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TOPOLOGY OF ALMOST CONTACT MANIFOLDS

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Introduction

In his Colloquium Lectures on G-structures [2], S. S. Chern asked for the conditions, both local and global, on a C^∞ manifold in order that a linear differential form η exist such that

$$\eta \wedge (d\eta)^p \neq 0$$

for a given value of p. The form η defines a differential system and it is important to study the local and global properties of its integral manifolds. To this end, the notion of a quasi-Sasakian structure on an almost contact metric manifold was introduced by one of the authors [1] and its main properties developed. In the present paper their topological properties are considered and it is shown that both compact Sasakian and cosymplectic manifolds have global properties similar to compact Kaehler manifolds. Examples are the unit hypersphere S^{2n+1} in Euclidean space, and in fact, the circle bundles over any compact Hodge variety. In the latter class, examples are provided by $M \times S^1$ where M is any compact Kaehler manifold. As one might expect, therefore, not only locally, but topologically as well, the compact cosymplectic spaces are the proper odd dimensional analogues of the compact Kaehler manifolds. A complete, but not compact, simply connected cosymplectic manifold is a product with one factor Kaehlerian.

The notation and terminology in this paper will be the same as that employed in [1].

1. Topology of Sasakian manifolds

Define two operators L and Λ, dual to each other, on a quasi-Sasakian manifold by $L = \varepsilon(\Phi)$ and $\Lambda = \iota(\Phi)$ where ε and ι are respectively the exterior and interior product operators. We say that a p-form $\alpha (p \geq 2)$ is *effective* if $\Lambda\alpha = 0$. Since $\iota(\Phi) = * \varepsilon(\Phi) *$ where $*$ is the Hodge star isomorphism, $\Lambda = * L *$.

An orthonormal basis of $\mathscr{E}^{2n+1}$ on an almost contact metric manifold M^{2n+1} of the form $\{\xi, X_i, X_{i^*} = \phi X_i\}$, $i = 1, \cdots, n$, is called a *ϕ-basis*. It is well known that such a basis always exists. For, let $V = \{X \in M_m \mid g(X, \xi) = 0\}$. Equations (1.1) and (1.2) of [1] show that $\phi|_V$ is an almost complex structure

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on V and $g|_V$ is a Hermitian metric. If an orthonormal basis of V of the form $\{X_i, (\phi|_V)X_i\}$, $i = 1, \cdots, n$ is then chosen, we obtain a ϕ-basis of M_m.

In terms of a ϕ-basis $\{\xi, X_i, X_{i^*}\}$ with dual basis $\{\eta, \omega_i, \omega_{i^*}\}$ we have

$$\Phi = \textstyle\sum_i \omega_i \wedge \omega_{i^*}, \quad \Lambda = \sum_i \iota(\omega_{i^*})\iota(\omega_i) .$$

Lemma 1.1. *On a quasi-Sasakian manifold M^{2n+1} the operators L and Λ satisfy*

$$(\Lambda L - L\Lambda)\alpha = (n - p)\alpha$$

for any p-form α.

Proof. By linearity it suffices to consider the decomposable forms $\omega_{i_1} \wedge \cdots \wedge \omega_{i_q} \wedge \omega_{j_1^*} \wedge \cdots \wedge \omega_{j_r^*}$, $q + r = p$ and $\eta \wedge \omega_{i_1} \wedge \cdots \wedge \omega_{i_q} \wedge \omega_{j_1^*} \wedge \cdots \wedge \omega_{j_r^*}$, $q + r = p - 1$. The result then follows by a long computation similar to that in [4] for almost hermitian manifolds.

S. Tachibana [6] proved that if α is a harmonic p-form with $1 \leq p \leq n$ on a compact Sasakian manifold M^{2n+1}, $n \geq 1$, then α is 'orthogonal' to ξ, that is, $\iota(\xi)\alpha = 0$.

