha'^{-1}(V')$, write $(y_1, \ldots, y_m, z_1, \ldots, z_n)$ for the coordinates on $\mathbb{C}^{m+n}$, and define $\alpha'' : W''' \to U$ and $\beta'' : W''' \to \mathbb{C}^{m+n}$ by $\alpha'' = \alpha \circ \alpha'|_{W'''}$ and $\beta'' = (\beta \circ \alpha'|_{W'''}) \times \beta'|_{W'''}$. Then

$$\begin{aligned} h|_{W'''} &= g \circ \alpha'|_{W'''} + (z_1^2 + \cdots + z_n^2) \circ \beta'|_{W'''} \\ &= f \circ \alpha \circ \alpha'|_{W'''} + (y_1^2 + \cdots + y_m^2) \circ \beta \circ \alpha'|_{W'''} + (z_1^2 + \cdots + z_n^2) \circ \beta'|_{W'''} \\ &= f \circ \alpha'' + (y_1^2 + \cdots + y_m^2 + z_1^2 + \cdots + z_n^2) \circ \beta''. \end{aligned}$$

The other conditions are easy to verify, so $U''', W''', \alpha'', \beta'', m + n$ are a possible outcome for Proposition 2.22 applied to $\Psi \circ \Phi : (R, U, f, i) \hookrightarrow (T, W, h, k)$ at x. Hence Proposition 2.25(a) with $R''' = i^{-1}(U''') = R' \cap S''$ gives

$$\tag{6.6} \begin{aligned} \hat{\beta}'' : \langle \mathrm{d}y_1, \ldots, \mathrm{d}y_m, \mathrm{d}z_1, \ldots, \mathrm{d}z_n \rangle_{W'''} \xrightarrow{\cong} N_{UW}^*|_{W'''}, \qquad \text{and} \\ q_{UW}|_{R'''} = i|_{W'''}^* \big[(S^2 \hat{\beta}'')(\mathrm{d}y_1 \otimes \mathrm{d}y_1 + \cdots + \mathrm{d}y_m \otimes \mathrm{d}y_m \\ + \mathrm{d}z_1 \otimes \mathrm{d}z_1 + \cdots + \mathrm{d}z_n \otimes \mathrm{d}z_n) \big]. \end{aligned}$$

The isomorphism $i^*(N_{UW})\cong i^*(N_{UV})\oplus j^*(N_{VW})|_R$ in (2.24) is now clear on the open subset $x\in R'''=R'\cap S''\subseteq R$, from the isomorphisms

$$\begin{aligned} i|_{R'''}^*(\hat\beta)&:\langle \mathrm{d}y_1,\ldots,\mathrm{d}y_m\rangle_{R'''}\xrightarrow{\cong} i^*(N_{UV})|_{R'''}^*,\\ j|_{R'''}^*(\hat\beta')&:\langle \mathrm{d}z_1,\ldots,\mathrm{d}z_n\rangle_{R'''}\xrightarrow{\cong} j^*(N_{VW})|_{R'''}^*,\\ i|_{R'''}^*(\hat\beta'')&:\langle \mathrm{d}y_1,\ldots,\mathrm{d}y_m\rangle_{R'''}\oplus\langle \mathrm{d}z_1,\ldots,\mathrm{d}z_n\rangle_{R'''}\xrightarrow{\cong} i^*(N_{UW})|_{R'''}^*.\end{aligned}$$

It is easy to see this isomorphism is compatible with (2.23), and (6.4)–(6.6) imply that the isomorphism induces equation (2.25) on R'''.

Now isomorphisms (2.24) compatible with (2.23) are in 1-1 correspondence with complementary vector subbundles to $i^*(\gamma_{UVW})\bigl(i^*(N_{UV})\bigr)$ in $i^*(N_{UW})$. In this case, the complementary vector subbundle is just the orthogonal subbundle to $i^*(\gamma_{UVW})\bigl(i^*(N_{UV})\bigr)$ using the complex inner product q_{UW} on $i^*(N_{UW})$. This orthogonal subbundle is complementary provided $q_{UW}|_{i^*(\gamma_{UVW})(i^*(N_{UV}))}$ is nondegenerate, which holds as this restriction is isomorphic to q_{UV}.

Thus, the isomorphism (2.24) exists locally on R, and is unique (even locally) if it exists. So we can glue local choices on an open cover of R by subsets R''' to get a unique global isomorphism (2.24) compatible with (2.23) and (2.25). This completes the proof of Proposition 2.25.

6.2. Proof of Proposition 2.27. Let $\Phi:(R,U,f,i)\hookrightarrow(S,V,g,j)$ be an embedding of critical charts on a complex analytic d-critical locus (X,s). Fix $x\in R\subseteq X$. As for (2.15), define the normal $\mathbb{C}$-vector spaces $N_{XU}|_x, N_{XV}|_x$ to X in U,V at x by the exact sequences

$$\begin{gathered} 0\longrightarrow T_xX\xrightarrow[\mathrm{d}i|_x]{} T_{i(x)}U\longrightarrow N_{XU}|_x\longrightarrow 0,\\ 0\longrightarrow T_xX\xrightarrow{\mathrm{d}j|_x} T_{j(x)}V\longrightarrow N_{XV}|_x\longrightarrow 0,\end{gathered}\tag{6.7}$$

where T_xX is the Zariski tangent space of X at x. Write $\dim T_xX=l$, $\dim U=l+m$ and $\dim V=l+m+n$, so that $\dim N_{XU}|_x=m$, $\dim N_{XV}|_x=m+n$, and $\dim N_{UV}|_{i(x)}=n$. As for (2.26), equation (6.7) induces isomorphisms

$$\begin{aligned} \rho_{XU}|_x&:\Lambda^lT_x^*X\otimes\Lambda^mN_{XU}^*|_x\longrightarrow\Lambda^{l+m}T_{i(x)}^*U=K_U|_{i(x)},\\ \rho_{XV}|_x&:\Lambda^lT_x^*X\otimes\Lambda^{m+n}N_{XV}^*|_x\longrightarrow\Lambda^{l+m+n}T_{j(x)}^*V=K_V|_{j(x)}.\end{aligned}\tag{6.8}$$

We also have a commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & T_xX \arrow[r, "\mathrm{d}i|_x"'] \arrow[d, "\mathrm{id}"'] & T_{i(x)}U \arrow[r, "\mathrm{Hess}_{i(x)}f"'] \arrow[d, "\mathrm{d}\Phi|_{i(x)}"] & T_{i(x)}^*U \arrow[r, "\mathrm{d}i|_x^*"'] \arrow[d, dotted] & T_x^*X \arrow[r] \arrow[d, "\mathrm{id}"] & 0\\ 0 \arrow[r] & T_xX \arrow[r, "\mathrm{d}j|_x"] & T_{j(x)}V \arrow[r, "\mathrm{Hess}_{j(x)}g"] & T_{j(x)}^*V \arrow[r, "\mathrm{d}j|_x^*"] & T_x^*X \arrow[r] & 0. \end{tikzcd}\tag{6.9}$$

Since $T_xX=\mathrm{Ker}(\mathrm{Hess}_{i(x)}f)$, by (6.7) $\mathrm{Hess}_{i(x)}f$ is the pullback to $T_{i(x)}U$ of a nondegenerate quadratic form $\mathrm{Hess}'_{i(x)}f$ on $N_{XU}|_x$. Then

$\det(\mathrm{Hess}'_{i(x)} f)$ is a nonzero element of $\Lambda^m N^*_{XU}|_x^{\otimes^2}$. Similarly, $\mathrm{Hess}_{j(x)}\, g$ is the pullback to $T_{j(x)}V$ of $\mathrm{Hess}'_{j(x)}\, g$ on $N_{XV}|_x$, and $0 \neq \det(\mathrm{Hess}'_{j(x)}\, g)$ in $\Lambda^{m+n} N^*_{XV}|_x^{\otimes^2}$.

In a similar way to (2.23), there is a natural exact sequence

$$0 \longrightarrow N_{XU}|_x \longrightarrow N_{XV}|_x \longrightarrow N_{UV}|_{i(x)} \longrightarrow 0, \tag{6.10}$$

and as for (2.24), there is a unique isomorphism $N_{XV}|_x \cong N_{XU}|_x \oplus N_{UV}|_{i(x)}$ compatible with (6.10) and identifying $\mathrm{Hess}'_{j(x)}\, g$ with $\mathrm{Hess}'_{i(x)} f \oplus q_{UV}|_x \oplus 0$ as in (2.25). From this, we deduce that the following diagram commutes:

$$\begin{tikzcd}
(\Lambda^{l+m}T^*_{i(x)}U)^{\otimes^2} = K_U^{\otimes^2}|_{i(x)} \arrow[r, "\mathrm{id}\otimes\det(q_{UV})|_x"'] \arrow[dr, "J_\Phi|_x"] & K_U^{\otimes^2}|_{i(x)} \otimes \Lambda^n N^*_{UV}|^{\otimes^2}_{i(x)} \arrow[d, "\rho^{\otimes^2}_{UV}|_x"'] \\
(\Lambda^l T^*_x X)^{\otimes^2} \otimes \Lambda^m N^*_{XU}|^{\otimes^2}_x \arrow[u, "\rho^{\otimes^2}_{XU}|_x"'] & (\Lambda^{l+m+n}T^*_{j(x)}V)^{\otimes^2} = K_V^{\otimes^2}|_{j(x)} \\
(\Lambda^l T^*_x X)^{\otimes^2} \arrow[u, "\mathrm{id}\otimes\det(\mathrm{Hess}'_{i(x)} f)"'] \arrow[r, "\mathrm{id}\otimes\det(\mathrm{Hess}'_{j(x)}\, g)"] & (\Lambda^l T^*_x X)^{\otimes^2} \otimes \Lambda^{m+n} N^*_{XV}|^{\otimes^2}_x . \arrow[u, "\rho^{\otimes^2}_{XV}|_x"]
\end{tikzcd} \tag{6.11}$$

Here the upper right triangle is the restriction of (2.27) to x, and depends on Φ. But the rest of the diagram depends on $x, (R, U, f, i)$, (S, V, g, j) but not on Φ. So (6.11) a diagram of commuting isomorphisms implies that $J_\Phi|_x$ is independent of the choice of Φ. Thus, if $\Phi, \tilde\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ are embeddings of critical charts, then $J_\Phi|_x = J_{\tilde\Phi}|_x$ for all $x \in R^{\mathrm{red}}$. As R^{red} is a reduced complex analytic space, this implies that $J_\Phi = J_{\tilde\Phi}$, as we want.

