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MANIFOLDS WITH HOLONOMY GROUP $Z_2 \oplus Z_2$ AND FIRST BETTI NUMBER ZERO

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In 1957 E. Calabi [1] announced an approach to the classification of flat compact Riemannian manifolds wherein he proved that any n-dimensional flat compact Riemannian manifold with first Betti number q can be constructed from a torus of dimension q and a flat compact Riemannian manifold of dimension $n - q$. The manifolds which are primitive in this approach are therefore those with first Betti number zero. In this paper we construct a rather large family of flat compact Riemannian manifolds with holonomy group isomorphic to $Z_2 \oplus Z_2$, all of which have first Betti number zero. The approach is algebraic, via well-known theorems concerning the classification of flat compact Riemannian manifolds by their fundamental groups.

1. Preliminaries

Let G be an abstract group. We will call G a *space form group of dimension* p if

i. G is torsion-free,
ii. G contains a normal subgroup N_G which is maximal abelian in G,
iii. N_G is free abelian of rank p,
iv. G/N_G is finite.

We will need the following theorems which are all paraphrased from [2, Chapter 3].

Theorem. *An abstract group G is the fundamental group of a flat compact Riemannian manifold of dimension p if and only if G is a space-form group of dimension p. If G is a space-form group, then N_G is uniquely determined and the holonomy group of the manifold is isomorphic to G/N_G.*

Theorem. *Two flat compact Riemannian manifolds are affinely equivalent if and only if they have isomorphic fundamental groups.*

We fix the following notation. Let n be a positive integer. A_n will denote a fixed set of cardinality $4n$, i.e., $A_n = \{a_1, \cdots, a_{4n}\}$. P_n will denote the group of permutations of the set A_n. (I.e., P_n is isomorphic to the symmetric group on $4n$ symbols.) T_n will denote the free abelian group generated by the set A_n. Finally, E_n will denote the semi-direct product of T_n with P_n via the obvious action of P_n on T_n.

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Our object is to identity certain subgroups of E_n as space-form groups. We will make use of the following subset of T_n:

$$V_n = \{u_i \mid u = r, s, \text{ or } t,\ 0 \leq i \leq n-1\}\ ,$$

where

$$r_i = a_{4i+1} + a_{4i+2} - a_{4i+3} - a_{4i+4}\ ,$$
$$s_i = a_{4i+1} - a_{4i+2} + a_{4i+3} - a_{4i+4}\ ,$$
$$t_i = a_{4i+1} - a_{4i+2} - a_{4i+3} + a_{4i+4}\ .$$

A trivial orthogonality argument yields the following lemma.

Lemma. *V_n is a linearly independent subset of T_n.*

2. The construction

Define X and Y in P_n to be the permutations:

$$X(a_i) = \begin{cases} a_{i+1}\ , & i \text{ odd}\ , \\ a_{i-1}\ , & i \text{ even}\ ; \end{cases}$$
$$Y(a_i) = \begin{cases} a_{i+2}\ , & i \equiv 1 \text{ or } 2 \pmod 4\ , \\ a_{i-2}\ , & i \equiv 0 \text{ or } 3 \pmod 4\ . \end{cases}$$

The subgroup of P_n generated by X and Y is easily seen to be isomorphic to $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$.

Since $E_n = T_n^* P_n$, we can define elements of E_n:

$$x = (a_1 - a_3, X)\ ; \qquad y = (a_1 - a_4, Y)\ .$$

Notice that $x^2 = r_0$ and $y^2 = s_0$.

Let U be a subset of V_n, such that $\{r_0, s_0, t_0\} \subseteq U$. Denote the cardinality of U by P_U, and define G_U to be the subgroup of E_n generated by x, y, and U, where we identify T_n with its image in E_n.

Proposition 1. *G_U is a space-form group of dimension p_U with holonomy group isomorphic to $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$ and first Betti number zero.*

Proof. The following relations can be verified by simple, direct computation (all operations are written multiplicatively):

1.1) $x^2 = r_0,\ y^2 = s_0,\ (yx)^2 = t_0^{-1}$;

1.2) $xy = yxr_0^{-1}s_0t_0$;

1.3) $r_ix = xr_i,\ s_ix = xs_i^{-1},\ t_ix = xt_i^{-1}$,
$r_iy = yr_i^{-1},\ s_iy = ys_i,\ t_iy = yt_i^{-1}$, for $0 \leq i \leq n-1$.

