J. DIFFERENTIAL GEOMETRY
10 (1975) 225-238

# SMOOTHNESS OF HOROCYCLE FOLIATIONS

MORRIS W. HIRSCH & CHARLES C. PUGH

## 1. Introduction

Let $SM$ denote the unit tangent bundle of a compact $C^\infty$ Riemannian manifold $M$. Suppose that $M$ has everywhere negative sectional curvature. In [1] Anosov proved that the geodesic flow $\varphi$ on $SM$ is of a certain type, called "Anosov" by later writers, and defined below.

Associated with any Anosov flow $\varphi$ is a foliation by "strong stable manifolds"; this is called the *horocycle foliation* in the special case where $\varphi$ is the geodesic flow on $SM$ and $M$ has negative curvature. The strong unstable manifolds provide another isomorphic horocycle foliation.

The *leaves* of these foliations are as smooth as the Anosov flow $\varphi$, but Anosov showed that the *foliations* are not in general of class $C^1$, even when $\varphi$ is real analytic.[1] However, when $M$ has dimension two or the curvature is $\frac{1}{4}$-pinched, we shall prove that the horocycle foliations (and even their tangent plane fields) *are* of class $C^1$. In the course of the proof, the fact that "negative curvature $\Rightarrow$ Anosov geodesic flow" falls out naturally. Our methods in §§ 5, 6 resemble those of Anosov and Sinai [2].

This smoothness result was suggested to us by an analogous situation we encountered in [8]; there, we showed that the strong stable manifold foliation of an Anosov diffeomorphism $f$ is of class $C^1$ provided that either the strong stable manifolds have codimension one in $M$ or the spectrum of $Tf$ is "bunched". These cases are analogous to (i), (ii) below.

Thanks are due to Pat Eberlein, Rob Gardiner, Leon Green, and Joe Wolf for helpful conversations.

## 2. The smoothness theorem

Let $M$ be a $C^\infty$ compact boundaryless manifold with a $C^\infty$ Riemann structure $\mathscr{R}$. The geodesics of $\mathscr{R}$ give rise to the geodesic flow $\varphi$ on the tangent bundle $TM$ of $M$:

Received February 22, 1974. The work of the first author was partially supported by NSF grant #GP-29073, and that of the second author by NSF grant #GP-14519.

[1]It is amusing that, to mean "generic", Russian mathematicians, such as Anosov, use a word translated from Russian to English as "rough". Here is an example where roughness is likely to *be* generic.

if $v \in TM$ and $t \to \gamma_v(t)$ is the unique $\mathscr{R}$-geodesic with
$$\dot\gamma_v(0) = v, \text{ then } \varphi_t(v) = \dot\gamma_v(t) \in T_{\gamma_v(t)}M .$$

$\varphi$ is tangent to a vector field $X$, called the *geodesic spray*. Geodesics have constant speed, so $\varphi$ preserves the unit sphere bundle $SM$ of $TM$.

The geodesic flow $\varphi$ on $SM$ is *Anosov* if there is a continuous splitting $T(SM) = E^u \oplus E^\varphi \oplus E^s$, invariant under the tangent flow $T\varphi$ on $T(SM)$, such that $E^\varphi$ is the subbundle spanned by the geodesic spray $X$, $T\varphi$ exponentially expands $E^u$, and $T\varphi$ exponentially contracts $E^s$. This means that for some (hence any) Riemann structure or Finsler on $T(SM)$, there are constants $C$, $c > 0$, $\lambda > 1$ such that

$$|T\varphi_t(x)| \geq c\lambda^t |x| \qquad \text{if } x \in E^u \text{ and } t \geq 0 ,$$
$$|T\varphi_t(x)| \leq C\lambda^{-t} |x| \qquad \text{if } x \in E^s \text{ and } t \geq 0 .$$

The subbundle $E^u, E^s$ are known to be uniquely integrable. They are tangent to the horocycle foliations. Thus, to prove the horocycle foliations are of class $C^1$, it suffices to prove $E^u, E^s$ are of class $C^1$.

The *sectional curvature* of $\mathscr{R}$ at a 2-plane $\Pi \subset T_pM$ is $K_p(\Pi) =$ the Gaussian curvature of $\exp_p(\Pi)$ at $p$ relative to the inclusion-induced Riemann structure. If $K_p(\Pi) < 0$ for all $p \in M$ and all 2-planes $\Pi \subset T_pM$, then $\mathscr{R}$ is said to have *negative curvature*.

**Definition.** The curvature of $\mathscr{R}$ is *absolutely $\alpha$-pinched* iff

$$\alpha < \inf |K_p(\Pi)/K_{p'}(\Pi')| .$$

The inf is taken over all $p, p' \in M$ and all 2-planes $\Pi, \Pi'$ in $T_pM$, $T_{p'}M$. The curvature of $\mathscr{R}$ is *relatively $\alpha$-pinched* iff

$$\alpha < \inf |K_p(\Pi)/K_p(\Pi')|$$

The inf is taken over all $p \in M$ and all 2-planes $\Pi, \Pi'$ in $T_pM$.

**Smoothness Theorem.** *Let $\mathscr{R}$ be a Riemann structure on $TM$. If either*

(i) *the curvature of $\mathscr{R}$ is negative and $M$ has dimension two or*

(ii) *the curvature of $\mathscr{R}$ is negative and absolutely $\frac{1}{4}$-pinched, then the Anosov splitting $T(SM) = E^u \oplus E^\varphi \oplus E^s$ for the geodesic flow is of class $C^1$. In particular, the horocycle foliations are of class $C^1$. Under natural uniformity assumptions on the curvature, compactness of $M$ can be relaxed to completeness.*

Under assumption (i), E. Hopf [10] proved this theorem. Under assumption (ii) Leon Green [4] announced the result, but later [3] found an error in its proof.

**Question.** Is this theorem true for relative $\frac{1}{4}$-pinching? If it is, then it includes (i) and (ii) as special cases. For negative curvature on a 2-manifold is always relatively $\alpha$-pinched for all $\alpha < 1$. Originally we were sure this would "follow easily" from the $C^r$ section theorem (see below), but now we doubt it. Also we conjecture that there are many cases when the horocycle foliation is *not* of class $C^1$. Even if the curvature is $\frac{1}{4}$-pinched, we expect the horocycle foliations are hardly ever of class $C^2$. Such results might follow from methods of R. Mañé who proved a converse to the $C^r$ section theorem [13]. Anosov said the horocycle foliation is "obviously not smooth in general" [1, p. 12].