To prove (2.28), consider the commutative diagram

$$\begin{tikzcd}
 & & \begin{array}{c} i^*\big(K_U^{\otimes^2}\big)\otimes \\ i^*\big(\Lambda^{\mathrm{top}} N^*_{UW}\big)^{\otimes^2}|_{R^{\mathrm{red}}} \end{array} \arrow[d, "\text{from (2.23)}"'] \arrow[ddd, bend left, "\rho^{\otimes^2}_{UW}"] \\
i^*\big(K_U^{\otimes^2}\big)|_{R^{\mathrm{red}}} \arrow[urr, bend left, "\mathrm{id}\otimes\det(q_{UW})"] \arrow[r, "\mathrm{id}\otimes\det(q_{UV})"] \arrow[dr, "J_\Phi"'] \arrow[ddrr, bend right, "J_{\Psi\circ\Phi}"'] & \begin{array}{c} i^*\big(K_U^{\otimes^2}\big)\otimes \\ i^*\big(\Lambda^{\mathrm{top}} N^*_{UV}\big)^{\otimes^2}|_{R^{\mathrm{red}}} \end{array} \arrow[r, "\mathrm{id}\otimes\det(q_{VW})"] \arrow[d, "\rho^{\otimes^2}_{UV}"] & \begin{array}{c} u^*\big(K_U^{\otimes^2}\big)\otimes \\ i^*\big(\Lambda^{\mathrm{top}} N^*_{UV}\big)^{\otimes^2}\otimes \\ i^*\big(\Lambda^{\mathrm{top}} N^*_{VW}\big)^{\otimes^2}|_{R^{\mathrm{red}}} \end{array} \arrow[d, "\rho^{\otimes^2}_{UV}\otimes\mathrm{id}"'] \\
 & i^*\big(K_V^{\otimes^2}\big)|_{R^{\mathrm{red}}} \arrow[r, "\mathrm{id}\otimes\det(q_{VW})"] \arrow[dr, "J_\Psi|_{R^{\mathrm{red}}}"'] & \begin{array}{c} i^*\big(K_V^{\otimes^2}\big)\otimes \\ i^*\big(\Lambda^{\mathrm{top}} N^*_{VW}\big)^{\otimes^2}|_{R^{\mathrm{red}}} \end{array} \arrow[d, "\rho^{\otimes^2}_{VW}"'] \\
 & & k^*\big(K_W^{\otimes^2}\big)|_{R^{\mathrm{red}}},
\end{tikzcd}$$

which includes (2.27) for Φ, Ψ and $\Psi \circ \Phi$, where the top left triangle commutes by (2.25). This proves Proposition 2.27 for complex analytic d-critical loci.

In the algebraic case, for (X, s) an algebraic d-critical locus over a field $\mathbb{K}$, the argument above shows that for each scheme-theoretic point x of the reduced $\mathbb{K}$-subscheme R^{red} we have $J_\Phi|_x = J_{\tilde{\Phi}}|_x$, where (6.7)–(6.11) are now diagrams of finite-dimensional vector spaces over the residue field of X at x, rather than over $\mathbb{C}$. Since R^{red} is a reduced $\mathbb{K}$-scheme and $J_\Phi, J_{\tilde{\Phi}}$ are isomorphisms of line bundles on R^{red}, $J_\Phi|_x = J_{\tilde{\Phi}}|_x$ for each $x \in R^{\mathrm{red}}$ implies that $J_\Phi = J_{\tilde{\Phi}}$. The rest of the proof is as for the complex analytic case.

6.3. Proof of Theorem 2.28. We first construct the line bundle $K_{X,s}$ in Theorem 2.28, and show it satisfies parts (i),(ii). Observe that Theorem 2.28(i),(ii) characterizing the sheaf $K_{X,s}$ are similar in structure to Theorem 2.1(i),(ii) characterizing the sheaf $\mathcal{S}_X$. We will follow the method of §3.1 to prove Theorem 2.28(i),(ii), handling the complex analytic and algebraic cases together. The analogues of Lemmas 3.1, 3.2 and 3.3 are Proposition 2.27, Theorem 2.20 and:

Lemma 6.1. *Let $(R, U, f, i), (S, V, g, j)$ be critical charts on (X, s). Then there exists a unique isomorphism*

$$J_{R,U,f,i}^{S,V,g,j} : i^*\big(K_U^{\otimes^2}\big)|_{R^{\mathrm{red}} \cap S^{\mathrm{red}}} \longrightarrow j^*\big(K_V^{\otimes^2}\big)|_{R^{\mathrm{red}} \cap S^{\mathrm{red}}} \tag{6.12}$$

such that if $x, U' \subseteq U, V' \subseteq V, (T, W, h, k)$ and $\Phi : (R', U', f', i') \hookrightarrow (T, W, h, k)$, $\Psi : (S', V', g', j') \hookrightarrow (T, W, h, k)$ are as in Theorem 2.20, and J_Φ, J_Ψ are as in Definition 2.26 for Φ, Ψ, then

$$J_{R,U,f,i}^{S,V,g,j}|_{R'^{\mathrm{red}} \cap S'^{\mathrm{red}}} = J_\Psi^{-1} \circ J_\Phi|_{R'^{\mathrm{red}} \cap S'^{\mathrm{red}}}. \tag{6.13}$$

Also, if (T, W, h, k) is any other critical chart on (X, s) then

$$\begin{aligned} &J_{R,U,f,i}^{R,U,f,i} = \mathrm{id}_{i^*(K_U^2)|_{R^{\mathrm{red}}}}, \qquad J_{S,V,g,j}^{R,U,f,i} = \big(J_{R,U,f,i}^{S,V,g,j}\big)^{-1}, \\ \textit{and} \quad &J_{S,V,g,j}^{T,W,h,k} \circ J_{R,U,f,i}^{S,V,g,j}|_{R^{\mathrm{red}} \cap S^{\mathrm{red}} \cap T^{\mathrm{red}}} = J_{R,U,f,i}^{T,W,h,k}|_{R^{\mathrm{red}} \cap S^{\mathrm{red}} \cap T^{\mathrm{red}}}. \end{aligned} \tag{6.14}$$

Proof. Suppose $x, U', V', \Phi, \Psi, (T, W, h, k)$ and $\check{x}, \check{U}', \check{V}', \check{\Phi}, \check{\Psi}, (\check{T}, \check{W}, \check{h}, \check{k})$ are two possible choices in Theorem 2.20. We will show that

$$J_\Psi^{-1} \circ J_\Phi|_{R'^{\mathrm{red}} \cap S'^{\mathrm{red}} \cap \check{R}'^{\mathrm{red}} \cap \check{S}'^{\mathrm{red}}} = J_{\check{\Psi}}^{-1} \circ J_{\check{\Phi}}|_{R'^{\mathrm{red}} \cap S'^{\mathrm{red}} \cap \check{R}'^{\mathrm{red}} \cap \check{S}'^{\mathrm{red}}}. \tag{6.15}$$

Let $\tilde{x} \in R'^{\mathrm{red}} \cap S'^{\mathrm{red}} \cap \check{R}'^{\mathrm{red}} \cap \check{S}'^{\mathrm{red}} \subseteq T \cap \check{T}$. Then applying Theorem 2.20 to $\tilde{x}, (T, W, h, k), (\check{T}, \check{W}, \check{h}, \check{k})$ yields open $\tilde{x} \in T' \subseteq T$, $\tilde{x} \in \check{T}' \subseteq \check{T}$, a critical chart $(\tilde{T}, \tilde{W}, \tilde{h}, \tilde{k})$ and embeddings $\Theta : (T', W', h', k') \hookrightarrow (\tilde{T}, \tilde{W}, \tilde{h}, \tilde{k})$ and $\Xi : (\check{T}', \check{W}', \check{h}', \check{k}') \hookrightarrow (\tilde{T}, \tilde{W}, \tilde{h}, \tilde{k})$.

Set $U'' = \Phi^{-1}(W') \cap \check{\Phi}^{-1}(\check{W}')$, $V'' = \Psi^{-1}(W') \cap \check{\Psi}^{-1}(\check{W}')$, and let $(R'', U'', f'', i'') \subseteq (R, U, f, i)$, $(S'', V'', g'', j'') \subseteq (S, V, g, j)$ be the corresponding subcharts. Then we have a diagram of embeddings of critical charts

$$\begin{tikzcd}
(R'',U'',f'',i'') \arrow[r, "\Phi|_{U''}"] \arrow[dr, "\check\Phi|_{U''}"'] & (T',W',h',k') \arrow[dr, "\Theta"] & \\
 & & (\tilde T,\tilde W,\tilde h,\tilde k). \\
(S'',V'',g'',j'') \arrow[r, "\check\Psi|_{V''}"] \arrow[ur, "\Psi|_{V''}"] & (\check T',\check W',\check h',\check k') \arrow[ur, "\Xi"'] &
\end{tikzcd}$$

Hence we have

$$\begin{aligned} J_\Theta\circ J_\Phi|_{R''^{\rm red}}&=J_{\Theta\circ\Phi}|_{R''^{\rm red}}=J_{\Xi\circ\check\Phi}|_{R''^{\rm red}}=J_\Xi\circ J_{\check\Phi}|_{R''^{\rm red}},\\ J_\Theta\circ J_\Psi|_{S''^{\rm red}}&=J_{\Theta\circ\Psi}|_{S''^{\rm red}}=J_{\Xi\circ\check\Psi}|_{S''^{\rm red}}=J_\Xi\circ J_{\check\Psi}|_{S''^{\rm red}}, \end{aligned} \tag{6.16}$$

using (2.28) in the first and third steps of each line, and the first part of Proposition 2.27 in the second steps. Restricting (6.16) to $R''^{\rm red}\cap S''^{\rm red}$, inverting the second line, and composing with the first line gives

$$\begin{aligned} J_\Psi^{-1}\circ J_\Phi|_{R''^{\rm red}\cap S''^{\rm red}}&=(J_\Theta\circ J_\Psi)^{-1}\circ(J_\Theta\circ J_\Phi)|_{R''^{\rm red}\cap S''^{\rm red}}\\ &=(J_\Xi\circ J_{\check\Psi})^{-1}\circ(J_\Xi\circ J_{\check\Phi})|_{R''^{\rm red}\cap S''^{\rm red}}=J_{\check\Psi}^{-1}\circ J_{\check\Phi}|_{R''^{\rm red}\cap S''^{\rm red}}. \end{aligned}$$

This proves (6.15) on the open neighbourhood $R''^{\rm red}\cap S''^{\rm red}$ of $\tilde x$ in $R'^{\rm red}\cap S'^{\rm red}\cap\check R'^{\rm red}\cap\check S'^{\rm red}$. As this works for all such $\tilde x$, equation (6.15) holds.