Define an operator C on p-forms in an almost contact manifold by

$$C\alpha(X_1, \cdots, X_p) = \alpha(\phi X_1, \cdots, \phi X_p) .$$

Lemma 1.2. *In a compact $(2n + 1)$-dimensional quasi-Sasakian manifold of rank $2n + 1$ or 1 the operator C sends harmonic p-forms into harmonic p-forms for $p \leq n$ in the Sasakian case and for $p = 1, \cdots, 2n$ in the cosymplectic case.*

Proof. The rank $2n + 1$ case is a consequence of the fact that harmonic p-forms are orthogonal to ξ for $1 \leq p \leq n$ (see [6]). The rank 1 case follows from Theorem 5.2 of [1] since then the linear transformation field ϕ is covariant constant. We give the proof for 1-forms only, the corresponding statement for p-forms, $p > 1$, following in an analogous manner. We require the following fact valid for p-forms. Since $\nabla\phi = 0$,

$$\nabla C\alpha = C\nabla\alpha .$$

For,

$$\begin{aligned} \nabla_X C\alpha(Y_1, \cdots, Y_p) &= X\alpha(\phi Y_1, \cdots, \phi Y_p) - \sum_{i=1}^{p} \alpha(\phi Y_1, \cdots, \phi\nabla_X Y_i, \cdots, \phi Y_p) \\ &= X\alpha(\phi Y_1, \cdots, \phi Y_p) - \sum_{i=1}^{p} \alpha(\phi Y_1, \cdots, \nabla_X\phi Y_i, \cdots, \phi Y_p) \\ &= \nabla_X\alpha(\phi Y_1, \cdots, \phi Y_p) = C\nabla_X\alpha(Y_1, \cdots, Y_p) . \end{aligned}$$

Applying the interchange formula to ϕ we see that the operator C and the Ricci curvature operator Q commute, that is,

$$QC = CQ .$$

Thus, since

$$Q\alpha = \langle \nabla\nabla\alpha, \cdot \rangle$$

is a necessary and sufficient condition for a 1-form α on a compact manifold to be harmonic (see [4, Theorem 3.2.3]), we obtain

$$\begin{aligned} \langle \nabla\nabla C\alpha, X \rangle &= \langle C\nabla\nabla\alpha, X \rangle \\ &= \langle \nabla\nabla\alpha, \phi X \rangle \\ &= Q\alpha(\phi X) \\ &= CQ\alpha(X) \\ &= \langle QC\alpha, X \rangle . \end{aligned}$$

It is known that the odd-dimensional betti numbers B_p (p: odd) of a compact Kaehler manifold are even [4]. Here we prove an analogous result for compact Sasakian manifolds not valid for cosymplectic manifolds since the first betti number of $S^1 \times PC_1$ is 1. (Observe that $C\eta$ vanishes on a quasi-Sasakian manifold of any rank, since $C\eta(X) = \eta(\phi X) = 0$. So, since η is harmonic on a cosymplectic manifold, $C\eta$ and η are not independent.)

Theorem 1.3. *The p-th betti number of a compact Sasakian manifold M^{2n+1} is even if p is odd and $p \leq n$. For $p \geq n+1$, B_p is even if p is even.*

Proof. The second statement follows from the first by Poincaré duality. So let α be a harmonic p-form with $p \leq n$; we shall show that α and $C\alpha$ are independent, that is, $C\alpha \neq \lambda\alpha$. First of all we have using equations (1.1) of [1]

$$\begin{aligned} C^2\alpha(X_1, \cdots, X_p) &= \alpha(\phi^2 X_1, \cdots, \phi^2 X_p) \\ &= \alpha(-X_1 + \eta(X_1)\xi, \cdots, -X_p + \eta(X_p)\xi) \\ &= (-1)^p\alpha(X_1, \cdots, X_p) \end{aligned}$$

since $\iota(\xi)\alpha = 0$. Hence if $C\alpha = 0$, α must also vanish. Suppose now that $C\alpha = \lambda\alpha$. Then $C^2\alpha = \lambda C\alpha = \lambda^2\alpha$. But $C^2\alpha = (-1)^p\alpha$, so if p is odd, $\lambda^2\alpha = -\alpha$, that is $\alpha = 0$.