## 3. Background

In [9] we proved, with Mike Shub, a general theorem giving sufficient conditions for an invariant section of a bundle to be smooth. Let $E$ be a $C^r$ finite dimensional vector bundle over the compact $C^r$ manifold $M$. Assume $E$ has a Finsler (= continuous family of norms on fibres). Let $D$ be a disc subbundle of $E$.

**Definition.** The *minimum norm* (also called the *conorm*) of an operator $A$ is $m(A) = \inf_{|x|=1} |Ax| = \|A^{-1}\|^{-1}$.

**Definition.** An *r-fiber contraction* is a $C^r$ fiber map $F: D \to D$ covering a $C^r$ diffeomorphism $f: M \to M$ such that for some Finslers on $E$ and $TM$

$$\sup_{p \in M} k_p \alpha_p^{-j} < 1 , \qquad 0 \leq j \leq r ,$$

where $k_p$ is the Lipschitz constant of $F|D_p, D_p$ is the $D$-fiber at $p \in M$, and $\alpha_p = m(T_p f)$.

$k_p$ is the fiber contraction rate; $\alpha_p$ is the base contraction rate. The assumption $\sup k_p \alpha_p^{-j} < 1$ implies $F$ uniformly contracts the $D$-fibers (let $j=0$) and contracts $D_p$ more sharply than $f$ contracts the base at $p$ (let $j = 1$).

**$C^r$ section theorem.** *If $F$ is an r-fiber contraction of $D$, $r \geq 0$ then there is a unique F-invariant section $\sigma: M \to D$. Besides, $\sigma$ is of class $C^r$.*

This is a central result of [9].

A second concept we use from [8], [9] is that of the "graph-transform" $F_\#$. If $F: D \to D$ is a fiber map as above, then $F$ induces a natural map $F_\#: \mathrm{Sec}\,(D) \circlearrowleft$ on the sections of $D$ defined by $F_\# \sigma(x) = F \circ \sigma \circ f^{-1}(x)$. This can be re-expressed as

$$\text{image } (F_\# \sigma) = F \text{ (image } \sigma) .$$

Finally, we use the uniqueness of the hyperbolic splitting of a hyperbolic bundle automorphism. This result is part of [9, 2.9].

## 4. Proof of (i)

Let $X$ be the geodesic spray generating the geodesic flow $\varphi$. Then $T\varphi$ preserves the subbundle of $T(SM)$ orthogonal to $X$ and, since the Anosov splitting is unique,

$$E = E^u_v \oplus E^s_v = X(v)^\perp , \qquad v \in SM .$$

Since $E$ is a smooth bundle, we can approximate $E^u, E^s$ by smooth subbundles $\tilde{E}^u, \tilde{E}^s$ of $E$. Let $\mathscr{G}$ be the smooth bundle over $SM$ whose fiber at $v$ is

$$\mathscr{G}_v = \{G \in L(\tilde{E}^u_v, \tilde{E}^s_v) : \|G\| \leq 1\} .$$

Put the "max Finsler" on $T(SM)$ so that

$$|z| = \max (|x|_{\mathscr{R}}, |w|_{\mathscr{R}}, |y|_{\mathscr{R}}) ,$$

where $z = x \oplus w \oplus y \in E^u_v \oplus \operatorname{span} X(v) \oplus E^s_v$, and $|\cdot|_{\mathscr{R}}$ is length respecting $\mathscr{R}$. This is a Finsler on the base-space of $\mathscr{G}$.

Since $T\varphi_t$ preserves $E^u \oplus E^s = \tilde{E}^u \oplus \tilde{E}^s$, the $T\varphi_1$-graph transform $(T\varphi_1)_\#$ is a fiber map $\mathscr{G} \to \mathscr{G}$ covering $\varphi_1$, the time-one map of the geodesic flow. $(T\varphi_1)_\#$ is defined by

$$(T_v\varphi_1)(\text{graph } G) = \text{graph}((T\varphi_1)_\# G) , \qquad G \in \mathscr{G}_v ,$$

where graph $G = \{x + G(\tilde{x}) \in \tilde{E}^u_v \oplus \tilde{E}^s_v\}$. Let $T^u\varphi = T\varphi | E^u$, $T^s\varphi = T\varphi | E^s$. The fiber $\mathscr{G}_v$ is contracted at a rate $\doteq \|T^s_v\varphi_1\| \cdot m(T^u_v\varphi_1)^{-1}$, and the base is contracted at the rate $\doteq m(T^s_v\varphi_1)$. (To say this about the base-map we need the max Finsler.) The hypothesis of the $C^r$ section theorem ($r = 1$) is that (fiber contraction) $\times$ (base contraction)$^{-1} < 1$, and we have shown this product to be $\doteq$

$$\|T^s_v\varphi_1\| \, m(T^u_v\varphi_1)^{-1} \cdot (m(T^s_v\varphi_1))^{-1} = m(T^u_v\varphi_1)^{-1} < 1 ,$$

since $E^s$ is one-dimensional. Hence the unique $(T\varphi_1)_\#$-invariant section of $\mathscr{G}$ is of class $C^1$. The section whose graphs give $E^u$ is clearly invariant, since $E^u$ is $T\varphi_1$-invariant. Hence $E^u \in C^1$. Symmetrically, $E^s \in C^1$.

**Remarks.** If for any other reason $\operatorname{bol}(T^s_v\varphi_1) m(T^u_v\varphi_1)^{-1} < 1$, then we get $E^u \in C^1$. By bol ( ) we mean the "bolicity" which measures how nonconformal an isomorphism is:

$$\operatorname{bol}(T) = \frac{\|T\|}{m(T)} = \sup_{|x|=1=|y|} \frac{|Tx|}{|Ty|} = \|T\| \, \|T^{-1}\| .$$

## 5. Second order linear differential equations

To prove (ii) we need good norm-estimates on $T^u\varphi_t, T^s\varphi_t$; the next lemma will provide them. By $\mathscr{S}(R^n) = \mathscr{S}$ we mean symmetric linear endomorphisms of $R^n$, i.e., self adjoint operators. By $\mathscr{S}^\pm(R^n)$ we mean the convex cone of positive or negative definite ones.