Thus, Theorem 2.20 shows that for each $x\in R^{\rm red}\cap S^{\rm red}$ we can choose an open neighbourhood $R'^{\rm red}\cap S'^{\rm red}$ on which the restriction of $J_{R,U,f,i}^{S,V,g,j}$ is defined by (6.13). These open neighbourhoods $R'^{\rm red}\cap S'^{\rm red}$ form an open cover of $R^{\rm red}\cap S^{\rm red}$, and on overlaps $(R'^{\rm red}\cap S'^{\rm red})\cap(\check R'^{\rm red}\cap\check S'^{\rm red})$ the corresponding values of $J_{R,U,f,i}^{S,V,g,j}$ agree. Therefore by properties of sheaves there is a unique morphism $J_{R,U,f,i}^{S,V,g,j}$ in (6.12) such that (6.13) holds for all applications of Theorem 2.20. Finally, we prove (6.14) by the method used for (3.6)–(3.7) in Lemma 3.3. q.e.d.

The existence of a line bundle $K_{X,s}$, unique up to canonical isomorphism, satisfying Theorem 2.28(i),(ii) now follows from Lemma 6.1 in the same way that the first part of Theorem 2.1 was deduced from Lemma 3.3 in §3.1.

For parts (iii),(iv), defining κ_x in (2.31) and showing that (2.33) commutes for all (R,U,f,i) with $x\in R$, first note that if (R,U,f,i) is a critical chart on (X,s) with $x\in X$, and $\alpha_{x,R,U,f,i}$ is as in Theorem 2.28(iv), then in the notation of §6.2 we have

$$\alpha_{x,R,U,f,i}=\rho_{XU}^{\otimes^2}\big|_x\circ\big[\mathrm{id}\otimes\det(\mathrm{Hess}'_{i(x)}\,f)\big]. \tag{6.17}$$

Thus, if $\Phi : (R,U,f,i) \hookrightarrow (S,V,g,j)$ is an embedding of critical charts then equation (6.11) shows the following commutes:

$$
\begin{tikzcd}
 & K_U^{\otimes^2}|_{i(x)} \arrow[d, "J_\Phi|_x"] \\
\bigl(\Lambda^{\mathrm{top}} T_x^* X\bigr)^{\otimes^2} \arrow[ur, "\alpha_{x,R,U,f,i}"] \arrow[r, "\alpha_{x,S,V,g,j}"'] & K_V^{\otimes^2}|_{j(x)}.
\end{tikzcd}
$$

It is now easy to see from Theorem 2.28(i)–(ii) that there is a unique isomorphism κ_x in (2.31) such that $\alpha_{x,R,U,f,i} \circ \kappa_x = \iota_{R,U,f,i}|_x$ for all critical charts (R,U,f,i) on (X,s) with $x \in R$. This proves Theorem 2.28(iii)–(iv).

6.4. Proof of Proposition 2.30. Let $\phi : (X,s) \to (Y,t)$ be a morphism of d-critical loci with $\phi : X \to Y$ smooth, fix $x_0 \in X$ with $\phi(x_0) = y_0 \in Y$, and set $m = \dim T_{x_0}X$ and $n = \dim T_{y_0}Y$, so that ϕ is smooth of relative dimension $m - n \geqslant 0$ near x_0.

As in the proof of Proposition 2.8 in §4.2, we may choose open $y_0 \in S \subseteq Y$ and $x_0 \in R \subseteq \phi^{-1}(S) \subseteq X$, closed embeddings $i : R \hookrightarrow U$, $j : S \hookrightarrow V$ for U,V complex manifolds (or smooth $\mathbb{K}$-schemes) with $\dim U = m$, $\dim V = n$, a morphism $\Phi : U \to V$ smooth of relative dimension $m-n$ with $\Phi \circ i = j \circ \phi|_R : R \to V$, and holomorphic $g : V \to \mathbb{C}$ and $f = g \circ \Phi : U \to \mathbb{C}$ (or regular $g : V \to \mathbb{A}^1$ and $f = g \circ \Phi : U \to \mathbb{A}^1$) with $\iota_{R,U}(s|_R) = i^{-1}(f) + I_{R,U}^2$ and $\iota_{S,V}(t|_S) = j^{-1}(g) + I_{S,V}^2$. Since $\dim T_{x_0}X = \dim U$ and $\dim T_{y_0}Y = \dim V$, the second part of Proposition 2.7 shows that making R,S,U,V smaller, we can suppose that (R,U,f,i) and (S,V,g,j) are critical charts on (X,s) and (Y,t).

We have a diagram of coherent sheaves on R with exact rows, where the bottom row is (2.34) restricted to R:

$$
\begin{tikzcd}
0 \arrow[r] & \begin{array}{c}(j\circ\phi|_R)^*(T^*V) \\ = (\Phi\circ i)^*(T^*V)\end{array} \arrow[r, "i^*(\mathrm{d}\Phi^*)"'] \arrow[d, "\phi|_R^*(\mathrm{d}j^*)"] & i^*(T^*U) \arrow[r] \arrow[d, "\mathrm{d}i^*"] & i^*(T^*_{U/V}) \arrow[r] \arrow[d, dotted, "\alpha"] & 0 \\
0 \arrow[r] & \phi|_R^*(T^*Y) \arrow[r, "\mathrm{d}\phi^*|_R"] & T^*X|_R \arrow[r] & T^*_{X/Y}|_R \arrow[r] & 0.
\end{tikzcd}
\tag{6.18}
$$

The left hand square commutes as $\Phi \circ i = j \circ \phi|_R$. So by exactness, there is a unique morphism α as shown making (6.18) commute. As $\Phi, \phi|_R$ are both smooth of relative dimension $m-n$, $i^*(T^*_{U/V})$ and $T^*_{X/Y}|_R$ are both vector bundles on R of rank $m-n$. But as i is an embedding, $\mathrm{d}i^*$ is surjective, so α is surjective, and thus α is an isomorphism.

Taking determinants in the top line of (6.18) gives an isomorphism

$$
\beta : (j\circ\phi|_R)^*(K_V) \otimes i^*\bigl(\Lambda^{\mathrm{top}} T^*_{U/V}\bigr) \xrightarrow{\cong} i^*(K_U). \tag{6.19}
$$

Define an isomorphism $\Upsilon_{\Phi,R} : \bigl[\phi|^*_{X^{\rm red}}(K_{Y,t})\otimes(\Lambda^{\rm top}T^*_{X/Y})|^{\otimes^2}_{X^{\rm red}}\bigr]\big|_{R^{\rm red}} \to K_{X,s}|_{R^{\rm red}}$ by the commutative diagram of isomorphisms

$$\begin{tikzcd}
\bigl[\phi|^*_{X^{\rm red}}(K_{Y,t})\otimes(\Lambda^{\rm top}T^*_{X/Y})|^{\otimes^2}_{X^{\rm red}}\bigr]\big|_{R^{\rm red}} \arrow[r, "\Upsilon_{\Phi,R}"'] \arrow[d, "\phi|^*_{R^{\rm red}}(\iota_{S,V,g,j})\otimes(\Lambda^{\rm top}\alpha^{-1}|_{R^{\rm red}})^{\otimes^2}"] & K_{X,s}|_{R^{\rm red}} \arrow[d, "\iota_{R,U,f,i}"'] \\
(j\circ\phi|_{R^{\rm red}})^*\bigl(K_V^{\otimes^2}\bigr)\otimes i^*\bigl(\Lambda^{\rm top}T^*_{U/V}\bigr)^{\otimes^2} \arrow[r, "\beta|^{\otimes^2}_{R^{\rm red}}"] & i^*\bigl(K_U^{\otimes^2}\bigr)|_{R^{\rm red}},
\end{tikzcd} \tag{6.20}$$

for $\iota_{R,U,f,i}, \iota_{S,V,g,j}$ as in (2.29), and α,β as in (6.18)–(6.19).