Theorem 1.4. *There are no covariant constant p-forms on a compact Sasakian manifold M^{2n+1} for $1 \leq p \leq 2n$.*

Proof. Let α be a covariant constant p-form with $1 \leq p \leq n$ and let $X, Y_2, \cdots, Y_p \in \mathscr{E}^{2n+1}$. Then since $\iota(\xi)\alpha = 0$ and $\nabla_X\xi = -\frac{1}{2}\phi X$ (see [1, Lemma 4.3]), we have

$$\begin{aligned} 0 &= (\nabla_{\phi X}(\iota(\xi)\alpha))(Y_2, \cdots, Y_p) = (\iota(\nabla_{\phi X}\xi)\alpha)(Y_2, \cdots, Y_p) \\ &= -\frac{1}{2}\alpha(\phi^2 X, Y_2, \cdots, Y_p) = \frac{1}{2}\alpha(X, Y_2, \cdots, Y_p) . \end{aligned}$$

Thus $\alpha = 0$. The same is true for forms of degree p, $n+1 \leq p \leq 2n$, since $*\alpha$ is covariant constant whenever α is, and $*$ is an isomorphism.

Applying [4, Theorem 3.2.2] we obtain

Corollary 1.5. *If the Ricci curvature of a compact Sasakian manifold is positive semi-definite,* $B_1 = 0$.

A contact manifold M is *homogeneous* if there is a connected Lie group which acts transitively and effectively on M as a group of diffeomorphisms and leaves the contact form invariant. A *contact symmetric space* is a homogeneous contact manifold which is Riemannian symmetric with respect to the contact metric structure.

The Ricci curvature of a compact homogeneous Sasakian manifold M may not be positive semi-definite. For, let $A(S^{2n+1})$ be the automorphism group of S^{2n+1} with the almost contact metric structure $\Sigma = (\phi, \xi, \eta, g)$. $A(S^{2n+1})$ is transitive. If $f \in A(S^{2n+1})$

$$f*(ag + (a^2 - a)\eta \otimes \eta) = ag + (a^2 - a)\eta \otimes \eta$$

where a is a constant. Hence, $A(S^{2n+1})$ is also the automorphism group of the Sasakian structure $\bar{\Sigma} = (\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ with

$$\bar{\phi} = \phi, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\eta} = a\eta, \quad \bar{g} = ag + (a^2 - a)\eta \otimes \eta .$$

A ϕ-basis for Σ can be modified to a ϕ-basis for $\bar{\Sigma}$, so from the sectional curvatures $K_{\alpha\beta}$ and $\bar{K}_{\alpha\beta}$ of Σ and $\bar{\Sigma}$,

$$\bar{K}_{ii^*} = \frac{1}{a}[K_{ii^*} + 3(1 - a)] = \frac{4 - 3a}{a} .$$

If we put $a = 2$, then $\bar{K}_{ii^*} = -1$.

Theorem 1.6. *The fundamental group $\pi_1(M)$ of a compact symmetric Sasakian manifold M is finite.*

Proof. Since a harmonic form on a compact symmetric space has vanishing covariant derivative, $B_1 = 0$ by Theorem 1.4. Let $M = G/K$ and assume that K is connected. Consider the exact homotopy sequence

$$0 \to \pi_1(K) \to \pi_1(G) \to \pi_1(M) \to 0 .$$

Since $\pi_1(G)$ is abelian, so is $\pi_1(M)$. Hence $H_1(M, Z) \approx \pi_1(M)/[\pi_1(M), \pi_1(M)] \approx \pi_1(M)$. Thus, since $B_1 = 0$, $H_1(M, Z)$ is a finite group since it is a finitely generated torsion group, so $\pi_1(M)$ is finite also.

If K is not connected, let K_0 be the connected component of the identity in K and consider the exact sequence

$$0 \to \pi_1(G/K_0) \to \pi_1(G/K) \to K/K_0 \to 0 .$$

Since K is compact, K/K_0 is finite. Hence, since $\pi_1(G/K)$ is an extension of $\pi_1(G/K_0)$ by K/K_0, it is finite.