**Lemma 1.** *Suppose $t \mapsto P_t$ is a continuous map $\boldsymbol{R} \to \mathscr{S}_+(\boldsymbol{R}^n)$, and $\alpha, \beta$ are positive constants with*

$$\alpha < \inf m(P_t) , \qquad \sup \|P_t\| < \beta .$$

*Let $\Phi$ be the flow on $\boldsymbol{R} \times \boldsymbol{R}^n \times \boldsymbol{R}^n$ generated by the artificially autonomous differential equation*

$$\dot{\tau} = 1,\ \dot{x} = y,\ \dot{y} = P_\tau x\ ; \qquad \tau \in \boldsymbol{R},\ x, y \in \boldsymbol{R}^n\ .$$

*Then there exists a unique $\Phi$-invariant splitting $E^u_\tau \oplus E^s_\tau = \tau \times R^{2n}$ such that $E^u_\tau, E^s_\tau$ are graphs of uniformly bounded linear maps $\boldsymbol{R}^n \to \boldsymbol{R}^n$. Besides*

$$E^u_\tau = graph\, G^u_\tau,\ G^u_\tau \in \mathscr{S}^+(\boldsymbol{R}^n)\ , \qquad \alpha^{1/2} < \langle G^u_\tau x, x\rangle < \beta^{1/2}\ ,$$
$$E^s_\tau = graph\, G^s_\tau,\ G^s_\tau \in \mathscr{S}^-(\boldsymbol{R}^n)\ , \qquad \alpha^{1/2} < \langle -G^s_\tau x, x\rangle < \beta^{1/2}$$

*for all $x \in R^n$ with $|x| = 1$. This splitting $E^u \oplus E^s$ of the product bundle $\boldsymbol{R} \times \boldsymbol{R}^{2n}$ exhibits the hyperbolicity of $\Phi$. Norms on $E^u, E^s$ can be chosen, which are uniformly equivalent to the induced norms and make*

$$e^{t\alpha^{1/2}} < m(\Phi^u_t) \leq \|\Phi^u_t\| < e^{t\beta^{1/2}}\ , \qquad e^{-t\beta^{1/2}} < m(\Phi^s_t) \leq \|\Phi^s_t\| < e^{-t\alpha^{1/2}}$$

*for all $t > 0$. If $P_\tau$ has period $\omega$, then so do $E^u$ and $E^s$.*

**Remark.** A special case of this lemma is enlightening. Consider the autonomous constant coefficient linear differential equation:

$$\dot{x} = y\ , \qquad \dot{y} = px\ , \qquad p > 0$$

arising from the second order equation $\ddot{x} = px$. This vector field on $\boldsymbol{R}^2$ generates the linear flow

$$t \to \Phi_t = \begin{bmatrix} \cosh(pt) & \dfrac{\sinh(pt)}{p} \\ p\sinh(pt) & \cosh(pt) \end{bmatrix} ,$$

which has the constant invariant splitting

$$E^u = \{(x, px) : x \in \boldsymbol{R}\}\ , \qquad E^s = \{(x, -px) : x \in \boldsymbol{R}\}\ .$$

It is a delightful coincidence that the hyperbolic trigonometric functions occur in a hyperbolic flow, and that this flow represents the tangent flow on the standard Poincaré hyperbolic plane (when $p = 1$).

*Proof of Lemma* 1. The flow $\Phi$ on $R \times \boldsymbol{R}^n \times \boldsymbol{R}^n$ naturally induces a (local) flow $\Phi_\#$ on $\boldsymbol{R} \times \mathrm{GL}(n)$ as follows. Fix $\tau \in \boldsymbol{R}$. For each $S \in \mathrm{GL}(n)$ put $\Phi_{\# t}(\tau, S) = (\tau + t, S_t)$. Here $S_t$ is the unique linear map $\boldsymbol{R}^n \to \boldsymbol{R}^n$ such that

$$(\tau + t) \times \mathrm{graph}\,(S_t) = \Phi_t(\tau \times \mathrm{graph}\, S)\ .$$

When $S = S_0$ is fixed and $t$ is small, $S_t$ is well defined.

Fix $\tau$ and consider the solution $W_t \equiv \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix}$ of

$$\dot{W} = \begin{bmatrix} 0 & I \\ P_{t+\tau} & 0 \end{bmatrix} W , \qquad W_0 = I .$$

Thus $\Phi_t | \tau \times \boldsymbol{R}^n \times \boldsymbol{R}^n = W_t$. If $t > 0$ is small, then

$$S_t = (C_t + D_t S) \circ (A_t + B_t S)^{-1} .$$

The tangent to the curve $S_t$ is

$$\frac{dS_t}{dt} = (\dot{C} + \dot{D}S_0)(A + BS_0)^{-1} - (C + DS_0)(A + BS_0)^{-1}(\dot{A} + \dot{B}S_0)(A + BS_0)^{-1} .$$

At $t = 0$ this reduces to $P_\tau - S^2$ since

$$\begin{bmatrix} \dot{A} & \dot{B} \\ \dot{C} & \dot{D} \end{bmatrix} = \begin{bmatrix} C & D \\ PA & PB \end{bmatrix} , \qquad \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} .$$

Thus the flow $\Phi_\#$ is tangent to the vector field (on $\boldsymbol{R} \times \mathrm{GL}(n)$) given by $(\tau, S) \mapsto (1, P_\tau - S^2)$. (Note that its integral curves are solutions to the Ricatti equation $\dot{S} = P - S^2$.) Since this vector field is tangent to $\boldsymbol{R} \times \mathscr{S}(\boldsymbol{R}^n)$ by inspection, the flow $\Phi_\#$ leaves $\boldsymbol{R} \times \mathscr{S}(\boldsymbol{R}^n)$ invariant.

We claim that all points of the boundary $\partial(\boldsymbol{R} \times \mathscr{S}_{\alpha\beta})$ are strict ingress points for $\Phi_\#$ where

$$\mathscr{S}_{\alpha\beta} = \{S \in \mathscr{S} : \alpha^{1/2} \leq \langle Sx, x \rangle \leq \beta^{1/2} \text{ for all } x \in \boldsymbol{R}^n,\ |x| = 1\} .$$

A boundary point $p$ of a region $U$ is a *strict ingress point* for a local flow $\varphi$ if $\varphi_t p \in \mathrm{Int}\,(U)$ for all small $t > 0$. This is an idea due to Ważewski.