We now claim that for each $x\in R$, equation (2.36) with $\Upsilon_{\Phi,R}$ in place of Υ_ϕ commutes. To see this, compare (2.33) for $\kappa_x,\kappa_{\phi(x)}$, (2.36), (6.17), the restriction of (6.20) to x, and the commutative diagram

$$\begin{tikzcd}
\bigl(\Lambda^{\rm top}T^*_{\phi(x)}Y\bigr)^{\otimes^2}\otimes\bigl(\Lambda^{\rm top}T^*_{X/Y}|_x\bigr)^{\otimes^2} \arrow[r, "\upsilon_x^{\otimes^2}"'] \arrow[d, "\mathrm{id}\otimes\det(\mathrm{Hess}'_{j\circ\phi(x)}\,g)"] & \bigl(\Lambda^{\rm top}T^*_xX\bigr)^{\otimes^2} \arrow[d, "\mathrm{id}\otimes\det(\mathrm{Hess}'_{i(x)}\,f)"'] \\
\begin{array}{l}\bigl(\Lambda^{\rm top}T^*_{\phi(x)}Y\bigr)^{\otimes^2}\otimes\bigl(\Lambda^{\rm top}T^*_{X/Y}|_x\bigr)^{\otimes^2}\\ \otimes\bigl(\Lambda^{\rm top}N^*_{YV}|_{\phi(x)}\bigr)^{\otimes^2}\end{array} \arrow[r, "\upsilon_x^{\otimes^2}\otimes(\Lambda^{\rm top}\gamma_x)^{\otimes^2}"'] \arrow[d, "(\rho_{YV}|^{\otimes^2}_{\phi(x)})\otimes(\Lambda^{\rm top}\alpha|_x^{-1})^{\otimes^2}"] & \begin{array}{l}\bigl(\Lambda^{\rm top}T^*_xX\bigr)^{\otimes^2}\\ \otimes\bigl(\Lambda^{\rm top}N^*_{XU}|_x\bigr)^{\otimes^2}\end{array} \arrow[d, "\rho_{XU}|_x^{\otimes^2}"'] \\
K_V^{\otimes^2}|_{j\circ\phi(x)}\otimes\bigl(\Lambda^{\rm top}T^*_{U/V}|_{i(x)}\bigr)^{\otimes^2} \arrow[r, "\beta|_x^{\otimes^2}"] & K_U^{\otimes^2}|_{i(x)}.
\end{tikzcd} \tag{6.21}$$

Here to prove (6.21) commutes, consider the commutative diagram, with exact rows and columns

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & N^*_{YV}|_{\phi(x)} \arrow[r, dotted, "\gamma_x", "\cong"'] \arrow[d] & N^*_{XU}|_x \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T^*_{\phi(x)}V \arrow[r, "\mathrm{d}\Phi^*|_{i(x)}"'] \arrow[d, "\mathrm{d}j^*|_{\phi(x)}"] & T^*_{i(x)}U \arrow[r] \arrow[d, "\mathrm{d}i^*|_x"] & T^*_{U/V}|_{i(x)} \arrow[r] \arrow[d, "\alpha|_x"] & 0 \\
0 \arrow[r] & T^*_{\phi(x)}Y \arrow[r, "\mathrm{d}\phi^*|_x"] \arrow[d] & T^*_xX \arrow[r] \arrow[d] & T^*_{X/Y}|_x \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0. &
\end{tikzcd} \tag{6.22}$$

The bottom two rows of (6.22) are (6.18) restricted to x. Exactness implies that there is a unique isomorphism γ_x as shown in the first row. One can show that γ_x identifies the nondegenerate quadratic forms $\mathrm{Hess}'_{j\circ\phi(x)}\,g$ and $\mathrm{Hess}'_{i(x)}\,f$ on $N^*_{YV}|_{\phi(x)}$ and $N^*_{XU}|_x$. This implies the upper rectangle of (6.21) commutes. As in equations (6.8), (6.19), and (2.36), the maps $\rho_{YV}|_{\phi(x)}, \rho_{XU}|_x, \beta|_x, \upsilon_x$ in (6.21) are obtained by taking top exterior powers in the first column, second column, second row, and third row of (6.22), respectively. Thus, taking top exterior powers in (6.22) shows that the bottom rectangle of (6.21) commutes.

To summarize the proof so far: for each $x_0 \in X$ we have constructed an open neighbourhood $x_0 \in R \subseteq X$ and an isomorphism $\Upsilon_{\Phi,R} : \big[\phi|^*_{X^{\rm red}}(K_{Y,t}) \otimes (\Lambda^{\rm top}T^*_{X/Y})|^{\otimes^2}_{X^{\rm red}}\big]\big|_{R^{\rm red}} \to K_{X,s}|_{R^{\rm red}}$, such that for each $x \in R^{\rm red}$, equation (2.36) with $\Upsilon_{\Phi,R}$ in place of Υ_ϕ commutes.

Suppose $x_0, R, \Upsilon_{\Phi,R}$ and $x_0', R', \Upsilon_{\Phi',R'}$ are two possible choices above. Equation (2.36) implies that $\Upsilon_{\Phi,R}|_x = \Upsilon_{\Phi',R'}|_x$ for all $x \in R^{\rm red} \cap R'^{\rm red}$. As $R^{\rm red} \cap R'^{\rm red}$ is reduced, this forces $\Upsilon_{\Phi,R}|_{R^{\rm red}\cap R'^{\rm red}} = \Upsilon_{\Phi',R'}|_{R^{\rm red}\cap R'^{\rm red}}$. Since such $R^{\rm red}$ form an open cover of $X^{\rm red}$, there is a unique isomorphism Υ_ϕ in (2.35) with $\Upsilon_\phi|_{R^{\rm red}} = \Upsilon_{\Phi,R}$ for all $x_0, R, \Upsilon_{\Phi,R}$ as above. Proposition 2.30 follows.

7. Equivariant d-critical loci

Next we prove Propositions 2.43 and 2.44 from §2.6.

7.1. Proof of Proposition 2.43. For part (a), suppose (X, s) is an algebraic d-critical locus over $\mathbb{K}$ equivariant under a good action $\mu : G \times X \to X$ of an algebraic $\mathbb{K}$-torus G, with character $\chi : G \to \mathbb{G}_m$, and let $x \in X$. As μ is good, there exists a G-invariant affine open neighbourhood R' of x in X. Choose a closed embedding $(y_1, \dots, y_K) : R' \hookrightarrow \mathbb{A}^K$. Then $y_a \circ \mu : G \times R' \to \mathbb{A}^1$ is regular, so as $H^0(\mathcal{O}_{G\times R'}) \cong H^0(\mathcal{O}_G) \otimes H^0(\mathcal{O}_{R'})$, we may write $y_a \circ \mu(\gamma, r) = \sum_{b=1}^{L_a} x_a^b(\gamma) y_a^b(r)$ with $x_a^b : G \to \mathbb{A}^1$ and $y_a^b : R' \to \mathbb{A}^1$ regular, for $a = 1, \dots, K$.

Define $V = \langle y_a^b : a = 1, \dots, K,\ b = 1, \dots, L_a\rangle_{\mathbb{K}} \subseteq H^0(\mathcal{O}_Y)$ to be the finite-dimensional $\mathbb{K}$-vector subspace of $H^0(\mathcal{O}_Y)$ generated by the y_a^b. Then V is G-invariant, and contains $y_1, \dots, y_K$ as $y_a = \sum_{b=1}^{L_a} x_a^b(1) \cdot y_a^b$. As G is a torus, V decomposes as a direct sum of 1-dimensional G-representations, so we may choose a basis $v_1, \dots, v_M$ for V with $v_a \circ \mu(\gamma, r) = \kappa_a(\gamma) v_a(r)$ for all $\gamma \in G$, $r \in R'$ and $a = 1, \dots, M$, where $\kappa_a : G \to \mathbb{G}_m$ is a character of G. Then $i' := (v_1, \dots, v_M) : R' \hookrightarrow \mathbb{A}^M = V^*$ is a closed embedding, since $(v_1, \dots, v_K) : R' \hookrightarrow \mathbb{A}^K$ is and $v_1, \dots, v_K \in V$, and is G-equivariant under the obvious linear G-action on $\mathbb{A}^M$ given by $\gamma : (z_1, \dots, z_M) \mapsto (\kappa_1(\gamma) z_1, \dots, \kappa_M(\gamma) z_M)$.

Let $I \subset \mathbb{K}[z_1, \dots, z_M]$ be the ideal of functions vanishing on $i'(R')$. Then I is G-invariant, as $i'(R')$ is, so as G is a torus we can choose a finite set of G-equivariant generators $h_1, \dots, h_N$ for I, with $h_a(\gamma \cdot r) = \lambda_a(\gamma) h_a(r)$ for all $\gamma \in G$, $r \in R'$ and $a = 1, \dots, N$, where $\lambda_a : G \to \mathbb{G}_m$ is a character of G. Then

$$T_{i'(x)}(i'(R')) = \mathrm{Ker}\big[\textstyle\bigoplus_{a=1}^N \mathrm{d}h_a|_{i'(x)} : T_{i'(x)}\mathbb{A}^M \longrightarrow \bigoplus_{a=1}^N \mathrm{d}h_a^*(T_0\mathbb{A}^1)|_{i'(x)}\big].$$

Choose a minimal subset $\{h_{a_1}, \dots, h_{a_P}\} \subseteq \{h_1, \dots, h_N\}$ such that

$$T_{i'(x)}(i'(R')) = \mathrm{Ker}\big[\textstyle\bigoplus_{b=1}^P \mathrm{d}h_{a_b}|_{i'(x)} : T_{i'(x)}\mathbb{A}^M \longrightarrow \bigoplus_{b=1}^P \mathrm{d}h_{a_b}^*(T_0\mathbb{A}^1)|_{i'(x)}\big].$$

Then $P = N - \dim T_{i'(x)} i'(R') = N - \dim T_x X$, and $\mathrm{d}h_{a_1}, \ldots, \mathrm{d}h_{a_P}$ are linearly independent at $i'(x)$. Define U' to be the closed $\mathbb{K}$-subscheme of $\mathbb{A}^M$ defined by $h_{a_1} = \cdots = h_{a_P} = 0$. Then $i'(R') \subseteq U'$, as $h_{a_1}, \ldots, h_{a_P} \in I$, and U' is G-invariant as the h_{a_b} are G-equivariant, and U' is smooth of dimension $N - P = \dim T_x X$ near $i'(x)$ as $\mathrm{d}h_{a_1}, \ldots, \mathrm{d}h_{a_P}$ are linearly independent at $i'(x)$.