Added in proof. If $M = G/K$ is simply connected, one of the authors has recently shown that M is (globally) isometric to a Euclidean sphere (see S. I. Goldberg, *On the topology of compact contact manifolds*, to appear in Tôhoku Math. J., August, 1968).

2. Topology of cosymplectic manifolds

In terms of a ϕ-basis $\{\xi, X_i, X_{i*}\}$ with dual basis $\{\eta, \omega_i, \omega_{i*}\}$ we define six new operators as follows

$$d' = \textstyle\sum_i \varepsilon(\omega_i)\nabla_{X_i}, \quad d'' = \sum_i \varepsilon(\omega_{i*})\nabla_{X_{i*}}, \quad d^\circ = \varepsilon(\eta)\nabla_\xi$$
$$\delta' = -\textstyle\sum_i \iota(\omega_i)\nabla_{X_{i*}}, \quad \delta'' = -\sum_i \iota(\omega_{i*})\nabla_{X_i}, \quad \delta^\circ = -\iota(\eta)\nabla_\xi .$$

Then, $d = d' + d'' + d^\circ$ and $\delta = \delta' + \delta'' + \delta^\circ$.

Lemma 2.1. *On a cosymplectic manifold*

$$\delta L - L\delta = d' - d'' .$$

The proof is a computation similar to the corresponding one for Kaehler manifolds (see [4]); it is important to note the role played by $\nabla_X \Phi = 0$ for every $X \in \mathscr{E}^{2n+1}$ in this computation. Thus, lemmas of this sort do not hold on non-cosymplectic quasi-Sasakian manifolds. We also make use of the fact that

$$\delta^\circ L - L\delta^\circ = 0 .$$

It should also be kept in mind that $\Phi \neq d\eta$ on a cosymplectic manifold but that $\Phi^n \neq 0$ and $d\eta = 0$.

The following lemmas are analogues of those for Kaehler manifolds [4].

Lemma 2.2. *On a cosymplectic manifold*

$$\begin{aligned} d'd' &= 0 , & d'd'' + d''d' &= 0 , \\ d''d'' &= 0 , & d^\circ d' + d'd^\circ &= 0 , \\ d^\circ d^\circ &= 0 , & d^\circ d'' + d''d^\circ &= 0 . \end{aligned}$$

Lemma 2.3. *L commutes with the Laplace-Beltrami operator Δ.*

Thus we see that L maps $\wedge_H^p$, the space of harmonic p-forms into $\wedge_H^{p+2}$, the space of harmonic $(p+2)$-forms.

Theorem 2.4. *The betti numbers of a compact cosymplectic manifold are non-zero.*

Proof. We deduce $B_{2p} \neq 0$ by showing the existence of a non-zero harmonic $2p$-form, $1 \leq p \leq n$; since the manifold is odd-dimensional, $B_{2p+1} \neq 0$ follows by Poincaré duality. Since $\nabla_Y \Phi = 0$ for every $Y \in \mathscr{E}^{2n+1}$, we see that $\Delta\Phi = 0$, that is Φ is harmonic. Now suppose Φ^{p-1} is harmonic, then

$$\Delta(\Phi^p) = \Delta(L\Phi^{p-1}) = L(\Delta\Phi^{p-1}) = 0$$

by Lemma 2.3. Thus Φ^p is harmonic for every p, $1 \leq p \leq n$, and since $\Phi^n \neq 0$, $\Phi^p \neq 0$ completing the proof.