For $x \in \boldsymbol{R}^n$ and $S \in \mathscr{S}$ we have

$$\begin{aligned} \dot{x}_t &= y_t , & x_0 &= x \in \boldsymbol{R}^n , \\ \dot{y}_t &= P_{t+\tau} x_t , & y_0 &= S_0 x_0 , \end{aligned}$$

and compute

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \frac{\langle S_t x_t, x_t \rangle}{\langle x_t, x_t \rangle} &= \{[\langle \dot{S}_0 x_0 + S_0 \dot{x}_0, x_0 \rangle + \langle S_0 x_0, \dot{x}_0 \rangle]\langle x_0, x_0 \rangle \\ &\qquad - \langle S_0 x_0, x_0 \rangle [2\langle x_0, \dot{x}_0 \rangle]\} / \langle x_0, x_0 \rangle^2 \\ &= [\langle (P_\tau - S^2)x + S(Sx), x \rangle + \langle Sx, Sx \rangle - 2\langle Sx, x \rangle^2] / |x|^4 \\ &= [\langle P_\tau x, x \rangle + \langle Sx, Sx \rangle - 2\langle Sx, x \rangle^2] / |x|^4 . \end{aligned} \tag{1}$$

For small $t$, $x \mapsto x_t$ defines an embedding of the unit sphere $S^{n-1}$ of $\boldsymbol{R}^n$ into $\boldsymbol{R}^n$ which is near the inclusion. Thus the mapping $S^{n-1} \circlearrowleft$

$$x \longmapsto x_t / \langle x_t, x_t \rangle^{1/2}$$

is near the identity; therefore it is surjective. This implies that

$$\inf_{|x|=1} \langle S_t x, x \rangle = \inf_{|x|=1} \frac{\langle S_t x_t, x_t \rangle}{\langle x_t, x_t \rangle} \tag{2}$$

for small $t$.

Choose $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that

$$\alpha < \alpha_1 < \alpha_2 \leq \inf_\tau m(P_\tau) \,, \qquad \sup \|P_\tau\| \leq \beta_2 < \beta_1 < \beta \,,$$

$$\alpha_1 - \alpha < \alpha_2 - \alpha_1 \,, \qquad \beta - \beta_1 < \beta_1 - \beta_2 \,.$$

Since $P_\tau$ is symmetric, $\langle P_\tau x, x \rangle \geq \alpha_2 |x|^2$.

Suppose $S \in \partial_{\alpha\beta}$ and consider the sets

$$X_\alpha(S) = \{x \in S^{n-1} \colon \alpha^{1/2} \leq \langle Sx, x \rangle < \alpha_1^{1/2}\} \,,$$
$$X_\beta(S) = \{x \in S^{n-1} \colon \beta_1^{1/2} < \langle Sx, x \rangle \leq \beta^{1/2}\} \,.$$

For each $x \in X_\alpha(S)$ we have from (1)

$$\frac{d}{dt}\bigg|_{t=0} \frac{\langle S_t x_t, x_t \rangle}{\langle x_t, x_t \rangle} = \langle P_\tau x, x \rangle + \langle Sx, Sx \rangle - 2\langle Sx, x \rangle^2 \,.$$

It follows from (2) that if $x \in X_\alpha(S)$, then

$$\langle S_t x, x \rangle > \alpha^{1/2} \qquad \text{for all small } t > 0 \,. \tag{3}$$

But if $x \in S^{n-1} - X_\alpha(S)$ and $t$ is small, then

$$\langle S_t x_t, x_t \rangle \doteq \langle Sx, x \rangle > \alpha^{1/2}$$

by continuity. Thus (3) holds for all $x \in S^{n-1}$, that is,

$$\inf_{|x|=1} \langle S_t x, x \rangle > \alpha^{1/2} \qquad \text{for all small } t > 0 \,.$$

The same reasoning proves that also

$$\sup_{|x|=1} \langle S_t x, x \rangle < \beta^{1/2} \qquad \text{for all small } t > 0 \,.$$

This shows that $\tau \times S$ is a strict ingress point of $\partial(\boldsymbol{R} \times \mathcal{S}_{\alpha\beta})$ for the local flow $\Phi_\#$.

The set $\mathscr{S}_{\alpha\beta}$ is a compact convex subset of the (finite dimensional) linear space $\mathscr{S}$. All the points of its boundary were shown to be strict ingress points. Since $\partial(R \times \mathscr{S}_{\alpha\beta})$ is not a retract of $\mathscr{S}_{\alpha\beta}$, Ważewski's Principle [6, p. 279] says there must be a trajectory of $\Phi_\#$ remaining in $\boldsymbol{R} \times \mathscr{S}$ for all time. Let $\tau \mapsto \tau \times G^u_\tau$ be such a trajectory, and set $E^u_\tau = \operatorname{graph} G^u_\tau$, $\tau \in \boldsymbol{R}$. Clearly $G^u_\tau$ is interior to $\mathscr{S}_{\alpha\beta}$, and $\Phi_t(E^u_\tau) = E^u_{t+\tau}$.

Let $\mathscr{S}^-_{\alpha\beta} = \{S \in \mathscr{S} : \alpha^{1/2} \leq \langle -Sx, x\rangle \leq \beta^{1/2}$ for all $x \in \boldsymbol{R}^n$, $|x| = 1\}$. Then all points of $\partial(\boldsymbol{R} \times \mathscr{S}^-_{\alpha\beta})$ are strict egress points. This can be seen by some reasoning similar to the above. Again by Ważewski's Principle, there is a $\Phi_\#$-trajectory remaining in $\mathscr{S}^-_{\alpha\beta}$ for all time. This gives $G^s_\tau, E^s_\tau$ as claimed and completes the existence part of Lemma 1.