Let $\tilde{U} \subseteq U'$ be the open $\mathbb{K}$-subscheme of points where U' is smooth of dimension $\dim T_x X$. Then $\tilde{U}$ is G-invariant, smooth of dimension $\dim T_x X$, and quasi-affine, so that $i'(x) \in \tilde{U}$. As $\tilde{U}$ is smooth it is normal, so by Lemma 2.42 we can choose a G-invariant affine open neighbourhood U'' of $i'(x)$ in $\tilde{U}$. Define $R'' = i'^{-1}(U'')$ and $i'' := i'|_{R''} : R'' \hookrightarrow U''$. Then R'' is an open neighbourhood of x in X, as R' is, $i'(R') \subseteq U'$, and U'' is an open neighbourhood of $i'(x)$ in U'. Also i'' is a closed embedding as i' is, so R'' is affine. And R'' is G-invariant as U'' is and i' is G-equivariant, so i'' is G-equivariant.

Write $I_{R'',U''} \subset H^0(\mathcal{O}_{U''})$ for the G-invariant ideal of functions on U'' vanishing on the closed subscheme $i''(R'') \subset U''$. Theorem 2.1(i) gives a sheaf morphism $\iota_{R'',U''}$. As R'', U'' are affine this descends to global sections, giving

$$(\iota_{R'',U''})_* : H^0(\mathcal{S}_X|_{R''}) \longrightarrow H^0(\mathcal{O}_{U''})/I^2_{R'',U''}.$$

Thus $(\iota_{R'',U''})_*(s|_{R''}) \in H^0(\mathcal{O}_{U''})/I^2_{R'',U''}$, so we can choose regular $f'' : U'' \to \mathbb{A}^1$ with $(\iota_{R'',U''})_*(s|_{R''}) = f'' + I^2_{R'',U''}$. Now by assumption s is G-equivariant with character $\chi : G \to \mathbb{G}_m$, so $(\iota_{R'',U''})_*(s|_{R''})$ is also G-equivariant with character χ as R'' is G-invariant and i'' G-equivariant. By averaging f'' over the G-action twisted by χ, we can suppose that f'' is also G-equivariant with character χ, that is, $f''(\gamma \cdot u'') = \chi(\gamma) \cdot f''(u'')$ for all $\gamma \in G$ and $u'' \in U''$.

Since $\dim U'' = \dim T_x X$, Proposition 2.7 now shows that we can choose Zariski open $U \subseteq U''$ and $R = i''^{-1}(U) \subseteq R''$ such that (R, U, f, i) is a critical chart on (X, s) with $x \in R$, where $f = f''|_U$ and $i = i''|_R$. The proof of this in §4.1 works by showing that the closed subschemes $i''(R'') \subseteq \mathrm{Crit}(f'') \subseteq U''$ satisfy $i''(R'') = \mathrm{Crit}(f'')$ near $i''(x)$ in U'', and restricting to an open neighbourhood U of x in U'' such that $U \cap i''(R'') = U \cap \mathrm{Crit}(f'')$. Take U to be the largest such neighbourhood, the union of all open $V \subseteq U''$ with $V \cap i''(R'') = V \cap \mathrm{Crit}(f'')$. Then U is G-invariant, as $i''(R''), \mathrm{Crit}(f'')$ are. Writing ρ for the G-action on U, it follows that $(R, U, f, i), \rho$ is a G-equivariant critical chart on (X, s) with $x \in R$ and $\dim U = \dim T_x X$. This proves Proposition 2.43(a).

For part (b), suppose that for all $x \in X$ there exists a G-equivariant critical chart $(R, U, f, i), \rho$ on (X, s) with $x \in R$. For such $x, (R, U, f, i)$, ρ, note that as U is smooth it is normal, so by Lemma 2.42 there is a G-invariant affine open neighbourhood U' of x in X. As $i : R \hookrightarrow U$ is a G-equivariant closed embedding, $R' = i^{-1}(U')$ is a G-equivariant affine open neighbourhood of x in X. Since such R' exist for all $x \in X$, μ is a good G-action.

7.2. Proof of Proposition 2.44. Proposition 2.44 is a G-equivariant version of the $\mathbb{K}$-scheme case of Theorem 2.20, which is proved in §5.2. The proof relies on Proposition 2.19, which is proved in §5.1. We will explain how to modify the proofs in §5.1–§5.2 to include G-equivariance throughout.

Suppose (X,s) is an algebraic d-critical locus over $\mathbb{K}$ equivariant under the action $\mu : G \times X \to X$ of an algebraic $\mathbb{K}$-torus G, with character $\chi : G \to \mathbb{G}_m$, and let $(R,U,f,i),\rho$ be a G-equivariant critical chart on (X,s). For the equivariant version of Proposition 2.19, we must show that for each $x \in X$ there exists a G-equivariant subchart $(R',U',f',i') \subseteq (R,U,f,i)$ with $x \in R'$, and a G-equivariant embedding $\Phi : (R',U',f',i') \hookrightarrow (S,V,g,j)$ into a G-equivariant critical chart $(S,V,g,j),\sigma$ on (X,s), such that G acts linearly on $\mathbb{A}^n$, inducing the G-action σ on the G-invariant Zariski open subset $V \subseteq \mathbb{A}^n$.

To prove this, modify §5.1 as follows. Take the affine open neighbourhood $\tilde U$ of x in U to be G-invariant, which is possible by Lemma 2.42. Then take $\tilde\Phi : \tilde U \to \mathbb{A}^n$ to be G-equivariant under a linear G-action on $\mathbb{A}^n$, which is possible as in §7.1. Choose $V \subseteq \mathbb{A}^n$ to be G-invariant, and the étale coordinates $(z_1,\dots,z_n)$ on V with $\tilde\Phi(\tilde U) \cap V = \{z_{m+1} = \cdots = z_n = 0\}$ to be G-equivariant, that is, they should satisfy $z_a(\gamma \cdot v) = \kappa_a(\gamma) z_a(v)$ for all $\gamma \in G$, $v \in V$ and $a = 1,\dots,n$, where $\kappa_a : G \to \mathbb{G}_m$ is a character of G. Choose $h : V \to \mathbb{A}^1$ to be G-equivariant with character χ, which is possible as for f'' in §7.1.

Now in the expression (5.1) for $g : V \to \mathbb{A}^1$, the terms z_a^2 for $a = m+1,\dots,n$ are not G-equivariant. To deal with this, replace V by $V' = V \times \mathbb{A}^{n-m}$, where $\mathbb{A}^{n-m}$ has coordinates $(w_{m+1},\dots,w_n)$ and G acts on $\mathbb{A}^{n-m}$ by

$$\gamma : (w_{m+1},\dots,w_n) \longmapsto \bigl(\kappa_{m+1}(\gamma)^{-1}\chi(\gamma) w_{m+1},\dots,\kappa_n(\gamma)^{-1}\chi(\gamma) w_n\bigr).$$

Replace $\Phi : U \to V$ by $\Phi' = \Phi \times 0 : U \to V' = V \times \mathbb{A}^{n-m}$, and $g : V \to \mathbb{A}^1$ in (5.1) by $g' : V' \to \mathbb{A}^1$ given by

$$\begin{aligned} g'\bigl(v,(w_{m+1},\dots,w_n)\bigr) &= h(v) - \textstyle\sum_{a=m+1}^n z_a(v) \cdot \frac{\partial h}{\partial z_a}(v) \\ &\quad + \tfrac12 \textstyle\sum_{a,b=m+1}^n z_a(v) z_b(v) \cdot \frac{\partial^2 h}{\partial z_a \partial z_b}(v) + \sum_{a=m+1}^n z_a(v) w_a. \end{aligned} \tag{7.1}$$

Then each term in (7.1) is G-equivariant with character χ. The rest of the proof in §5.1 may be made G-equivariant in a similar way to §7.1.

To modify the $\mathbb{K}$-scheme case of §5.2 to include G-equivariance, we choose $i(x) \in U' \subseteq U$ to be G-invariant and $\Theta : U' \to V$ with $\Theta \circ i' = j|_{R'}$ to be G-equivariant, which is possible as $V \subseteq \mathbb{A}^m$ is open and G-invariant with G-action induced from a linear G-action on $\mathbb{A}^m$. Then $f' - g \circ \Theta \in (I'_{R',U'})^2$ is G-equivariant with character χ. So when as in (5.6) we write $f' = g \circ \Theta + r_1 s_1 + \cdots + r_n s_n$, we choose the r_a to be G-equivariant with some character $\lambda_a : G \to \mathbb{G}_m$, and the s_a to be G-equivariant with the complementary character $\lambda_a^{-1}\chi$, for $a = 1, \ldots, n$. Then $W = V \times \mathbb{A}^{2n}$ has G-action

$$\begin{aligned}\gamma : \bigl(v, (r_1, \ldots, r_n, s_1, \ldots, s_n)\bigr) \longmapsto \\ \bigl(\gamma \cdot v, (\lambda_1(\gamma) r_1, \ldots, \lambda_n(\gamma) r_n, \lambda_1^{-1}(\gamma)\chi(\gamma) s_1, \ldots, \lambda_n^{-1}(\gamma)\chi(\gamma) s_n)\bigr),\end{aligned}$$

and the rest of the proof in §5.2 may be made G-equivariant in a similar way to §7.1. This proves Proposition 2.44.

8. Extension to Artin stacks

Finally we prove Proposition 2.54 and Theorem 2.56 from §2.8.

8.1. Proof of Proposition 2.54. Part (i) is a general property of global sections of sheaves on Artin stacks, applied to the sheaf $\mathcal{S}_X$. For (ii), the 'only if' and 'and then' parts are immediate from Definition 2.53. To prove the 'if' part, suppose $s \in H^0(\mathcal{S}_X^0)$ with $\bigl(T, t^*(s)\bigr)$ an algebraic d-critical locus, and let $v : V \to X$ be smooth with V a $\mathbb{K}$-scheme. Then $T \times_{t,X,v} V$ is an algebraic $\mathbb{K}$-space. Choosing a surjective étale morphism $W \to T \times_{t,X,v} V$ with W a $\mathbb{K}$-scheme gives a 2-commutative diagram in $\mathrm{Art}_{\mathbb{K}}$:

$$\begin{tikzcd} W \arrow[r, "\pi_V"] \arrow[d, "\pi_T"'] \arrow[dr, phantom, "\zeta \Uparrow"] & V \arrow[d, "v"] \\ T \arrow[r, "t"] & X, \end{tikzcd}$$

where $\pi_T : W \to T$ and $\pi_V : W \to V$ are smooth as v, t are, and $\pi_V : W \to V$ is surjective as t and $W \to T \times_{t,X,v} V$ are.