Let H_U be the subgroup of G_U generated by U. By the lemma, H_U is free abelian of rank p_U, and H_U is normal in G_U by 1.3). If z is in G_U, it follows easily from 1.1), 1.2), and 1.3) that z can be written uniquely in the form:

1.4) $z = wh$, with $w \in \{1, x, y, yx\}$ and $h \in H_U$.

Hence G_U/H_U is isomorphic to $Z_2 \oplus Z_2$.

From 1.3) and 1.4) it follows that H_U is its own centralizer in G_U and, a fortiori, maximal abelian in G_U.

$H_1(G_U; Z) \cong G_U/[G_U, G_U]$, but 1.1) and 1.3) imply that $G_U/[G_U, G_U] \cong (Z_4)^2 + (Z_2)^{p_U - 3}$. Hence $H_1(G_U; Z)$ is a finite group so that G_U has first Betti number zero. The proposition will therefore be established if we can show that G_U is torsion-free.

Suppose the contrary. Then there exists z in G_U, $z \neq 1$, and $n \geq 2$ such that $z^n = 1$. Using 1.4) and the fact that H_U is torsion-free, we can assume that $z = wh$ where h is in H_U and $w = x, y$, or yx. Then $z^2 = w^2(w^{-1}hw)h$ which is in H_U. Since H_U is torsion-free, it follows that n must be even and that in fact $z^2 = 1$. Let u_0 denote w^2. Then u_0 is in U, and the elements of U form a basis for H_U. Let q be the exponent of u_0 when h is expressed as a word in the elements of U. Using 1.3) one can easily see that $z^2 = whwh = w^2 u_0^{2q} h' = u_0^{2q+1} h'$, where h' is in the span of $U - \{u_0\}$. But u_0^{2q+1} cannot equal 1, hence $z^2 \neq 1$. This is a contradiction. Hence G_U is torsion-free.

3. Remarks

We wish to investigate the manner in which the choice of n and U in the above construction affects the isomorphism class of G_U.

Let $\mathcal{U} = \{(n, U) \mid n \geq 1; \{r_0, s_0, t_0\} \subseteq U \subseteq V_n\}$. Let J denote the set of unordered triples of positive integers. Define a function $d: \mathcal{U} \to J$ by the rule; $d(n, U) = \langle d_r(U), d_s(U), d_t(U) \rangle$, where $d_r(U)$ (resp. $d_s(U), d_t(U)$) denotes the number of r_i's (resp. s_i's, t_i's) in U.

Proposition 2. *Let (n, U) and (n', U') be in $\mathcal{U}$. Then G_U is isomorphic to $G_{U'}$ if and only if $d(n, U) = d(n', U')$.*

Proof. If $d(n, U) = d(n', U')$, one can easily construct an isomorphism from G_U to $G_{U'}$ using an appropriate bijection from U to U' and an appropriate automorphism of $Z_2 \oplus Z_2$. We omit the details.

To prove the other implication, notice that G_U determines a unique integral representation of $Z_2 \oplus Z_2$ in $H_U \cong Z^{p_U}$. The relations 1.3) tell us that this representation is completely reducible to a direct sum of one-dimensional representations. But it is clear that these representations cannot be equivalent if $d(n, U) \neq d(n', U')$, since the eigenvalue patterns will be different.

Corollary. *The number of affine equivalence classes of n-dimensional flat compact Riemannian manifolds with first Betti number zero and holonomy group $Z_2 \oplus Z_2$ is at least equal to the cardinality of the set $\{(n_1, n_2, n_3) \in Z^3 \mid 1 \leq n_1 \leq n_2 \leq n_3, n_1 + n_2 + n_3 = n\}$.*

References

[1] E. Calabi, *Closed, locally euclidean, 4-dimensional manifolds,* Bull. Amer. Math. Soc. **63** (1957) 135.

[2] J. A. Wolf, *Spaces of constant curvature,* McGraw-Hill, New York, 1967.

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