In this regard let us discuss briefly the case of a compact quasi-Sasakian manifold M^{2n+1} of rank $r = 2p + 1$ under the hypothesis that $\nabla_X \theta = 0$ for every $X \in \mathscr{E}^{2n+1}$. An example of such a space can be given by taking the direct product of a compact Sasakian manifold M^{2p+1} and a compact Kaehler manifold M^{2q} (see [1, Theorem 3.2]). Then, $\nabla_X \Theta = 0$ for every $X \in \mathscr{E}^{2n+1}$ and $\Theta(X, Y) = g(X, \theta Y)$. Hence Θ is harmonic and $\Phi = \Theta + d\eta$. Now Θ restricted to the Kaehler manifold M^{2q}, $q = n - p$, is the fundamental 2-form of the Kaehler structure [1], and $\Theta(X, Y) = 0$ if either X or Y is in $\mathscr{E}^{2p+1}$. Hence, $\Theta^i \neq 0$ for $1 \leq i \leq q$ giving us the following theorem which generalizes Theorem 2.4.

Theorem 2.5. *In the locally decomposable case the first q even-dimensional betti numbers of a compact quasi-Sasakian manifold M^{2n+1} of rank $2p + 1$, $p + q = n$, are different from zero.*

Several lemmas leading to a monotonicity condition on the betti numbers of a cosymplectic manifold are now given. These are valid for any quasi-Sasakian manifold no matter what its rank.

Lemma 2.6. *The operators L and Λ satisfy*

$$(\Lambda L^k - L^k \Lambda)\alpha = k(n - p - k + 1)L^{k-1}\alpha$$

for any p-form α.

Proof. By induction using Lemma 1.1.

Lemma 2.7. *Every p-form α with $p \leq n + 1$ may be written uniquely as a sum*

$$\alpha = \sum_{k=0}^{r} L^k \beta_{p-2k}$$

where the β_{p-2k}'s are effective forms of degree $p - 2k$ and $r = \left[\frac{p}{2}\right]$.

Proof. The proof is analogous to that of the corresponding result for Hermitian manifolds (see [4, Theorem 5.7.1]), and so is omitted.

Lemma 2.8. *ΛL is an automorphism of the space of p-forms $\wedge^p$ for $p \leq n - 1$. Furthermore, L is an isomorphism of $\wedge^p$ into $\wedge^{p+2}$ for $p \leq n - 1$.*

Proof. Analogous to the proofs of the corresponding results for Hermitian manifolds [4, Corollaries 5.7.1 and 5.7.2].

A theorem due to S. S. Chern [3] says that if M is a compact Riemannian manifold with G the structural group of its tangent bundle, $W_1, \cdots, W_l$ the irreducible invariant subspaces of $\wedge_H^p$ under the action of G, and P_{W_i} the projection map of $\wedge_H^p$ into W_i, then if a p-form α is harmonic so is $P_{W_i}\alpha$. Now, let $\tilde{\wedge}_H^p$ denote the subspace of $\wedge_H^p$ of effective harmonic p-forms. In our case, since Φ is invariant under the action of $G \subset U(n) \times 1$, each $L^k \tilde{\wedge}_H^{p-2k}$ is an invariant subspace of $\wedge_H^p$. Thus, each $L^k \tilde{\wedge}_H^{p-2k}$ is a sum of W_i's and hence the projection of a harmonic form into $L^k \tilde{\wedge}_H^{p-2k}$ is again harmonic.

We therefore have the following statement.

Proposition 2.9. *Every harmonic p-form α on a cosymplectic manifold M^{2n+1} with $p \leq n + 1$ may be written uniquely as a sum*

$$\alpha = \sum_{k=0}^{r} L^k \beta_{p-2k}$$

where the β_{p-2k}'s are effective harmonic forms of degree $p - 2k$ and $r = \left[\frac{p}{2}\right]$.

(Since a harmonic p-form $p \leq n$, on a compact Sasakian manifold is easily seen to be effective, Proposition 2.9 is trivially true in that case. For cosymplectic manifolds, the effective harmonic forms are not devoid of geometric content.)

From Lemmas 2.3 and 2.8, it is seen that L is an isomorphism of $\wedge_H^p$ into $\wedge_H^{p+2}$. Thus,

Theorem 2.10. *The betti numbers B_p of a compact cosymplectic manifold M^{2n+1} satisfy*

$$B_p \leq B_{p+2}$$

for $1 \leq p \leq n - 1$.

The difference $B_p - B_{p-2}$ may be measured in terms of the number of effective harmonic forms of degree p, $p \leq n + 1$.