We have $\pi_T^\star(s(T,t)) = s(W, t \circ \pi_T) = s(W, v \circ \pi_V) = \pi_V^\star(s(V,v))$ as $s \in H^0(\mathcal{S}_X)$. Since $\bigl(T, t^*(s)\bigr) = \bigl(T, s(T,t)\bigr)$ is an algebraic d-critical locus and $\pi_T : W \to T$ is smooth, the first part of Proposition 2.8 shows that $\bigl(W, \pi_T^\star(s(T,t))\bigr)$ is an algebraic d-critical locus. Thus $\bigl(W, \pi_V^\star(s(V,v))\bigr)$ is an algebraic d-critical locus, and $\pi_V : W \to V$ is smooth and surjective, so the second part of Proposition 2.8 shows that $\bigl(V, s(V,v)\bigr)$ is an algebraic d-critical locus. As this holds for all smooth $v : V \to X$, (X, s) is a d-critical stack, proving the 'if' part.

8.2. Proof of Theorem 2.56. Let $u : U \to X$ be a smooth atlas for X. Then $U \times_{u,X,u} U$ is an algebraic $\mathbb{K}$-space, so we may choose a surjective étale morphism $V \to U \times_{u,X,u} U$ with V a $\mathbb{K}$-scheme. Composing with the projections $U \times_{u,X,u} U \to U$ gives smooth surjective morphisms $\pi_1, \pi_2 : V \to U$ and a 2-morphism $\eta : u \circ \pi_1 \Rightarrow u \circ \pi_2$. We may now form a unique 2-commutative diagram in $\mathrm{Art}_{\mathbb{K}}$:

$$
\begin{tikzcd}
 & V \arrow[drr, "\pi_2"] \arrow[dd, "\pi_1"] & & \\
W \arrow[ur, "\pi_{23}"] \arrow[drr, "\pi_{31}"] \arrow[dd, "\pi_{12}"] & & & U \arrow[dd, "u"] \\
 & U \arrow[drr, "u"] & V \arrow[ur, "\pi_1"] \arrow[dd, "\pi_2"] & \\
V \arrow[ur, "\pi_2"] \arrow[drr, "\pi_1"] & & & X, \\
 & & U \arrow[ur, "u"] &
\end{tikzcd}
\tag{8.1}
$$

such that W is the limit of the rest of the diagram. The 2-morphisms in (8.1) are η in the square faces with X a vertex, and identities in the faces with W a vertex. That is, we have a diagram of 1- and 2-morphisms in $\mathrm{Art}_{\mathbb{K}}$:

$$
\begin{tikzcd}
u\circ\pi_1\circ\pi_{12} \arrow[r, Rightarrow, "\eta*\mathrm{id}_{\pi_{12}}"'] \arrow[d, equal] & u\circ\pi_2\circ\pi_{12} \arrow[r, equal] & u\circ\pi_1\circ\pi_{23} \arrow[d, Rightarrow, "\eta*\mathrm{id}_{\pi_{23}}"'] \\
u\circ\pi_2\circ\pi_{31} & u\circ\pi_1\circ\pi_{31} \arrow[l, Rightarrow, "\eta*\mathrm{id}_{\pi_{31}}"'] \arrow[r, equal] & u\circ\pi_2\circ\pi_{23},
\end{tikzcd}
\tag{8.2}
$$

and W being the limit implies that composition around the rectangle gives the identity 2-morphism. By writing W as an iterated fibre product $W \simeq (V \times_U V) \times_{(U\times_X U)} V$, we see that W is represented by a $\mathbb{K}$-scheme. Also $\pi_{12}, \pi_{23}, \pi_{31}$ are smooth and surjective, as u, π_1, π_2 are.

Taking reduced $\mathbb{K}$-subschemes and $\mathbb{K}$-substacks in (8.1) gives another 2-commutative diagram in $\mathrm{Art}_{\mathbb{K}}$:

$$
\begin{tikzcd}
 & V^{\mathrm{red}} \arrow[drr, "\pi_2^{\mathrm{red}}"] \arrow[dd, "\pi_1^{\mathrm{red}}"] & & \\
W^{\mathrm{red}} \arrow[ur, "\pi_{23}^{\mathrm{red}}"] \arrow[drr, "\pi_{31}^{\mathrm{red}}"] \arrow[dd, "\pi_{12}^{\mathrm{red}}"] & & & U^{\mathrm{red}} \arrow[dd, "u^{\mathrm{red}}"] \\
 & U^{\mathrm{red}} \arrow[drr, "u^{\mathrm{red}}"] & V^{\mathrm{red}} \arrow[ur, "\pi_1^{\mathrm{red}}"] \arrow[dd, "\pi_2^{\mathrm{red}}"] & \\
V^{\mathrm{red}} \arrow[ur, "\pi_2^{\mathrm{red}}"] \arrow[drr, "\pi_1^{\mathrm{red}}"] & & & X^{\mathrm{red}}, \\
 & & U^{\mathrm{red}} \arrow[ur, "u^{\mathrm{red}}"] &
\end{tikzcd}
\tag{8.3}
$$

where again $u^{\mathrm{red}} : U^{\mathrm{red}} \to X^{\mathrm{red}}$ is a smooth atlas, $\pi_1^{\mathrm{red}}, \pi_2^{\mathrm{red}} : V^{\mathrm{red}} \to U^{\mathrm{red}}$ are smooth, and so on.

Now line bundles on X^{red} are examples of *Cartesian* (quasi-coherent) sheaves on X^{red} in the lisse-étale topology [20, Def. 12.3], which in the notation of Proposition 2.49 means that $\mathcal{A}(\phi, \eta)$ is an isomorphism for all diagrams (2.39). As in [20, Prop. 12.4.5], Cartesian sheaves can be described completely in terms of the diagram (8.3) for $u^{\mathrm{red}} : U^{\mathrm{red}} \to X^{\mathrm{red}}$ a smooth atlas. For line bundles on X^{red}, this means that the following functor is an equivalence of categories:

(8.4)

$$
\begin{aligned}
&F:\bigl(\text{category of line bundles }\mathcal{L}\text{ on }X^{\mathrm{red}}\bigr)\longrightarrow\\
&\bigl(\text{category of pairs }(L,\lambda),\text{ where }L\to U^{\mathrm{red}}\text{ is a line bundle and}\\
&\quad\lambda:(\pi_1^{\mathrm{red}})^*(L)\to(\pi_2^{\mathrm{red}})^*(L)\text{ an isomorphism of line bundles on }V^{\mathrm{red}},\\
&\quad\text{with }(\pi_{31}^{\mathrm{red}})^*(\lambda)\circ(\pi_{23}^{\mathrm{red}})^*(\lambda)\circ(\pi_{12}^{\mathrm{red}})^*(\lambda)=\mathrm{id}_{(\pi_1^{\mathrm{red}}\circ\pi_{12}^{\mathrm{red}})^*(L)}\text{ on }W^{\mathrm{red}}\bigr),\\
&\quad\text{mapping }\;F:\mathcal{L}\longmapsto\bigl(\mathcal{L}(U,u),\mathcal{L}(\pi_2,\eta)^{-1}\circ\mathcal{L}(\pi_1,\mathrm{id}_{u\circ\pi_1})\bigr)\;\text{ on objects.}
\end{aligned}
$$

Define a line bundle L on U^{red} by

$$
L=K_{U,s(U,u)}\otimes\bigl(\Lambda^{\mathrm{top}}T^{u\,*}_{U/X}\bigr)\big|^{\otimes^{-2}}_{U^{\mathrm{red}}}, \tag{8.5}
$$

as in (2.44). In an analogue of (2.46), for each point $p\in U^{\mathrm{red}}\subseteq U$ define an isomorphism μ_p by the commutative diagram

$$
\begin{tikzcd}
L|_p \arrow[rr, equal] \arrow[d, "\mu_p"] && K_{U,s(U,u)}|_p\otimes\bigl(\Lambda^{\mathrm{top}}T^*_{U/X}\bigr)\big|_p^{\otimes^{-2}} \arrow[d, "\kappa_p\otimes\mathrm{id}"'] \\
\begin{array}{l}\bigl(\Lambda^{\mathrm{top}}T^*_{u(p)}X\bigr)^{\otimes^2}\otimes\\ \bigl(\Lambda^{\mathrm{top}}\mathfrak{Iso}_{u(p)}(X)\bigr)^{\otimes^2}\end{array} \arrow[rr, "\alpha_p^2"] && \bigl(\Lambda^{\mathrm{top}}T_p^*U\bigr)^{\otimes^2}\otimes\bigl(\Lambda^{\mathrm{top}}T^*_{U/X}\bigr)\big|_p^{\otimes^{-2}},
\end{tikzcd} \tag{8.6}
$$

where κ_p is as in (2.31), and α_p as in Theorem 2.56(c).