Uniqueness of $E^u, E^s$ follows from hyperbolicity of $\Phi$ and Hirsch-Pugh-Shub [9, 2.9]. To prove hyperbolicity and the asserted estimates on its strength, we introduce the new inner product in $\boldsymbol{R}^n \times \boldsymbol{R}^n$ by setting

$$\langle z^1, z^2\rangle_* = \langle x^1, x^2\rangle,\ z^j = (x^j, y^j) \in \boldsymbol{R}^n \times \boldsymbol{R}^n\ ; \qquad j = 1, 2\ .$$

By restriction we get new inner products on each $E^u_\tau, E^s_\tau$ ($\tau \in \boldsymbol{R}$). This makes $x \mapsto (x, G^u x)$, $x \mapsto (x, G^s x)$ isometries of $\boldsymbol{R}^n$ onto $E^u_\tau, E^s_\tau$.

Denote $\Phi_\tau(t, z)$ by $(\tau + t, z_t)$ and put $z_t = (x_t, y_t) \in \boldsymbol{R}^n \times \boldsymbol{R}^n$. Then

$$\dot{x}_t = y_t\ , \qquad \dot{y}_t = P_{\tau+t} x_t\ ,$$

and so

$$\begin{aligned} \frac{d}{dt}\langle z_t, z_t\rangle_* &= \frac{d}{dt}\langle x_t, x_t\rangle = 2\langle x_t, \dot{x}_t\rangle \\ &= 2\langle x_t, y_t\rangle = 2\langle x_t, G^u_{\tau+t}(x_t)\rangle \end{aligned}$$

by invariance of $E^u_\tau$. Since $G^u_\tau \in \mathscr{S}^+_{\alpha\beta}$, this last quantity lies between $2\alpha^{1/2}$ and $2\beta^{1/2}$. Hence $\langle z_t, z_t\rangle_*$ satisfies the differential inequality

$$2\alpha^{1/2} < \frac{d}{dt}\langle z_t, z_t\rangle < 2\beta^{1/2}\ , \qquad t > 0\ ,$$

while

$$\langle z_0, z_0\rangle_* = |z|^2_*\ , \qquad 0 \neq z \in E^u_t\ .$$

From Hartman [6, p. 24] we conclude that

$$e^{2t\alpha^{1/2}}|z|^2_* < \langle z_t, z_t\rangle_* < e^{2t\beta^{1/2}}|z|^2_*$$

for all $t > 0$. Taking square roots gives the growth estimate on $\Phi^u_t$ in Lemma 1. Similarly, if $z \in E^s_\tau$ then

$$\frac{d}{dt}\langle z_t, z_t\rangle_* = 2\langle x_t, G^s_{\tau+t}(x_t)\rangle ,$$

which lies between $-2\alpha^{1/2}$ and $-2\beta^{1/2}$ since $G^s_\tau \in \mathscr{S}^-_{\alpha\beta}$. This gives the growth estimate on $\Phi^s_t$ in Lemma 1.

As remarked before, hyperbolicity of $\Phi$ implies the uniqueness of $E^u, E^s$. Suppose $P_\tau$ has period $\omega$. Set $F^u_\tau = E^u_{\tau+\omega}$, $F^s_\tau = E^s_{\tau+\omega}$. Then $F^u \oplus F^s$ is a $\Phi$-invariant splitting of $\boldsymbol{R} \times \boldsymbol{R}^{2n}$ since $\Phi_t(\tau + \omega, z) \equiv \Phi_t(\tau, z) + (\omega, 0)$. Clearly $F^u \oplus F^s$ also exhibits the hyperbolicity of $\Phi$ so by [9, 2.9] $E^u \equiv F^u$, $E^s \equiv F^s$, and $\omega$-periodicity of $E^u, E^s$ is proved. This completes the proof of Lemma 1.

**Remark.** An alternative proof that $E^u, E^s$ exist can be devised by showing that the flow $\Phi_\#$ contracts $\mathscr{S}^+_{\alpha\beta}$, instead of using Ważewski's principle. Contractiveness of $\Phi_\#$ on $\mathscr{S}^+_{\alpha\beta}$ follows from considering the first variation equation of $\dot{S} = P - S^2$, along a $\Phi$-trajectory $S_t$, namely, $\dot{V} = -(VS_t + S_tV)$. While $S_t$ is in $\mathscr{S}_{\alpha\beta}$, it is a positive operator so the above $\dot{V}$ is "negative", showing that $\Phi_{\#t}$ contracts infinitesimally, $t > 0$. Contractiveness of $\Phi_{\#t}$ in the large follows by the mean value theorem since $\mathscr{S}_{\alpha\beta}$ is convex. The details of this argument involve use of the inner product

$$\langle A, B\rangle = \text{trace}\,(A^tB)$$

on $L(\boldsymbol{R}^n, \boldsymbol{R}^n)$ and its corresponding norm. This is not the operator norm on $L(\boldsymbol{R}^n, \boldsymbol{R}^n)$, and it does not have an analogue for an infinite dimensional real Hilbert space $E$. The estimates in the proof of Lemma 1 remain valid for $E$, but Ważewski's Principle fails because $\partial\mathscr{S}_{\alpha\beta}$ probably *is* a retract of $\mathscr{S}_{\alpha\beta}$; compare Klee [11]. Thus the generalization of Lemma 1 to Hilbert space remains unproved by us.

## 6. Fermi coordinates

The next lemma concerns a special coordinate system along a geodesic, called a "Fermi chart". For the geodesic flow, the bundle-chart over a Fermi chart serves the same purpose as a flowbox does for a flow. Let $\mathscr{R}$ be a smooth Riemann structure on $TM$, and let $v \in S_pM$ be given, $p \in M$. Let $X$ be the geodesic spray of $\mathscr{R}$. Let $e_1, \cdots, e_m$ be an orthonormal basis for $T_pM$ with $v = e_1$, and let $\gamma$ be the geodesic initially tangent to $v$. Parallel translation down $\gamma$ gives smooth orthonormal vector fields $e_1(t), \cdots, e_m(t)$ on $\gamma$ such that $e_1(t) \equiv \dot{\gamma}(t)$. Since exp is tangent to the identity,

$$f_v(\sum a_ie_i) = \exp_{\gamma(a_1)}\Big(\sum_{i\ge 2} a_ie_i(t)\Big)$$

defines an immersion $f_v$, called the *Fermi chart* associated with $\mathscr{R}$ and $v \in S_pM$. The domain of $f_v$ includes

$$\mathscr{D}_v = \{\tau v + v' \in T_pM : v' \perp v, |v'| \leq c, \ \tau \in \boldsymbol{R}\} ,$$

where $c$ is some positive constant. $f_v$ sends span $(v)$ isometrically onto $\gamma$. Since $f_v$ is an immersion, $\mathscr{R}$ pulls back to a Riemann structure $f_v^*\mathscr{R}$ on $T\mathscr{D}_v = \mathscr{D}_v \times T_pM$. Thus $f_v^*\mathscr{R}$ is $\mathscr{R}$ expressed in the $f_v$-chart. Let $g_{ab}, \Gamma^\sigma_{\alpha\beta}$ and $R^i_{kjl}$ be the components of $f_v^*\mathscr{R}$, its Christoffel symbols and its Riemannian curvature tensor in the $f_v$-chart.