Theorem 2.11. *On a compact cosymplectic manifold, the dimension of the space of effective harmonic p-forms is $B_p - B_{p-2}$, $p \leq n + 1$.*

Proof. Analogous to the proof of the corresponding result for Kaehler manifolds [4, Theorem 5.7.2].

Observe that on a quasi-Sasakian manifold of any rank $\iota(\xi)\Phi^n = 0$, so $\iota(\xi)(\eta \wedge \Phi^n) = \Phi^n$, from which

$$\eta \wedge \Phi^n \neq 0$$

—a statement resembling the definition of a contact manifold. (A $(2n + 1)$-dimensional manifold admitting a global 1-form η and 2-form Φ such that $\eta \wedge \Phi^n \neq 0$ is called almost contact by S. Takizawa, and cosymplectic if, moreover, Φ is closed [7]. They have been studied by means of sheaf theory but, in view of their generality, no examples were provided other than the contact manifolds where $\Phi = d\eta$.)

A relation between η and Φ is suggested. Since $\iota(\eta)\varepsilon(\eta)\Phi^n = \Phi^n$,

$$* \Phi^n = * \iota(\eta) ** \varepsilon(\eta) ** \Phi^n = \varepsilon(\eta)\iota(\eta) * \Phi^n = f\eta$$

where $f = \iota(\eta) * \Phi^n$. We show that $|f| = |\Phi^n|$. To this end, observe that $c^2 * 1 = \Phi^n \wedge * \Phi^n = f\eta \wedge \Phi^n$ where $c^2 = \langle \Phi^n, \Phi^n \rangle$. Thus, $c^2 = f * \varepsilon(\eta) ** \Phi^n = f\iota(\eta) * \Phi^n = f^2$.

Proposition 2.12. *On a quasi-Sasakian manifold the forms η and Φ are related by*

$$\eta = \pm \frac{1}{|\Phi^n|} * \Phi^n .$$

Corollary 2.13. *On a quasi-Sasakian manifold*

$$\eta \wedge \Phi^n = \pm |\Phi^n| * 1 .$$

By Lemma 4.3 and Theorem 5.2 of [1] it is seen that on a cosymplectic manifold the 1-forms η and $*\Phi^n$ have vanishing covariant derivatives. Thus, $|\Phi^n|$ is a constant.

Corollary 2.14. *On a cosymplectic manifold the* 1*-form* $*\Phi^n$ *is a constant multiple of* η.

Note that on a cosymplectic manifold harmonic forms are not, in general, orthogonal to the 'vertical' vector field ξ. For $\iota(\xi)\eta = 1$. In addition, $\iota(\xi) * \Phi \neq 0$; in fact, $\eta \wedge \Phi^n \neq 0$ implies $\eta \wedge \Phi \neq 0$.

Observe that from Theorem 3.1 of [1] that a cosymplectic manifold is locally the product of a Kaehler manifold with a circle or a line.

For complete simply connected cosymplectic manifolds M the only examples are given by $M = M' \times M''$ where M'' is Kaehlerian. Indeed, since $\nabla\Phi = 0$, $M'_m = \{X \in M_m \mid \Phi(X, M_m) = 0\}$ defines a parallel distribution. Therefore, the orthogonal complement M''_m (with respect to the Riemannian metric) also gives a parallel distribution. Applying the de Rham decomposition theorem [5] we obtain $M = M' \times M''$ where $\Phi = 0$ on M' and Φ has maximal rank on M''. Thus, since Φ is closed, M'' is symplectic. In fact, since $\nabla\Phi$ vanishes, M'' is a Kaehler manifold. However these manifolds are not compact by Theorem 2.4.

To construct the normal almost contact structure on the cosymplectic manifold $M \times S^1$ given in the Introduction take any point (m, t) of $M \times S^1$ and set $\phi(X, Y) = (JX, 0)$, $X \in M_m$, $Y \in S^1_t$, $\xi = (0, d/dt)$ and $\eta = (0, dt)$ where J is the complex structure of M.

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