By Remark 2.51(i)–(v) we have exact sequences of vector bundles of mixed rank on V, and an isomorphism η_*:

$$
\begin{tikzcd}
0 \arrow[r] & \pi_1^*(T^{u\,*}_{U/X}) \arrow[r] & T^{u\circ\pi_1\,*}_{V/X} \arrow[r] \arrow[d, "\eta_*", "\cong"'] & T^{\pi_1\,*}_{V/U} \arrow[r] & 0 \\
0 \arrow[r] & \pi_2^*(U^{u\,*}_{U/X}) \arrow[r] & T^{u\circ\pi_2\,*}_{V/X} \arrow[r] & T^{\pi_2\,*}_{V/U} \arrow[r] & 0.
\end{tikzcd} \tag{8.7}
$$

Define an isomorphism $\lambda:(\pi_1^{\mathrm{red}})^*(L)\to(\pi_2^{\mathrm{red}})^*(L)$ by the commutative diagram of line bundles on V^{red}:

$$
\begin{tikzcd}
(\pi_1^{\mathrm{red}})^*(L) \arrow[rr, "\lambda"'] \arrow[d, equal] && (\pi_2^{\mathrm{red}})^*(L)=\begin{array}{l}(\pi_2^{\mathrm{red}})^*\bigl(K_{U,s(U,u)}\bigr)\otimes\\ (\pi_2^{\mathrm{red}})^*\bigl((\Lambda^{\mathrm{top}}T^{u\,*}_{U/X})^{\otimes^{-2}}\bigr)\end{array} \arrow[d, "\mathrm{id}\otimes(\Delta_{\pi_2}|^{\otimes^{-2}}_{V^{\mathrm{red}}})"'] \\
\begin{array}{l}(\pi_1^{\mathrm{red}})^*\bigl(K_{U,s(U,u)}\bigr)\otimes\\ (\pi_1^{\mathrm{red}})^*\bigl((\Lambda^{\mathrm{top}}T^{u\,*}_{U/X})^{\otimes^{-2}}\bigr)\end{array} \arrow[d, "\mathrm{id}\otimes(\Delta_{\pi_1}|^{\otimes^{-2}}_{V^{\mathrm{red}}})"] && \begin{array}{l}(\pi_2^{\mathrm{red}})^*\bigl(K_{U,s(U,u)}\bigr)\otimes\\ (\Lambda^{\mathrm{top}}T^{\pi_2\,*}_{V/U})|^{\otimes^2}_{V^{\mathrm{red}}}\otimes(\Lambda^{\mathrm{top}}T^{u\circ\pi_2\,*}_{V/X})|^{\otimes^{-2}}_{V^{\mathrm{red}}}\end{array} \arrow[d, "\Upsilon_{\pi_2}\otimes\mathrm{id}"'] \\
\begin{array}{l}(\pi_1^{\mathrm{red}})^*\bigl(K_{U,s(U,u)}\bigr)\otimes\\ (\Lambda^{\mathrm{top}}T^{\pi_1,*}_{V/U})|^{\otimes^2}_{V^{\mathrm{red}}}\otimes(\Lambda^{\mathrm{top}}T^{u\circ\pi_1\,*}_{V/X})|^{\otimes^{-2}}_{V^{\mathrm{red}}}\end{array} \arrow[rr, "\Upsilon_{\pi_1}\otimes(\Lambda^{\mathrm{top}}\eta_*)"] && \begin{array}{l}K_{V,s(V,u\circ\pi_1)}\otimes\\ \bigl(\Lambda^{\mathrm{top}}T^{u\circ\pi_2\,*}_{V/X}\bigr)\big|^{\otimes^{-2}}_{V^{\mathrm{red}}},\end{array}
\end{tikzcd} \tag{8.8}
$$

with $\Upsilon_{\pi_1}, \Upsilon_{\pi_2}$ as in (2.35), and η_* as in (8.7), and $\Delta_{\pi_i} : \pi_i^*(\Lambda^{\mathrm{top}} T^{u\,*}_{U/X}) \to (\Lambda^{\mathrm{top}} T^{\pi_i,*}_{V/U})^{-1} \otimes \Lambda^{\mathrm{top}} T^{u\circ\pi_i\,*}_{V/X}$ for $i = 1, 2$ are induced by taking top exterior powers in the rows of (8.7).

Let $v \in V^{\mathrm{red}} \subseteq V$, so that $\pi_1(v), \pi_2(v) \in U$ with $\eta(v) : \pi_1(v) \Rightarrow \pi_2(v) \in X$. We claim that the following diagram commutes:

$$\begin{tikzcd} (\pi_1^{\mathrm{red}})^*(L)|_v = L|_{\pi_1(v)} \arrow[r, "\mu_{\pi_1(v)}"'] \arrow[d, "\lambda|_v"] & \big(\Lambda^{\mathrm{top}} T^*_{u\circ\pi_1(v)} X\big)^{\otimes^2} \otimes \big(\Lambda^{\mathrm{top}} \mathfrak{Iso}_{u\circ\pi_1(v)}(X)\big)^{\otimes^2} \arrow[d, "\eta(v)_*"'] \\ (\pi_2^{\mathrm{red}})^*(L)|_v = L|_{\pi_2(v)} \arrow[r, "\mu_{\pi_2(v)}"] & \big(\Lambda^{\mathrm{top}} T^*_{u\circ\pi_2(v)} X\big)^{\otimes^2} \otimes \big(\Lambda^{\mathrm{top}} \mathfrak{Iso}_{u\circ\pi_2(v)}(X)\big)^{\otimes^2}. \end{tikzcd} \tag{8.9}$$

To see this, combine (8.6), the restriction of (8.8) to v, equation (2.36) applied to give expressions for $\Upsilon_{\pi_1}|_v$ and $\Upsilon_{\pi_2}|_v$, and natural compatibilities between the isomorphisms obtained by taking top exterior powers in the sequences (2.34), (2.45), and (8.7).

Now let $w \in W^{\mathrm{red}}$. Using (8.9) at $\pi_{12}(w), \pi_{23}(w), \pi_{31}(w)$ and restriction to w of composition round (8.2) being the identity shows that

$$\lambda|_{\pi_{31}(w)} \circ \lambda|_{\pi_{23}(w)} \circ \lambda|_{\pi_{12}(w)} = \mathrm{id} : L|_{\pi_1\circ\pi_{12}(w)} \longrightarrow L|_{\pi_1\circ\pi_{12}(w)}.$$

This is the restriction of $(\pi_{31}^{\mathrm{red}})^*(\lambda) \circ (\pi_{23}^{\mathrm{red}})^*(\lambda) \circ (\pi_{12}^{\mathrm{red}})^*(\lambda) = \mathrm{id}$ in (8.4) to w. Since W^{red} is reduced, the equation is implied by its restriction to each point w of W^{red}. Thus $(\pi_{31}^{\mathrm{red}})^*(\lambda) \circ (\pi_{23}^{\mathrm{red}})^*(\lambda) \circ (\pi_{12}^{\mathrm{red}})^*(\lambda) = \mathrm{id}_{(\pi_1^{\mathrm{red}}\circ\pi_{12}^{\mathrm{red}})^*(L)}$. So by (8.4), there exists a line bundle $K_{X,s}$ on X^{red}, unique up to canonical isomorphism, with an isomorphism $\chi : F(K_{X,s}) \xrightarrow{\cong} (L, \lambda)$.

We claim that $K_{X,s}$ is independent up to canonical isomorphism of the choices of smooth atlas $u : U \to X$ above and étale cover $V \to U \times_{u,X,u} U$ above. To see this, note that if u', U', V' are alternative choices, then setting $U'' := U \amalg U'$ and $u'' := u \amalg u'$ gives a third atlas $u'' : U'' \to X$, and we may choose $V'' \to U'' \times_{u'',X,u''} U''$ to be V over $U \times_{u,X,u} U$ and V' over $U' \times_{u',X,u'} U'$ and arbitrary over $U \times_{u,X,u'} U'$ and $U' \times_{u',X,u} U$, yielding $K''_{X,s} \to X^{\mathrm{red}}$. Now $K''_{X,s}$ satisfies properties on the analogue of (8.3) for $U'' = U \amalg U'$. Restricting to the subdiagram generated by $U \subseteq U''$ shows $K''_{X,s}$ satisfies the same properties as $K_{X,s}$, and restricting to that generated by U' shows $K''_{X,s}$ satisfies the same properties as $K'_{X,s}$. So we have canonical isomorphisms $K_{X,s} \cong K''_{X,s} \cong K'_{X,s}$.

For Theorem 2.56(a), let $x \in X$. As $u : U \to X$ is surjective, there exists $p \in U$ and a 2-isomorphism $\varphi : x \Rightarrow u(p)$. Define an isomorphism κ_x as in (2.43) by the commutative diagram of isomorphisms

$$\begin{tikzcd} K_{X,s}|_x \arrow[rr, "\kappa_x"'] \arrow[d, "\varphi_*"] & & \big(\Lambda^{\mathrm{top}} T^*_x X\big)^{\otimes^2} \otimes \big(\Lambda^{\mathrm{top}} \mathfrak{Iso}_x(X)\big)^{\otimes^2} \arrow[d, "\varphi_*"'] \\ K_{X,s}|_{u(p)} \arrow[r, "\chi|_p"] & L|_p \arrow[r, "\mu_p"] & \big(\Lambda^{\mathrm{top}} T^*_{u(p)} X\big)^{\otimes^2} \otimes \big(\Lambda^{\mathrm{top}} \mathfrak{Iso}_{u(p)}(X)\big)^{\otimes^2}, \end{tikzcd} \tag{8.10}$$

for μ_p as in (8.6). To see κ_x is independent of the choice of p, φ, suppose p', φ' are alternate choices. As $V \to U \times_{u,X,u} U$ is surjective, there exists $v \in V^{\mathrm{red}} \subseteq V$ with $\pi_1(v) = p$, $\pi_2(v) = p'$ and $\eta(v) = \varphi' \circ \varphi^{-1}$. Compare (8.10) for p, φ and p', φ' with (8.9) for v, and use the commutative diagram

$$\begin{tikzcd} K_{X,s}|_{u(p)} \arrow[rr, "\chi|_p"] \arrow[d, "(\varphi'\circ\varphi^{-1})_*"'] && L|_p \arrow[d, "\lambda|_v"] \\ K_{X,s}|_{u(p')} \arrow[rr, "\chi|_{p'}"] && L|_{p'}, \end{tikzcd}$$

as $\chi : F(K_{X,s}) \xrightarrow{\cong} (L, \lambda)$ is an isomorphism in the lower category in (8.4). By the same argument as for $K_{X,s}$, we can show κ_x is independent of the choice of u, U, V above. This proves Theorem 2.56(a).