**Lemma 2.** *The Fermi chart $f_v$ has the following properties at all points of* span $(v)$:

(0-*th order*) $\qquad g_{ab} = \delta_{ab}$ ,

(1*st order*) $\qquad \Gamma^\sigma_{\alpha\beta} = 0$ ,

(2*nd order*) $\qquad R^1_{k1l} = -\dfrac{1}{2} \dfrac{\partial^2 g_{11}}{\partial x^k \partial x^l} = \dfrac{\partial \Gamma^k_{11}}{\partial x^l}$ .

*Proof.* The 0-th and 1st order assertions are proved in Gromoll-Klingenberg-Mayer [5]. In any chart

$$\Gamma^\sigma_{\alpha\beta} = \frac{1}{2} \sum_r g^{\sigma r}(\partial_\alpha \beta_{r\beta} + \partial_\beta g_{r\alpha} - \partial_r g_{\alpha\beta}) ,$$

where $(g^{\sigma r})$ is the matrix inverse to $(g_{ab})$. By $\partial_\alpha$ etc. we mean $\partial/\partial x^\alpha$ where $x^1, \cdots, x^m$ are the coordinates in the chart. Juggling indices and summing as in Weatherburn [15] we get

$$\partial_\sigma g_{\alpha\beta} = 0 , \qquad 1 \leq \alpha, \beta, \sigma \leq m$$

at any point of a chart where $\Gamma = 0$ and $(g_{ab}) = (\delta_{ab})$. This means the map

$$x \longmapsto (g_{ab}(x)) \in \{\text{real } m \times m \text{ matrices}\}$$

has zero derivative at all points of span $(v)$ in the Fermi chart. By the chain rule the same is true of

$$x \longmapsto (g_{ab}(x))^{-1} = (g^{\sigma r}(x)) .$$

Thus all first partials of $g_{ab}$ and $g^{\sigma r}$ vanish along span $(v)$. From this constancy we conclude $\partial_1\partial_l g_{ab} = \partial_1\partial_l g^{\sigma r} = 0$ along span $(v) = x^1$-axis.

In any chart the components $R^i{}_{kjl}$ are related to the $\Gamma^\sigma_{\alpha\beta}$ by

$$R^i{}_{kjl} = \partial_j \Gamma^i_{kl} - \partial_l \Gamma^i_{kj} + \sum_r (\Gamma^i_{rj}\Gamma^r_{kl} - \Gamma^i_{rl}\Gamma^r_{kj})$$

(see Hicks [7]), so in the Fermi chart along span $(v)$

$$\begin{aligned} R^1{}_{k1l} &= \partial_1 \Gamma^1_{kl} - \partial_l \Gamma^1_{k1} \\ &= \tfrac{1}{2} \sum_r \partial_1(g^{1r})(\partial_k g_{rl} + \partial_l g_{rk} - \partial_r g_{kl}) \\ &\quad + \tfrac{1}{2} \sum_r g^{1r}(\partial_1\partial_k g_{rl} + \partial_1\partial_l g_{rk} - \partial_1\partial_r g_{kl}) \\ &\quad - \tfrac{1}{2} \sum_r \partial_l(g^{1r})(\partial_k g_{r1} + \partial_1 g_{rk} - \partial_r g_{1k}) \\ &\quad - \tfrac{1}{2} \sum_r g^{1r}(\partial_l\partial_k g_{r1} + \partial_l\partial_1 g_{rk} - \partial_l\partial_r g_{1k}) \\ &= -\tfrac{1}{2}(\partial_l\partial_k g_{11} + \partial_l\partial_1 g_{1k} - \partial_l\partial_1 g_{1k}) = -\frac{1}{2}\frac{\partial^2 g_{11}}{\partial x^l \partial x^k} . \end{aligned}$$

For along span $(v)$: $\partial_1(g^{1r})$ vanishes, $\partial_1\partial_k g_{rl}$ etc. vanish, $\partial_l(g^{1r})$ vanishes, and $g^{1r} = \delta^{1r}$. For the same reasons

$$\begin{aligned} \frac{\partial \Gamma^k_{11}}{\partial x^l} &= \tfrac{1}{2} \sum_r \partial_l(g^{kr})(\partial_1 g_{r1} + \partial_1 g_{1r} - \partial_r g_{11}) \\ &\quad + \tfrac{1}{2} \sum_r g^{kr}(\partial_l\partial_1 g_{r1} + \partial_l\partial_1 g_{1r} - \partial_l\partial_r g_{11}) \\ &= -\tfrac{1}{2}\partial_l\partial_k g_{11} = -\frac{1}{2}\frac{\partial^2 g_{11}}{\partial x^l \partial x^k} \end{aligned}$$

along span $(v)$. This completes the proof of Lemma 2.

## 7. Proof of (ii)

Let $\mathscr{R}$ be the given Riemann structure on $TM$. Let $v \in S_pM$, $p \in M$, and choose an orthonormal basis of $T_pM$, $e_1, \cdots, e_m$ with $e_1 = v$. Let $f_v$ be the Fermi chart determined by $e_1, \cdots, e_m$, and let $F_v$ be the bundle chart of $TM$ tangent to $f_v$:

$$\begin{aligned} \mathscr{D}_v \times T_pM &\xrightarrow{F_v} TM \\ (x, \xi) &\longmapsto T_x f_v(\xi) \in T_{f_v x}M . \end{aligned}$$

$\mathscr{D}_v$ is the domain of $f_v$. The geodesic spray $X$ is represented in any $TM$-bundle-chart for $TM$ as the first order ordinary differential equation

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \xi \\ -\Gamma(x)(\xi, \xi) \end{bmatrix} , \tag{1}$$

where $\Gamma(x) \colon T_pM \times T_pM \to T_pM$ is the symmetric bilinear map such that

$$\Gamma(x)(e_i, e_j) = \sum_k \Gamma^k_{ij}(x) e_k , \qquad x \in \mathscr{D}_v .$$

The $\Gamma^k_{ij}$ are the Christoffel symbols of $\mathscr{R}$ expressed in the $f_v$-chart.