For part (b), suppose $t : T \to X$ is a smooth 1-morphism, and set $U' := T \amalg U$ and $u' := t \amalg u$. Then $u' : U' \to X$ is another smooth atlas for X, so as above replacing U, u by U', u' yields a canonically isomorphic line bundle $K'_{X,s} \cong K_{X,s}$, with an isomorphism $\chi' : F(K'_{X,s}) \to (L', \lambda')$ for L', λ' given by the analogues of (8.5), (8.8) for U', u', so that in line bundles on $T^{\mathrm{red}} \amalg U^{\mathrm{red}}$ we have

$$\begin{aligned} \chi' : K'_{X,s}(T \amalg U, t \amalg u) = K'_{X,s}(T, t) \amalg K'_{X,s}(U, u) \xrightarrow{\cong} \\ \bigl[K_{T,s(T,t)} \otimes (\Lambda^{\mathrm{top}} T^{t\,*}_{T/X})|^{\otimes^{-2}}_{T^{\mathrm{red}}}\bigr] \amalg \bigl[K_{U,s(U,u)} \otimes (\Lambda^{\mathrm{top}} T^{u\,*}_{U/X})|^{\otimes^{-2}}_{U^{\mathrm{red}}}\bigr]. \end{aligned}$$

Since $K_{X,s}$ was only determined up to canonical isomorphism anyway, we may take $K'_{X,s} = K_{X,s}$, and $\chi'|_{U^{\mathrm{red}}} = \chi$. Define $\Gamma_{T,t} := \chi'|_{T^{\mathrm{red}}}$, as in (2.44). It is a natural isomorphism, proving Theorem 2.56(b).

For part (c), continuing to use the same notation, let $p \in T^{\mathrm{red}} \subseteq T$, so that $t(p) := t \circ p \in X$. As above the definition of $\kappa_{t(p)}$ is independent of the choice of atlas $u : U \to X$, so we can define it using $u' : U' \to X$ for $U' = T \amalg U$ and $u' = t \amalg u$, for the point $p \in T^{\mathrm{red}} \subseteq T^{\mathrm{red}} \amalg U^{\mathrm{red}} = U'^{\mathrm{red}}$ and 2-morphism $\varphi = \mathrm{id}_{t(p)} : t(p) \Rightarrow t(p)$. Combining (8.10) at $x = t(p)$ with $\Gamma_{T,t} := \chi'|_{T^{\mathrm{red}}}$ and the definition (8.6) of μ_p shows that (2.46) commutes. This completes the proof of Theorem 2.56.

References

[1] M. Artin, *Grothendieck topologies*, Harvard University Notes, 1962.

[2] K. Behrend, *Donaldson–Thomas type invariants via microlocal geometry*, Ann. of Math. **170** (2009), 1307–1338, MR 2600874, Zbl 1191.14050. math.AG/0507523.

[3] K. Behrend & B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), 45–88, MR 1437495, Zbl 0909.14006. alg-geom/9601010.

[4] O. Ben-Bassat, C. Brav, V. Bussi, & D. Joyce, *A 'Darboux Theorem' for shifted symplectic structures on derived Artin stacks, with applications*, to appear in Geometry and Topology, 2014. arXiv:1312.0090.

[5] C. Brav, V. Bussi, D. Dupont, D. Joyce, & B. Szendrői, *Symmetries and stabilization for sheaves of vanishing cycles*, arXiv:1211.3259, 2012.

[6] C. Brav, V. Bussi & D. Joyce, *A Darboux theorem for derived schemes with shifted symplectic structure*, arXiv:1305.6302, 2013.

[7] V. Bussi, *Categorification of Lagrangian intersections on complex symplectic manifolds using perverse sheaves of vanishing cycles*, arXiv:1404.1329, 2014.

[8] V. Bussi, D. Joyce & S. Meinhardt, *On motivic vanishing cycles of critical loci*, arXiv:1305.6428, 2013.

[9] A. Dimca & B. Szendrői, *The Milnor fibre of the Pfaffian and the Hilbert scheme of four points on* $\mathbb{C}^3$, Math. Res. Lett. **16** (2009), 1037–1055, MR 2644372, Zbl 1191.14013. arXiv:0904.2419.

[10] K. Fukaya, Y.-G. Oh, H. Ohta & K. Ono, *Lagrangian intersection Floer theory — anomaly and obstruction*, Parts I & II. AMS/IP Studies in Advanced Mathematics, 46.1 & 46.2, A.M.S./International Press, 2009. MR 2553465, Zbl 1181.53002.

[11] T.L. Gómez, *Algebraic stacks*, Proc. Indian Acad. Sci. Math. Sci. **111** (2001), 1–31. MR 1818418, Zbl 0982.14005. math.AG/9911199.

[12] H. Grauert & R. Remmert, *Coherent analytic sheaves*, Grundlehren der math. Wiss. 265, Springer-Verlag, Berlin, 1984. MR 0755331, Zbl 0537.32001.

[13] R.C. Gunning & H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, NJ, 1965. MR 0180696, Zbl 1204.01045.

[14] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math. 52, Springer, New York, 1977. MR 0463157, Zbl 0367.14001.

[15] L. Illusie, *Complexe cotangent et déformations. I*, Springer Lecture Notes in Math. 239, Springer-Verlag, Berlin, 1971. MR 0491680, Zbl 0224.13014.

[16] L. Illusie, *Cotangent complex and deformations of torsors and group schemes*, pages 159–189 in Springer Lecture Notes in Math. 274, Springer-Verlag, Berlin, 1972. MR 0491682, Zbl 0238.14016.

[17] D. Joyce & Y. Song, *A theory of generalized Donaldson–Thomas invariants*, Mem. Amer. Math. Soc. **217** (2012), no. 1020. MR 2951762, Zbl 1259.14054. arXiv:0810.5645.

[18] M. Kontsevich & Y. Soibelman, *Stability structures, motivic Donaldson–Thomas invariants and cluster transformations*, arXiv:0811.2435, 2008.

[19] M. Kontsevich & Y. Soibelman, *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson–Thomas invariants*, Commun. Number Theory Phys. **5** (2011), 231–352. MR 2851153, Zbl 1248.14060. arXiv:1006.2706.

[20] G. Laumon & L. Moret-Bailly, *Champs algébriques*, Ergeb. der Math. und ihrer Grenzgebiete 39, Springer-Verlag, Berlin, 2000. MR 1771927, Zbl 0945.14005.

[21] Y. Laszlo & M. Olsson, *The six operations for sheaves on Artin stacks. I* and *II*, Publ. Math. Inst. Hautes Études Sci. 107 (2008), 109–168 and 169–210, MR 2434692, Zbl 1191.14002. math.AG/0512097 and math.AG/0603680.

[22] Y. Laszlo & M. Olsson, *Perverse t-structure on Artin stacks*, Math. Z. 261 (2009), 737–748, MR 2480756, Zbl 1188.14002. math.AG/0606175.

[23] D. Maulik, *Motivic residues and Donaldson–Thomas theory*, in preparation, 2013.

[24] M. Olsson, *Sheaves on Artin stacks*, J. Reine Angew. Math. **603** (2007), 55–112, MR 2312554, Zbl 1137.14004.

[25] V.P. Palamodov, *Deformations of complex spaces*, Russian Math. Surveys **31** (1976), 129–197, MR 0508121, Zbl 0347.32009.

[26] V.P. Palamodov, *Cohomology of analytic algebras*, Trans. Moscow Math. Soc. 1983, No. 2. (1983), 1–61, MR 0656282, Zbl 0596.32012.

[27] V.P. Palamodov, *The tangent complex of an analytic space*, A.M.S. Translations **122** (1984), 119–171, MR 0834175, Zbl 0542.32007.

[28] V.P. Palamodov, *Deformations of complex spaces*, pages 105–194 in S.G. Gindikin and G.M. Khenkin, editors, *Several complex variables IV*, Encyclopaedia of Mathematical Sciences 10, Springer-Verlag, Berlin, 1990. MR 0894264, Zbl 0781.32024.

[29] R. Pandharipande & R.P. Thomas, *Almost closed* 1*-forms*, Glasgow Math. J. **56** (2014), 169–182, MR 3137857, Zbl 1287.1300. arXiv:1204.3958.

[30] T. Pantev, B. Toën, M. Vaquié & G. Vezzosi, *Shifted symplectic structures*, Publ. Math. I.H.E.S. **117** (2013), 271–328, MR 3090262, Zbl 06185246. arXiv:1111.3209.

[31] J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier **6** (1956), 1–42, MR 0082175, Zbl 0075.30401.

[32] H. Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14** (1974), 1–28, MR 0337963, Zbl 0277.14008.

[33] R.P. Thomas, *A holomorphic Casson invariant for Calabi–Yau* 3*-folds, and bundles on* $K3$ *fibrations*, J. Diff. Geom. **54** (2000), 367–438, MR 1818182, Zbl 1034.14015. math.AG/9806111.

[34] B. Toën, *Higher and derived stacks: a global overview*, pages 435–487 in *Algebraic Geometry — Seattle 2005*, Proc. Symp. Pure Math. 80, Part 1, A.M.S., Providence, RI, 2009. MR 2483943, Zbl 1183.14001. math.AG/0604504.

[35] B. Toën & G. Vezzosi, *Homotopical Algebraic Geometry II: Geometric Stacks and Applications*, Mem. Amer. Math. Soc. **193** (2008), no. 902. MR 2394633, Zbl 1145.14003. math.AG/0404373.

[36] B. Toën & G. Vezzosi, *From HAG to DAG: derived moduli stacks*, pages 173–216 in *Axiomatic, enriched and motivic homotopy theory*, NATO Sci. Ser. II Math. Phys. Chem., 131, Kluwer, Dordrecht, 2004. MR 2061855, Zbl 1076.14002. math.AG/0210407.

[37] G. Vezzosi, *Derived critical loci I: Basics*, arXiv:1109.5213, 2011.

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