The geodesic flow $\varphi$ of $\mathscr{R}$, represented in the $F_v$-chart, is the solution of (1). The assertion of the smoothness theorem concerns the tangent flow $T\varphi$ on

$T(TM)$. When represented in the $TF_v$-chart, $T\varphi$ is the solution of the first variation equation of (1):

$$(2) \qquad \dot{W} = D(F_v^*X)_{w_t}W\ , \qquad W(0) = I$$

for $w_t = F_v^{-1} \circ \varphi_t \circ F_v(w)$, $w \in \mathscr{D}_v \times T_pM$. By $F_v^*X$ we mean the vector field $X \circ TF_v^{-1}$ on $\mathscr{D}_v \times T_pM$. At $F_v^{-1}(\varphi_t v) = (tv, e_1)$ we calculate

$$D(F_v^*X)_{(tv,e_1)} = D\begin{pmatrix} \xi \\ -\Gamma(x)(\xi,\xi) \end{pmatrix}_{(tv,e_1)} = \begin{bmatrix} 0 & I \\ -\dfrac{\partial \Gamma}{\partial x}(\ ,\xi,\xi) & -2\Gamma(x)(\ ,\xi) \end{bmatrix}_{(tv,e_1)}$$

$$= \begin{bmatrix} 0 & I \\ \dfrac{1}{2}\,\dfrac{\partial^2 g_{11}(x)}{\partial x^l \partial x^k} & 0 \end{bmatrix}_{x=tv} = \begin{bmatrix} 0 & I \\ -R^1_{k1l}(tv) & 0 \end{bmatrix}$$

by Lemma 2 since

$$\left(\frac{\partial \Gamma}{\partial x}(e_l, \xi, \xi)\right)_{(x,\xi)=(tv,e_1)} = \sum_k \left(\frac{\partial \Gamma^k_{11}(x)}{\partial x^l}\right)_{x=tv} e_k\ .$$

(The $R^i{}_{kjl}$ are the components of the curvature tensor in the $f_v$-chart.) Thus, along $F_v^{-1}(\varphi_t v)$, (2) becomes

$$(3) \qquad \dot{W} = \begin{bmatrix} 0 & I \\ -R^1{}_{k1l}(tv) & 0 \end{bmatrix} W\ , \qquad W(0) = I\ .$$

In general, $R^i{}_{kjl}$ is skew-symmetric in $jl$ and $R^i{}_{ijl} = 0$, so we see that

$$(R^1{}_{k1l}) = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & R^1{}_{k1l} & \\ 0 & & \end{bmatrix}, \qquad 2 \leq k,\ l \leq m\ .$$

These extra zeros indicate that $T\varphi$ preserves $X$ (as does any tangent flow) and that $T\varphi$ preserves $X^\perp$ (as does any tangent geodesic flow). Let $E = X^\perp \cap T(SM)$. Then $T\varphi$ preserves $E$ and $\Phi_t = T_v\varphi_t | E$, expressed in the $F_v$-chart, solves

$$\dot{\Phi} = \begin{bmatrix} 0 & I \\ P_t & 0 \end{bmatrix} \Phi\ , \qquad \Phi_0 = I\ ,$$

where

$$P_t = [-R^1{}_{k1l}(tv)]_{2 \leq k, l \leq m}\ .$$

$\Phi$ is a linear flow on $\operatorname{span}(v) \times H_v \times V_v \approx \boldsymbol{R} \times \boldsymbol{R}^{m-1} \times \boldsymbol{R}^{m-1}$ where $H_v = \{(x, 0) \in T_pM \times T_pM, x \perp v\}$, $V_v = \{(0, \xi) \in T_pM \times T_pM : \xi \perp v\}$.

In any chart at a point where the coordinates are orthonormal, the sectional curvature of a pair of vectors $Y, Z \in T_pM$ is

$$K_p(Y,Z) = \langle R(Y,Z)Z, Y\rangle , \qquad Y = \sum y_i e_i ,$$
$$= \sum_{i,j,k,l} R^i{}_{kjl} y_i y_j z_k z_l , \qquad Z = \sum z_i e_i ,$$

and thus finally using the negative curvature hypothesis we have

$$(4) \qquad \langle P_t Z, Z\rangle = -\sum_{k,l} R^1{}_{k1l} z_k z_l = -K(e_1, Z) > 0 ,$$

where $R^1{}_{k1l} = R^1{}_{k1l}(tv)$.

Choose constants $K > k > 0$ such that every sectional curvature lies strictly between $-K^2$ and $-k^2$. By (4), in applying Lemma 1 we can take $\alpha = k$, $\beta = K$.

By Lemma 1, $\Phi$ is hyperbolic and the strength of its hyperbolicity can be estimated. Using the $F_v$-chart we get a well defined $T\varphi$-invariant splitting $E^u \oplus E^s$ of $E$ over the $\varphi$-orbit of $v$. (If $t \mapsto \varphi_t v$ is periodic in $t$, then $P_t$ is periodic and, by Lemma 1, so is the $\Phi$-invariant-splitting. Hence $E^u \oplus E^s$ *is* well defined.) Choose one $v$ on each $\varphi$-orbit and make the preceding construction. This gives a well defined $T\varphi$-invariant splitting of $E$ over all $SM$.

Since the Finsler on span $(v) \times H_v \times V_v$ adapted to $\Phi$ is uniformly equivalent to the standard Finsler, and since $f_v$ is a Fermi-chart, we see that the estimates

$$e^{tk} < m(\Phi^u_t) \leq \|\Phi^u_t\| < e^{tK} , \qquad e^{-tK} < m(\Phi^s_t) \leq \|\Phi^s_t\| < e^{-tk} ,$$

which are valid for all $t > 0$—when the adapted Finsler is used—imply

$$(5) \quad e^{tk} < m(T^u_v\varphi_t) \leq \|T^u_v\varphi_t\| < e^{tK} , \qquad e^{-tK} < m(T^s_v\varphi_t) \leq \|T^s_v\varphi_t\| < e^{-tk}$$

respecting the $\mathscr{R}$-norms for all *large t*. By $T^u_v\varphi_t$, $T^s_v\varphi_t$ we mean $T\varphi_t \,|\, E^u_v$, $T\varphi_t \,|\, E^s_v$. Thus, respecting the fixed $\mathscr{R}$-norms, $T\varphi \,|\, E$ is a linear uniformly hyperbolic flow and so, by [9, (2.9)], $E^u$ and $E^s$ are automatically continuous and independent of which $v$ was chosen on each $\varphi$-orbit. *Hence $\varphi$ is Anosov.*

By (5) we get

$$\operatorname{bol}(T^u_v\varphi_t) < e^{t(K-k)} , \qquad m(T^u\varphi_t) > e^{tk} ,$$
$$\operatorname{bol}(T^s_v\varphi_t) < e^{t(K-k)} , \qquad \|T^s_v\varphi_t\| < e^{-tk}$$

for all large $t$. Now return to the proof of (ii). Since $E$ is a smooth bundle we can approximate $E^u, E^s$ by smooth subbundles $\tilde{E}^u, \tilde{E}^s$ of $E$. Then we can consider, for a large fixed $t$, the $\mathscr{G}$-map $(T\varphi_t)_\#: \mathscr{G} \to \mathscr{G}$ where $\mathscr{G}_v = \{G \in L(\tilde{E}^u_v, \tilde{E}^s_v) : \|G\| \leq 1\}$. As in the proof of (i), $(T\varphi_t)_\#$ is a fiber contraction with

$$\begin{aligned}&(\text{fiber contraction})\cdot(\text{base contraction})^{-1}\\ &\doteq (\|T^s_v\varphi_t\|(m(T^u\varphi_t))^{-1})(m(T^s_v\varphi_t))^{-1}\end{aligned}$$

$$= \text{bol}\,(T^s_v\varphi_t)/m(T^u\varphi_t) < e^{t(K-k)}/e^{tk} = e^{t(K-2k)} .$$

Since the curvature is $\frac{1}{4}$-pinched, we have $K - 2k < 0$ and the hypothesis of the $C^r$ section theorem is satisfied; therefore the unique $(T\varphi_t)_\#$-invariant section of $\mathscr{G}$ is of class $C^1$. Since $E^u$ gives such a section, $E^u$ is of class $C^1$. Working with the reverse flow and $\mathscr{G}^-_v = \{G \in L(\tilde{E}^s_v, \tilde{E}^u_v) : \|G\| \leq 1\}$, (5) gives the same result for $E^s$. This completes the proof of (ii).

**Remarks on the smoothness of $\mathscr{R}$.** For simplicity, we assumed the Riemann structure $\mathscr{R}$ was $C^\infty$. However, the above constructions work equally naturally when $\mathscr{R}$ is $C^4$, the smoothness theorem holds when $\mathscr{R}$ is $C^3$, and $\varphi$ is Anosov when $\mathscr{R}$ is $C^2$ with negative curvature. This can be seen by $C^2$-approximating $\mathscr{R}$ by a $C^\infty$ Riemann structure $\tilde{\mathscr{R}}$ and using the uniformities in the hyperbolicity estimates. Alternatively, the Fermi chart could be smoothed as were flow boxes in Pugh-Robinson [14].

**Standard question.** If the geodesic flow $\varphi$ of $\mathscr{R}$ is Anosov, then does $M$ admit a Riemann structure $\mathscr{R}'$ with negative curvature? Wilhelm Klingenberg showed in [12], [16] that all known topological properties of $M$ which are implied by negative curvature are equally implied by $\varphi$ being Anosov.

## References


[1] D. Anosov, *Geodesic flows on closed Riemann manifolds with negative curvature,* Trudy Mat. Inst. Steklov **90** (1967), (Russian); Proc. Steklov Inst. Math., Amer. Math. Soc., 1969, (English translation).

[2] D. Anosov & Y. Sinai, *Some smooth ergodic systems,* Upseki Mat. Nauk **22** (1967) 107–172=Russian Math. Surveys **22** (1967) 103–167.

[3] L. Green, *The generalized geodesic flow,* Duke Math. J. **41** (1974) 115–126.

[4] ——, *Group-like decompositions of Riemann bundles,* Proc. Conf. in Topological Dynamics, Lecture Notes in Math. Vol. 318, Springer, Berlin, 1973, 126–139.

[5] D. Gromoll, W. Klingenberg & W. Mayer, *Riemannsche Geometrie im Grossen,* Lecture Notes in Math. Vol. 55, Springer, Berlin, 1968, 113–114.

[6] P. Hartman, *Ordinary differential equations,* John Wiley and Sons, New York, 1964.

[7] N. Hicks, *Notes on differential geometry,* van Nostrand, Princeton, 1965, 63.

[8] M. Hirsch & C. Pugh, *Stable manifolds and hyperbolic sets,* Proc. Sympos. Pure Math. Vol. 14, Amer. Math. Soc., 1970, 133–163.

[9] M. Hirsch, C. Pugh & M. Shub, *Invariant Manifolds,* Lecture Notes in Math., Springer, Berlin.

[10] E. Hopf, *Statistik der Lösungen geodätischer Probleme vom unstabilen Typus* **II**, Math. Ann. **117** (1940) 590–608.

[11] V. Klee, *Convex bodies and periodic homeomorphism in Hilbert space,* Trans. Amer. Math. Soc. **74** (1953) 10–43.

[12] W. Klingenberg, *Riemannian manifolds with geodesic flow of Anosov type,* Ann. of Math. (2) **99** (1974) 1–13.

[13] R. Mañé, *Persistent manifolds are normally hyperbolic,* IMPA Rio de Janeiro, preprint.

[14] C. Pugh & R. C. Robinson, *The* $C^1$ *closing lemma including Hamiltonians,* to appear in Pacific J. Math.

[15] C. Weatherburn, *An introduction to Riemannian geometry and the tensor calculus,* Cambridge University Press, Cambridge, 1938.

[16] M. Peixoto (Ed.), *Dynamical systems,* Academic Press, New York, 1973, 155–164.


UNIVERSITY OF CALIFORNIA, BERKELEY